Towards an Algebraic Network Information Theory

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Joint work with Sung Hoon Lim (EPFL), Chen Feng (UBC),
Adriano Pastore (EPFL), and Michael Gastpar (EPFL).

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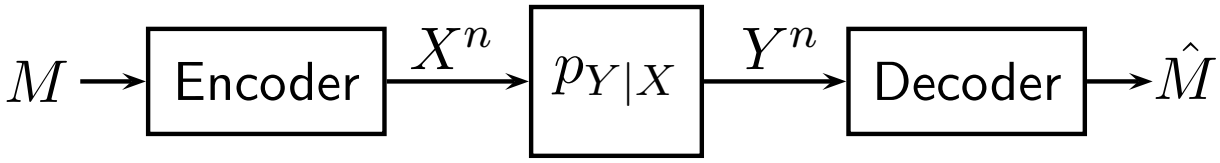
- Codes with algebraic structure are sought after to mimic the performance of random i.i.d. codes with low implementation complexity.

Point-to-Point Channels



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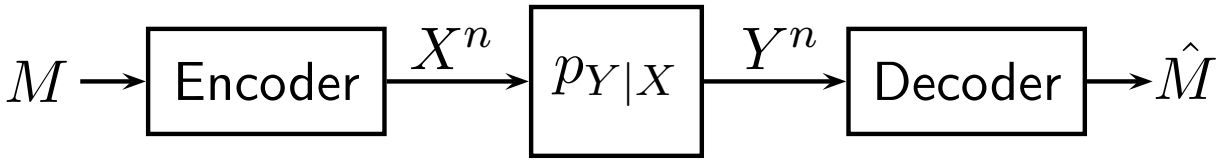

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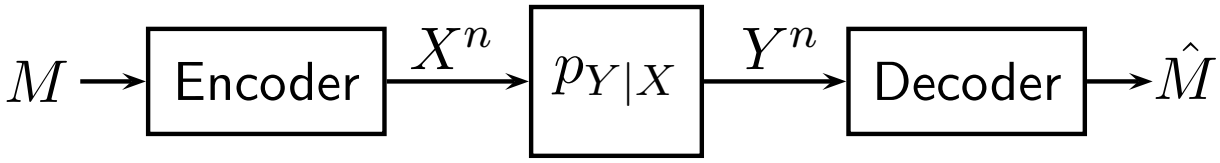

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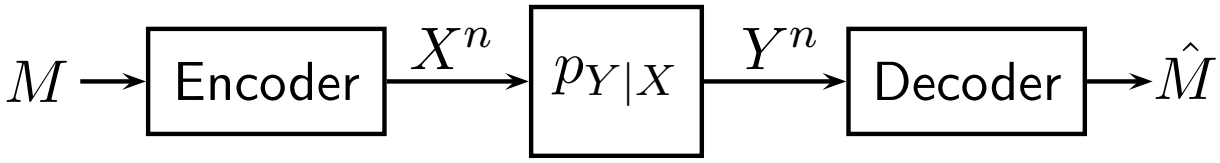

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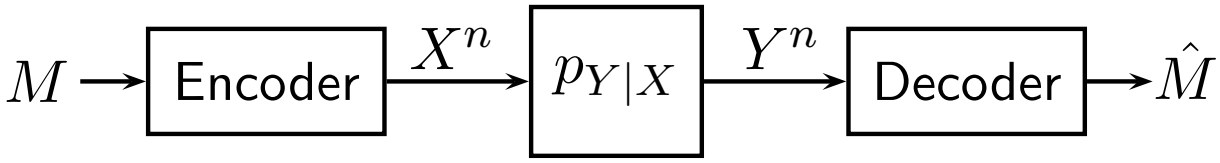

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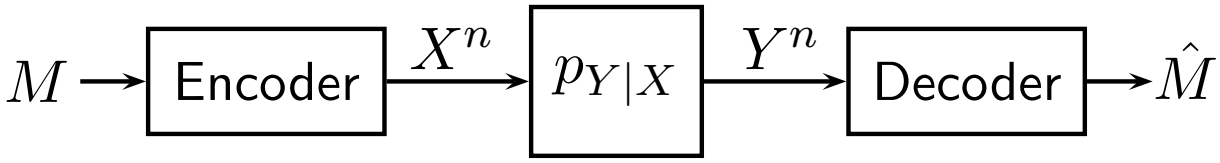

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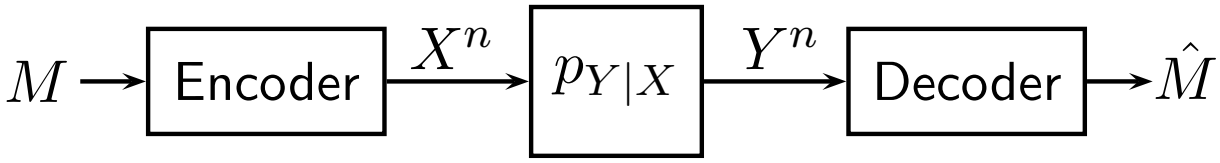
$$C = \max_{p_X(x)} I(X;Y)$$



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- Proof relies on random i.i.d. codebooks combined with joint typicality decoding.

Typicality

- x^n is a length-n sequence with elements from finite alphabet $\mathcal{X}$
- The empirical pmf (i.e., type) of x^n is

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- This motivates the typical set

$$\mathcal{T}_\epsilon^{(n)}(X) = \left\{x^n : |\pi(x|x^n) - p_X(x)| \le \epsilon p_X(x) \text{ for all } x \in \mathcal{X}\right\}$$

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- We can generalize this definition to pairs of sequences (X^n, Y^n) that are i.i.d. according to $p_{XY}(x, y)$ and so on...

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Intuition: Probability that i.i.d. $\tilde{X}^n$ looks jointly typical $\approx 2^{-nI(X;Y)}$

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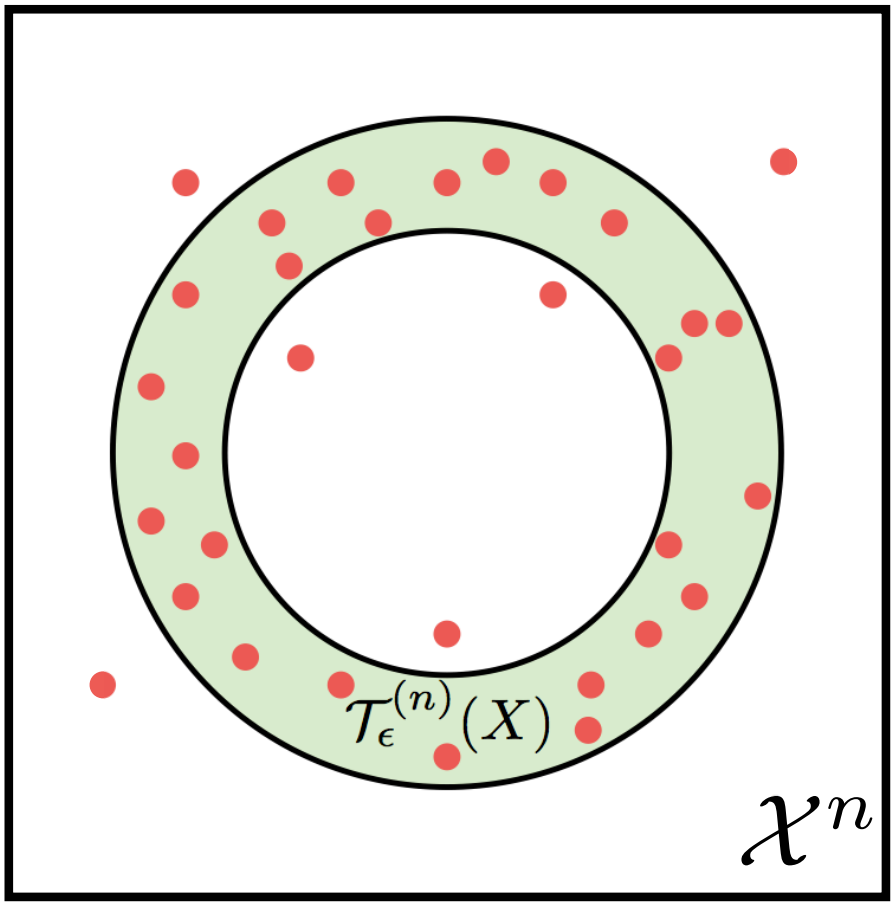
  - Some other codeword is jointly typical,

