Integer-Forcing Source Coding

Or Ordentlich (MIT)
Joint work with Uri Erez (TAU)

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Wireless and Number Theory Workshop
York, England

The Communication Problem

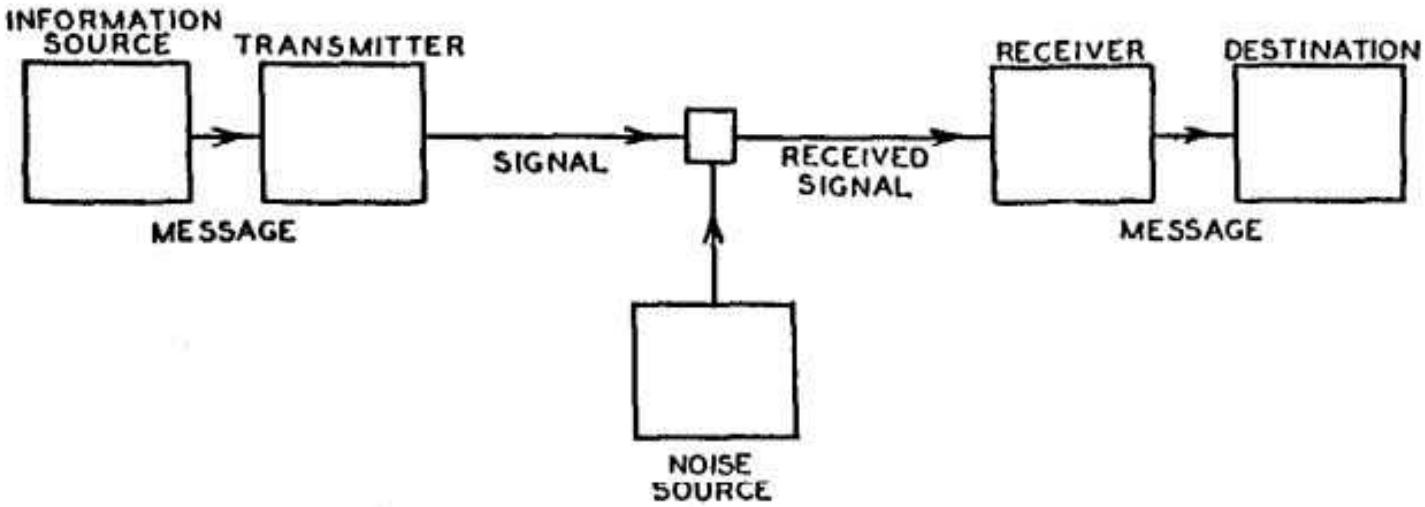
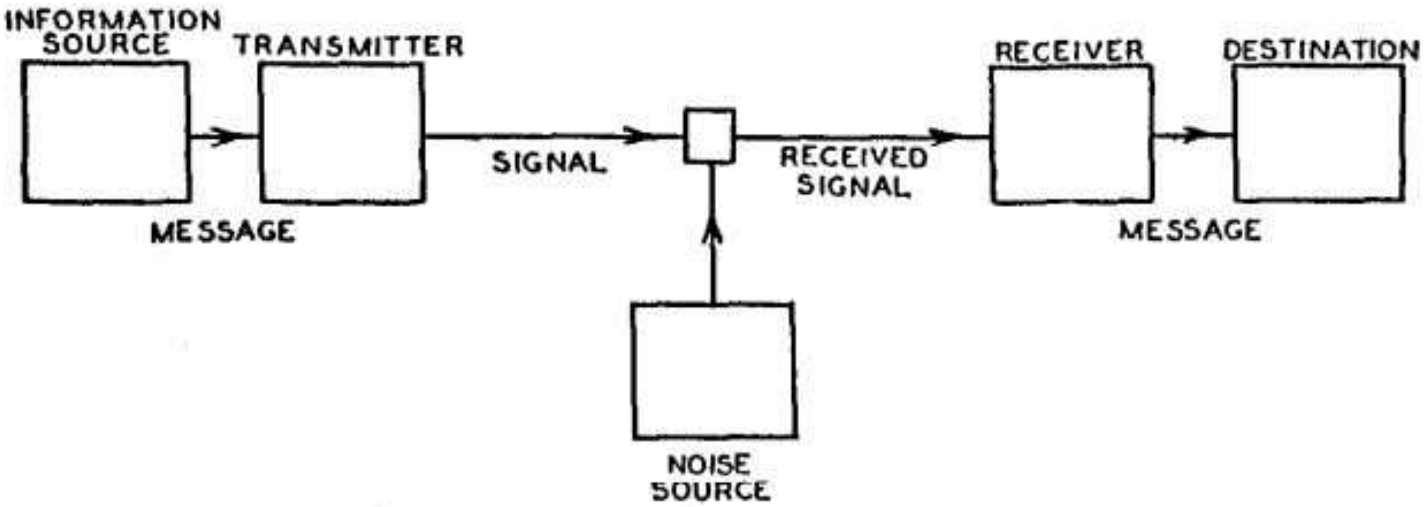


Fig. 1—Schematic diagram of a general communication system.

The Communication Problem



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Channel coding: sending i.i.d. Bernoulli(1/2) bits over a noisy channel

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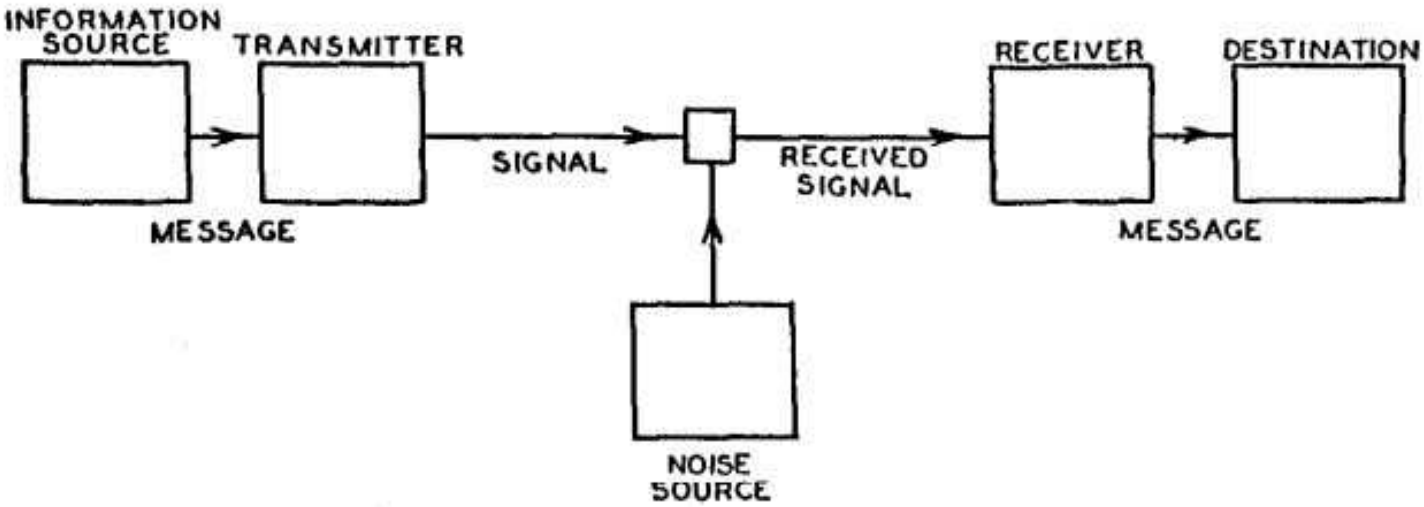

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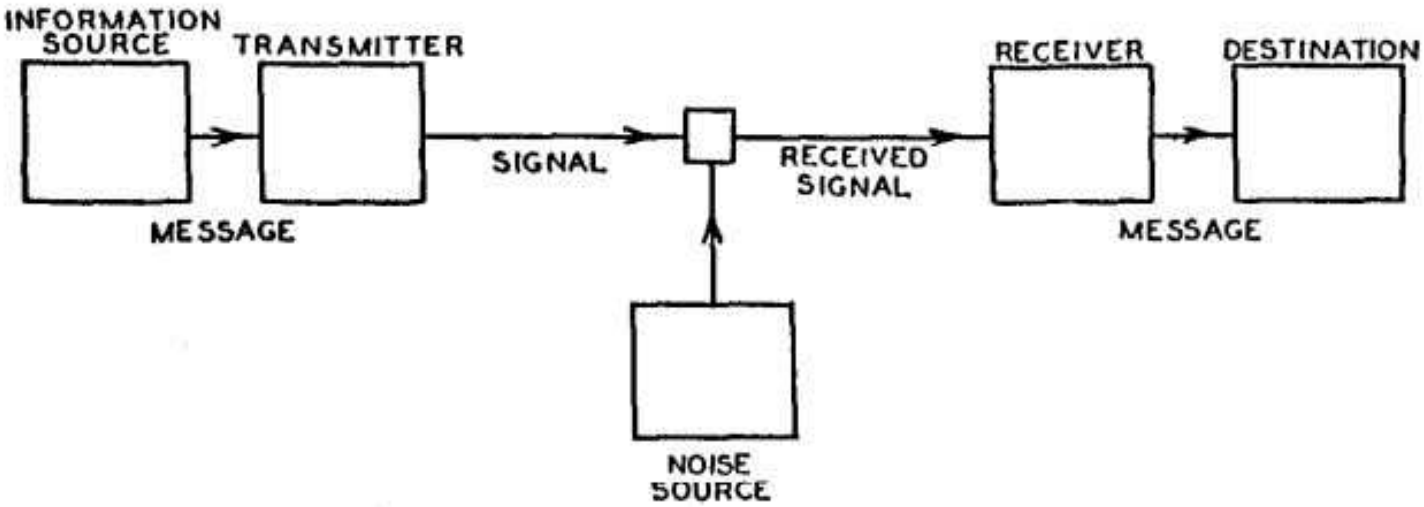
Shannon

For i.i.d. information source and discrete memoryless channels there is no loss (asymptotically) in solving the source and channel coding separately

The Communication Problem



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Most talks in the workshop deal with channel coding

