Workshop on Interactions between Number Theory and Wireless Communication,
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Quadratic forms, lattice points and interference alignment

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Plan

Motivation

- Let $m, n \geq 1$ be integers and let $\gamma, c > 0$ be real numbers. Define

$$\mathcal{H}_{m,n}(\gamma, c) := \left\{ H \in \mathbb{R}^{n \times m} \ : \ \det\left(\gamma I_m + H^T \cdot H\right) = c \right\}.$$

Initial Problem

Assume that the set $\mathcal{H}_{m,n}(\gamma, c)$ is equipped with a "uniform" probability measure. Let $s \geq 0$.

What is the probability that the quantity

$$\min_{\mathbf{a} \in \mathbb{Z}^m \setminus \{\mathbf{0}\}} \mathbf{a}^T \cdot \left(\gamma I_m + H^T \cdot H\right) \cdot \mathbf{a} \qquad (1)$$

should be less than s ?

In other words, what is the cumulative distribution function of (1) seen as a random variable ?

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Rephrasing the Initial Problem in a more general context

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- Given $H \in \mathcal{H}_{m,n}(\gamma, c)$, let

$$\Sigma_H := c^{-1/m} \cdot \left(\gamma I_m + H^T \cdot H\right) \in \Sigma_m^{++},$$

where Σ_m^{++} is the set of positive definite matrices *with determinant one.*
- Given $\Sigma \in \Sigma_d^{++}$, set

$$M_d(\Sigma) := \min_{\mathbf{a} \in \mathbb{Z}^d \setminus \{\mathbf{0}\}} \mathbf{a}^T \cdot \Sigma \cdot \mathbf{a}$$

(which is easily seen to be well-defined).

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The non–probabilistic case : the Hermite constant

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- Hermite proved that there exists a constant $\gamma_d > 0$ such that, for *any* $\Sigma \in \Sigma_d^{++}$,

$$M_d(\Sigma) \leq \gamma_d.$$

d (dimension)	1	2	3	4	5	6	7	8	24
γ_d^d	1	4/3	2	4	8	64/3	64	256	4^{24}

TABLE : Known values of the Hermite constant γ_d

- One can also prove for instance that

$$V_d^{-2/d} \leq \gamma_d \leq 4 \cdot V_d^{-2/d},$$

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How to tackle the Main Problem ?

- Two possible approaches :
  - a purely algebraic one :
    - essentially based on the Cholesky decomposition of an element in Σ_d^{++},
    - will provide an answer to the Initial Problem ;
  - a purely geometric one :
    - based on the spectral decomposition of an element in Σ_d^{++},
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Two preliminary remarks

- The problem is $SL_d(\mathbb{Z})$–invariant in the sense that for any $\Sigma \in \Sigma_d^{++}$ and any $A \in SL_d(\mathbb{Z})$,

$$M_d\left(A^T \cdot \Sigma \cdot A\right) = M_d(\Sigma)$$

- Any $\Sigma \in \Sigma_d^{++}$ can be decomposed as

$$\Sigma = L^T \cdot L,$$

where L belongs to the set of upper triangular matrices.
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Some heuristics with an abstract formulation of the problem

- The problem of estimating $M_d(\Sigma)$ for $\Sigma \in \Sigma_d^{++}$ is thus well–defined when Σ is seen as a element of the set

$$X_d := SO_d(\mathbb{R})\backslash SL_d(\mathbb{R})/SL_d(\mathbb{Z})$$

(a so–called *locally symmetric space*).

- The latter set can be equipped with a natural probability measure μ_{X_d} (coming from a so–called *Haar measure*).

Theorem (Kleinbock & Margulis, 1998)

Let $\delta > 0$. Denote by $p_{X_d}(\delta)$ the probability (w.r.t μ_{X_d}) of the event $M_d(\Sigma) \leq \delta$ in the space X_d. Then :

$$\frac{V_d}{2\zeta(d)}\delta^{d/2} - c_d\frac{V_d^2}{4}\delta^d \leq p_{X_d}(\delta) \leq \frac{V_d}{2\zeta(d)}\delta^{d/2}. \qquad (2)$$

Here, ζ denotes the Riemann zeta function, c_d is an explicit strictly positive constant and V_d denotes again the volume of the unit Euclidean ball in dimension d.

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- Resulting heuristics : for a "typical measure", one should expect the probability of the event $M_d(\Sigma) \leq \delta$ to grow like the volume of the Euclidean ball of radius $\sqrt{\delta}$ in dimension d.