$$\begin{aligned} \mathsf{P}\left\{ \bigcup_{\tilde{m} \neq m} \left\{ (X^n(\tilde{m}), Y^n) \in \mathcal{T}_\epsilon^{(n)} \right\} \right\} &\leq \sum_{\tilde{m} \neq m} \mathsf{P}\left\{ (X^n(\tilde{m}), Y^n) \in \mathcal{T}_\epsilon^{(n)} \right\} \\ &\leq \sum_{\tilde{m} \neq m} 2^{-nI(X;Y)-\delta(\epsilon)} \\ &< 2^{nR}\, 2^{-nI(X;Y)-\delta(\epsilon)} . \end{aligned}$$

  Probability goes to zero if $R < I(X;Y) - \delta(\epsilon)$.



Random i.i.d. Codes

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This Talk: We build on previous work and propose a joint typicality approach to algebraic network information theory.

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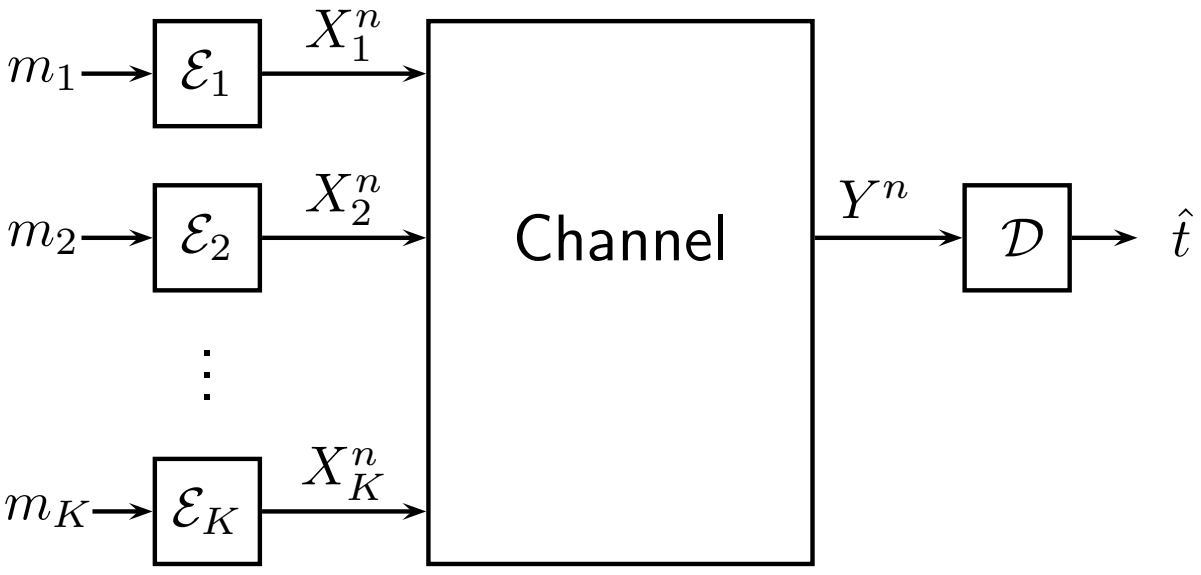

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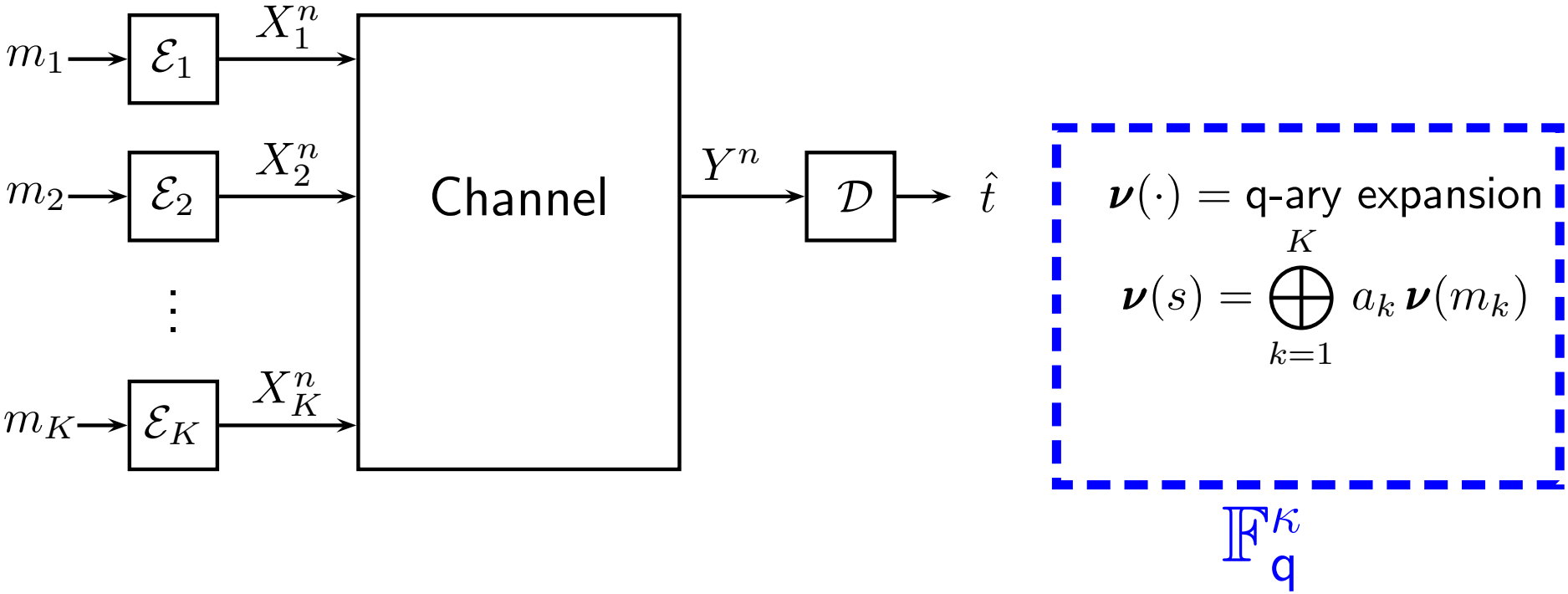