This talk is about source coding

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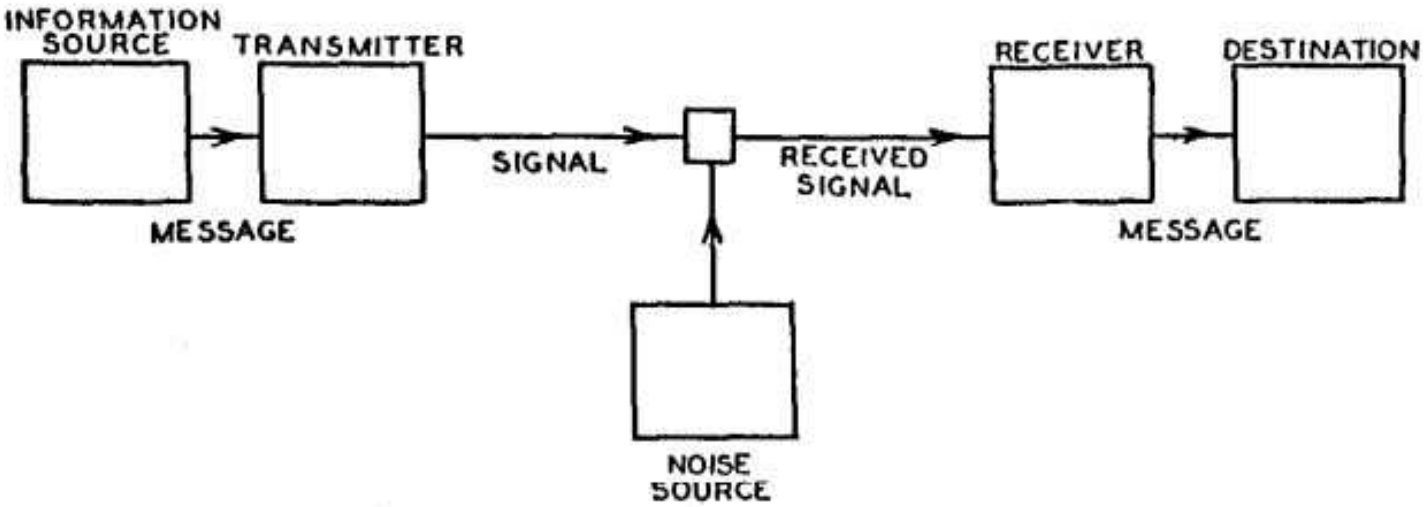

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watch out!

Channel coding: large rate is good

Source coding: small rate is good

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audio, video, pictures, EM waves

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Analog-to-Digital Conversion

- Continuous-to-discrete (sampling)
  Can represent any band-limited signal by a discrete sequence (Nyquist's Theorem)
- Quantization of samples to a finite-alphabet (w.l.o.g. bits)
  Storage is finite, so average number of bits/sample is limited

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The more bits we allocate for storing the signal, the smaller the loss

Rate-Distortion Theory

- We can restrict attention to discrete signals
- Let $x[k]$, $k = 1, \ldots, n$, be the discrete-time sampled signal

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This is done using two functions:

- Encoder: $\mathcal{E} : \mathbb{R}^n \mapsto \{1, \ldots, 2^{nR}\}$
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Goal: design $\mathcal{E}$ and $\mathcal{D}$ to minimize D

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Common assumption: $\mathbf{x}$ is a random vector with i.i.d. entries

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Rate-distortion theorem (Shannon)

The minimum number of bits/sample for attaining avg. distortion D is

$$R(D) = \frac{1}{2}\log\left(\frac{\sigma^2}{D}\right)$$

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$$Q(x) = \begin{cases} (\Delta - 1)/2 & \text{if } \Delta/2 < x \\ \text{round}(x) & \text{if } -\Delta/2 \leq x \leq \Delta/2 \\ -(\Delta - 1)/2 & \text{if } x < -\Delta/2 \end{cases}$$

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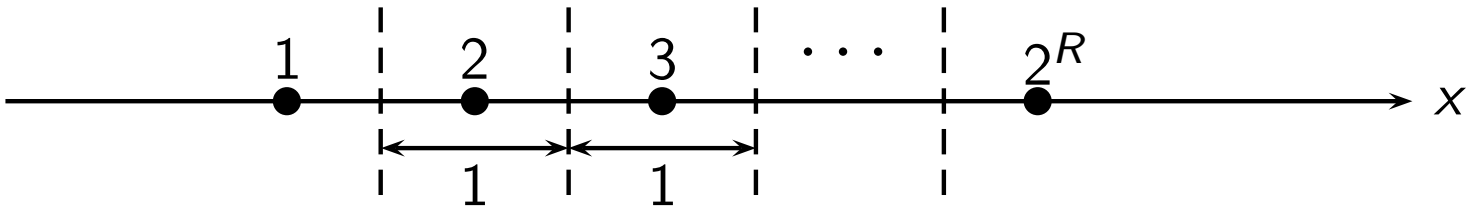

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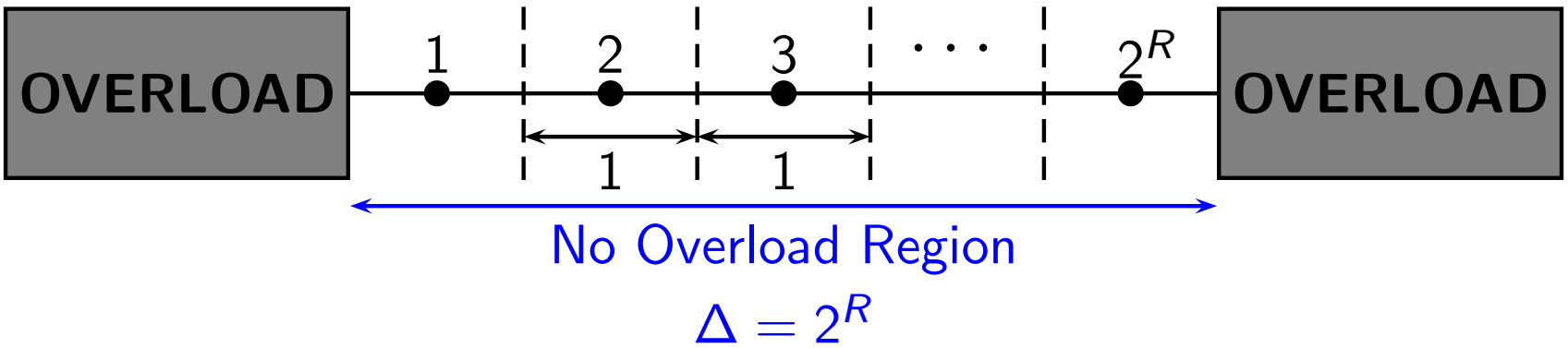

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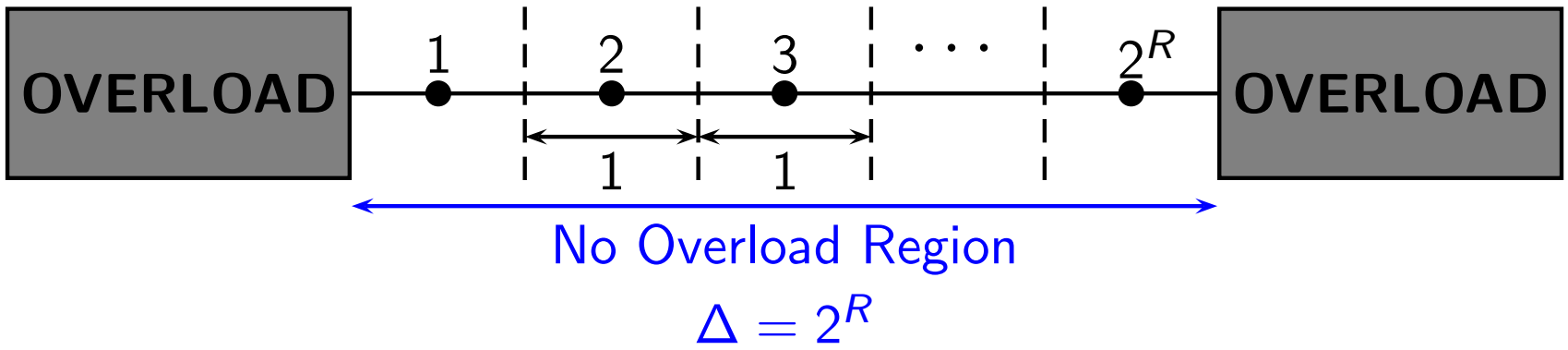

Dynamic range$= [-\Delta/2, \Delta/2]$
Overload probability$= \Pr(|X| > \Delta/2) = 2\mathrm{erfc}(\Delta/(2\sigma))$

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Assuming no overload, we have

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It is easy to see that $\mathrm{round}(x + u) - (x + u)$ is $\mathrm{Uniform}\left(-\frac{1}{2}, \frac{1}{2}\right)$ and $\perp\!\!\!\perp x$

Scalar Uniform Quantization 2

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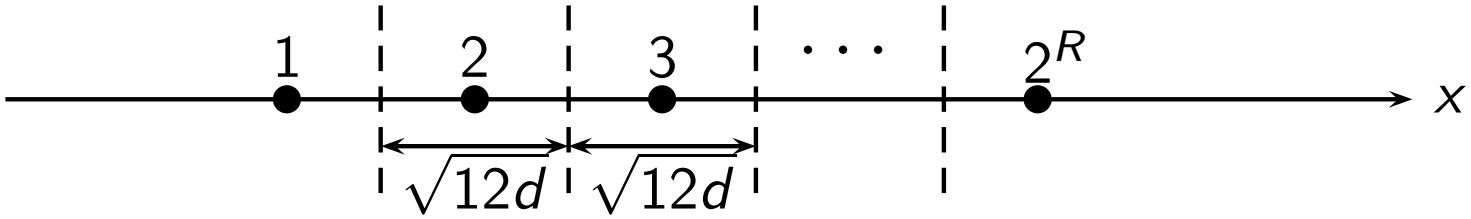

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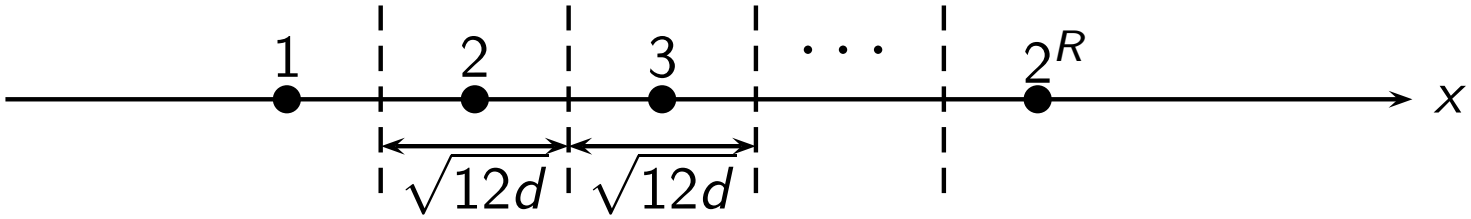
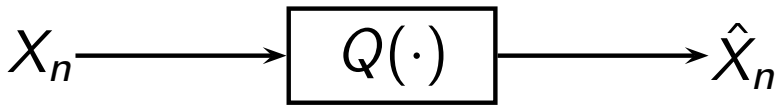