Strategy to tackle the Main Problem with the algebraic approach

- Assume that f is some density function on Σ_d^{++} : the problem boils down to estimating the integral :

$$\mathfrak{m}_f(\delta) \;:=\; \int_{\Sigma_d^{++}} \chi_{[M_d(\Sigma)>\delta]} \cdot f(\Sigma) \cdot \mathrm{d}\Sigma$$

for a given $\delta > 0$.

- If $\Sigma \in \Sigma_d^{++}$ is decomposed in its Cholesky form as $\Sigma = L^T \cdot L$, then :

$$(M_d(\Sigma) > \delta) \quad \Longleftrightarrow \quad \left(L \cdot \mathbb{Z}^d \cap B_2(\mathbf{0}, \sqrt{\delta}) = \{\mathbf{0}\}\right).$$

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A key–observation

- Denote by $\mathcal{T}_d^{++}$ the set of upper triangular matrices of determinant one with strictly positive diagonal entries.
- The diagonal entries of a generic elements $L \in \mathcal{T}_d^{++}$ will hereafter be denoted by $(\beta_1, \dots, \beta_d) \in (\mathbb{R}_{>0})^d$. In particular,

$$\beta_d = \left(\prod_{i=1}^{d-1} \beta_i \right)^{-1}.$$

Lemma

With the previous notation, the following hold :

- *if $\beta_i > \eta$ for all $i = 1, \dots, d$, then*

$$L \cdot \mathbb{Z}^d \cap B_2(\mathbf{0}, \eta) = \{\mathbf{0}\};$$

- *conversely, if $L \cdot \mathbb{Z}^d \cap B_2(\mathbf{0}, \eta) = \{\mathbf{0}\}$, then $\beta_1 > \eta$.*

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Implication for the Main Problem

- By change of variables, given $\delta > 0$,

$$\mathfrak{m}_f(\delta) := \int_{\Sigma_d^{++}} \chi_{[M_d(\Sigma)>\delta]} \cdot f(\Sigma) \cdot \mathrm{d}\Sigma = \int_{\mathcal{T}_d^{++}} \chi_{[M_d(\varphi_{chol}(L))>\delta]} \cdot G_f(L) \cdot \mathrm{d}L,$$

with

$$G_f(L) := f(\varphi_{chol}(L)) \cdot \mathrm{Jac}_L(\varphi_{chol}).$$

Here, $\mathrm{Jac}_L(\varphi_{chol})$ is the Jacobian at $L \in \mathcal{T}_d^{++}$ of the Cholesky map

$$\varphi_{chol} : L \in \mathcal{T}_d^{++} \mapsto L^T L \in \Sigma_d^{++}.$$

- From the previous lemma, one has therefore

$$\int_{\mathcal{T}_d^{(1)}(\delta)} G_f(L) \cdot \mathrm{d}L \le \mathfrak{m}_f(\delta) \le \int_{\mathcal{T}_d^{(2)}(\delta)} G_f(L) \cdot \mathrm{d}L,$$

where :
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Conclusions regarding the theoretical part of the algebraic approach

- Given a density function f on the space Σ_d^{++} and given $\delta > 0$, the probability $\mathfrak{m}_f(\delta)$ of the event $M_d(\Sigma) > \delta$ can be estimated both by above and by below.
- The lower bound for $\mathfrak{m}_f(\delta)$ is more accurate (can be seen as nearly optimal for "not too wild" density functions).

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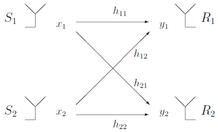
Application to Signal Processing : the general set–up

- Assume that two *users* (or *transmitters*) S_1 and S_2 want to *transmit* messages (or *signals*) x_1 (for S_1) and x_2 (for S_2) along a communication channel (e.g., a cable or a radio channel) simultaneously to two *receivers* R_1 and R_2.



- In the simplest case that they use an additive channel, the message y_i received by R_i ($i \in \{1, 2\}$) is represented by the system of equations

$$\begin{cases} y_1 = h_{11}x_1 + h_{12}x_2 + z_1 \\ y_2 = h_{21}x_1 + h_{22}x_2 + z_2, \end{cases} \tag{3}$$

where z_1 and z_2 are the noise and h_{ij} ($1 \leq i, j \leq 2$) the *channel coefficients* representing a certain degree of *fading* in the transmission.

The general set–up (bis)

- In the case of $m \geq 1$ users and $n \geq 1$ receivers, the model generalises in an obvious way :

$$\boldsymbol{y} \; = \; H \cdot \boldsymbol{x} + \boldsymbol{z},$$

where $H \in \mathbb{R}^{n \times m}$, $\boldsymbol{x} \in \mathbb{R}^m$ and $\boldsymbol{y}, \boldsymbol{z} \in \mathbb{R}^n$.