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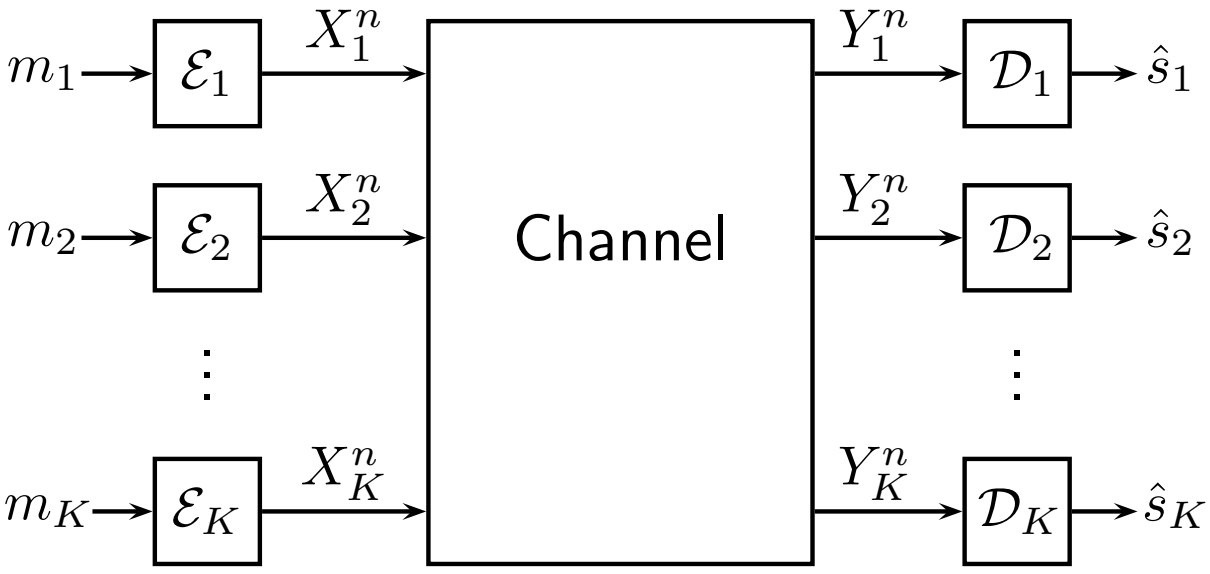

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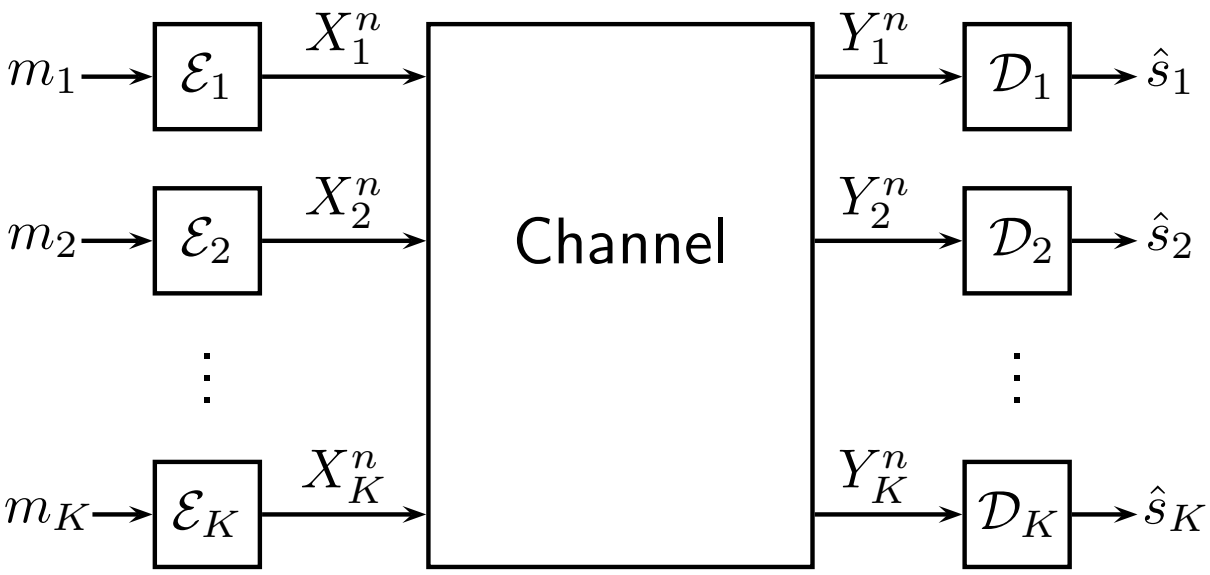

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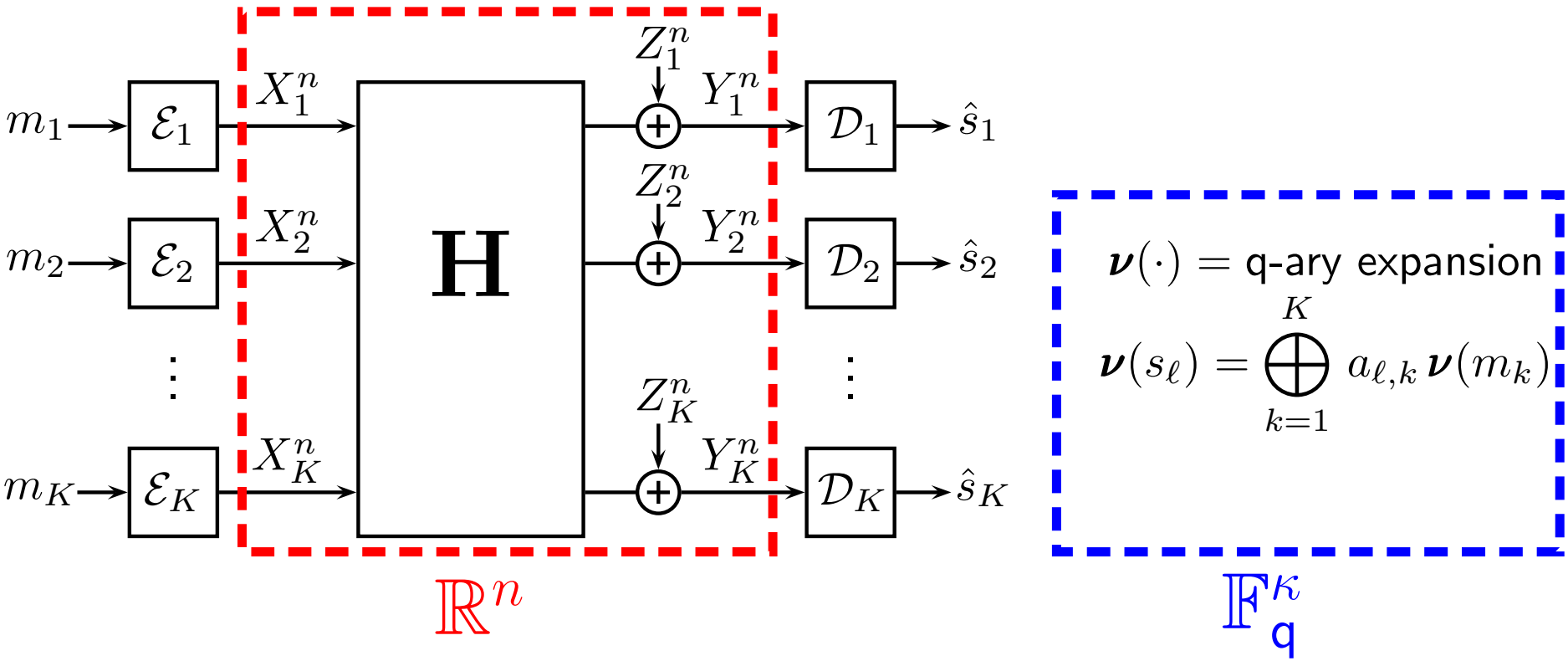