Under no overload + dithered quantization



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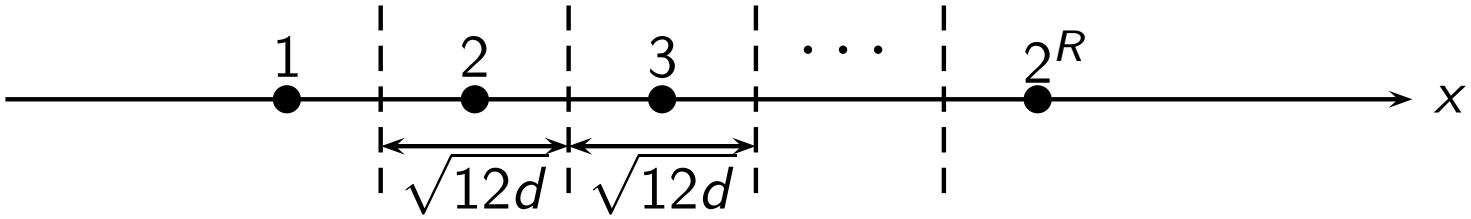

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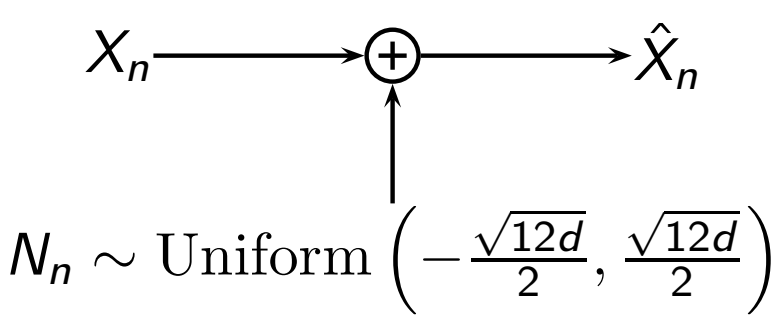

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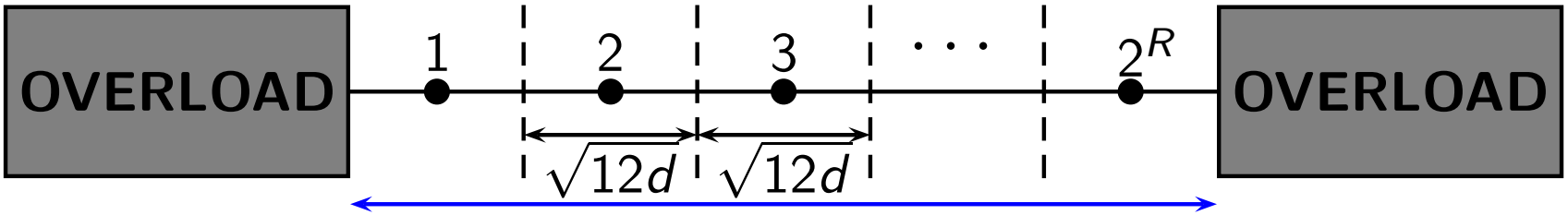

$$\Delta = 2^R \sqrt{12d}$$

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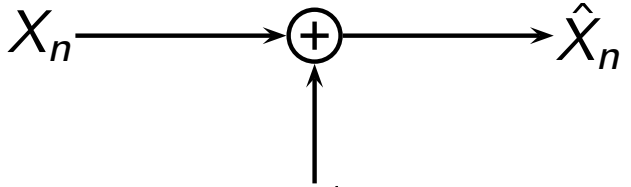

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$$\Pr(\text{OL}) = 2\text{erfc}\left(\frac{\Delta}{2\sigma}\right) \leq e^{-\frac{\Delta^2}{8\sigma^2}} = e^{-\frac{3}{2}\cdot 2^{2\left(R-\frac{1}{2}\log\left(\frac{\sigma^2}{d}\right)\right)}}$$

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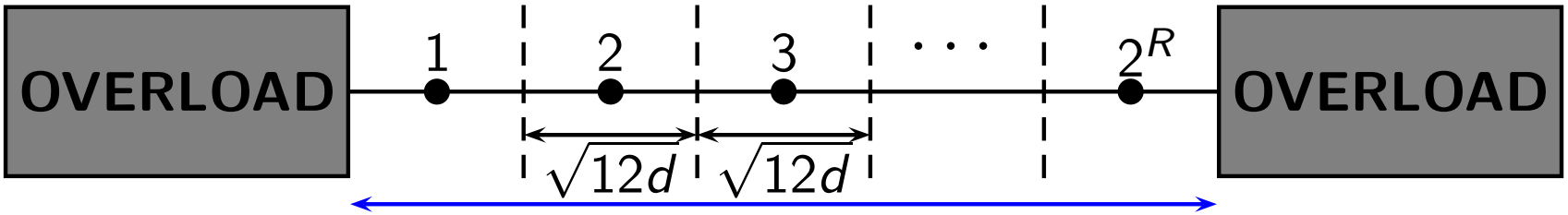

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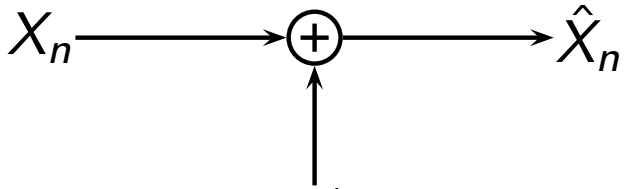

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$$N_n \sim \text{Uniform}\left(-\frac{\sqrt{12d}}{2}, \frac{\sqrt{12d}}{2}\right)$$

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Scalar Uniform Quantization 2

Conclusion:

For $x \sim \mathcal{N}(0, \sigma^2)$, a scalar uniform quantizer with rate

$$R = \delta + \underbrace{\frac{1}{2} \log \left(\frac{\sigma^2}{d} \right)}_{\text{Shannon}}$$

achieves distortion $\approx d$ whenever overload does not occur, and

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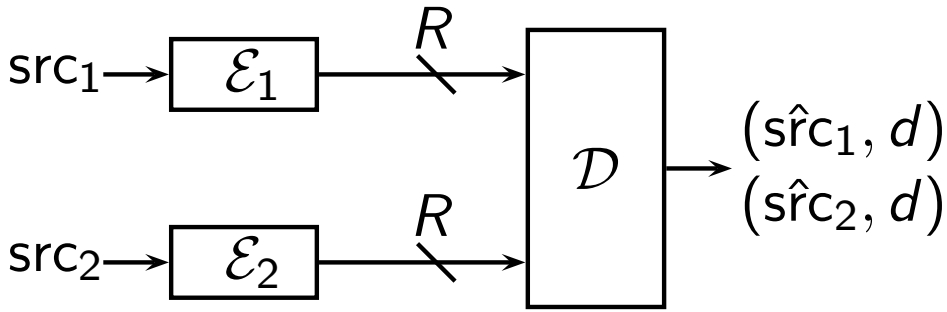
Main Goal:

Can we get close-to-optimal performance using scalar quantizers also when the source is a Gaussian *vector* $\mathbf{x} \sim \mathcal{N}(\mathbf{0}, \mathbf{K}_{xx})$?

Quantization of Gaussian Vectors

$$\begin{bmatrix} \mathrm{src}_1 \\ \mathrm{src}_2 \end{bmatrix} \sim \mathcal{N}\left(\mathbf{0}, \begin{bmatrix} \sigma^2 & \rho\sigma^2 \\ \rho\sigma^2 & \sigma^2 \end{bmatrix}\right)$$



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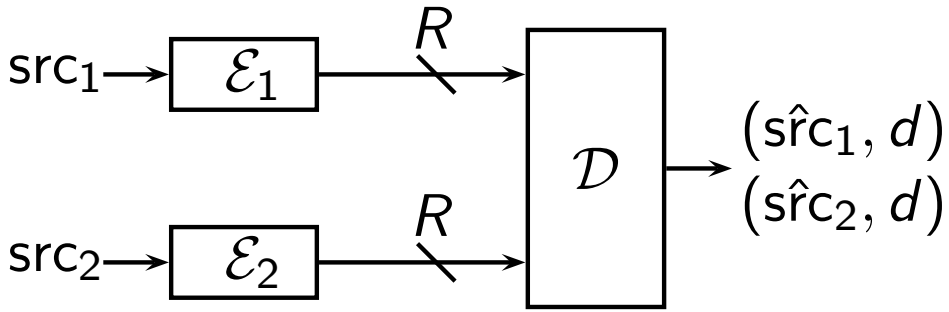

Naive Approach (typically used in practice)

Ignore correlation and use scalar quantizers with $\Delta \propto \sigma$ so that with high probability both $\text{src}_1, \text{src}_2 \in [-\frac{\Delta}{2}, \frac{\Delta}{2}]$

$$\Rightarrow R \propto \frac{1}{2} \log\left(\frac{\sigma^2}{d}\right)$$

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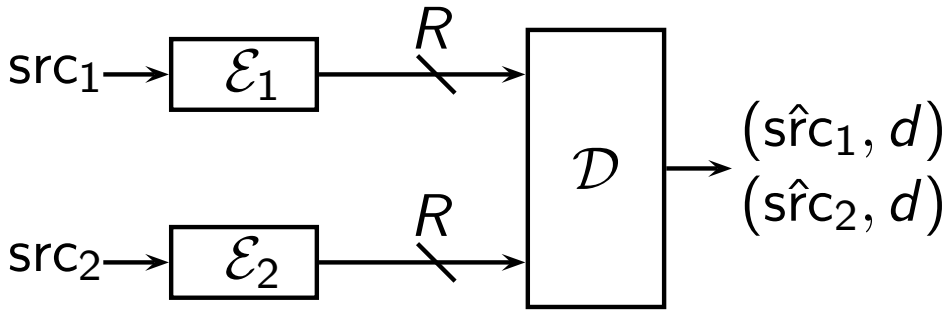

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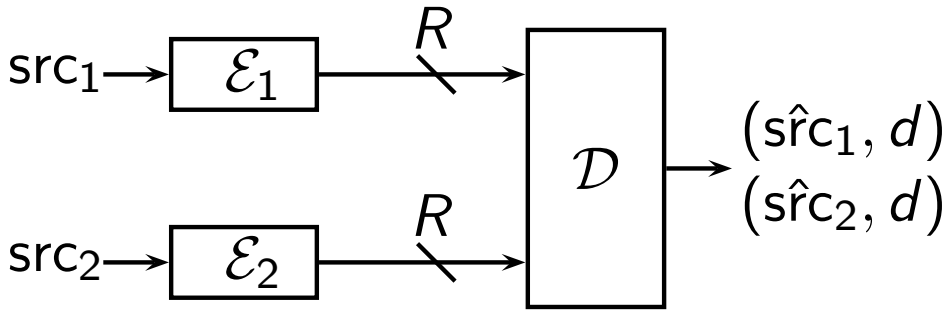

