Wireless Network Coding over Finite Rings

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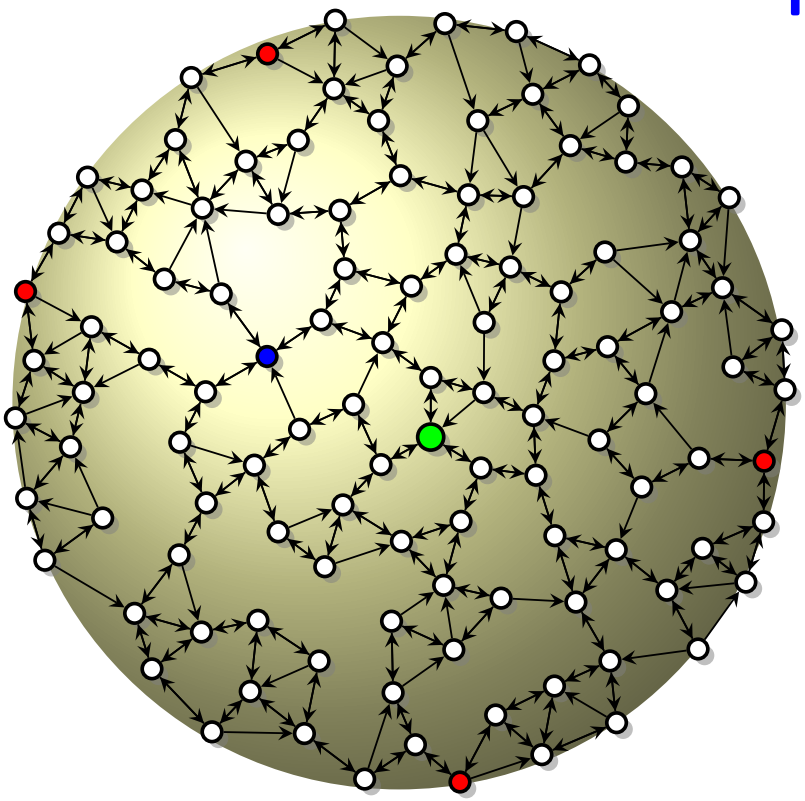
July 7, 2016

Heslington, York, United Kingdom

Network Coding over Finite Fields



Packet Network

Random Linear Network Coding

- **Transmitter** injects *packets*: vectors from $\mathbb{F}_q^m$, the rows of a matrix X
- Intermediate nodes forward random $\mathbb{F}_q$-linear combinations of packets
- **Errors** may also be injected, which randomly mix with the legitimate packets
- (Each) **receiver** gathers as many packets as possible, forming the rows of matrix Y

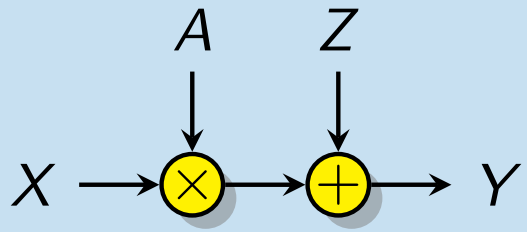
At any particular receiver:

$$Y = AX + Z$$

where: X is $n \times m$; Y, Z are $N \times m$; and A is $N \times n$.

A Basic Model

Multiplicative-Additive Matrix Channel (MAMC)



Previous work[1] considered a basic stochastic linear matrix channel model:

$$Y = AX + Z$$

where

- X and Y are $n \times m$ matrices over $\mathbb{F}_q$;
- A is $n \times n$, nonsingular, drawn uniformly at random;
- Z is $n \times m$ with rank t, drawn uniformly at random;
- X, A, and Z are independent.

[1] Silva, Kschischang, and Kötter, "Communication over Finite-Field Matrix Channels," *IEEE Trans. Inf. Theory*, vol. 56, pp. 1296–1305, Mar. 2010.

MAMC: Capacity

Theorem (upper bound)

For $n \leq m/2$,

$$C_{\mathrm{MAMC}} \leq (n-t)(m-n) + \log_q 4(n+1)(t+1).$$

Theorem (lower bound)

Assume $n \leq m$. For any $\epsilon \geq 0$, we have

$$C_{\mathrm{MAMC}} \geq (n-t-\epsilon t)(m-n) - \log_q 4 - \frac{2tnm}{q^{1+\epsilon t}}.$$

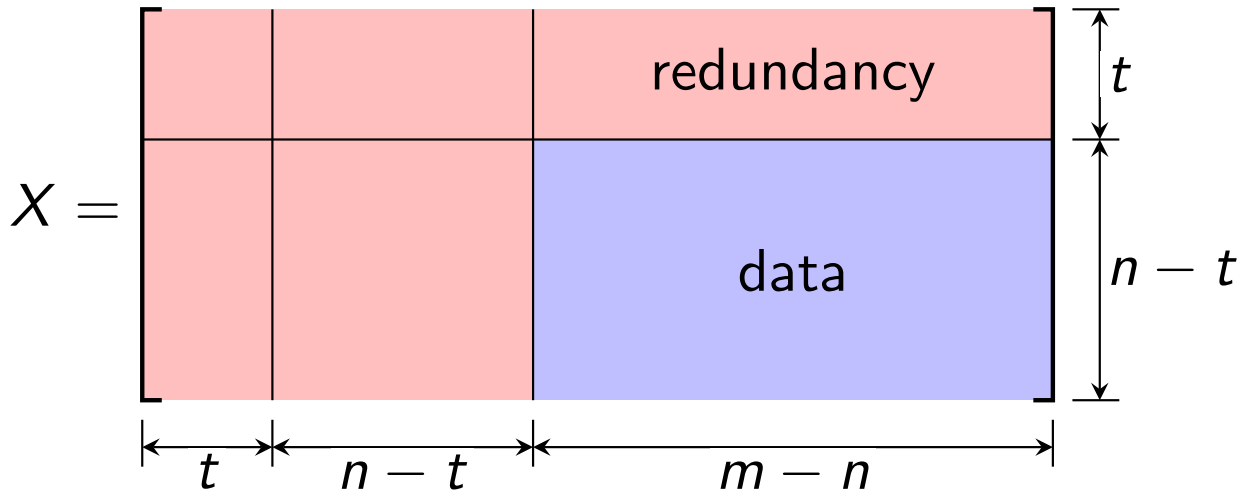
These upper and lower bounds match when $q \to \infty$ or $n \to \infty$ (with t/n and m/n fixed).

