Number-Theoretic Fast-Decodable Space–Time Codes for Multiuser Communications

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Overview

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1. **Space–Time Coding**
   - Space–Time Codes from Cyclic Division Algebras
   - On Fast-Decodability
   - Iterative Code Construction
2. **Amplify-and-Forward Relaying**
3. **Explicit Constructions**
   - Code Construction for SIMO-NAF
   - Code Construction for MIMO-NAF
4. **Further Applications and Conclusions**

Need for diversity

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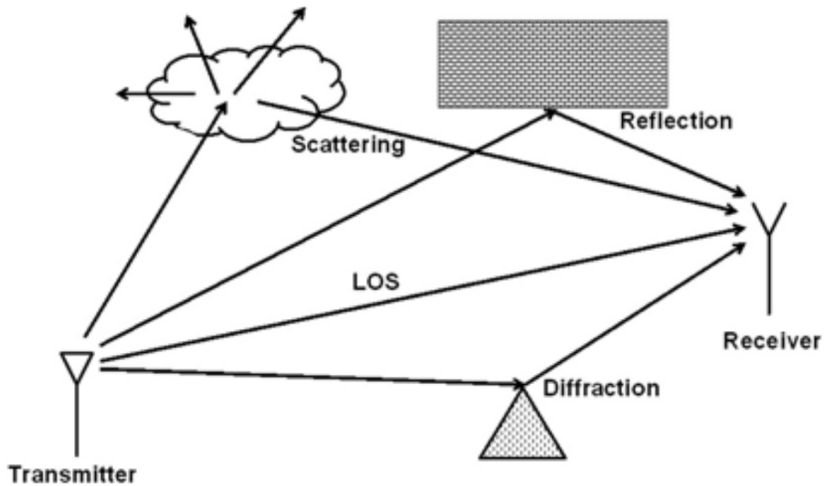

Waves transmitted over a wireless channel suffer from environmental effects and fading. Combat those effects through *diversity*!

- *Spatial diversity:* Multiple antennas at the transmitter and/or receiver.
- *Temporal diversity:* Use different time slots for transmission.

$\Rightarrow$ space–time codes.

MIMO channel model

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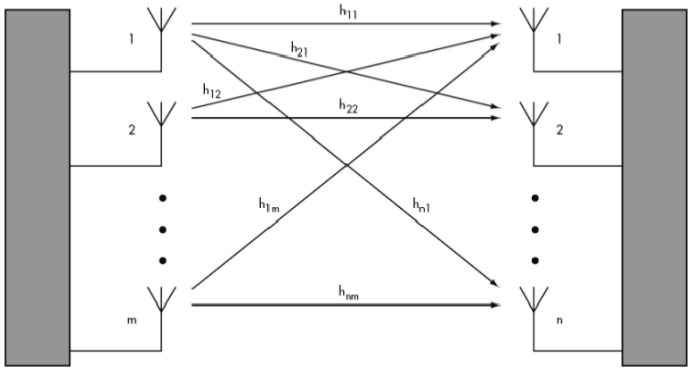
Consider an $n_s \times n_d$-antenna system:



$$\Leftrightarrow \quad H = \begin{bmatrix} h_{11} & h_{12} & \cdots & h_{1n_s} \\ h_{21} & h_{22} & \cdots & h_{2n_s} \\ \vdots & \vdots & \ddots & \vdots \\ h_{n_d1} & h_{n_d2} & \cdots & h_{n_dn_s} \end{bmatrix}$$

Transmission using T time slots can be modeled as

$$Y = HX + N.$$

- The channel matrix $H \in \mathrm{Mat}(n_d \times n_s, \mathbb{C})$ models Rayleigh fading.
- $X \in \mathrm{Mat}(n_s \times T, \mathbb{C})$ is a codeword matrix.
- $N \in \mathrm{Mat}(n_d \times T, \mathbb{C})$ a noise matrix, whose entries are complex Gaussian with zero mean.

We will focus on square codes, $n_s = T$.

F. Oggier, J-C. Belfiore, and E. Viterbo. "Cyclic division algebras: a tool for space-time coding". In: *Foundations and trends in communications and information theory* 4.1 (2007), pp. 1–95

Algebraic setup

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- L/K degree n cyclic Galois extension of number fields with respective ring of integers $\mathcal{O}_K$ and $\mathcal{O}_L$.
- Cyclic Galois group $\Gamma(L/K) = \langle\sigma\rangle$.
- Choose an $\mathcal{O}_K$-basis $\{\omega_0, \ldots, \omega_{n-1}\}$ of $\mathcal{O}_L$ (assume K is a PID).
- Fix $\gamma \in K^\times$ such that $\gamma^i \notin \mathrm{Nm}_{L/K}(L^\times)$, $i = 1, \ldots, n-1$.

The triple

$$\mathcal{C} = (L/K, \sigma, \gamma) \cong \bigoplus_{i=0}^{n-1} u^i L$$

is a *cyclic division algebra* (CDA) of degree n, and multiplication is determined by the relations

$$u^n = \gamma, \quad lu = u\sigma(l) \text{ for all } l \in L.$$

$$\begin{array}{ccc} \mathcal{C} & & \mathcal{C} \\ {\scriptstyle n}\,| & & |\,\{u^i\}_{i=0}^{n-1} \\ L & & \mathcal{O}_L \\ {\scriptstyle n}\,| & & |\,\{\omega_i\}_{i=0}^{n-1} \\ K & & \mathcal{O}_K \\ {\scriptstyle m}\,| & & \\ \mathbb{Q} & & \end{array}$$

The left-regular representation

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We need to identify elements $x = \sum_{j=0}^{n} u^j x_j \in \mathcal{C}$ with matrices.