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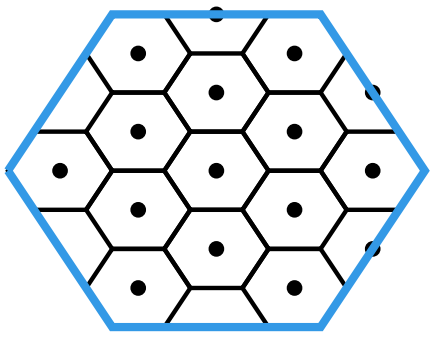

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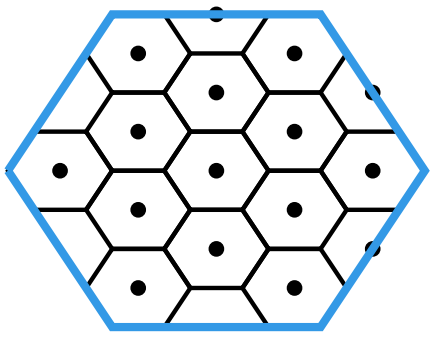

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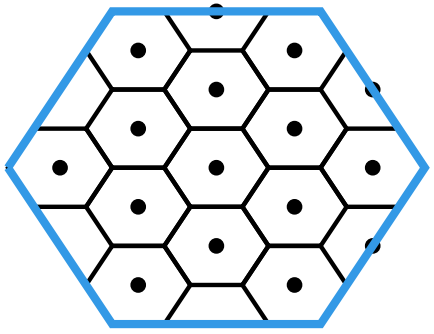

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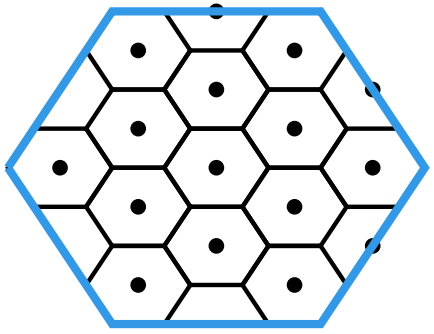

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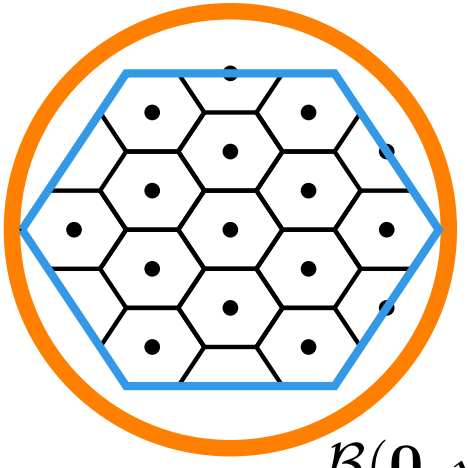

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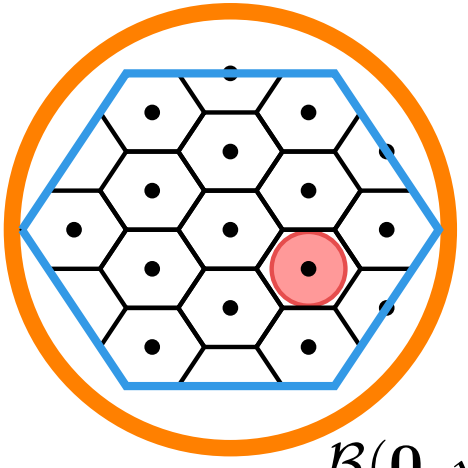

Disclaimer

- In the interest of conveying the key intuitions, I will now switch to heuristic arguments for the next few slides.

- Although the steps I will show do not constitute a formal proof, all of the conclusions we will reach can be shown rigorously (with some extra steps in between).

- Interested in the details? See
  B. Nazer and M. Gastpar, Compute-and-Forward: Harnessing Interference through Structured Codes, IEEE Transactions on Information Theory, vol. 57, no. 10, pp. 6463-6486, October 2011.

Nested Lattice Code Heuristics

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- Can show that the achievable rate satisfies $R > \frac{1}{2}\log\left(\frac{P}{\sigma^2_{\mathsf{eff}}}\right) - \delta$.