MAMC: Capacity

Corollary

For large n or large q,

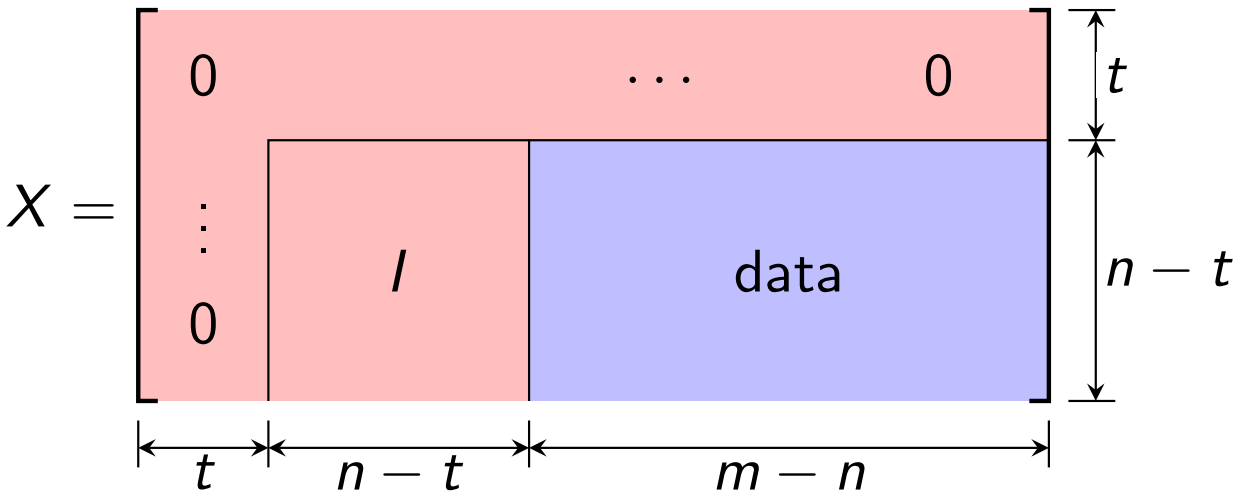
$$C_{\text{MAMC}} \approx (n-t)(m-n).$$



A Simple Coding Scheme

Strategy: Channel Sounding + Error Trapping

Use channel sounding "inside" and error trapping "outside" (but not the opposite!)



MAMC: A Coding Scheme

First, rewrite the channel model as

$$Y = AX + Z = A(X + A^{-1}Z) = A(X + W), \quad \text{where } W = A^{-1}Z.$$

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Suppose a "genie" gives the receiver $X + W$.
Let data matrix D be $(n - t) \times (m - n)$.
We have:

$$X = \begin{bmatrix} 0 & 0 & 0 \\ 0 & I & D \end{bmatrix} \qquad W = \begin{bmatrix} W_1 & W_2 & W_3 \\ W_4 & W_5 & W_6 \end{bmatrix}$$

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Assume that rank $W_1 = t = \text{rank}\, W$ $(= \text{rank}\, Z)$. In this case, for some matrix B, we have

$$W = \begin{bmatrix} W_1 & W_2 & W_3 \\ BW_1 & BW_2 & BW_3 \end{bmatrix}$$

Now convert $X + W$ to reduced row echelon (RRE) form:

$$
\begin{aligned}
X + W &= \begin{bmatrix} W_1 & W_2 & W_3 \\ BW_1 & I + BW_2 & D + BW_3 \end{bmatrix} \\
&\xrightarrow{\text{row op.}} \begin{bmatrix} I & W_1^{-1}W_2 & W_1^{-1}W_3 \\ BW_1 & I + BW_2 & D + BW_3 \end{bmatrix} \\
&\xrightarrow{\text{row op.}} \begin{bmatrix} I & W_1^{-1}W_2 & W_1^{-1}W_3 \\ 0 & I & D \end{bmatrix} \\
&\xrightarrow{\text{row op.}} \begin{bmatrix} I & 0 & \tilde{W}_3 \\ 0 & I & D \end{bmatrix} = \mathrm{RRE}(X + W).
\end{aligned}
$$

But we have Y, not $X + W$!

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But we have Y, not $X + W$!

Observation

$Y = A(X + W)$, A is full rank, so Y and $X + W$ have the same row space, which implies that

$$\text{RRE}(Y) = \text{RRE}(X + W).$$

Thus, D is exposed by reducing Y to RRE form!

MAMC: A Coding Scheme

- Decoding amounts to performing full Gaussian elimination on the received matrix Y.

  Complexity: $\mathcal{O}(n^2 m)$ operations in $\mathbb{F}_q$ to recover $(n-t)(m-n)$ symbols. Defining $R = (n-t)(m-t)/mn$, we have a complexity of $\mathcal{O}(n/R)$ operations per decoded symbol.

- The scheme fails if W_1 is not invertible.
  The probability of failure falls exponentially (for fixed n) in the number of bits per field-element, or exponentially (for fixed q) in n (assuming fixed aspect ratio of m/n and fixed t/n).

Theorem

This coding scheme can achieve the capacity of the MAMC when either $q \to \infty$ or $n \to \infty$.

This Talk:

Generalize
from
finite-field matrix channels
to
finite-ring matrix channels.

Why?

A: it could be useful for **compute-and-forward**[2]

[2]Bobak Nazer and Michael Gastpar, "Compute-and-forward: Harnessing interference through structured codes," *IEEE Trans. Inf. Theory*, vol. 57, pp. 6463–6486, Oct. 2011.

From Network Coding to Compute-and-Forward

Conventional network coding

Message space is $\mathbb{F}_q^m$.

Compute-and-forward

Message space is

$$T/\langle a_1 \rangle \times \cdots \times T/\langle a_m \rangle$$

for some nonzero, non-unit $a_1, \ldots, a_m \in T$ such that $a_1 \mid \cdots \mid a_m$.

- T is a discrete subring of $\mathbb{C}$ forming a principal ideal domain[3]

[3] F., D. Silva, F. Kschischang, "An Algebraic Approach to Physical-Layer Network Coding," *IEEE Trans. Inf. Theory*, vol. 59, pp. 7576–7596, Nov. 2013.

Message Space: Examples

Example 1: [Ordentlich, Zhan, Erez, Gastpar, Nazer, ISIT'11]

- Λ is obtained using Construction A applied to binary $(n = 64800, k = 54000)$ LDPC code C, with mod-4 shaping:

$$\Lambda = C + 2\mathbb{Z}^n, \quad \Lambda' = 4\mathbb{Z}^n.$$

- Message space: $\mathbb{Z}_2^{10800} \times \mathbb{Z}_4^{54000}$

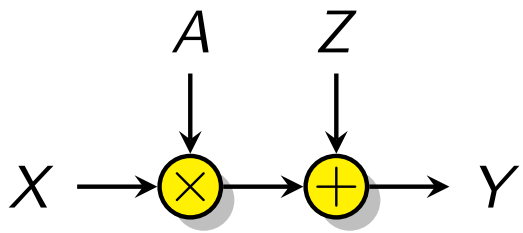
Example 2: Turbo Lattices [Sakzad, Sadeghi, Panario, Allerton'10]