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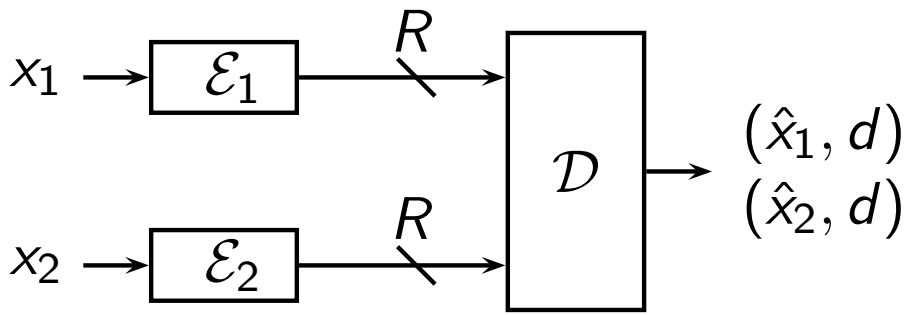
Fundamental IT Limits (Berger-Tung, Wagner et al)

With unlimited delay and computational complexity, it is possible to achieve* $R \approx \frac{1}{2} \cdot \frac{1}{2} \log\det\left(\mathbf{I} + \frac{1}{d} K_{xx}\right)$

Goal 1 - Universal Quantization

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \sim \mathcal{N}(\mathbf{0}, \mathbf{K}_{xx})$$



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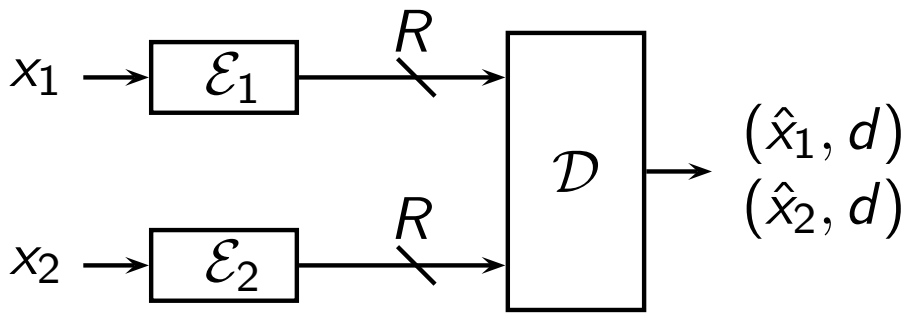

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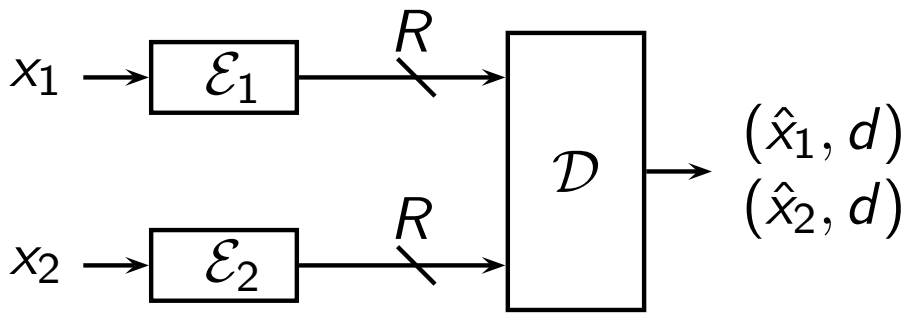

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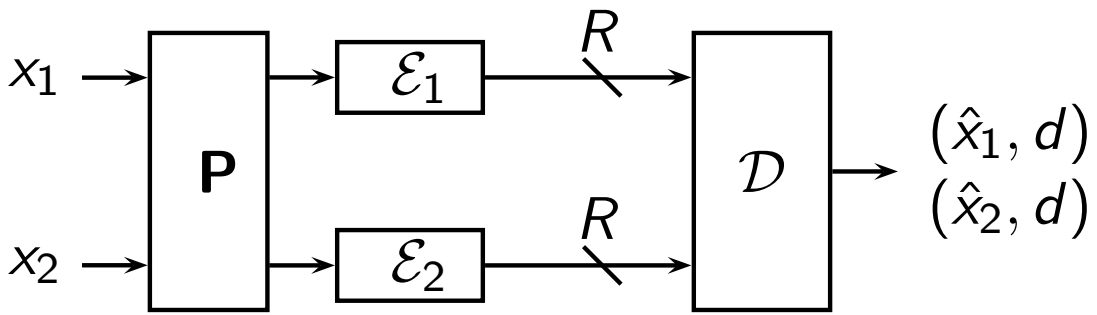

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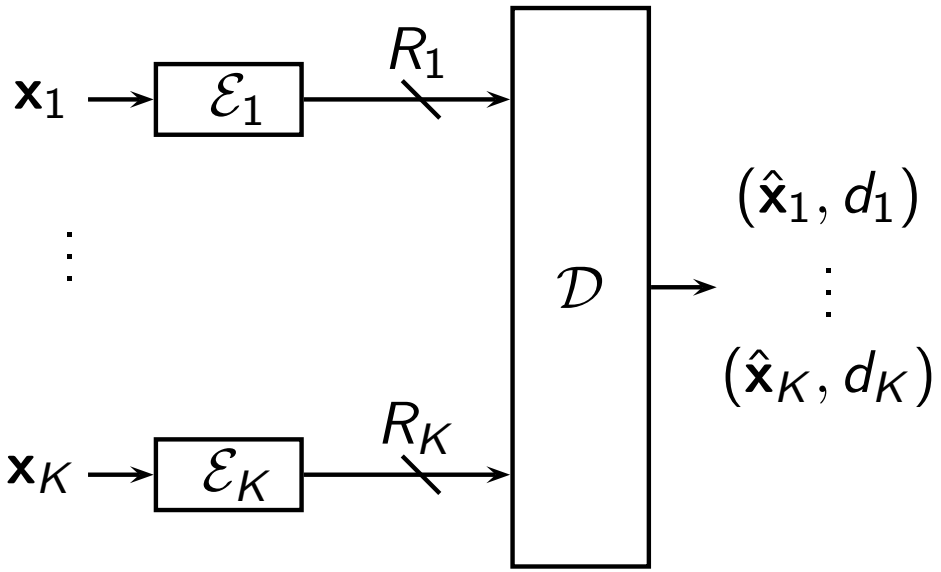
Willing to apply a **universal** linear transformation before quantization

Goal 2 - Distributed Lossy Compression



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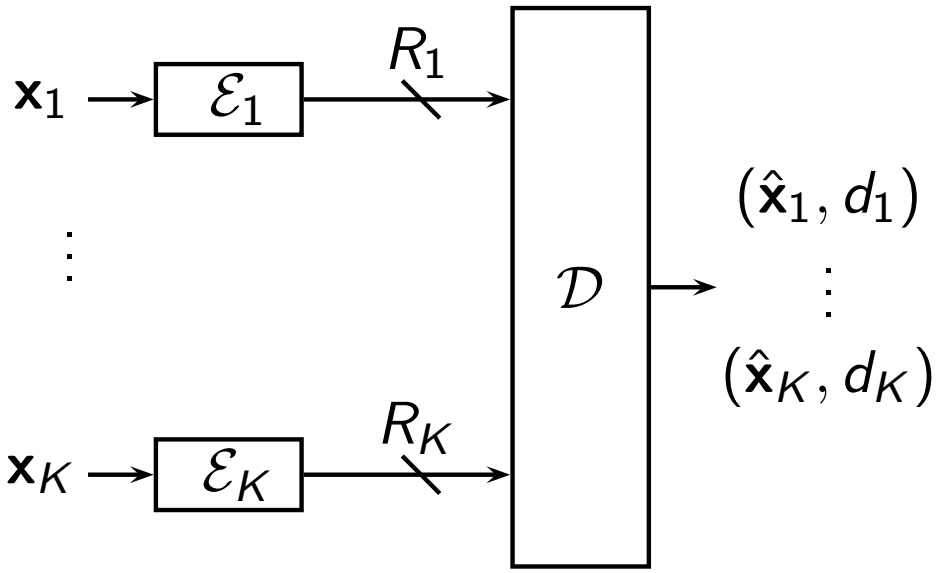

- Fundamental limits understood in some cases
- Inner and outer bounds known

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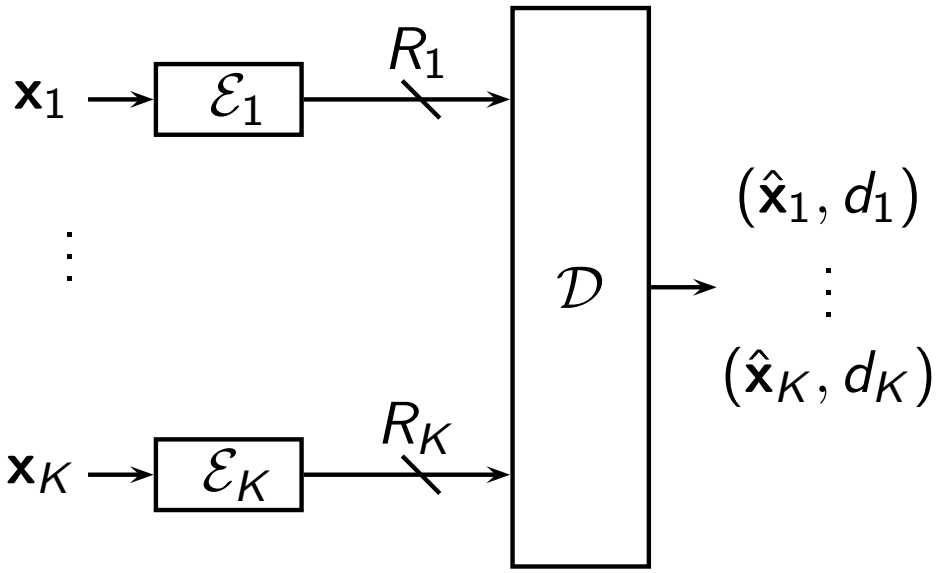