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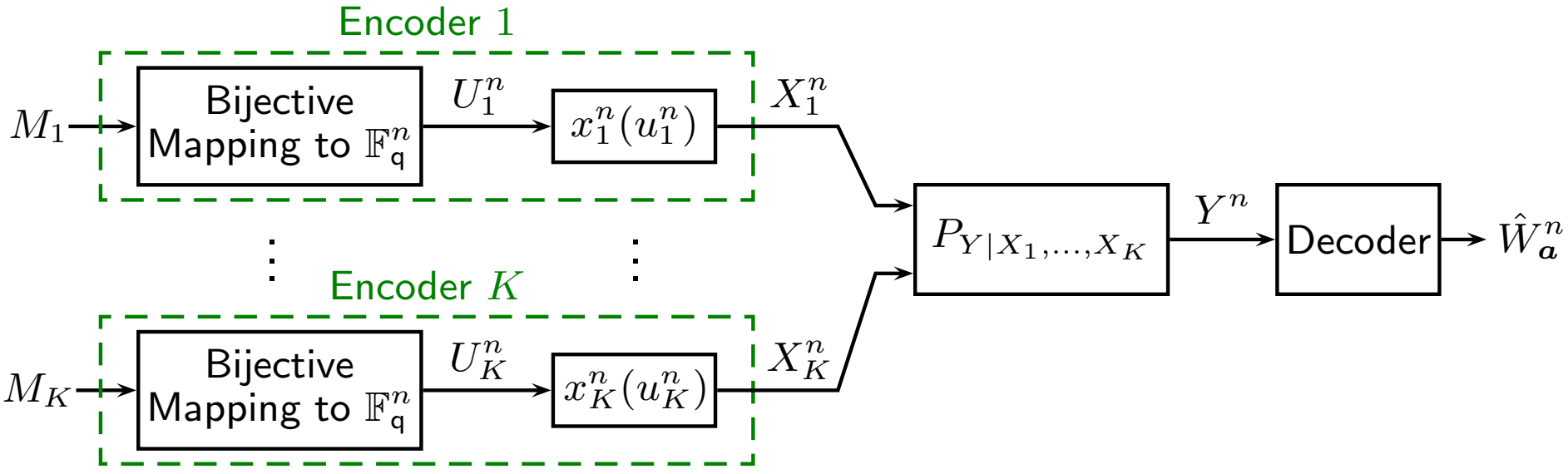
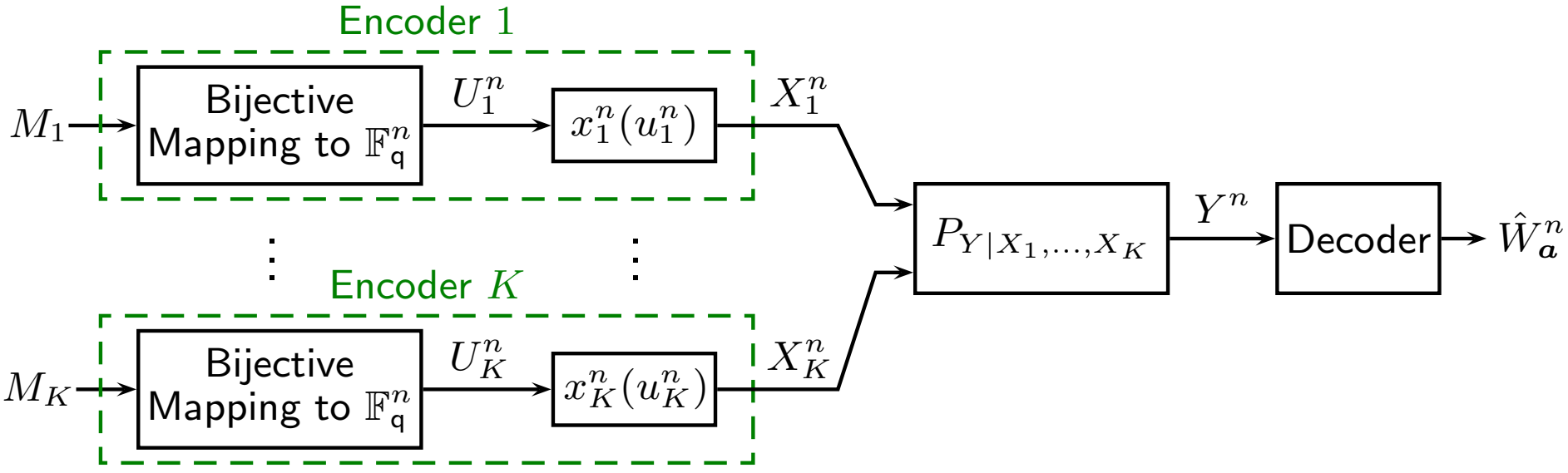
- Finding the best $\boldsymbol{a}$ corresponds to finding the shortest vector in the lattice $(\text{SNR}^{-1}\mathbf{I} + \mathbf{h}\mathbf{h}^{\mathsf{T}})^{-1/2}\mathbb{Z}^K$.

Compute-and-Forward: Beyond Gaussian Channels





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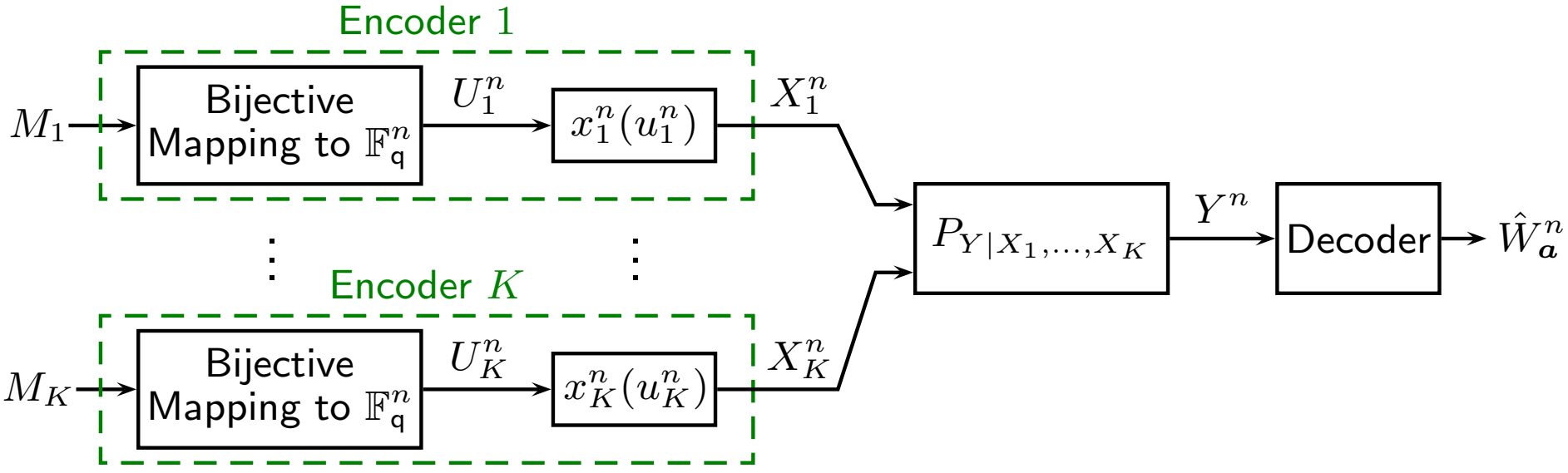

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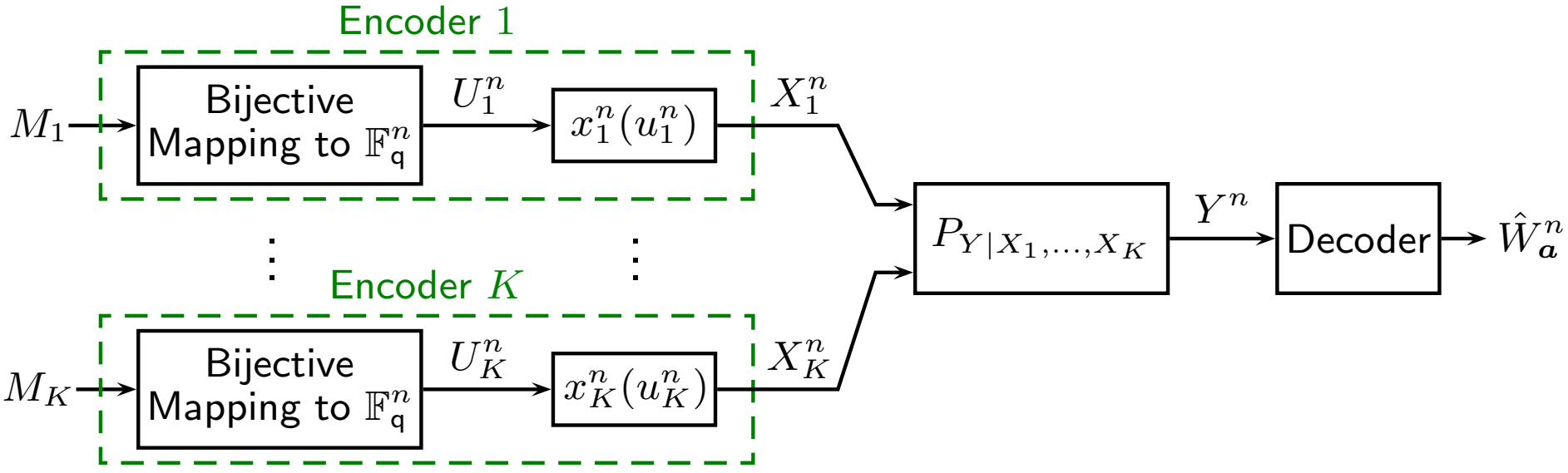