- Λ is obtained using Construction D applied to nested turbo codes $C_2 : (n = 10131, k_2 = 3377)$ and $C_1 : (n = 10131, k_1 = 5065)$;

$$\Lambda = C_2 + 2C_1 + 4\mathbb{Z}^n, \quad \Lambda' = 4\mathbb{Z}^n.$$

- Message space: $\mathbb{Z}_2^{1688} \times \mathbb{Z}_4^{3377}$

Objective



$$Y = AX + Z$$

where each row of X, Y, and Z is from the message space

$$T/\langle a_1 \rangle \times \cdots \times T/\langle a_m \rangle$$

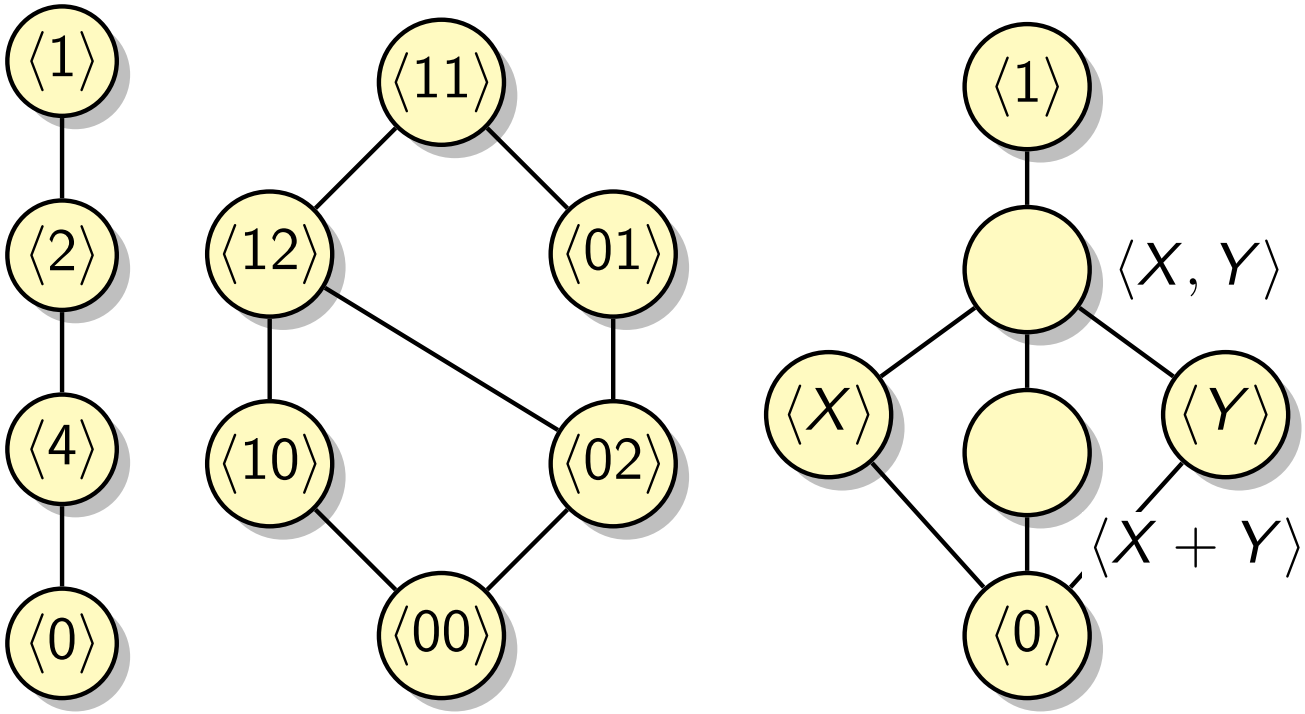
Objective:

- Capacity bounds
- Simple coding schemes

Finite-Ring Toolkit

Commutative Rings with Identity $1 \neq 0$

- Ideals in a ring can be partially ordered by subset inclusion.
- The resulting poset is called the lattice of ideals of the ring.



Chain ring: ideals are linearly ordered. Ex: $\mathbb{Z}_8$.

Principal ring: every ideal generated by 1 element. Ex: $\mathbb{Z}_8$, $\mathbb{Z}_2 \times \mathbb{Z}_4$.

Local ring: unique maximal proper ideal. Ex: $\mathbb{Z}_8$, $\mathbb{Z}_2[X, Y]/\langle X, Y\rangle^2$.

Finite Rings: Important Facts

Proposition

If R is a ring and N is a **maximal** ideal of R, then R/N is a **field**.

This is called a residue field.

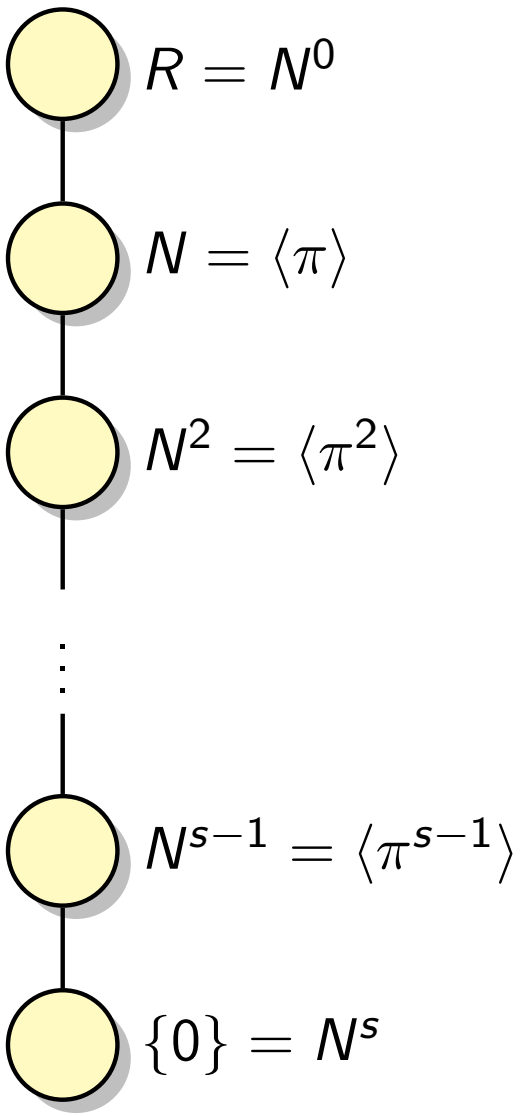
Proposition

A finite ring is a **chain** ring if and only if it is both **local** and **principal**.

Proposition

Every finite **principal** ring is a product of finite **chain** rings.

Finite Chain Rings: The Ideals



Let R be a finite chain ring, where

- N is the unique maximal ideal,
- π is any generator for N,
- q is the order of the residue field,
- s is the number of proper ideals.