The (transposed) *left-regular representation* (LRR) $\lambda : \mathcal{C} \to \mathrm{Mat}(n, \mathbb{C})$ is an injective algebra homomorphism, given by

$$\lambda(x) = \begin{bmatrix} x_0 & x_1 & \cdots & x_{n-2} & x_{n-1} \\ \gamma\sigma(x_{n-1}) & \sigma(x_0) & \cdots & \sigma(x_{n-3}) & \sigma(x_{n-2}) \\ \vdots & \vdots & & \vdots & \vdots \\ \gamma\sigma^{n-2}(x_2) & \gamma\sigma^{n-2}(x_3) & \cdots & \sigma^{n-2}(x_0) & \sigma^{n-2}(x_1) \\ \gamma\sigma^{n-1}(x_1) & \gamma\sigma^{n-1}(x_2) & \cdots & \gamma\sigma^{n-1}(x_{n-1}) & \sigma^{n-1}(x_0) \end{bmatrix}.$$

Write $x_j = \sum_{i=0}^{n-1} \omega_i x_{ji} \in \mathcal{O}_L$, where $x_{ji} \in \mathcal{O}_K$. If we further expand using a $\mathbb{Z}$-basis, we see that $\lambda(x)$ carries mn^2 independent integers (e.g., PAM symbols).

Space–time codes and their properties

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Definition

A *space–time* (ST) code constructed from a CDA $\mathcal{C}$ is a subset

$$\mathcal{X} \underset{\text{finite}}{\subset} \mathrm{Im}(\lambda(\mathcal{C})) = \{\lambda(x) | \, x \in \mathcal{C}\}.$$

Often, a *shaping element* $\alpha \in L^\times$ is introduced to restrict the entries to an ideal $(\alpha) \subset \mathcal{O}_L$ and improve the performance of the code, which can have many desirable properties, such as

- Full-rank codeword matrices (cf. diversity gain),
- Non-vanishing determinants (cf. coding gain),
- Balanced energy among transmitters (cf. PAPR),
- Optimal DMT,
- *Fast-decodability*, etc.

Equivalent representation

We can give a more explicit presentation of a ST code in terms of its generating matrices as follows.

Definition

Let $\{B_i\}_{i=1}^{k}$ be a set of fixed $n_s \times T$ complex *weight matrices*. A *linear space–time block code* of rank k is a set of the form

$$\mathcal{X} = \left\{ \sum_{i=1}^{k} s_i B_i \middle| s_i \in S \cap \mathbb{Z} \right\},$$

where $S \subset \mathbb{Z}$ is the finite *signaling alphabet* in use.

Remark

This definition agrees with the previous one by setting $n_s = T = n$ and choosing the set of weight matrices $\{B_i\}$ to be a basis of $\mathcal{C}$ over $\mathbb{Q}$.

Full-diversity codes with NVD

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Motivation for the use of division algebras in space–time coding:

Lemma

Let $\mathcal{D}$ be a division algebra and K a field. If $\phi : \mathcal{D} \to \mathrm{Mat}(n, K)$ is a ring homomorphism and $\mathcal{X} \subseteq \phi(\mathcal{D})$ is any finite subset, then $\mathrm{rank}(X - X') = n$ *for any distinct* $X, X' \in \mathcal{X}$.

λ is a ring homomorphism, and we defined $\mathcal{X} \subset \mathrm{Im}(\lambda(\mathcal{C}))$ to be finite.

Lemma

Let $\mathcal{X}$ be a space–time code coming from an order $\mathcal{O}$ of a CDA $\mathcal{C} = (L/K, \sigma, \gamma)$ with center $K = \mathbb{Q}$ or $K = \mathbb{Q}(\sqrt{-m})$, $m \in \mathbb{Z}_{>1}$ squarefree. Then $\mathcal{X}$ has the non-vanishing determinant property.

Example: the Golden code

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Fix $n = 2$. The algebraic setup for the Golden code is as follows:

- $L/K = \mathbb{Q}(i, \sqrt{5})/\mathbb{Q}(i)$.
- Ring of integers $\mathcal{O}_K = \mathbb{Z}[i]$, $\mathcal{O}_L = \mathcal{O}_K[\omega]$, where $\omega = \frac{1+\sqrt{5}}{2}$.
- Cyclic Galois group $\Gamma(L/K) = \langle \sigma : \sqrt{5} \mapsto -\sqrt{5} \rangle$.
- Our CDA is the triple $\mathcal{C}_G = (L/K, \sigma, i) \cong L \oplus uL$, with $u^2 = i$.
- Choose the shaping element $\alpha = 1 + i - i\omega$.

Restricting the entries to $\mathcal{O}_L$, the Golden code is a finite subset

$$\mathcal{X}_G \subset \left\{ \frac{1}{\sqrt{5}} \begin{bmatrix} \alpha & \\ & \sigma(\alpha) \end{bmatrix} \begin{bmatrix} x_0 & x_1 \\ i\sigma(x_1) & \sigma(x_0) \end{bmatrix} \middle| \, x_i \in \mathcal{O}_L \right\}$$

$$= \left\{ \frac{1}{\sqrt{5}} \begin{bmatrix} \alpha & \\ & \sigma(\alpha) \end{bmatrix} \begin{bmatrix} x_{00} + \omega x_{01} & x_{10} + \omega x_{11} \\ i(x_{10} + \sigma(\omega) x_{11}) & x_{00} + \sigma(\omega) x_{01} \end{bmatrix} \middle| \, x_{ij} \in \mathcal{O}_K \right\}.$$

A slight modification

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We will mostly consider field extensions L/K of degree 2. To have balanced energy and good decodability, we slightly modify the matrix representation of the elements in $\mathcal{C} = (L/K, \sigma, \gamma)$.

Instead of representing $x = c + \sqrt{\gamma}d \in \mathcal{O} \subset \mathcal{C}$ using the representation $\lambda(x)$ over the maximal subfield K, we define

$$\tilde{\lambda} : x \mapsto \begin{bmatrix} c & -\sqrt{-\gamma}\sigma(d) \\ \sqrt{-\gamma}d & \sigma(c) \end{bmatrix}.$$

This function is commonly used, and maintains the original determinant.