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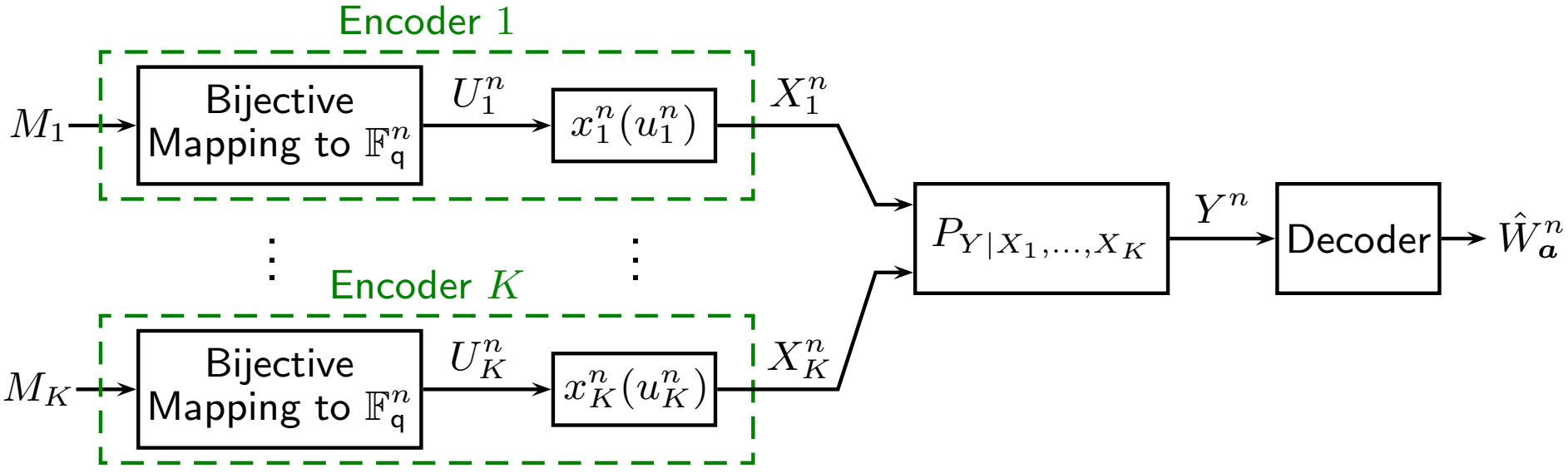

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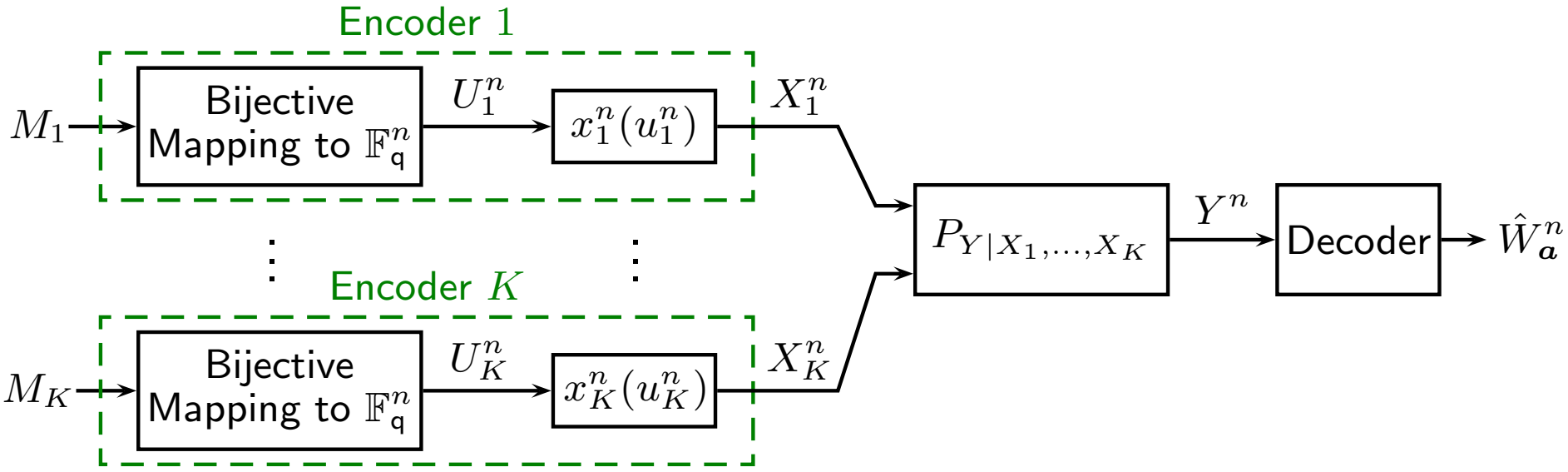

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- True Codeword: Want $(W_{\boldsymbol{a}}^n, Y^n)$ to look jointly typical. (Proof is actually a bit involved.)
- Decoder searches for sequences $\tilde{w}_{\boldsymbol{a}}^n$ that are jointly typical with Y^n. There are $\approx 2^{nH(W_{\boldsymbol{a}}|Y)}$ possible sequences. If only one such sequence is jointly typical, declare it as the estimate $\hat{W}_{\boldsymbol{a}}^n$ of the linear combination $W_{\boldsymbol{a}}^n = a_1 U_1^n \oplus \cdots \oplus a_K U_K^n$.

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$$R_k < H(U_k) - H(W_{\boldsymbol{a}}|Y).$$

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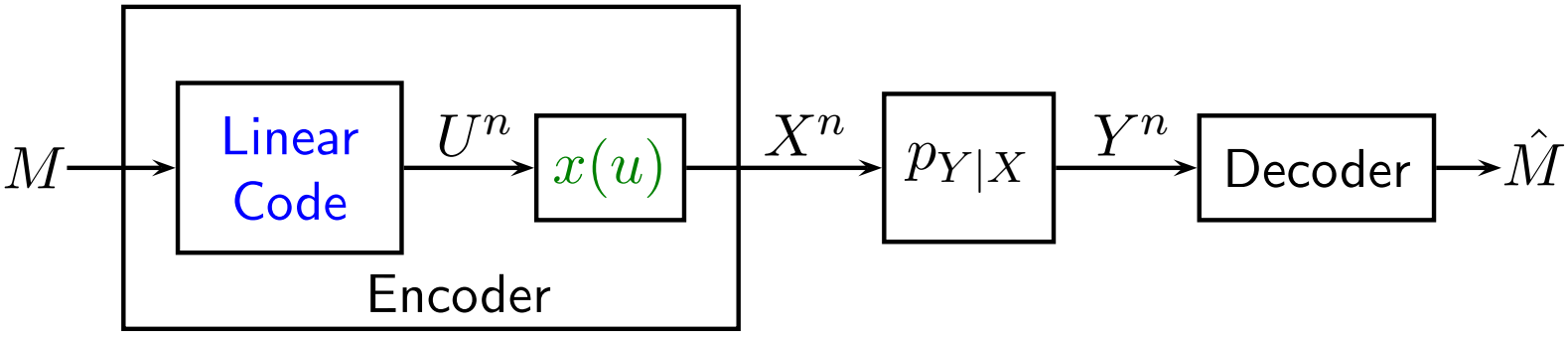
  (Not a mutual information and can be negative.)