Proposition

The lattice of ideals of R is

$$R = N^0 \supset N \supset \cdots \supset N^{s-1} \supset N^s = \{0\}.$$

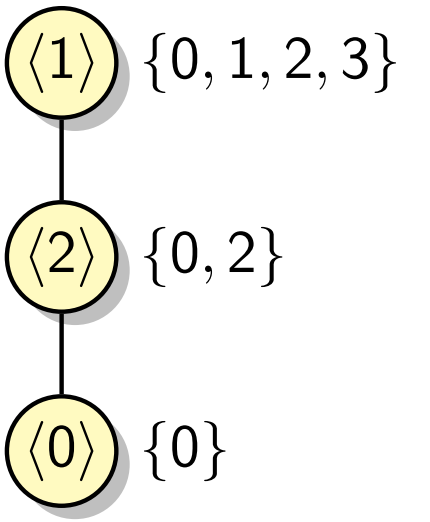
The size of each ideal is $|N^i| = q^{s-i}$.

Notation: (q, s) chain ring.

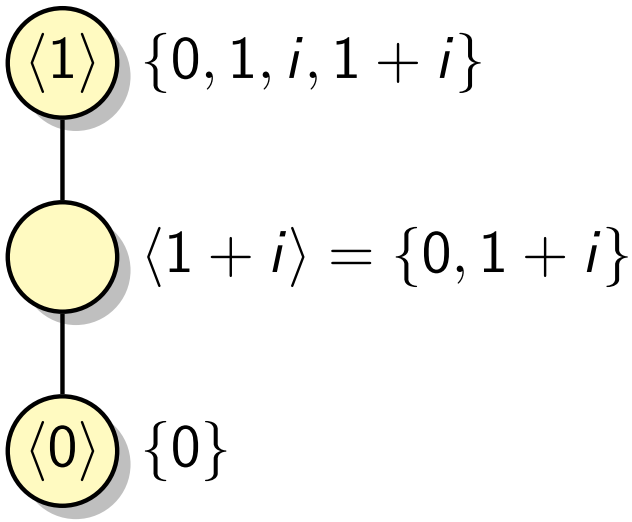
Finite Chain Rings: Examples

The following are two non-isomorphic $(q = 2, s = 2)$ chain rings.

$\mathbb{Z}_4 = \{0, 1, 2, 3\}$



$\mathbb{Z}_2[i] = \{0, 1, i, 1 + i\}$



Finite Chain Rings: The π-adic Decomposition

Let R be a (q, s) chain ring, and N be its maximal ideal.

Proposition

Fix the following:

$\pi \in R$, a generator for the maximal ideal N (i.e., $\langle \pi \rangle = N$).

$F \subseteq R$, a set of coset representatives for the residue field R/N.

Then, every element $r \in R$ can be written uniquely as

$$r = r_0 + r_1\pi + r_2\pi^2 + \cdots + r_{\ell-1}\pi^{s-1}$$

where $r_i \in F$.

This is known as the π-adic decomposition.

Ex: over $\mathbb{Z}_8$, $5 = 1 + 0 \cdot 2 + 1 \cdot 2^2$.

Finite Chain Rings: Element Degree

Definition

The degree, $\deg(r)$, of a nonzero element $r \in R^*$, where

$$r = r_0 + r_1\pi + \cdots + r_{s-1}\pi^{s-1},$$

is defined as the *least* index j for which $r_j \neq 0$.

- by convention, $\deg(0) = s$
- units have degree zero
- a divides b if and only if $\deg(a) \leq \deg(b)$

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For example, over $\mathbb{Z}_8$:

deg.	elements
0	$\{1, 3, 5, 7\}$
1	$\{2, 6\}$
2	$\{4\}$
3	$\{0\}$

Matrices over Finite Chain Rings: Reduced Row Echelon Form

We need an appropriate generalization of RREF.

The presence of zero divisors complicates the matters...

- Over a field, two matrices in echelon form with the same row span will have the same number of nonzero rows—the rank.
- Over a chain ring, this is not the case.

For example, the matrices

$$\begin{bmatrix} 2 & 1 & 1 & 2 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ and } \begin{bmatrix} 2 & 1 & 1 & 2 \\ 0 & 4 & 0 & 4 \\ 0 & 0 & 2 & 2 \end{bmatrix} \text{ over } \mathbb{Z}_8$$

have the same row span but not the same number of nonzero rows. So, generalization of RREF seems non-trivial.

Matrices over Finite Chain Rings: Reduced Row Echelon Form

We need an appropriate generalization of RREF.

There are two matrix canonical forms that generalize RREF:

- Fuller, "A canonical set for matrices over a principal ideal ring modulo m," Canad. J. Math, 54–59, 1954.
- Howell, "Spans in the module $\mathbb{Z}_m^s$," Linear and Multilinear Algebra, 19:1, 67–77, 1986.

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are Fuller and Howell canonical forms, respectively.

Matrices over Finite Chain Rings: Row Canonical Form

Definition

A matrix A is in **row canonical form** if it satisfies the following conditions.

1. Nonzero rows of A are above any zero rows.
2. The pivot of a row is of the form π^ℓ, and is the leftmost entry of the least degree.
3. For every pivot (say π^ℓ), all entries below and in the same column as the pivot are zero, and all entries above and in the same column as the pivot are residues of π^ℓ.
4. If A has two pivots of the same degree, the one that occurs earlier is above the one that occurs later. If A has two pivots of different degree, the one with smaller degree is above the one with larger degree.

For example, over $\mathbb{Z}_8$,

$$A = \begin{bmatrix} 0 & 2 & 0 & \bar{1} \\ \bar{2} & 2 & 0 & 0 \\ 0 & 0 & \bar{2} & 0 \\ 0 & \bar{4} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

is in row canonical form.

∗For details, see our paper and/or Kiermaier's thesis (in German).

Reduction to Row Canonical Form: Example

Reduction is a variant of **Gaussian elimination**.

An example over $\mathbb{Z}_8$:

$$A = \begin{bmatrix} 4 & 6 & 2 & \bar{1} \\ 0 & 0 & 0 & 2 \\ 2 & 4 & 6 & 1 \\ 2 & 0 & 2 & 1 \end{bmatrix} \rightarrow A_1 = \begin{bmatrix} 4 & 6 & 2 & 1 \\ 0 & 4 & 4 & 0 \\ \bar{6} & 6 & 4 & 0 \\ 6 & 2 & 0 & 0 \end{bmatrix} \rightarrow$$

$$A_1' = \begin{bmatrix} 4 & 6 & 2 & 1 \\ \bar{2} & 2 & 4 & 0 \\ 0 & 4 & 4 & 0 \\ 6 & 2 & 0 & 0 \end{bmatrix} \rightarrow A_2 = \begin{bmatrix} 0 & 2 & 2 & 1 \\ 2 & 2 & 4 & 0 \\ 0 & \bar{4} & 4 & 0 \\ 0 & 4 & 4 & 0 \end{bmatrix} \rightarrow$$

$$A_3 = \begin{bmatrix} 0 & 2 & 2 & \bar{1} \\ \bar{2} & 2 & 4 & 0 \\ 0 & \bar{4} & 4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$ which is in row canonical form.