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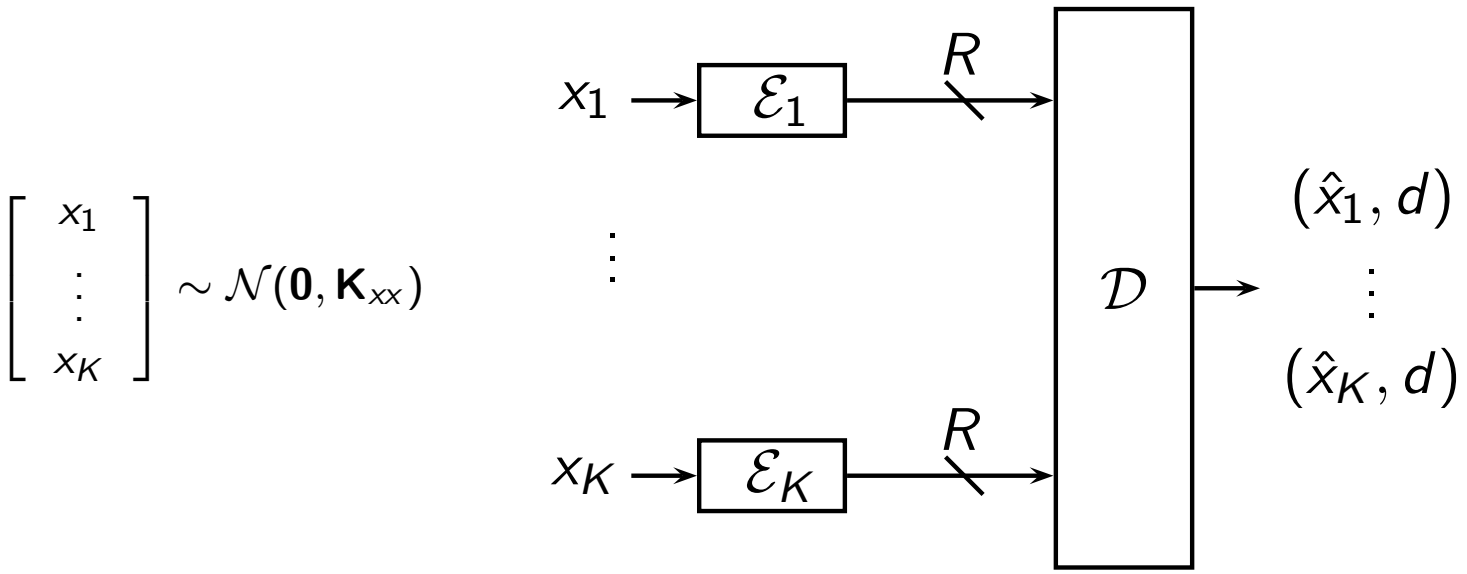
Some applications require

- Extremely simple encoders/decoder
- $n = 1$

Goal 2 -Distributed Lossy Compression



We restrict attention to:

- Gaussian sources $\mathbf{x} \sim \mathcal{N}(\mathbf{0}, \mathbf{K}_{xx})$
- One-shot compression - block length is 1
- Symmetric rates $R_1 = \cdots = R_K = R$
- Symmetric distortions $d_1 = \cdots = d_K = d$
- MSE distortion measure: $E(x_k - \hat{x}_k)^2 \leq d$

Goal and Means

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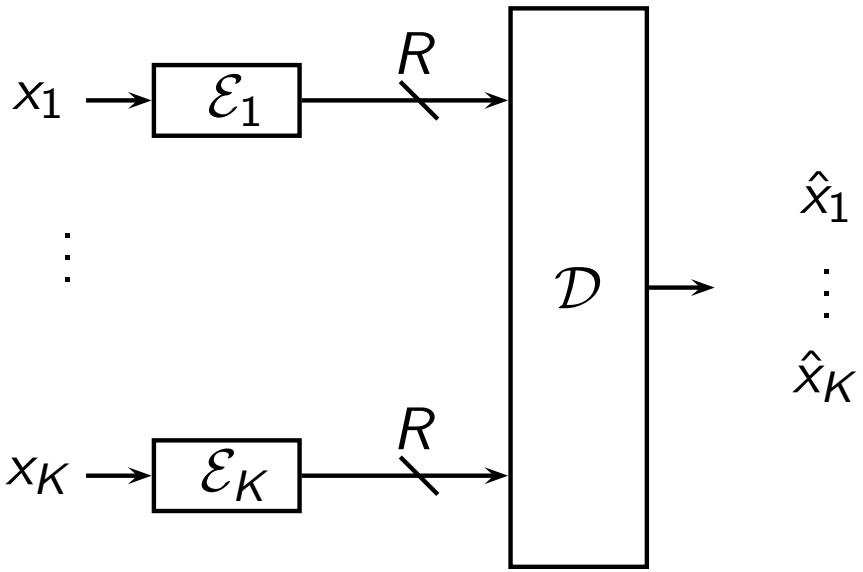
Scalar Modulo

- A simple 1-D binning operation
- Allows for efficient decoding using integer-forcing

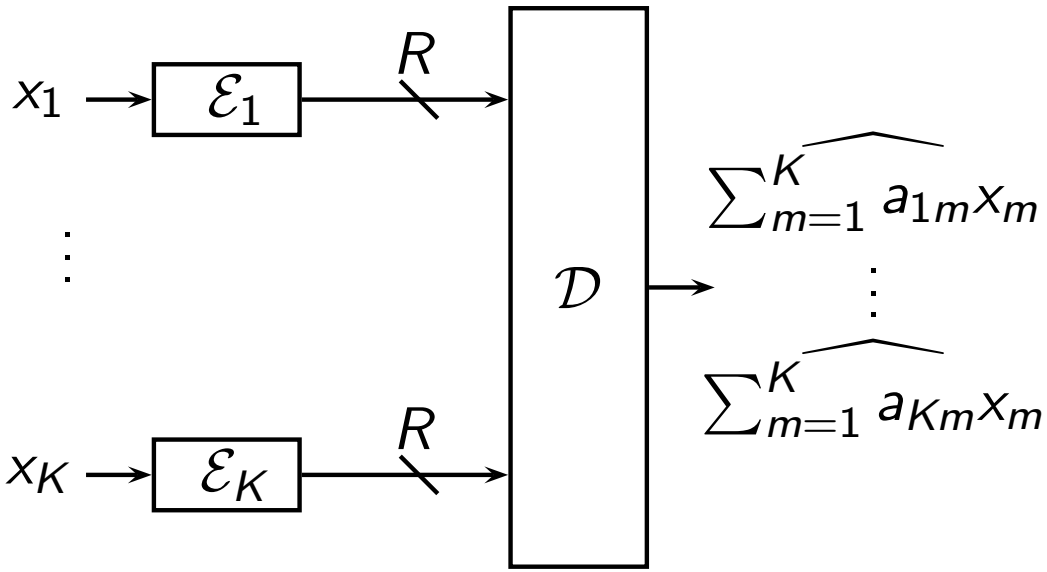
Integer-Forcing Source Coding: Overview

Basic Idea: Rather than solving the problem



Integer-Forcing Source Coding: Overview

First solve



and then invert equations to get $\hat{x}_1, \ldots, \hat{x}_K$

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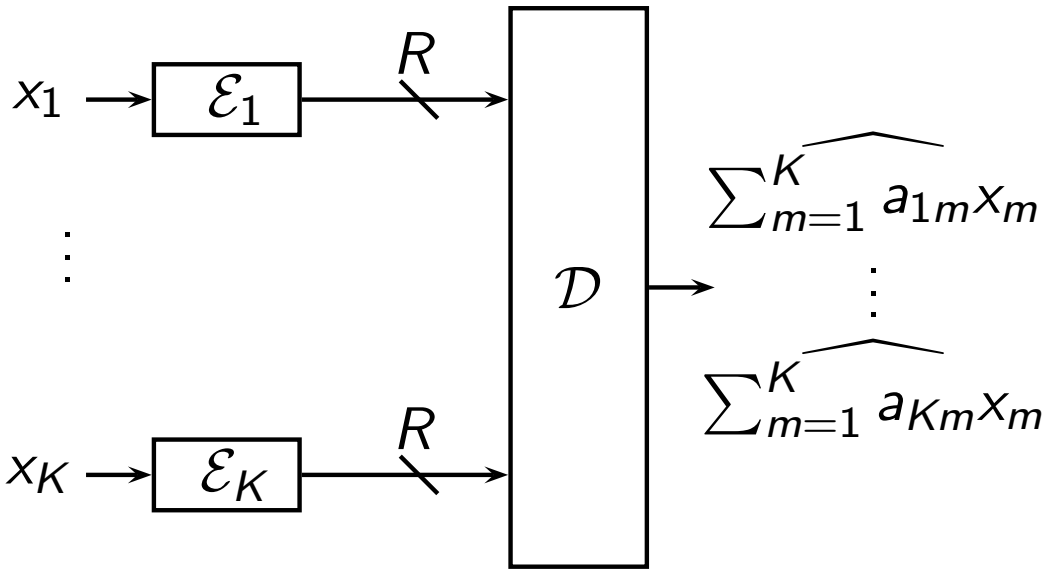
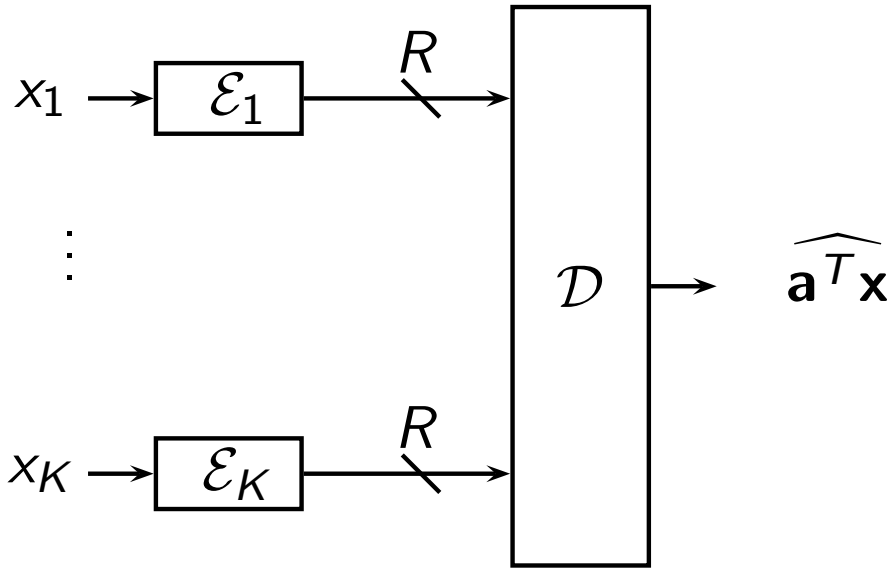