G. Berhuy, N. Markin, and B. A. Sethuraman. "Bounds on Fast Decodability of Space-Time Block Codes, Skew-Hermitian Matrices, and Azumaya Algebras". In: *IEEE Trans. Inf. Theory* 61.4 (2015), pp. 1959–1970

E. Biglieri, Y. Hong, and E. Viterbo. "On fast-decodable space-time block codes". In: *IEEE Trans. Inf. Theory* 55.2 (2009), pp. 524–530

Complexity of ML-decoding

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Given a space–time code $\mathcal{X}$, *Maximum-Likelihood* (ML) decoding amounts to finding the codeword $X \in \mathcal{X}$ that minimizes

$$\delta(X) := ||Y - HX||_F^2.$$

By defining the real-valued matrix $B := \begin{bmatrix} \text{vec}(HB_1) & \dots & \text{vec}(HB_k) \end{bmatrix}$, we can reduce the decoding problem to read

$$\begin{aligned} \arg\min_{X \in \mathcal{X}} \left\{ ||Y - HX||_F^2 \right\} &\rightsquigarrow \arg\min_{s \in S^k} \left\{ || \,\text{vec}(Y) - Bs||_E^2 \right\} \\ (B = QR) \ &\rightsquigarrow \arg\min_{s \in S^k} \left\{ ||Q^\dagger \,\text{vec}(Y) - Rs||_E^2 \right\}. \end{aligned}$$

Definition

The *ML decoding complexity* of a rank-k ST code $\mathcal{X}$ is upper bounded by the worst-case (ML) complexity $|S|^k$ corresponding to an exhaustive search. A ST code $\mathcal{X}$ is said to be *fast-decodable* if its worst-case ML decoding complexity is of the form $|S|^{k'}$ for $k' < k - 1$.

Hurwitz-Radon quadratic form

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Introducing the R-matrix in the decoding process permits to directly read out the decoding complexity of a given code.

Definition

The *Hurwitz-Radon Quadratic Form* (HRQF) is the map

$$\mathcal{Q} : \mathcal{X} \to \mathbb{R}; \qquad X \mapsto \sum_{1 \leq i \leq j \leq k} s_i s_j m_{ij},$$

where $m_{ij} := ||B_i B_j^\dagger + B_j B_i^\dagger||_F^2$ and $s_i \in S$.

- Define the matrix $M = (m_{ij})$. Then $m_{ij} = 0$ if and only if $B_i B_j^\dagger + B_j B_i^\dagger = \mathbf{0}$, that is, if B_i and B_j are *mutually orthogonal*.
- Premultiplication of the weight matrices by H does not affect the zero structure of M, whereas it does affect that of R.
- Yet, the zero structure of the R and M matrices are conveniently related to each other.

Hurwitz-Radon quadratic form

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Introducing the R-matrix in the decoding process permits to directly read out the decoding complexity of a given code.

Definition

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Breaking News!

Recently, the criteria for fast-decodability have been revisited:

A. Mejri, M. Kshiba, and G. Rekaya. "Reduced-Complexity ML Decodable STBCs: Revisited Design Criteria". In: *IEEE ISWCS*. 2015, pp. 666–670.

The authors show that the HRQF method does not capture all possible fast-decodable code families, and propose a relaxed criterion on the mutual orthogonality condition.

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- Yet, the zero structure of the R and M matrices are conveniently related to each other.

(Conditional) g-group decodability

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Definition

A rank-k ST code $\mathcal{X}$ is *conditionally g-group decodable* if there exists a partition of $\{1,\dots,k\}$ into $g+1 \geq 3$ disjoint subsets $\Gamma_1,\dots,\Gamma_g,\Gamma^{\mathcal{X}}$, such that $B_iB_j^\dagger + B_jB_i^\dagger = 0$ for $i \in \Gamma_p$, $j \in \Gamma_q$, $1 \leq p < q \leq g$.
$\mathcal{X}$ is *g-group decodable* if $|\Gamma^{\mathcal{X}}| = 0$.

The R-matrices related to these families of codes are of the form

$$R_{\text{cond}} = \begin{bmatrix} D_1 & & & G_1 \\ & \ddots & & \vdots \\ & & D_g & G_g \\ & & & G \end{bmatrix} \quad \text{resp.} \quad R = \begin{bmatrix} D_1 & & \\ & \ddots & \\ & & D_g \end{bmatrix},$$

where $D_i \in \text{Mat}(|\Gamma_i|,\mathbb{R})$, $G \in \text{Mat}(|\Gamma^{\mathcal{X}}|,\mathbb{R})$ upper triangular.

Lemma

The ML-decoding complexity of $\mathcal{X}$ is $|S|^{|\Gamma^{\mathcal{X}}|+\max\limits_{1\leq i\leq g}|\Gamma_i|}$.

N. Markin and F. Oggier. "Iterated space-time code constructions from cyclic algebras". In: *IEEE Trans. Inf. Theory* 59.9 (2013), pp. 5966–5979

Iterative codes from CDAs

Let $\mathcal{C} = (L/K, \sigma, \gamma)$ be a CDA or degree n. Fix $\theta = \zeta\theta' \in \mathcal{C}$, $\tau \in \mathrm{Aut}_{\mathbb{Q}}(K)$. For $X = \lambda(x)$, $Y = \lambda(y)$ (LRR), define the following function:

$$\alpha_{\tau,\theta} : \mathcal{C} \times \mathcal{C} \to \mathrm{Mat}(2n, K)$$
$$(x, y) \mapsto \begin{bmatrix} X & \zeta\sqrt{\theta'}\tau(Y) \\ \sqrt{\theta'}Y & \tau(X) \end{bmatrix}.$$

If the matrices $\{B_i\}_{i=1}^{k}$ define a ST code $\mathcal{X}$, the iterated ST code $\mathcal{X}_{\mathrm{it}}$ is double the rank and defined by the matrices

$$\{\alpha_{\tau,\theta}(B_i, 0), \alpha_{\tau,\theta}(0, B_i)\}.$$

The importance of this construction is that by making certain assumptions, we can guarantee that the code $\mathcal{X}_{\mathrm{it}}$ will inherit desirable properties from $\mathcal{X}$, such as *fast-decodability*.