Row canonical form is not necessarily an echelon form!

Matrices over Finite Chain Rings: Smith Normal Form

Theorem

For any $A \in R^{n\times m}$, A can be decomposed as

$$A = PDQ,$$

for some invertible $n \times n$ and $m \times m$ matrices P and Q, where $D = \text{diag}(\pi^{t_1}, \ldots, \pi^{t_k})$ with $t_1 \leq \cdots \leq t_k$ and $k = \min\{n, m\}$.

Ex: $R = \mathbb{Z}_8$

$$D = \begin{bmatrix} 3 & & \\ 3 & 1 & \\ & & 1 \end{bmatrix} \begin{bmatrix} 4 & 6 & 3 \\ 4 & 6 & 7 \\ & & \end{bmatrix} \begin{bmatrix} & & 1 \\ & 1 & \\ 1 & 6 & 4 \end{bmatrix} = \begin{bmatrix} 1 & & \\ & 0 & \\ & & 0 \end{bmatrix}.$$

Remark: The diagonal entries in D are precisely the pivots of the row canonical form of A

Shape of a Matrix

The shape is a tuple of non-decreasing integers.

Example: $\mu = (3,5)$

$$\begin{matrix} * & * & * \\ * & * & * & * & * \end{matrix}$$

$$R^{\mu} = \underbrace{\langle 1\rangle \times \cdots \times \langle 1\rangle}_{\mu_1} \times \underbrace{\langle \pi\rangle \times \cdots \times \langle \pi\rangle}_{\mu_2-\mu_1} \times \cdots \times \underbrace{\langle \pi^{s-1}\rangle \times \cdots \times \langle \pi^{s-1}\rangle}_{\mu_s-\mu_{s-1}}.$$

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The shape of a module generalizes the concept of dimension.

Theorem

For any finite R-module M, there is a unique μ such that $M \cong R^\mu$.

We call μ the shape of M, and write $\mu = \operatorname{shape} M$.

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The shape of a matrix generalizes the concept of rank.

Definition

The shape of a matrix A is defined as the shape of the row span of A, i.e., $\operatorname{shape} A = \operatorname{shape}(\operatorname{row}(A))$.

Properties of the Shape

Matrix shape has several properties similar to matrix rank:

1. shape A = shape A^T.
2. For any invertible P, Q, shape A = shape PAQ.
3. shape $AB \preceq$ shape A, shape $AB \preceq$ shape B.
4. For any submatrix C of A, shape $C \preceq$ shape A.

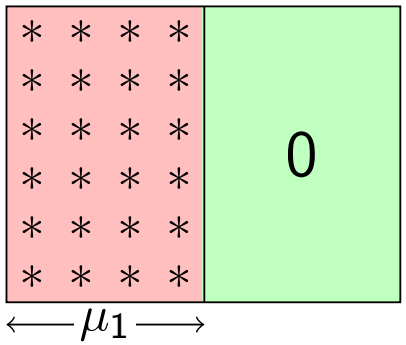
Enumeration Results

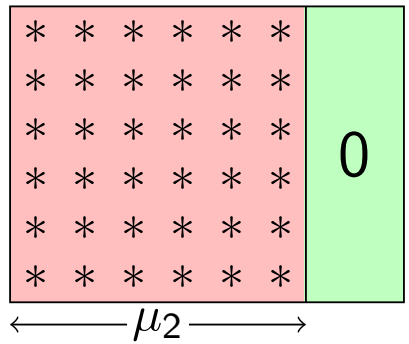
Notation

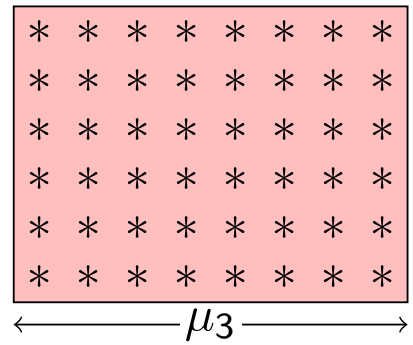
$R^{n\times\mu}$: The subset of matrices in $R^{n\times m}$ whose rows $\in R^{\mu}$.

- $|R^{n\times\mu}| = q^{n(\mu_1+\cdots+\mu_s)}$.

Example: $R = \mathbb{Z}_8$, $n = 6$, $\mu = (4, 6, 8)$, $X = X_0 + 2X_1 + 4X_2$

$X_0 =$ 

$X_1 =$ 

$X_2 =$ 

Enumeration Results

Notation

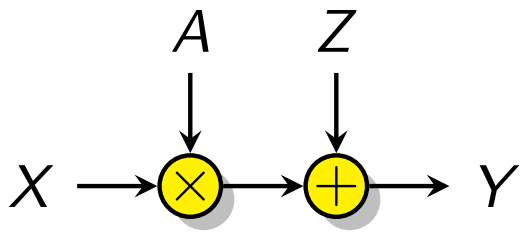
$\mathcal{T}_\tau(R^{n\times\mu})$: The subset of matrices in $R^{n\times\mu}$ whose shape is τ.

- $|\mathcal{T}_\tau(R^{n\times\mu})| = \begin{bmatrix}\mu\\ \tau\end{bmatrix}_q |R^{n\times\tau}| \prod_{i=0}^{\tau_s-1}(1-q^{i-n})$, where

$$\begin{bmatrix}\mu\\ \tau\end{bmatrix}_q = \prod_{i=1}^{s} q^{(\mu_i-\tau_i)\tau_{i-1}} \begin{bmatrix}\mu_i-\tau_{i-1}\\ \tau_i-\tau_{i-1}\end{bmatrix}_q,$$

and $\begin{bmatrix}m\\ k\end{bmatrix}_q$ is the Gaussian coefficient.

Back to Our Objective



$$Y = AX + Z$$

where

$$X, Y, Z \in R^{n \times \mu}$$

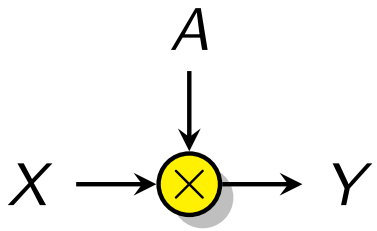
Objective:

- Capacity bounds
- Simple coding schemes

Multiplicative Matrix Channel

First warmup problem

The multiplicative matrix channel (MMC):



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where

- $X, Y \in R^{n \times \mu}$;
- A: invertible, uniform;
- A and X are independent.