and then invert equations to get $\hat{x}_1, \ldots, \hat{x}_K$

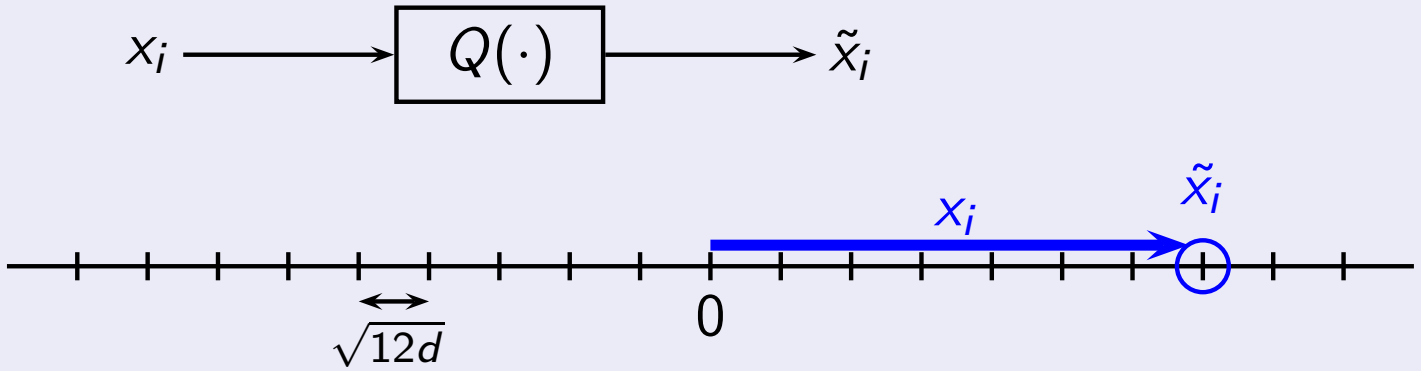
- Problem reduces to simultaneous distributed compression of K linear combinations
- Can be efficiently solved with small rates for certain choices of coefficients
- Equation coefficients can be chosen to optimize performance

Distributed Compression of Integer Linear Combination



Distributed Compression of Integer Linear Combination

Scalar Quantization



- High resolution/dithered quantization:

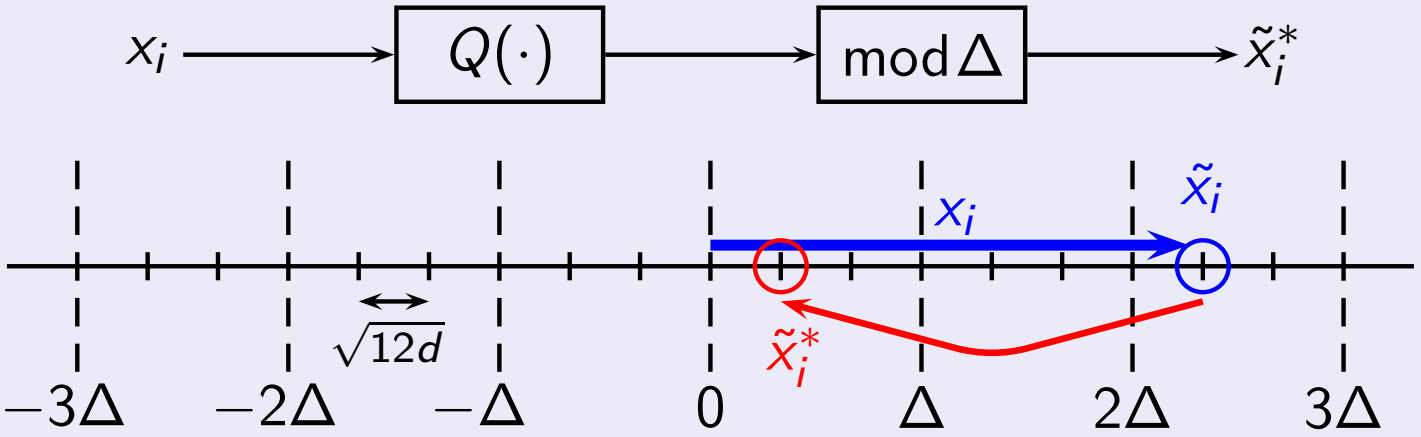
$$\tilde{x}_i = x_i + u_i$$

where $u_i \sim \text{Uniform}\left(\left[-\frac{\sqrt{12d}}{2}, \frac{\sqrt{12d}}{2}\right)\right)$, $u_i \perp\!\!\!\perp x_i$

- $\mathbb{E}(\tilde{x}_i - x_i)^2 = d$

Distributed Compression of Integer Linear Combination

Modulo Scalar Quantization



- $\Delta = 2^R\sqrt{12d} \Longrightarrow$ Compression rate is R
- High resolution/dithered quantization:

$$\tilde{x}_i^* = [x_i + u_i]^*$$

Distributed Compression of Integer Linear Combination

Encoders

Each encoder is a modulo scalar quantizer with rate R : produces $\tilde{x}_k^*$

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Simple modulo property

For any set of integers $a_1, \ldots, a_K$ and real numbers $\tilde{x}_1, \ldots, \tilde{x}_K$

$$\left[\sum_{k=1}^{K} a_k \tilde{x}_k\right]^* = \left[\sum_{k=1}^{K} a_k \tilde{x}_k^*\right]^*$$

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Decoder

- Gets: $\tilde{x}_1^*, \ldots, \tilde{x}_K^*$
- Outputs:

$$\widehat{\mathbf{a}^T \mathbf{x}} = \left[\sum_{k=1}^{K} a_k \tilde{x}_k^*\right]^* = \left[\sum_{k=1}^{K} a_k \tilde{x}_k\right]^* = \left[\mathbf{a}^T(\mathbf{x}+\mathbf{u})\right]^*$$

Compression of Integer Linear Combination - P_e

$$\widehat{\mathbf{a}^T\mathbf{x}} = \left[\mathbf{a}^T(\mathbf{x}+\mathbf{u})\right]^*$$

$$\widehat{\mathbf{a}^T\mathbf{x}} = \begin{cases} \mathbf{a}^T\mathbf{x} + \mathbf{a}^T\mathbf{u} & \text{if } \mathbf{a}^T(\mathbf{x}+\mathbf{u}) \in \left[-\frac{\Delta}{2}, \frac{\Delta}{2}\right) \\ \text{error} & \text{otherwise} \end{cases}$$

- P_e is small if $\frac{\Delta}{\sqrt{\mathrm{Var}\left(\mathbf{a}^T(\mathbf{x}+\mathbf{u})\right)}}$ is large
- Δ grows exponentially with R

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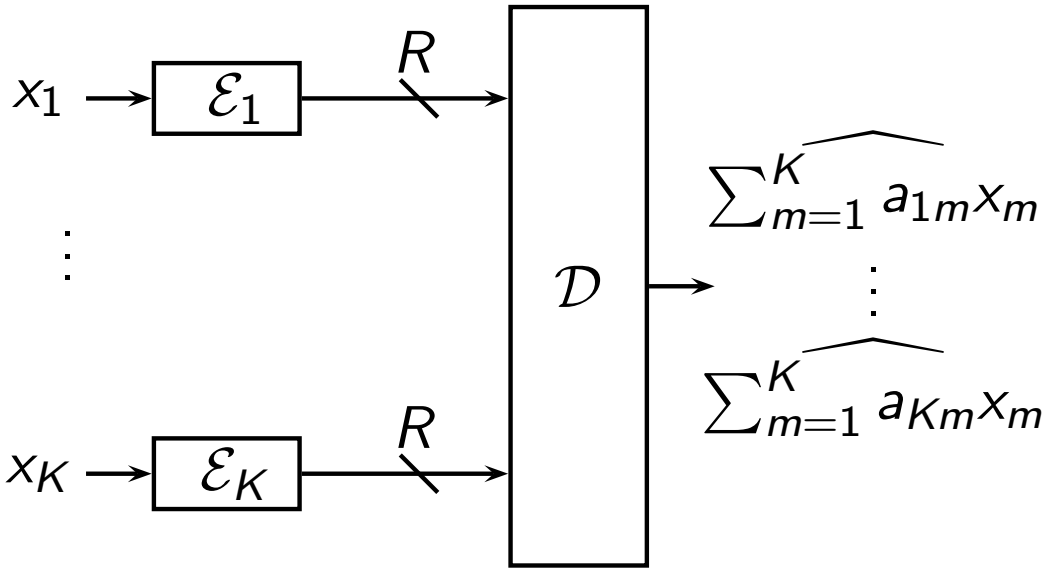
For $\mathbf{a}$ with small $\mathrm{Var}\left(\mathbf{a}^T(\mathbf{x}+\mathbf{u})\right)$ we can take small R

Integer-Forcing Source Coding



- Need to estimate K linearly independent integer linear combinations
- If all combinations estimated without error, can compute

$$\hat{\mathbf{x}} = \mathbf{A}^{-1}\widehat{\mathbf{Ax}} = \mathbf{A}^{-1}(\mathbf{Ax} + \mathbf{Au}) = \mathbf{x} + \mathbf{u}$$

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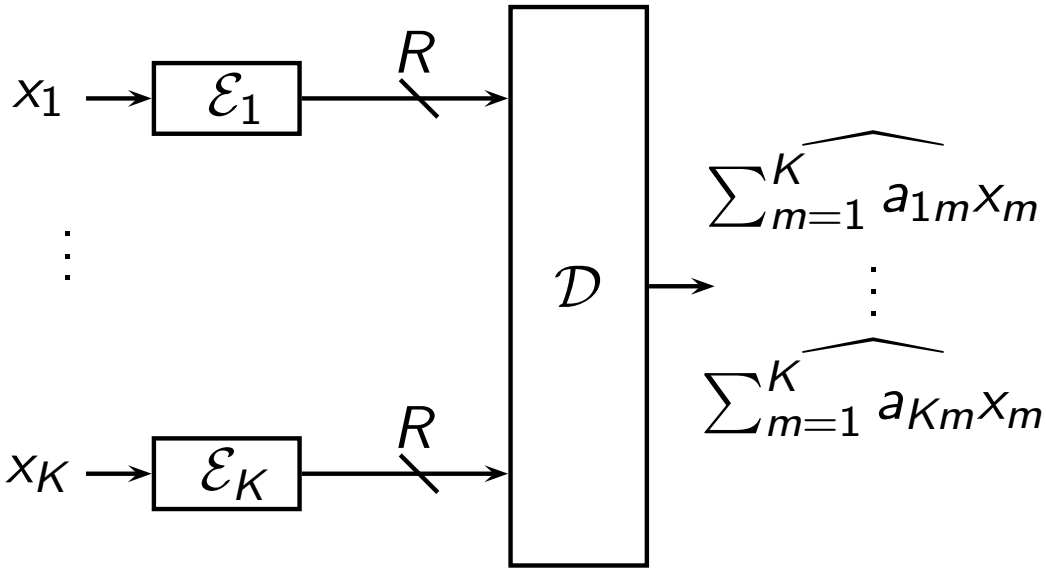