S. Yang and J-C. Belfiore. "Optimal space-time codes for the MIMO amplify-and-forward cooperative channel". In: *IEEE Trans. Inf. Theory* 53.2 (2007), pp. 647–663

N-relay MIMO NAF channel

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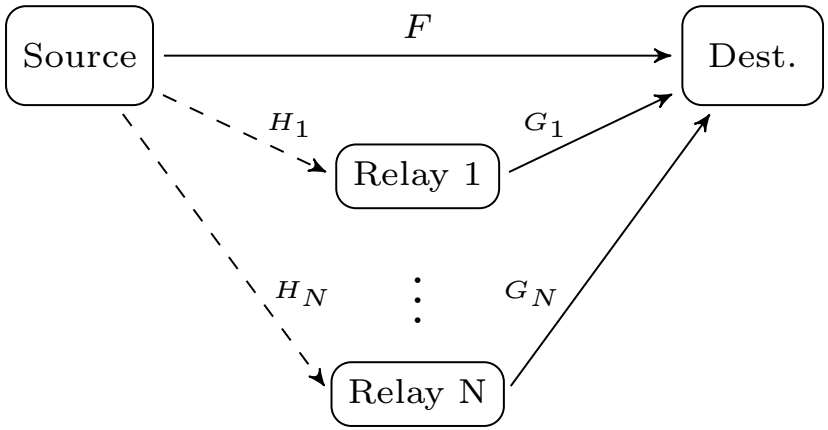

n_s antennas at the source.
$n_r \leq n_s$ antennas at each relay.
n_d antennas at the destination.

We impose the *half-duplex* constraint, i.e., a relay cannot receive and transmit at the same time.

From the destination's point of view this can be viewed as a virtual single-user MIMO channel

$$Y_{n_d \times n} = H_{n_d \times n} X_{n \times n} + V_{n_d \times n},$$

where $n = 2Nn_s$ and H has a special structure.

Frame model

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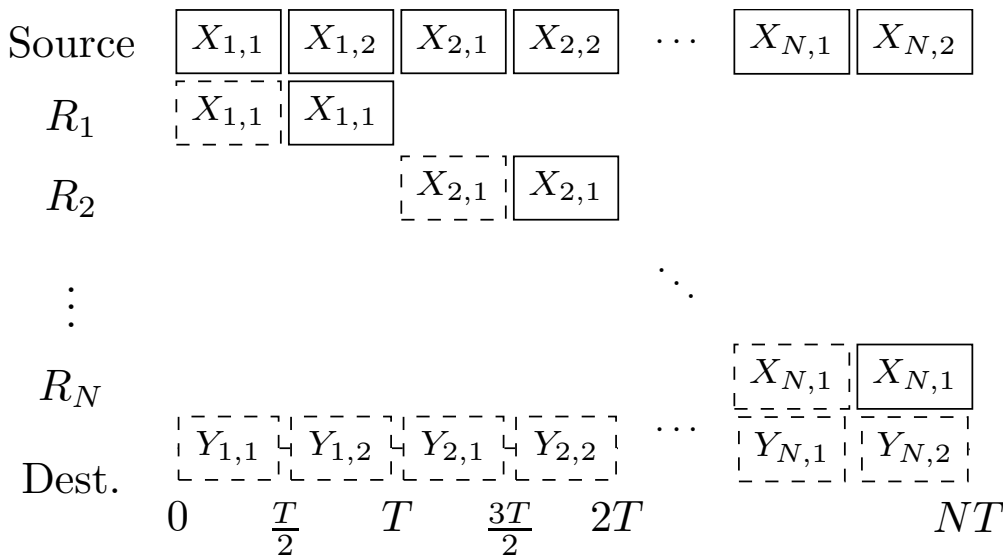
- N consecutive cooperation frames define a *superframe*.
- The relays take turns to cooperate with the transmitter in their respective cooperation frame.
- All channels remain static for the entire superframe.



Transmitted and received signals are represented by solid and dashed boxes, respectively.

Codeword structure

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The overall (equivalent) codewords are of the form

$$X = \operatorname{diag}\left\{\Xi_i\right\}_{i=1}^{N} = \begin{bmatrix} \Xi_1 & & \\ & \ddots & \\ & & \Xi_N \end{bmatrix},$$

where $\Xi_i \in \mathrm{Mat}(2n_s, \mathbb{C})$.

It would be desirable to construct block-diagonal codes which have

- "full" rate $2n_d$ (real) symbols per channel use (rspcu), i.e., the number of independent real information symbols (e.g., PAM) per codeword equals $4n_d n_s N$,
- full rank $2n_s N$,
- non-vanishing determinants,
- fast(er) decoding.

A. Barreal, C. Hollanti, and N. Markin. "Fast-Decodable Space–Time Codes for the N-Relay and Multiple-Access MIMO Channel". In: *IEEE Trans. Wireless Commun.* 15.3 (2016), pp. 1754–1767

The map that does the trick

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Definition

Consider an N-relay NAF channel. Given a ST code $\mathcal{X} \subset \mathrm{Mat}(2n_s, \mathbb{C})$ and a suitable function η of order N (*i.e.*, $\eta^N(X) = X$), define the function

$$\Psi_{\eta,N} : \mathcal{X} \to \mathrm{Mat}(nN, \mathbb{C})$$

$$X \mapsto \mathrm{diag}\left\{\eta^i(X)\right\}_{i=0}^{N-1} = \begin{bmatrix} X & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \eta(X) & \cdots & \mathbf{0} \\ \vdots & & \ddots & \vdots \\ \mathbf{0} & \cdots & & \eta^{N-1}(X) \end{bmatrix}.$$

Algebraic framework – SIMO

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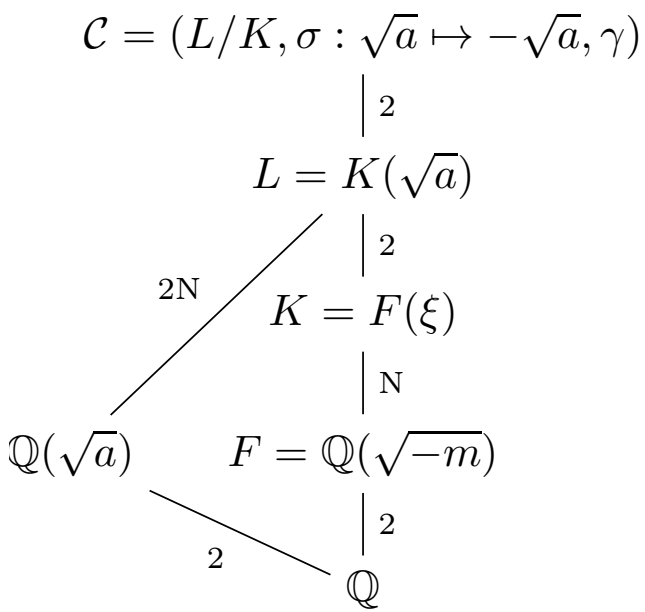
Assume $n_s = n_r = 1$, and $n_d \geq 2$, and consider the tower



We assume $m \in \mathbb{Z}_{\geq 1}$ and $a \in \mathbb{Z} \setminus \{0\}$, both square-free.