MMC: Review of [SKK10]

When R reduces to $\mathbb{F}_q$ and $R^{n\times\mu}$ reduces to $\mathbb{F}_q^{n\times m}$:

1. Exact capacity: A preserves the row span, so

$$C_{\mathsf{MMC}} = \log_q \left(\#\text{ of subspaces of } \mathbb{F}_q^m\right)$$

2. Capacity-achieving code: reduced row echelon form (RREF)

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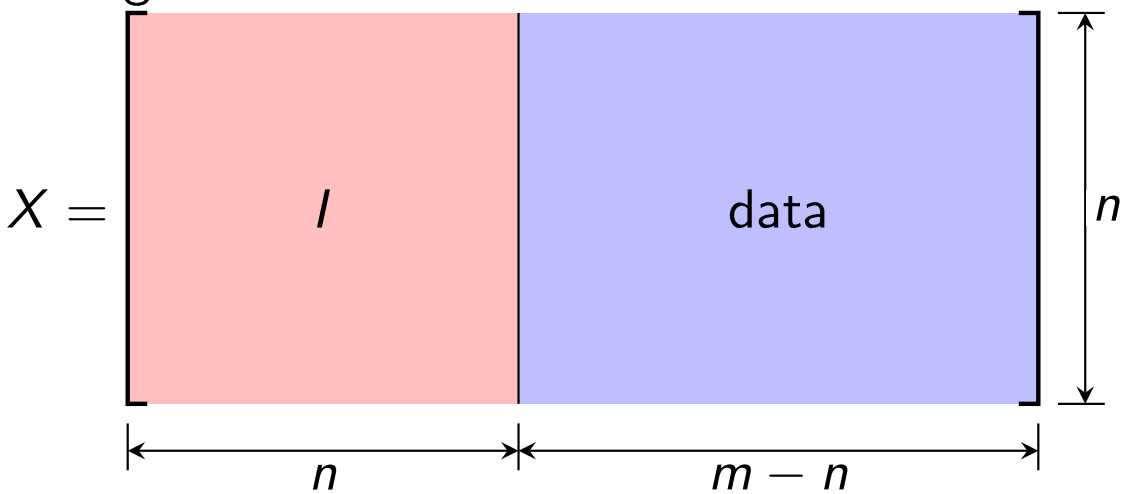
When R reduces to $\mathbb{F}_q$ and $R^{n\times\mu}$ reduces to $\mathbb{F}_q^{n\times m}$:

1. Exact capacity: A preserves the row span, so

$$C_{\mathrm{MMC}} = \log_q \left(\#\text{ of subspaces of } \mathbb{F}_q^m\right)$$

2. Capacity-achieving code: reduced row echelon form (RREF)
3. Efficient encoding-decoding:
   - encoding:



   - decoding: Gaussian elimination (reduction to RREF)

MMC: Exact Capacity

Theorem

The capacity of the MMC, in q-ary symbols per channel use, is

$$C_{\mathrm{MMC}} = \log_q \left(\#\text{ of submodules of } R^{\mu}\right).$$

$\#$ of submodules of R^{μ} is $\sum_{\lambda \preceq n,\mu} \left[\!\!\left[{\mu \atop \lambda} \right]\!\!\right]_q$ (see, e.g., [HL00][4]}), where

$$\left[\!\!\left[{\mu \atop \lambda} \right]\!\!\right]_q = \prod_{i=1}^{s} q^{(\mu_i - \lambda_i)\lambda_{i-1}} \begin{bmatrix} \mu_i - \lambda_{i-1} \\ \lambda_i - \lambda_{i-1} \end{bmatrix}_q,$$

and $\begin{bmatrix} m \\ k \end{bmatrix}_q$ is the Gaussian coefficient.

- note: $\lambda \preceq n, \mu$ means $\forall i,\ \lambda_i \leq n, \mu_i$

[4] Honold and Landjev, "Linear Codes over Finite Chain Rings," *The Electronic J. of Combinatorics*, vol. 7, 2000.

MMC: Efficient Encoding-Decoding

First attempt:

- Encoding: transmit a row canonical form (RCF)
- Decoding: reduction to RCF

The decoding complexity is $\mathcal{O}(n^2 m)$, but the encoding is hard.

Solution:

- Encoding: transmit a **principal** RCF
- Decoding: reduction to RCF

The encoding complexity is $\mathcal{O}(nm)$.
Principal RCFs occupy a significant portion of all RCFs.

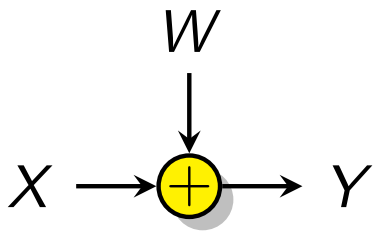
Hence,

The simple coding scheme asymptotically achieves the capacity.

Additive Matrix Channel

Second warmup problem

The additive matrix channel (AMC):



$$Y = X + W$$

where

- $X, Y \in R^{n \times \mu}$;
- W: shape τ, uniform;
- W and X are independent.

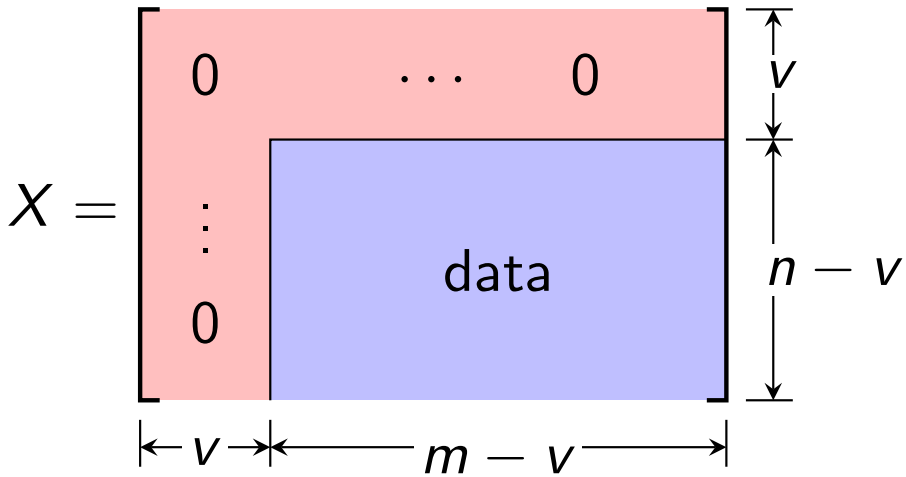
AMC: Review of [SKK10]

When R reduces to $\mathbb{F}_q$ and $R^{n\times\mu}$ reduces to $\mathbb{F}_q^{n\times m}$, shape τ reduces to rank t:

1. Exact capacity: a discrete symmetric channel

$$C_{\mathrm{AMC}} = nm - \log_q\left(\#\text{ of matrices of rank } t \text{ in } \mathbb{F}_q^{n\times m}\right)$$

2. Capacity-approaching code: v is a parameter



3. Efficient encoding-decoding:
   - encoding: error trapping
   - decoding: matrix completion

AMC: Exact Capacity

The AMC is an example of a discrete symmetric channel.