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$$P_e \leq 2K \exp\left\{ -\frac{3}{2} 2^{2\left(R - \frac{1}{2}\log\left(\frac{\max_{m=1,\ldots,K} \mathbf{a}_m^T(\mathbf{K_{xx}} + d\mathbf{I})\mathbf{a}_m}{d} \right) \right)} \right\}$$

Integer-Forcing Source Coding - Performance

Let

$$R_{\mathrm{IF}}(\mathbf{A}, d) \triangleq \frac{1}{2} \log \left(\max_{m=1,\ldots,K} \mathbf{a}_m^T \left(\mathbf{I} + \frac{1}{d} \mathbf{K}_{\mathbf{xx}} \right) \mathbf{a}_m \right)$$

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Theorem

Let $R = R_{\mathrm{IF}}(\mathbf{A}, d) + \delta$. IF source coding produces estimates with average MSE distortion d for all $x_1, \ldots, x_K$ with probability $> 1 - 2K \exp\left\{ -\frac{3}{2} 2^{2\delta} \right\}$

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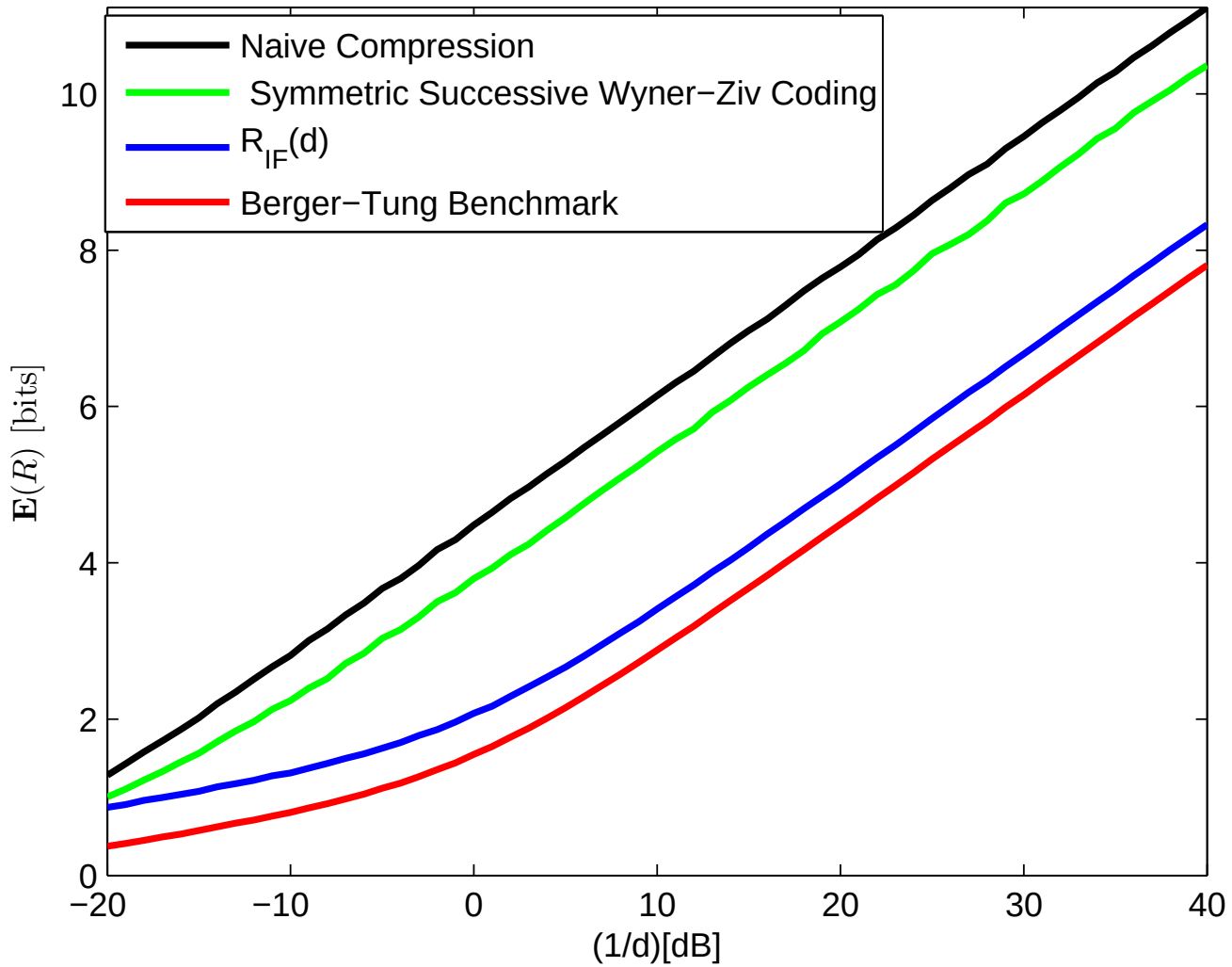
Can minimize compression rate by minimizing $R_{\mathrm{IF}}(\mathbf{A}, d)$ w.r.t. $\mathbf{A}$

Integer-Forcing Source Coding: Example

$\mathbf{x} \sim \mathcal{N}(\mathbf{0}, \mathbf{K_{xx}})$, $\mathbf{K_{xx}} = \mathbf{I} + \mathrm{SNR}\mathbf{H}\mathbf{H}^T$, $\mathrm{SNR} = 20\mathrm{dB}$ and $\mathbf{H} \in \mathbb{R}^{8\times 2}$



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However, if we change the setting...

this obstacle can be overcome.

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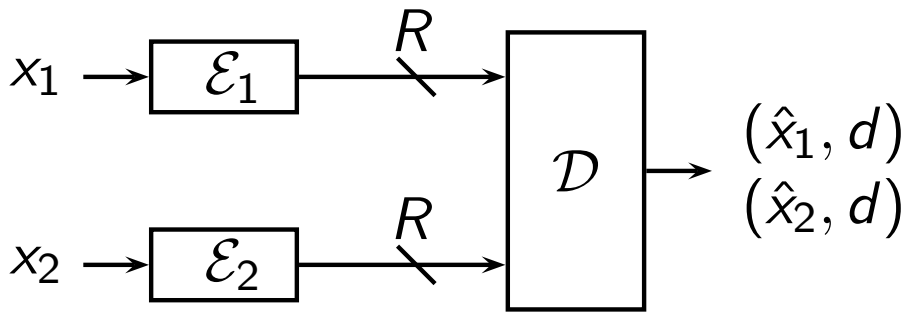

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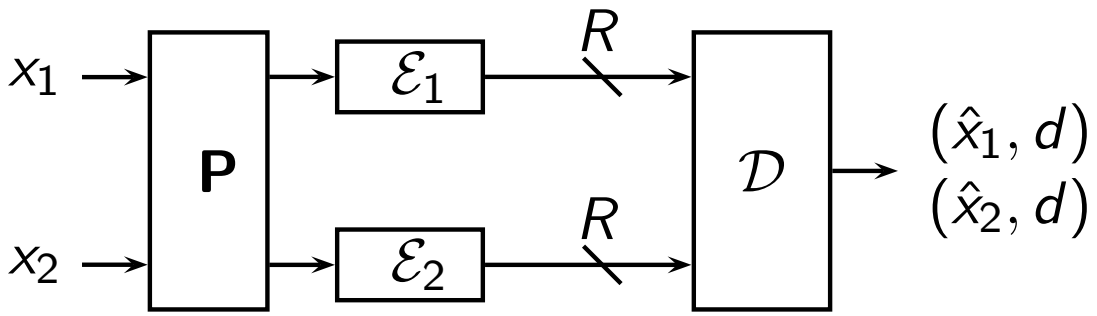

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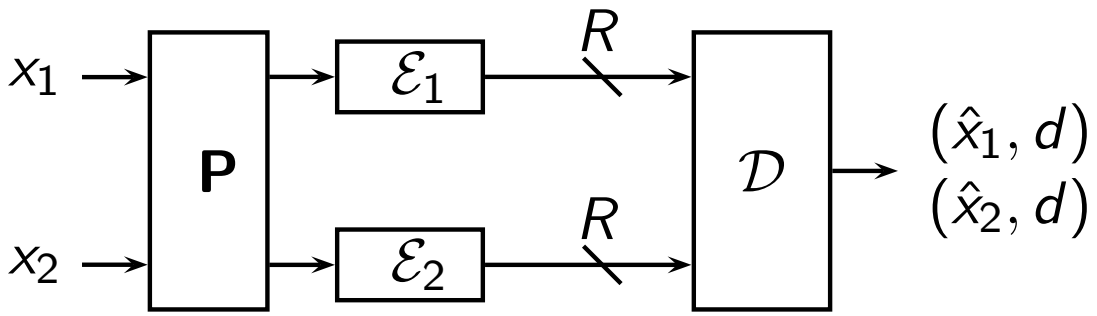

Requirements

- Universal precoding matrix $\mathbf{P}$ (does not depend on $\mathbf{K}_{xx}$)
- $R_{\mathrm{IF}}(d) \leq \text{const} + \frac{1}{2K} \log\det(\mathbf{I} + \frac{1}{d}\mathbf{K}_{xx})$ for all $\mathbf{K}_{xx}$

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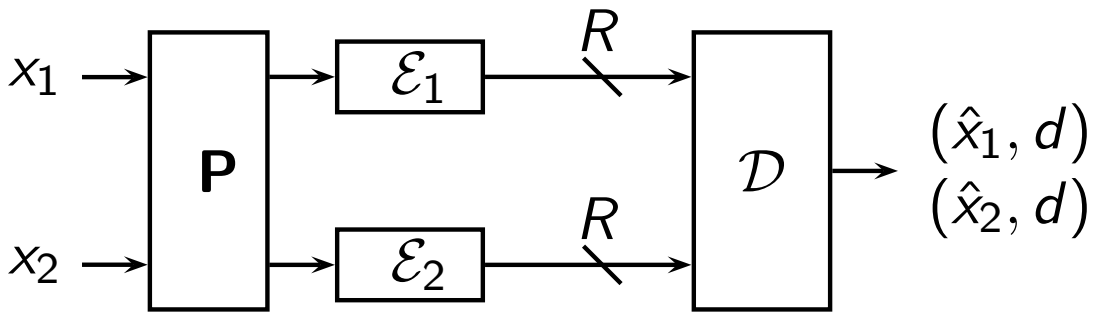

