Littlewood's Conjecture (1930)

Littlewood's Conjecture is at the heart of multiplicative Diophantine approximation and has motivated many recent breakthrough developments such as the work of Einsiedler, Katok and Lindenstrauss [5] that contributed to Lindenstrauss' Fields Medal in 2010. The conjecture is well known for its strong links with dynamical systems and ergodic theory (indeed, the measure rigidity conjecture of Margulis [7] regarding the dynamics on $\mathrm{SL}_3(\mathbb{R})/\mathrm{SL}_3(\mathbb{Z})$ implies Littlewood's Conjecture) and is currently a part of a major research trend world-wide. It has been in the spotlight at many recent major workshops and conferences including the 2010 ICM in Hyderabad.

Littlewood's Conjecture simply states that for any real α and β and any $\varepsilon > 0$ there are infinitely many positive integers q such that

$$q\|q\alpha\|\|q\beta\| < \varepsilon, \qquad (1)$$

where $\|x\|$ denotes the distance of x from the nearest integer. This trivially holds if either α or β is not a badly approximable number. For α and β lying in the same cubic field the conjecture was verified by Cassels and Swinnerton-Dyer [2] and subsequently by Peck [8] using different techniques in the stronger 'logarithmic' form

$$\liminf_{q\to\infty} q\log q\|q\alpha\|\|q\beta\| < \infty.$$

On the metrical side, Pollington and Velani [9] showed that the stronger form of the conjecture holds on a set of badly approximable pairs (α, β) of full Hausdorff dimension. More recently, Einsiedler et al [5] proved the measure rigidity conjecture of Margilus under the assumption of positive entropy. As a consequence, the set of possible exceptions to (1) is of Hausdorff dimension zero. The results of [5] on Littlewood's Conjecture are currently the best. Nevertheless, many 'simple' instances such as $(\alpha, \beta) = (\sqrt{2}, \sqrt{3})$ remain open. Considerable attention has also been given to other problems of similar nature. These include, for example, the so-called mixed Littlewood Conjecture of de Mathan and Teuli (see [1, 3, 4, 6]) and Cassels' problem [10] on inhomogeneous multiplicative Diophantine approximation.

References

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