- The input $\boldsymbol{x}$ (seen as a random variable) satisfies a power constraint of the form :

$$\mathbb{E}\left(\boldsymbol{x}^T \cdot \boldsymbol{x}\right) \; \leq \; m \cdot SNR,$$

where SNR stands for the *Signal–to–Noise Ratio* (expressed in *decibels*).

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The general set–up (ter)

- Basic problem : to determine whether the received information is reliable.
- Naive solution : to reduce the rate of new data sent by the users (for instance, by repeating each string of message several times).
- In 1948, Shannon proved that this intuition is surprisingly incorrect :
  - It is actually possible to exchange information at a *strictly positive* data rate keeping at the same time the error probability as small as desired.
  - There is nevertheless a maximal rate, the *capacity of the channel*, above which this cannot be done any more. The latter quantity is usually expressed in bits.

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The general set–up (quater)

- A crucial remark : the performances of a channel depends heavily on whether or not the transmitter knows the channel coefficients matrix H.
  - => If such information is available, they can for instance allocate more power to the stronger antennas to minimise the effect of fading.
  - => If this information is not known to the transmitter a reasonable strategy is to allocate equal power to each of the sub–channels. In this situation, the capacity of the channel is rather referred to as the *mutual information*.

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The general set–up (quinquies)

- There is no single expression for the capacity/mutual information C of a channel, but in all the cases we will be interested in :

$$C = \log\det\left(I_m + SNR \cdot H^T \cdot H\right).$$

- In the more specific case of a channel where a so–called Integer Forcing Technique is applied at the Receiver, the “quality” of the channel is determined by the so–called Effective–Signal–to–Noise Ratio SNR_{eff} which satisfies the estimates (Erez & Ordentlich, 2015) :

$$\frac{1}{4m^2} \cdot M_m\left(I_m + \mathrm{SNR} \cdot H^T \cdot H\right) < \mathrm{SNR}_{\mathrm{eff}} \leq M_m\left(I_m + \mathrm{SNR} \cdot H^T \cdot H\right).$$

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A possible Interpretation of the Initial Problem

- Recall that

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and

Initial Problem

Assume that the set $\mathcal{H}_{m,n}(\gamma, c)$ is equipped with a "uniform" probability measure. Let $s \geq 0$.
What is the cumulative distribution function of the quantity $M_m\left(\gamma I_m + H^T \cdot H\right)$ seen as a random variable ?

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Strategy to tackle the Initial Problem

1/ Define properly the concept of a "uniform" probability measure on $\mathcal{H}_{m,n}(\gamma, c)$.

2/ Push this measure forward to the space of positive definite matrices of determinant one Σ_d^{++} with the help of the map

$$f \ : H \in \mathcal{H}_{m,n}(\gamma, c) \ \mapsto \ c^{-1/m} \cdot \left(\gamma I_m + H^T \cdot H\right) \in \Sigma_m^{++}.$$

=> This essentially requires a change of variables $\Sigma = f(H)$ and the calculation of the corresponding Jacobian.

$$\mathcal{H}_{m,n}(\gamma, c) \xrightarrow{f} \Sigma_m^{++} \xrightarrow{\widetilde{\varphi}_{chol}^{-1}} \mathcal{T}_m^{++}$$

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=> It may be assumed *without loss of generality* that

$$d := \min\{m, n\} = m.$$

- The set $\mathcal{H}_{m,n}(\gamma, c)$ is empty if $c < \gamma^m$ and reduced to the zero matrix if $c = \gamma^m$. Assume therefore from now on that

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Definition

A measure is uniform (if it is Borelian and) if the measure of a ball depends only on its radius but not on the position of its center.

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A bounded subset in $\mathbb{R}^k$ ($k \geq 1$) carries a uniform measure only if it is contained in a sphere.

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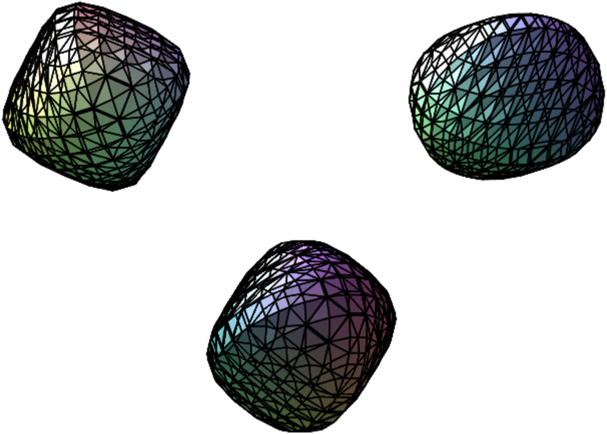
A bounded subset in $\mathbb{R}^k$ ($k \geq 1$) carries a uniform measure only if it is contained in a sphere.