Infinite family for SIMO

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Theorem

Consider the algebraic setup from above with $a < 0$, $\gamma < 0$. Fix a generator $\langle \eta \rangle = \Gamma(K/F)$ and define the set

$$\mathcal{X} = \{\Psi_{\eta,N}(X)\}_{X \in \tilde{\lambda}(\mathcal{O})} = \left\{ \operatorname{diag}\left\{\eta^i(X)\right\}_{i=0}^{N-1} \middle| \, X \in \tilde{\lambda}(\mathcal{O}) \right\}.$$

The code $\mathcal{X}$ is of rank $8N$, rate $R = 4$ rspcu and has the NVD property. It is full-rate if $n_d = 2$. Moreover, $\mathcal{X}$ is conditionally 4-group decodable, and its decoding complexity (up to a constant) can be reduced from $|S|^{8N}$ to $|S|^{5N}$, where S is the real constellation used, resulting in a complexity order reduction of 37.5%.

Moreover, full-rate codes constructed using this method will achieve the optimal DMT of the channel.

A two-relay example code

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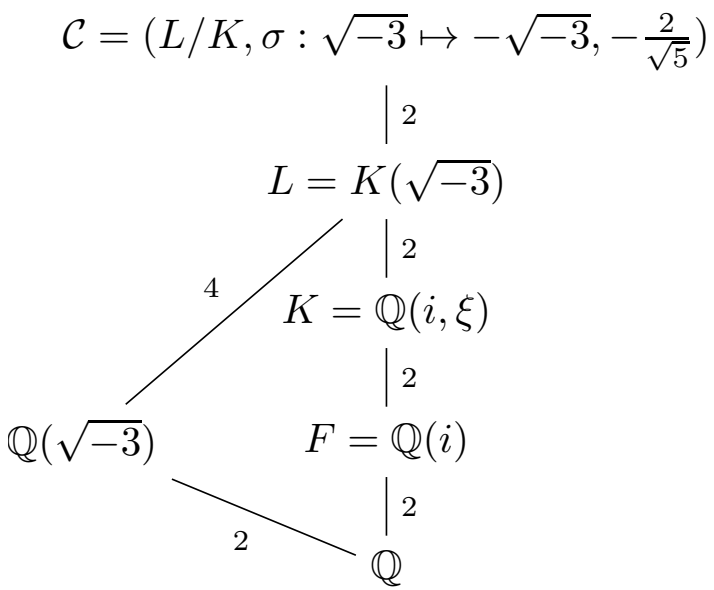
Consider $N = 2$ relays and the following tower of extensions.



- For $\xi = \sqrt{5}$, the algebra $\mathcal{C}$ is division.
- Let $\langle \eta \rangle = \Gamma(K/F)$.
- $X \in \tilde{\lambda}(\mathcal{O}_L)$ is of the form

$$X = \begin{bmatrix} c & -\sqrt{-\gamma}\sigma(d) \\ \sqrt{-\gamma}d & \sigma(c) \end{bmatrix}$$

Define the 2-relay code

$$\mathcal{X} = \left\{ \Psi_{\eta,2}(X) \middle| \, X \in \tilde{\lambda}(\mathcal{O}_L) \right\} = \left\{ \begin{bmatrix} X & \\ & \eta(X) \end{bmatrix} \middle| \, X \in \tilde{\lambda}(\mathcal{O}_L) \right\}.$$

This is a fully diverse NVD code of rank 16. It is conditionally 4-group decodable, and its decoding compexity is $|S|^{10}$ in contrast to $|S|^{16}$.

What about performance?

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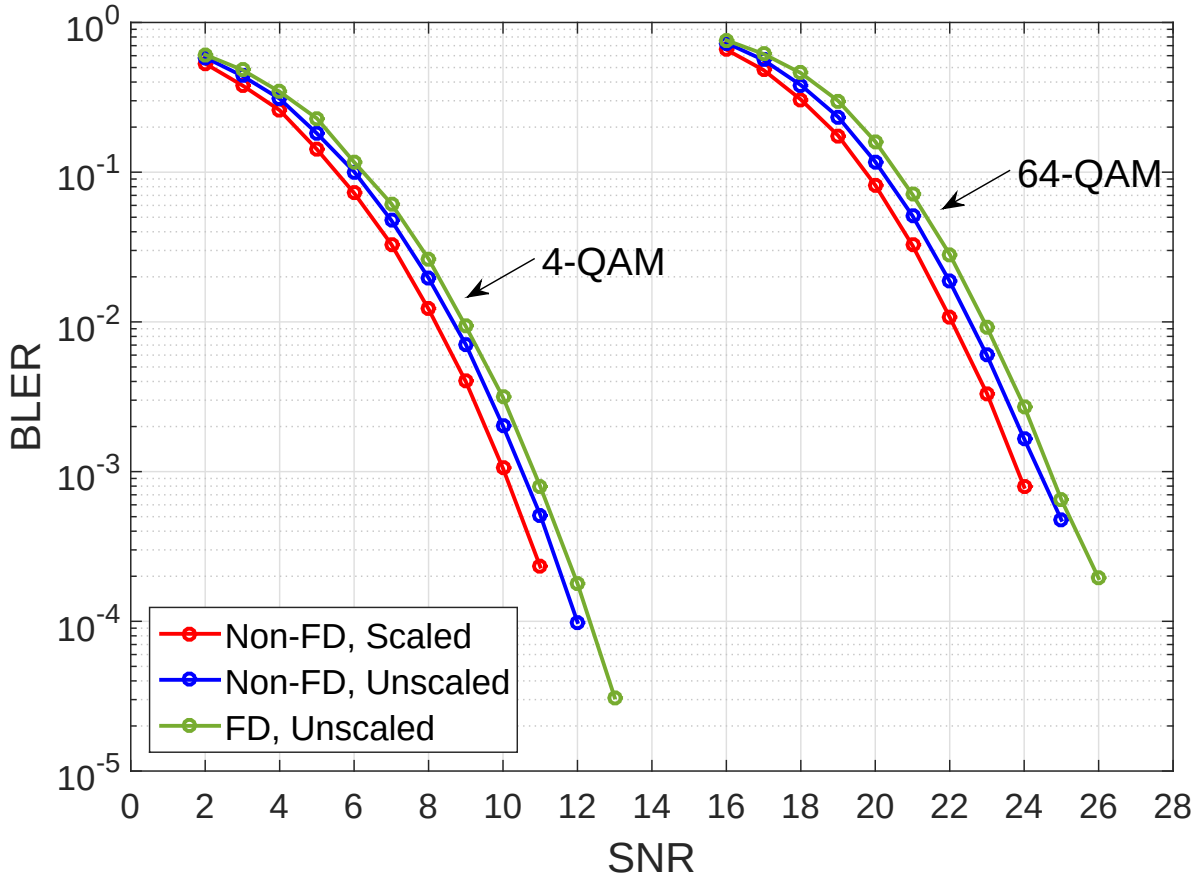
We compare the performance of the example code with the optimal code proposed in [Yang]) – a lifted version of the Golden code – and further an unshaped version of the same. ($n_s = n_r = 1$, $n_d = 4$)