Theorem

The capacity of the AMC, in q-ary symbols per channel use, is

$$C_{\text{AMC}} = \log_q |R^{n\times\mu}| - \log_q |\mathcal{T}_\tau(R^{n\times\mu})|.$$

We need the following:

- $|R^{n\times\mu}| = q^{n(\mu_1+\cdots+\mu_s)}$.
- $|\mathcal{T}_\tau(R^{n\times\mu})| = \left[\!\!\left[\begin{matrix} \mu \\ \tau \end{matrix} \right]\!\!\right]_q |R^{n\times\tau}| \prod_{i=0}^{\tau_s-1}(1-q^{i-n})$, where

$$\left[\!\!\left[\begin{matrix} \mu \\ \tau \end{matrix} \right]\!\!\right]_q = \prod_{i=1}^{s} q^{(\mu_i-\tau_i)\tau_{i-1}} \begin{bmatrix} \mu_i - \tau_{i-1} \\ \tau_i - \tau_{i-1} \end{bmatrix}_q.$$

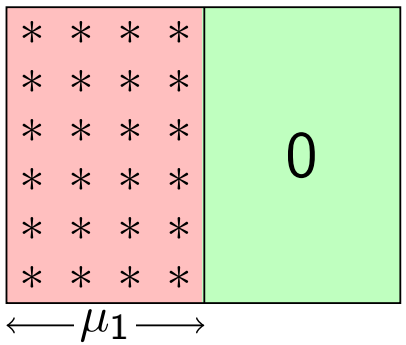
AMC: Capacity-Approaching Code Design

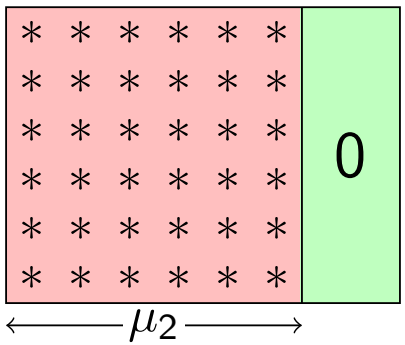
code design problem $\Rightarrow$ a generalization of error-trapping

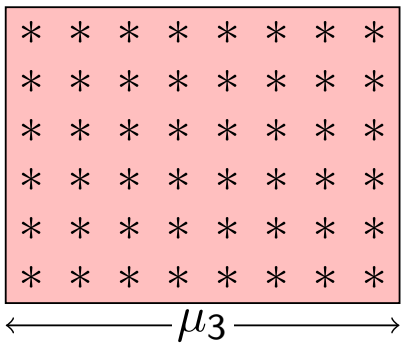
Solution: layered error-trapping

Note that every matrix in $R^{n\times\mu}$ admits a π-adic decomposition.

Example: $R = \mathbb{Z}_8$, $n = 6$, $\mu = (4, 6, 8)$, $X = X_0 + 2X_1 + 4X_2$

$X_0 =$ 

$X_1 =$ 

$X_2 =$ 

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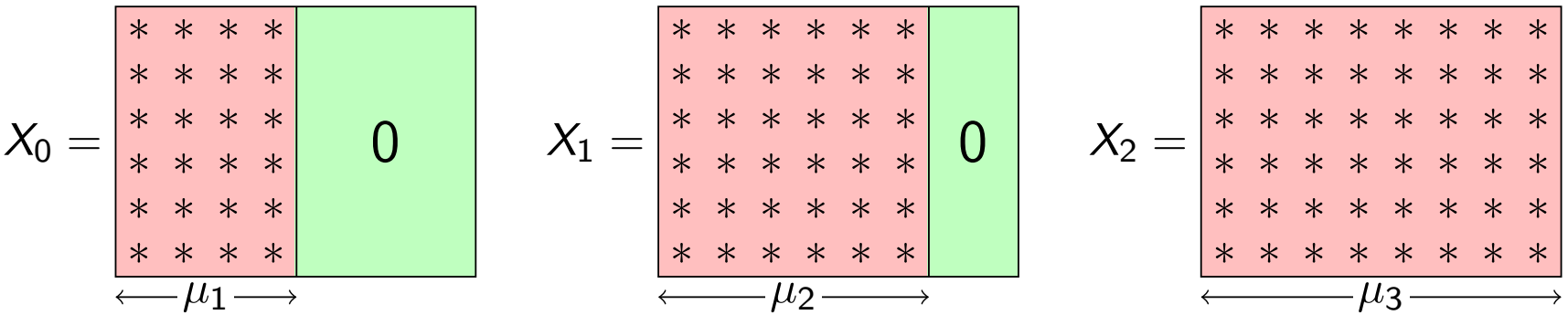
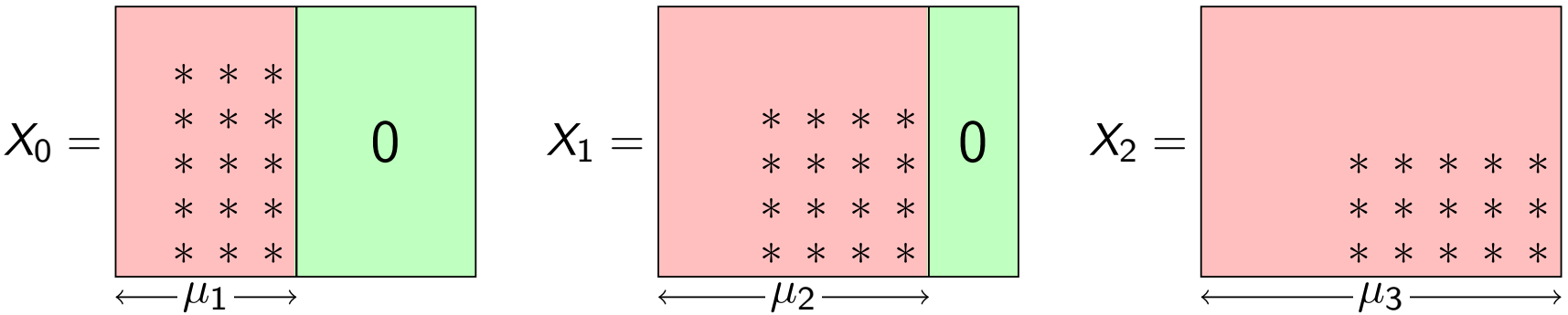


after error-trapping...