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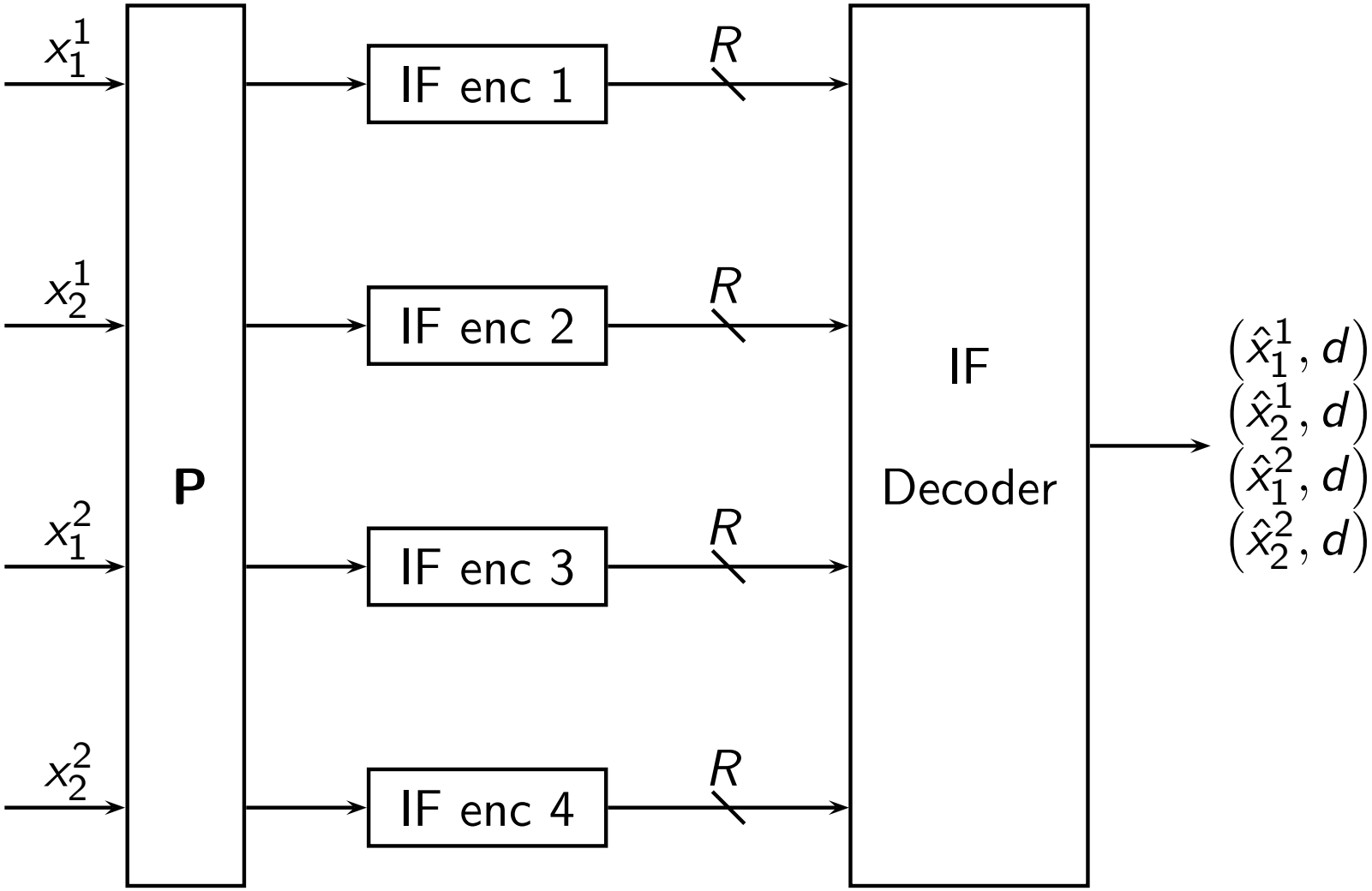
Price of universality - need to jointly encode K realizations

Space-Time Source Coding



Space-Time Source Coding - Performance Guarantees

Let $\mathbf{P}$ be a generating matrix of a "perfect" linear dispersion space-time code, with minimum det $\delta_{\min}(\mathcal{C}_{\infty}^{\mathrm{ST}})$

Theorem

For any source with covariance matrix $\mathbf{K_{xx}}$, the rate-distortion function of space-time integer-forcing source coding with precoding matrix $\mathbf{P}$ is bounded by

$$R_{\mathrm{IF}}(d) < \frac{1}{2K} \log\det\left(\mathbf{I} + \frac{1}{d}\mathbf{K_{xx}}\right) + \Gamma\left(K, \delta_{\min}(\mathcal{C}_{\infty}^{\mathrm{ST}})\right)$$

where $\Gamma\left(K, \delta_{\min}(\mathcal{C}_{\infty}^{\mathrm{ST}})\right) \triangleq 2K^2 \log(2K^2) + K \log \frac{1}{\delta_{\min}(\mathcal{C}_{\infty}^{\mathrm{ST}})}$

Remark: For $K = 2$ the golden-code precoding matrix has $\delta_{\min}(\mathcal{C}_{\infty}^{\mathrm{ST}}) = 1/5$

Example

$$\mathbf{K}_{xx}^1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \mathbf{K}_{xx}^2 = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}, \text{ and } \mathbf{K}_{xx}^3 = \begin{bmatrix} b & b \\ b & b \end{bmatrix}$$

$$\frac{1}{2K} \log \left(\mathbf{I} + \frac{1}{d} \mathbf{K}_{xx}^1 \right) = \frac{1}{2K} \log \left(\mathbf{I} + \frac{1}{d} \mathbf{K}_{xx}^2 \right) = \frac{1}{2K} \log \left(\mathbf{I} + \frac{1}{d} \mathbf{K}_{xx}^3 \right)$$

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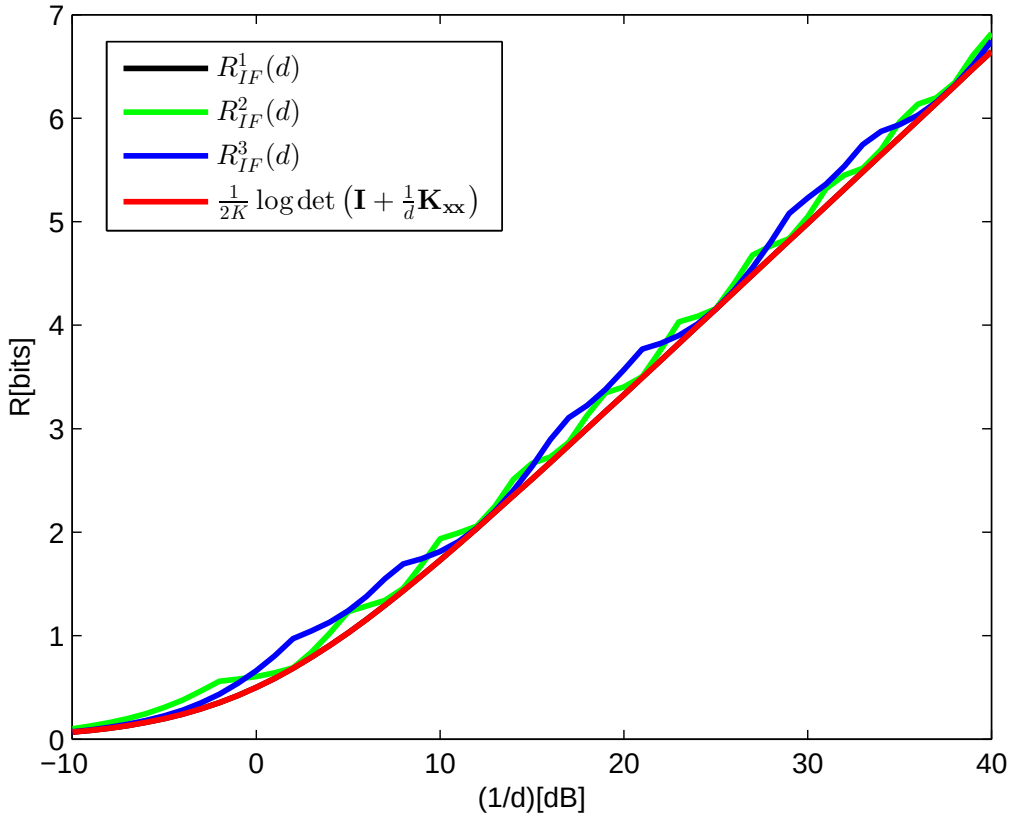

Further Applications of Integer-Forcing Source Coding:

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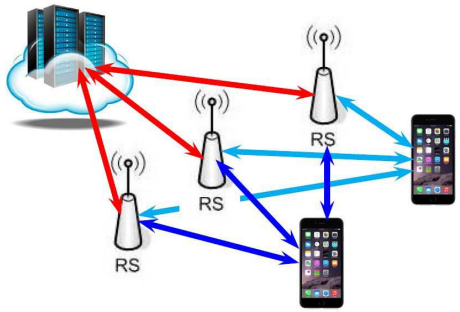
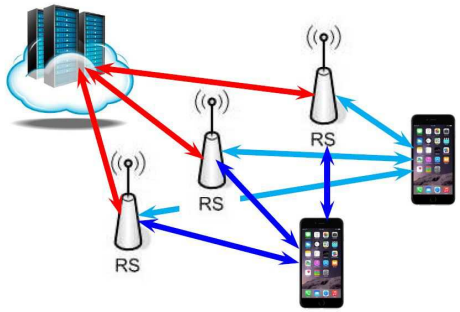
Compress-and-forward for relay networks (C-RAN)

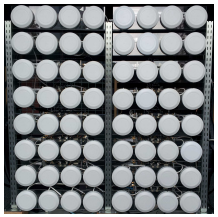


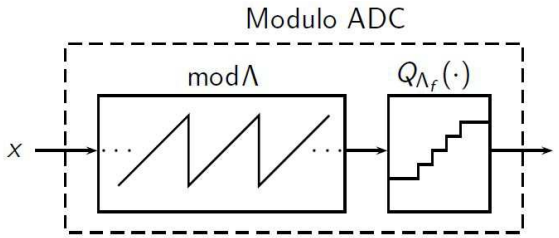
Further Applications of Integer-Forcing Source Coding:

Compress-and-forward for relay networks (C-RAN)



Analog-to-digital converters





Thanks for your attention!