S. Yang and J-C. Belfiore. "Optimal space-time codes for the MIMO amplify-and-forward cooperative channel". In: *IEEE Trans. Inf. Theory* 53.2 (2007), pp. 647–663

Algebraic framework – MIMO

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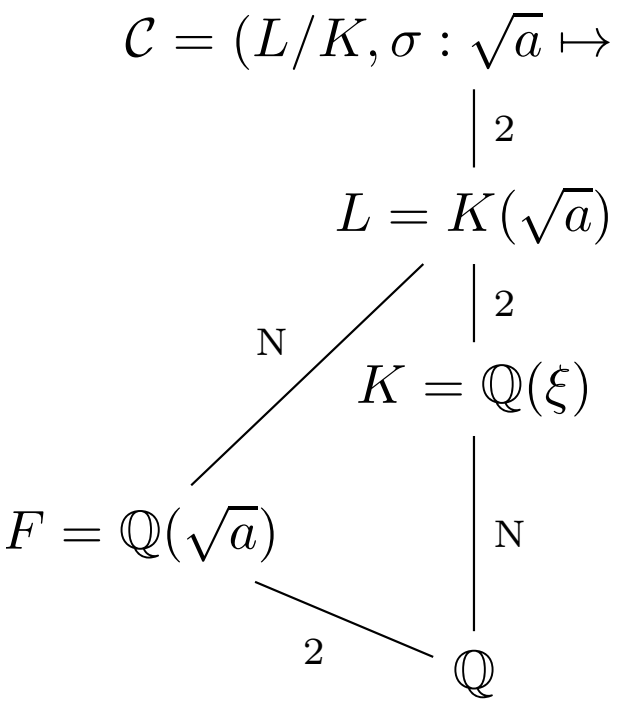
Assume $n_s = 2$, $n_d \geq 1$ and $N = (p-1)/2$ relays ($p \geq 5$ prime) equipped with $n_r \leq 2$ antennas.



- $K = \mathbb{Q}(\xi) = \mathbb{Q}^+(\zeta_p) \subset \mathbb{Q}(\zeta_p)$ maximal real subfield of the p^{th} cyclotomic field. ($\xi = \zeta_p + \zeta_p^{-1}$)
- $a \in \mathbb{Z} \backslash \{0\}$ is square-free.
- $\langle \sigma \rangle = \Gamma(L/K)$ and $\langle \eta \rangle = \Gamma(L/F)$.

Infinite family for MIMO

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Theorem

Consider the algebraic setup from above, and choose $a \in \mathbb{Z}_{<0}$ *such that* $\mathfrak{p} = a\mathcal{O}_K$ *is a prime ideal. Fix further* $\gamma < 0$ *and* $\theta \in \mathcal{O}_K \cap \mathbb{R}^\times = \mathbb{Z}[\xi] \cap \mathbb{R}^\times$ *such that*

- γ *and* θ *are both nonsquare* $\bmod \mathfrak{p}$,
- *the quadratic form* $\langle \gamma, -\theta \rangle_L$ *is anisotropic,*

and further let $\tau = \sigma$. *For* $\mathcal{O} \subset \mathcal{C}$ *an order, the distributed ST code*

$$\mathcal{X} = \left\{ \Psi_{\eta,N}(\alpha_{\tau,\theta}(X,Y)) = \operatorname{diag}\left\{\eta^i(\alpha_{\tau,\theta}(X,Y))\right\}_{i=0}^{N-1} \middle| \, X, Y \in \tilde{\lambda}(\mathcal{O}) \right\}$$

is a full-diversity ST code of rank $8N$, *rate* $R = 2$ *rspcu, exhibits the NVD property and is FD. Its decoding complexity is* $|S|^{k'}$, *where*

$$k' = \begin{cases} 4N & \text{if } a \equiv 1 \bmod 4, \\ 2N & \text{if } a \not\equiv 1 \bmod 4, \end{cases}$$