AMC: Efficient Encoding-Decoding

- Encoding: layered error-trapping, $\mathcal{O}(nm)$ complexity
- Decoding: multistage matrix completion, $\mathcal{O}(n^2 m)$ complexity

Example: $R = \mathbb{Z}_8$, $X = X_0 + 2X_1 + 4X_2$. Note that

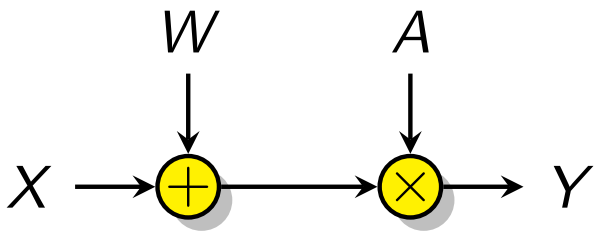
$$Y = X + W = X_0 + 2X_1 + 4X_2 + W.$$

1. Take mod 2: $[Y]_2 = X_0 + [W]_2$.
2. Decode X_0 by completing $[W]_2$.
3. Clear X_0 from Y: $Y' = Y - X_0 = 2X_1 + 4X_2 + W$.
4. Take mod 4: $[Y']_4 = 2X_1 + [W]_4$.
5. Decode $2X_1$ by completing $[W]_4$.
6. Clear X_1 from Y': $Y'' = Y' - 2X_1 = 4X_2 + W$.
7. We have $Y'' = 4X_2 + W$.
8. Decode $4X_2$ by completing W.

Additive-Multiplicative Matrix Channel

Now to the main event:

The additive-multiplicative matrix channel (AMMC):



$$Y = A(X + W)$$

where

- $X, Y \in R^{n \times \mu}$;
- A: invertible, uniform;
- W: shape τ, uniform;
- A, X and W are independent.

Remark: This model is statistically identical to $Y = AX + Z$.

AMMC: Upper Bound on Capacity

Theorem

The capacity of the AMMC, in q-ary symbols per channel use, is upper-bounded by

$$C_{\mathrm{AMMC}} \leq \sum_{i=1}^{s}(\mu_i - \xi_i)\xi_i + \sum_{i=1}^{s}(n-\mu_i)\tau_i + 2s\log_q 4 + \log_q \binom{n+s}{s}$$

$$+ \log_q \binom{\tau_s+s}{s} - \log_q \prod_{i=0}^{\tau_s-1}(1-q^{i-n}), \text{ where } \xi_i = \min\{n, \lfloor \mu_i/2 \rfloor\}.$$

In particular, when $\mu \succeq 2n$, the upper bound reduces to

$$C_{\mathrm{AMMC}} \leq \sum_{i=1}^{s}(n-\tau_i)(\mu_i - n) + 2s\log_q 4$$

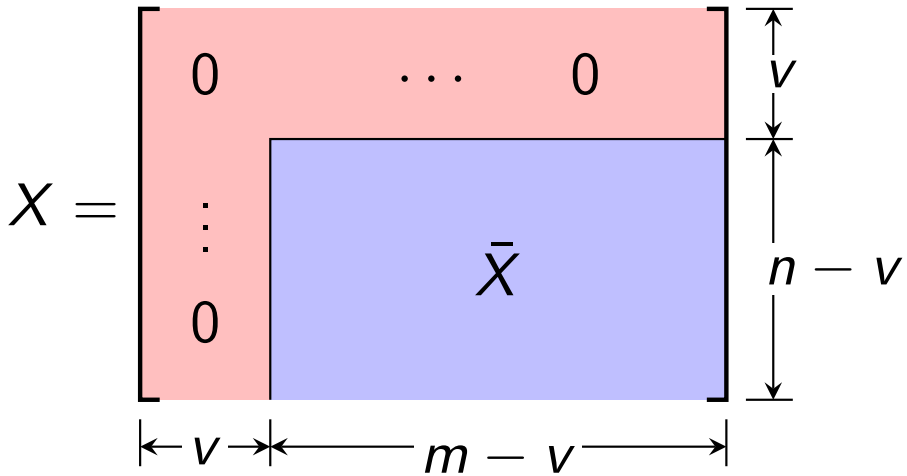
$$+ \log_q \binom{n+s}{s} + \log_q \binom{\tau_s+s}{s} - \log_q \prod_{i=0}^{\tau_s-1}(1-q^{i-n}).$$

AMMC: Coding Scheme

coding scheme = principal RCFs + layered error-trapping

However, the combination turns out to be non-trivial.
Hence, we focus on the special case when $\tau = (t, \ldots, t)$.

- Encoding:



- Decoding: upon receiving $Y = A(X + W)$, the decoder simply computes the RCF of Y, which exposes $\bar{X}$ with high probability.

This simple coding scheme asymptotically achieves the capacity for the special case when $\tau = (t, \ldots, t)$ and $\mu \succeq 2n$.

Concluding Remarks

1. New motivations
2. "User-friendly" tools
3. Need more workshops like this

Construction of Principal RCFs

Definition

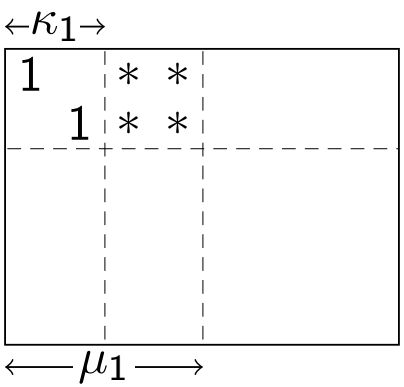
A row canonical form in $\mathcal{T}_\kappa(R^{n\times\mu})$ is called *principal* if its diagonal entries $d_1, d_2, \ldots, d_r$ $(r = \min\{n, m\})$ have the following form:

$$d_1, \ldots, d_r = \underbrace{1, \ldots, 1}_{\kappa_1}, \underbrace{\pi, \ldots, \pi}_{\kappa_2-\kappa_1}, \ldots, \underbrace{\pi^{s-1}, \ldots, \pi^{s-1}}_{\kappa_s-\kappa_{s-1}}, \underbrace{0, \ldots, 0}_{r-\kappa_s}.$$

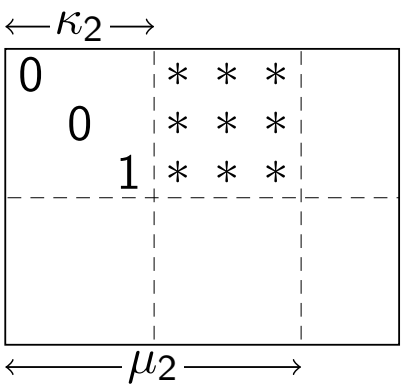
All principal RCFs in $\mathcal{T}_\kappa(R^{n\times\mu})$ can be constructed via a π-adic decomposition $X = X_0 + \pi X_1 + \cdots + \pi^{s-1}X_{s-1}$.

Example: $s = 3$, $n = 6$, $\mu = (4, 6, 8)$, and $\kappa = (2, 3, 4)$

$X_0 =$



$X_1 =$



$X_2 =$