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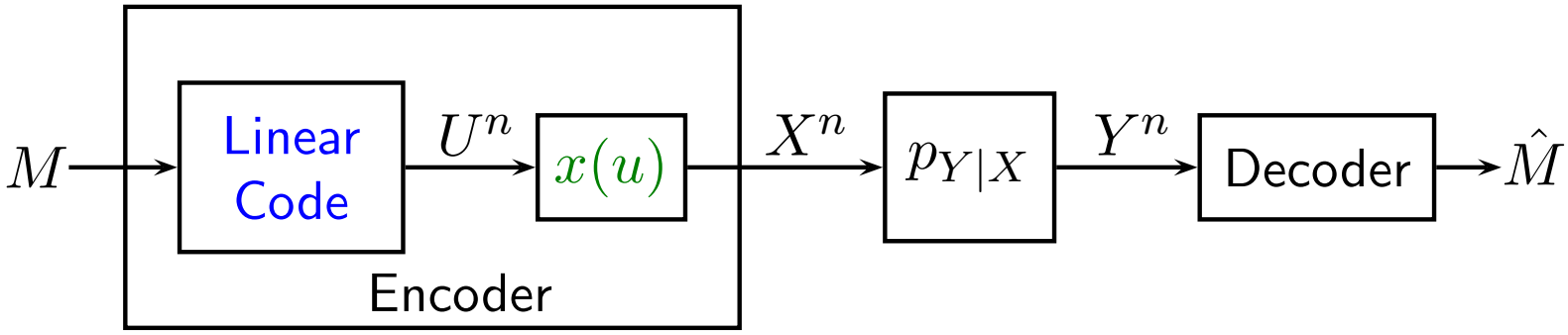

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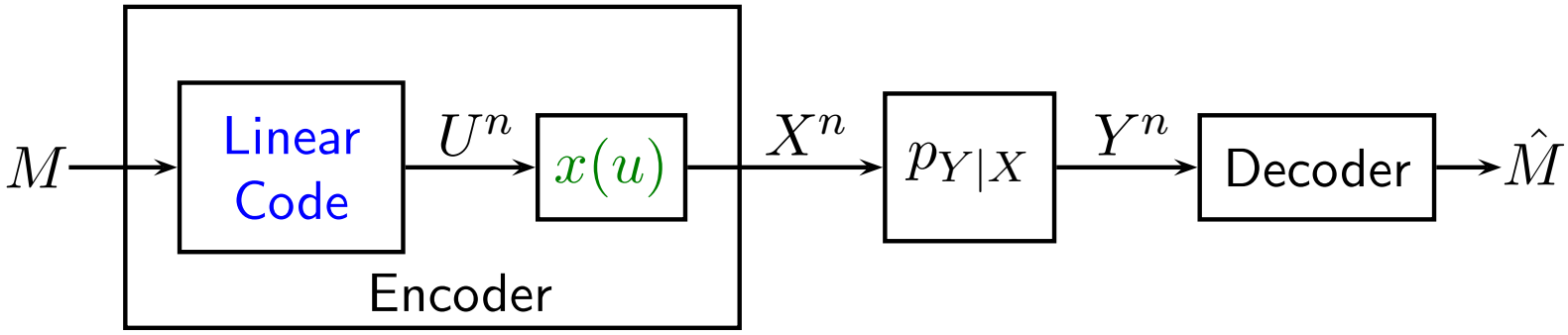

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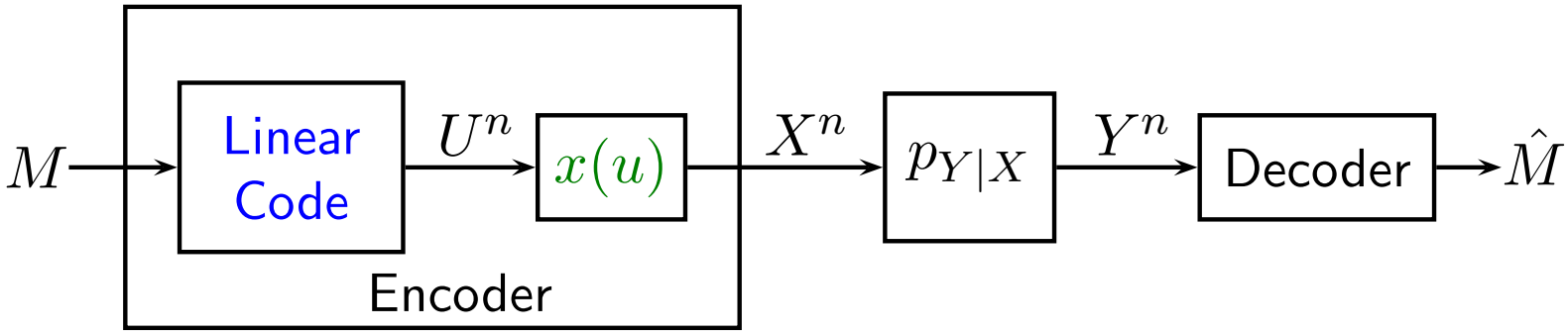

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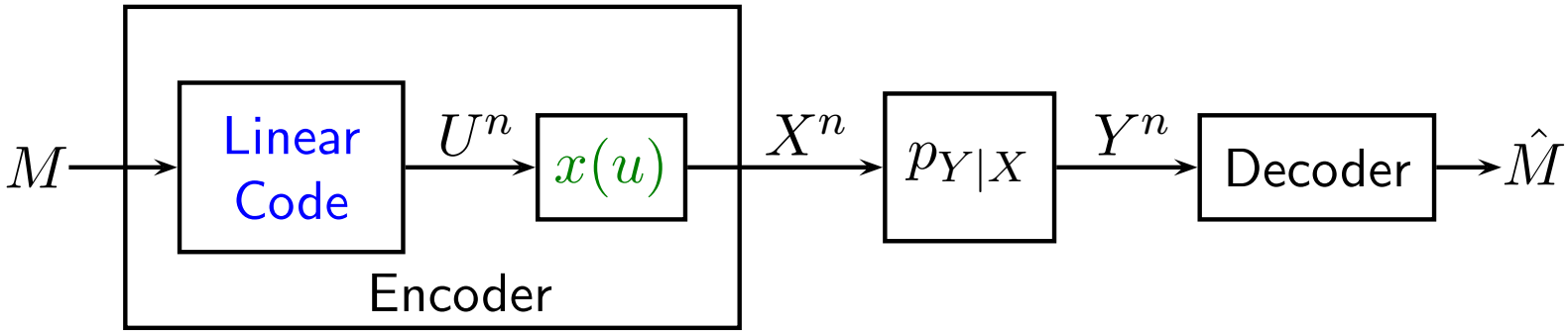