resulting in a reduction in complexity of 50% *and* 75%, *respectively.*

Two 3-relay example codes

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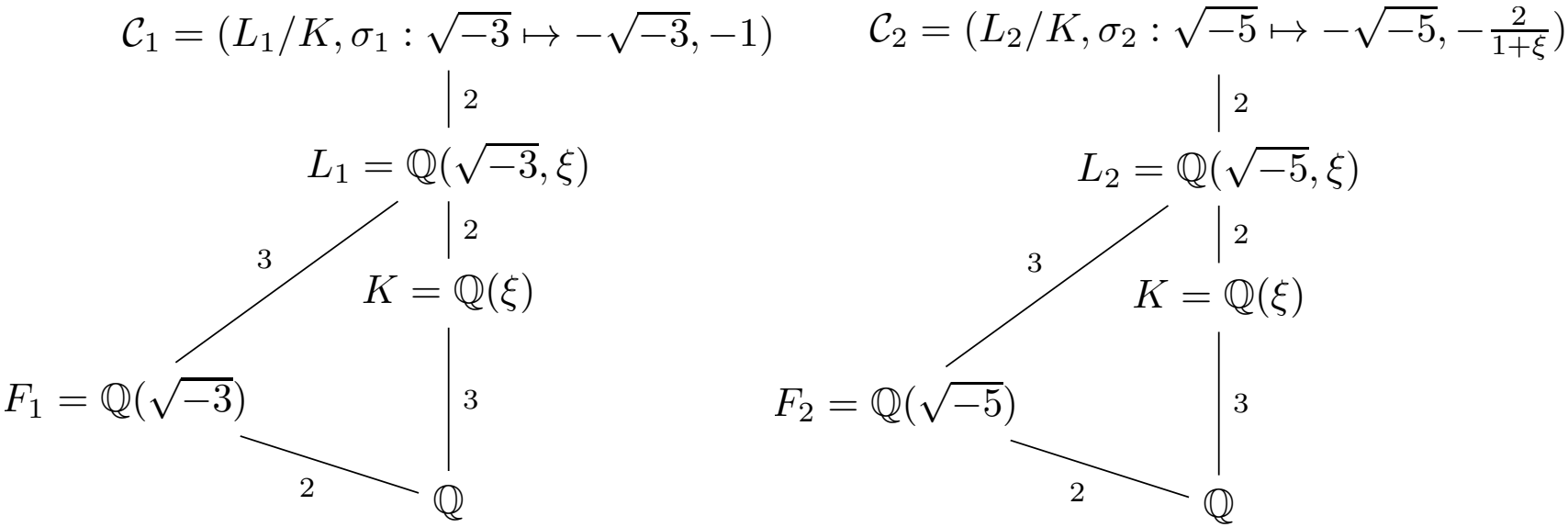
Consider $N = 3$ relays and the following towers of extensions.



- Both algebras are division for $\xi = \zeta_7 + \zeta_7^{-1}$.
- For $i = 1, 2$, set $\tau_i = \sigma_i$ and $\langle \eta_i : \xi \mapsto \xi^2 - 2 \rangle = \Gamma\left(L_i/F_i\right)$. ($\eta_1 \neq \eta_2$, since they have distinct fixed fields.)
- Let $\mathcal{O}_i \subset \mathcal{C}_i$, and set $\omega_1 = \frac{1+\sqrt{-3}}{2}$, $\omega_2 = \sqrt{-5}$.

Codeword structure

The original codes (before iteration an diagonalization) consist of codewords of the form

$$X_i = \tilde{\lambda}(x_i) = \begin{bmatrix} x_{1,i}+x_{2,i}\omega_i & -\sqrt{-\gamma_i}(x_{3,i}+x_{4,i}\sigma_i(\omega_i)) \\ \sqrt{-\gamma_i}(x_{3,i}+x_{4,i}\omega_i) & x_{1,i}+x_{2,i}\sigma_i(\omega_i) \end{bmatrix},$$

respectively.

Using these codes as building blocks, the overall 3-relay codes can be constructed as

$$\mathcal{X}_i = \left\{ \Psi_{\eta_i,3}(\alpha_{\tau_i,\theta_i}(X,Y)) = \mathrm{diag}\left\{ \eta_i^j(\alpha_{\tau_i,\theta_i}(X,Y)) \right\}_{j=0}^{2} \middle| \, X,Y \in \tilde{\lambda}(\mathcal{O}_i) \right\}$$

The code $\mathcal{X}_1$ is 2-group decodable and enjoys a reduction in decoding complexity of 50%, from $|S|^{24}$ to $|S|^{12}$.

The complexity of the 4-group decodable code $\mathcal{X}_2$ is reduced by 75% to $|S|^6$.

Comparing the performance

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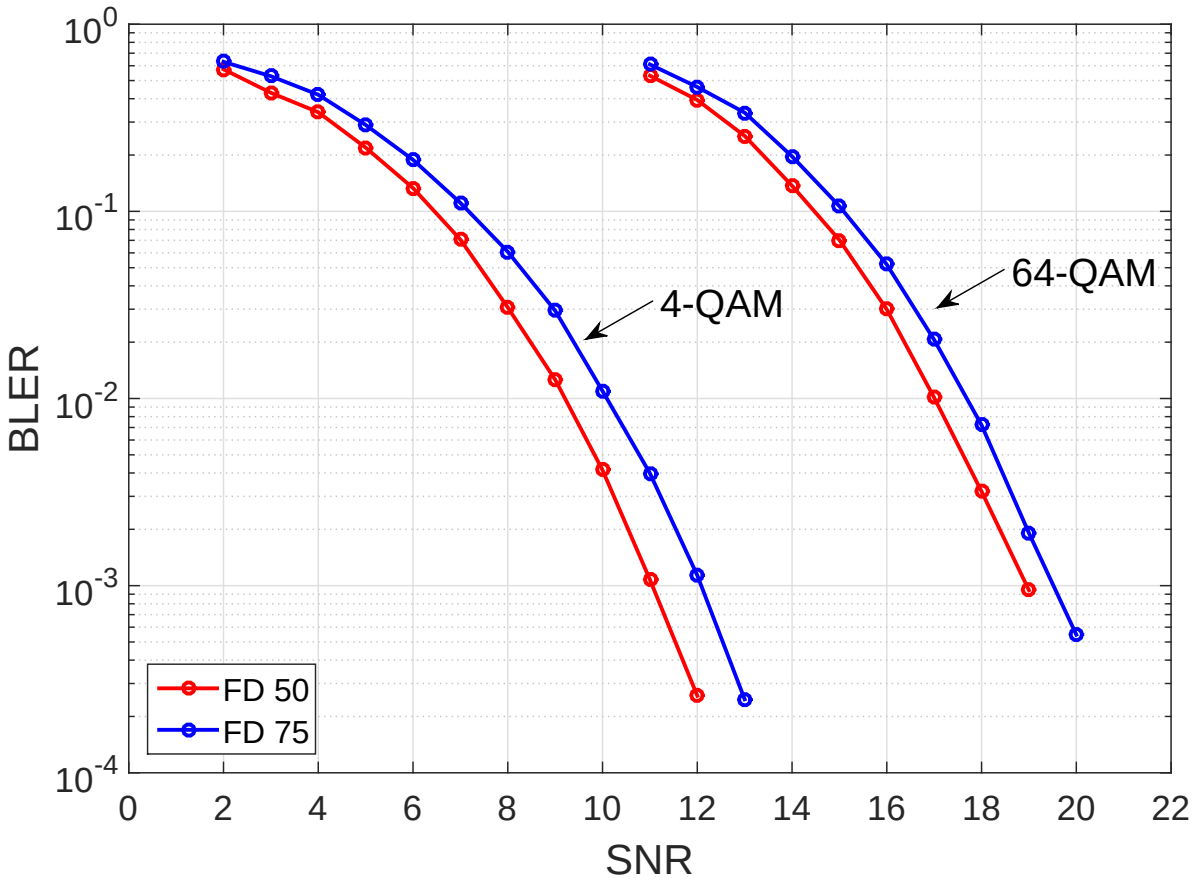
No (ST) codes can be found in the literature for $N \geq 3$ relays. Thus, we compare the two example codes. $(n_s = 2, n_d = 6)$.