Projection of the manifold $\mathcal{H}_{2,2}(1,2)$ in $\mathbb{R}^3$ (various angles of view)



=> This manifold cannot be contained in a sphere. This holds for the more general manifold $\mathcal{H}_{m,n}(\gamma, c)$ as soon as $\min\{m, n\} \geq 2$.

How to render the idea of a uniform measure ?

- Recall that

$$\mathcal{H}_{m,n}(\gamma, c) := \left\{ H \in \mathbb{R}^{n\times m} \;:\; \det\left(\gamma I_m + H^T \cdot H\right) = c \right\}.$$

- Clearly, given any $U \in \mathcal{O}_m(\mathbb{R})$, this set is invariant under the map $H \mapsto U \cdot H$ — this is just saying that

$$\det\left(\gamma I_m + H^T \cdot H\right) = \det\left(\gamma I_m + (UH)^T \cdot (UH)\right).$$

- From the QR decomposition, for each $H \in \mathcal{H}_{m,n}(\gamma, c)$, there exists an (essentially unique) orthogonal matrix $U \in \mathcal{O}_m(\mathbb{R})$ such that

$$UH = \begin{pmatrix} T \\ \mathbf{0} \end{pmatrix}$$

with T lying in the set $\Theta_d(\mathbb{R})$ of upper triangular matrices with size $d = m$. Furthermore, in this case,

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How to render the idea of a uniform measure ? (bis)

- Idea : defining equivalently the "uniform" measure taking into account the geometric properties of the manifold

$$\mathcal{H}'_{m,n}(\gamma, c) := \left\{ T \in \Theta_d(\mathbb{R}) \; : \; \det\left(\gamma I_d + T^T \cdot T\right) = c \right\}.$$

- This is precisely the idea behind the concept of *volume element* $\mathrm{d}\,\mathrm{vol}(T)$ on a manifold : it defines a probability measure ν_d deduced from the formula

$$\nu_d(\mathcal{B}) = \int_{\mathcal{B}} \mathrm{d}\,\mathrm{vol}(T)$$

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Some numerical values

- In the case that there are $d = m = 2$ users and $n = 2$ receivers, assume that the capacity of the channel is $C = 30$ bits and that $SNR = 5$ db.
- Let $\mathfrak{m}_2(s)$ denote the probability that $SNR_{eff} \geq s$:

s	1	3/2	2
$\mathfrak{m}_2(s) \geq$	0.672723	0.560289	0.489859

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Plan

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- Idea : Any matrix $\Sigma \in \Sigma_d^{++}$ can be decomposed as

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where $P \in \mathcal{O}_d(\mathbb{R})$ and D is diagonal with determinant 1 (spectral decomposition).

- Defining a suitable class of probability measures on the set of orthogonal matrices and (above all) on the set of diagonal matrices with determinant 1 thus determines a class of probability measures in Σ_d^{++}.
- The problem amounts to determining whether a random ellipsoid centered at the origin contains a non–zero lattice points.
- The results obtained are sharp in the sense that we recover the growth in $\delta^{d/2}$ from the Theorem of Kleinbock & Margulis.
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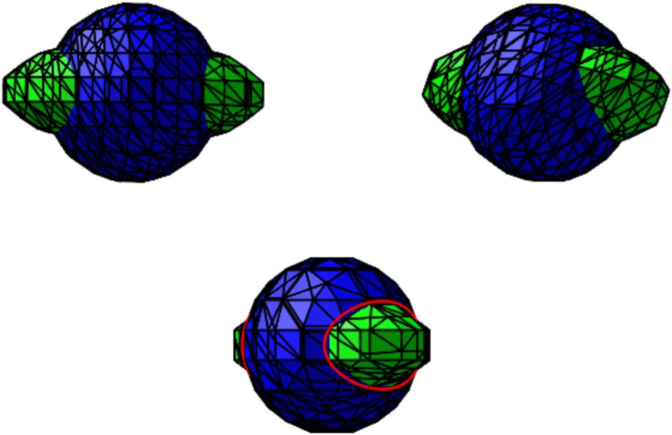
Intersection of an ellipsoid with the unit sphere



- **Goal :** To compute the area (on the unit sphere) of this intersection.


Thank you