A two-user 2 Tx MIMO-MAC example I

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The previous constructions can also yield fast-decodable codes for a K-user MIMO-MAC.

- $K = 2$ with $n_s = 2$ and a single destination with $n_d = 4$.
- Both transmitters carve their ST codes from the algebra $\mathcal{C} = \left(\mathbb{Q}(\sqrt{-2}, \sqrt{-3}, i)/\mathbb{Q}(\sqrt{-2}, i), \sigma : \sqrt{-3} \mapsto -\sqrt{-3}, -\frac{2}{\sqrt{5}}\right)$.
- Let $\langle \tau : i \mapsto -i \rangle = \Gamma(\mathbb{Q}(\sqrt{-2}, i)/\mathbb{Q}(\sqrt{-2}))$.
- Codewords (for each transmitter) are of the form $U_k = [\, X_k \ \tau(X_k) \,]$, where for $x_k \in \mathcal{O} \subset \mathcal{C}$ and $\theta = \frac{1+\sqrt{-3}}{2}$,

$$X_k = \tilde{\lambda}(x_k) = \begin{bmatrix} x_{k,1}+x_{k,2}\theta & -\sqrt{-\gamma}(x_{k,3}+x_{k,4}\sigma(\theta)) \\ \sqrt{-\gamma}(x_{k,3}+x_{k,4}\theta) & x_{k,1}+x_{k,2}\sigma(\theta) \end{bmatrix}.$$

A two-user 2 Tx MIMO-MAC example II

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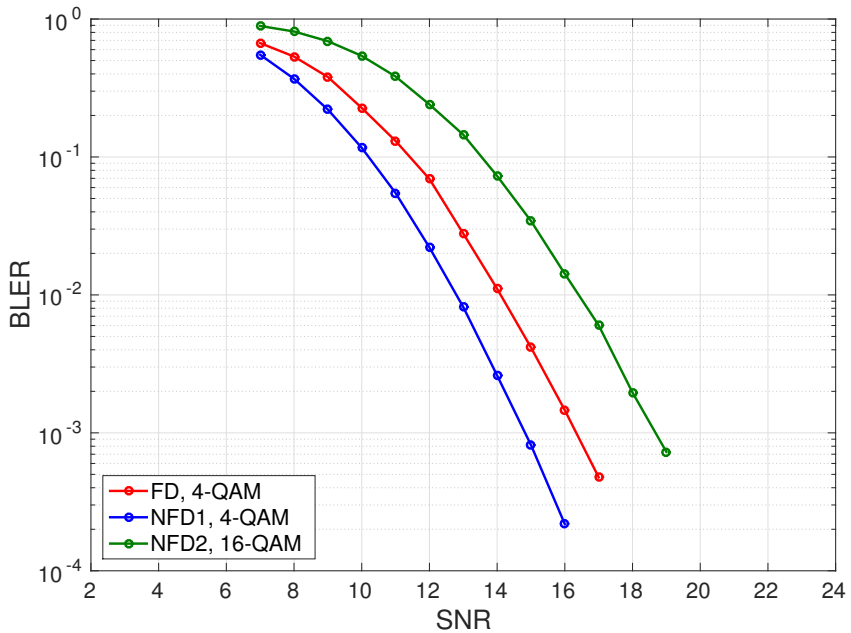
The overall transmitted codewords are of the form

$$X = \begin{bmatrix} X_1 & \tau(X_1) \\ X_2 & \tau(X_2) \end{bmatrix}.$$

This code has the *conditional* NVD property, and its reduction in decoding complexity is 50%, from $|S|^{32}$ to $|S|^{16}$.



The two codes chosen from comparison were adapted to match the data rate, and can be found in [Lu].

H. f. Lu et al. "New space–time code constructions for two-user multiple access channels". In: *IEEE J. Sel. Top. Signal Process.* 3.6 (2009), pp. 939–957

Conclusions and future research

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- Constructions of fast-decodable space–time codes for the N-relay NAF amplify-and-forward channel.
- Full-diversity and NVD.
- SIMO: 37% reduction in decoding complexity order.
  Achieves the DMT of the channel.
- MIMO: up to 75% reduction in decoding complexity order.
- Constructions can be adapted to the N-user MIMO-MAC, resulting in fast-decodable codes with CNVD for that setting.

- In future, xtensions to $n_s > 2$, which means dealing with higher-degree algebras.
- Maybe the new relaxed conditions on fast-decodability facilitate new constructions.
- More accurate complexity comparison by counting floating point operations.

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Kiitos!