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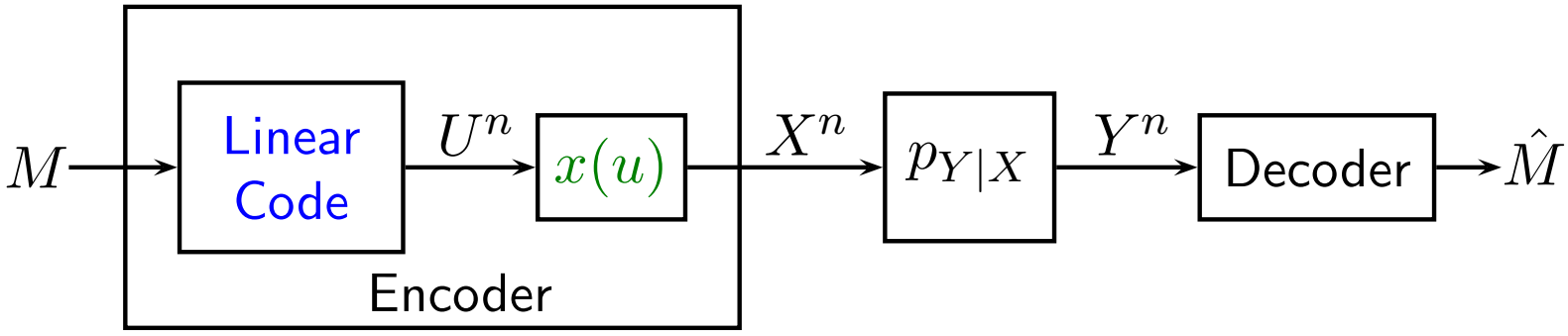

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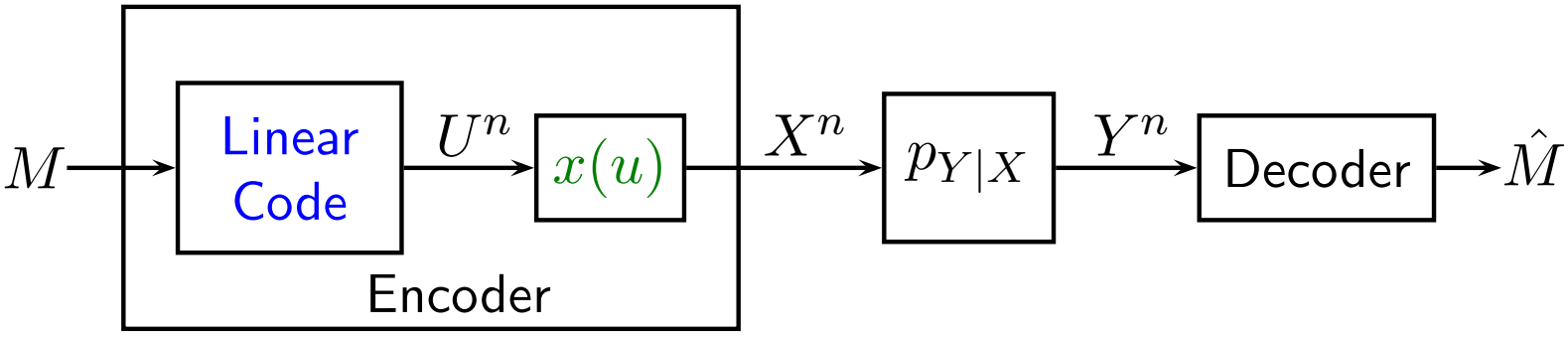

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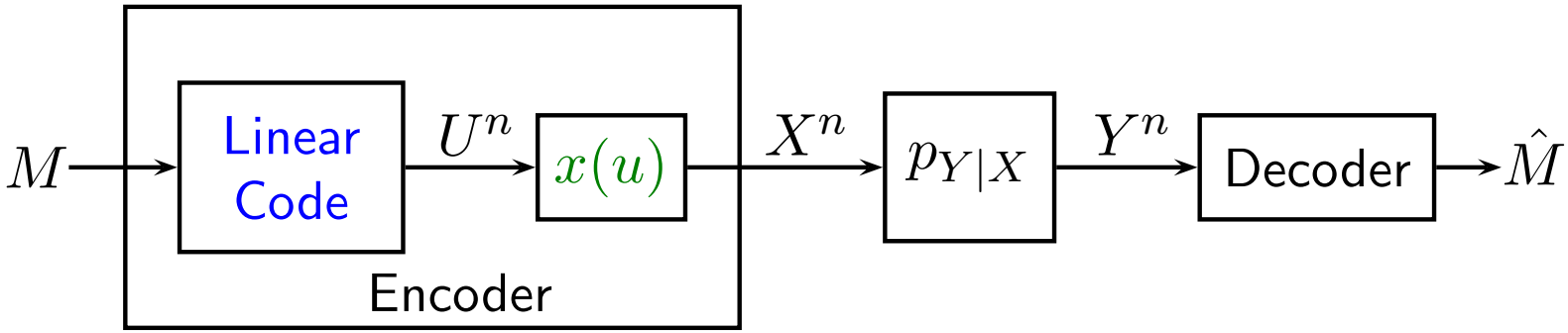

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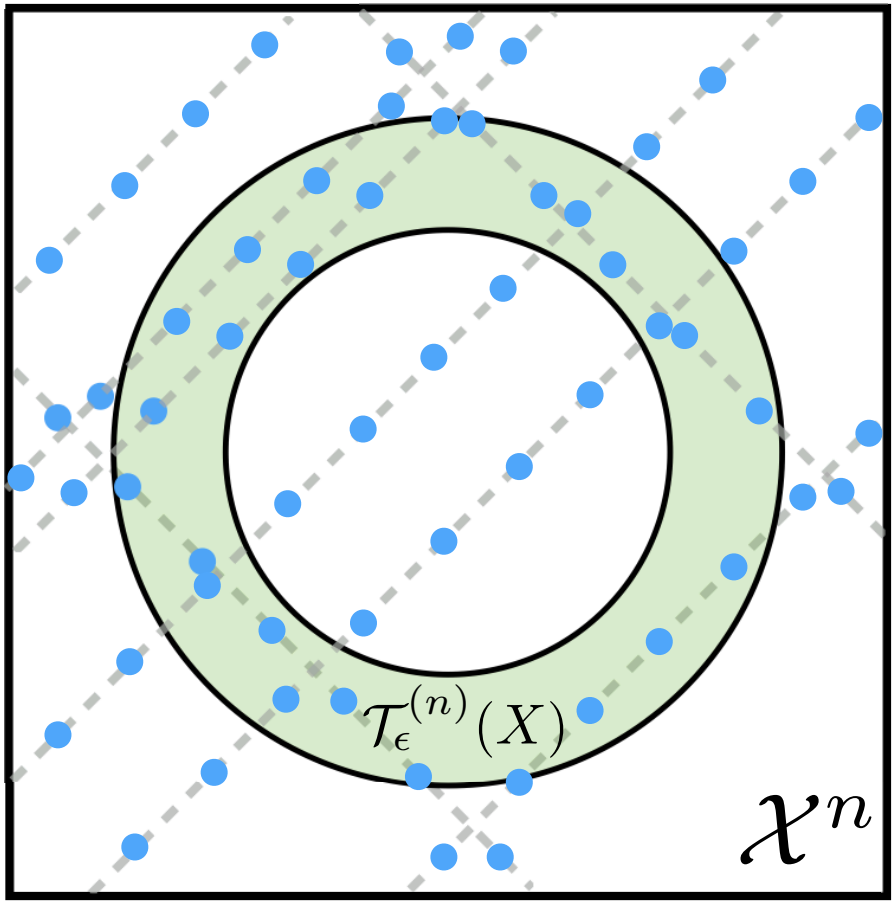
- Channel input at time i is $x_i(m) = x(u_i(m))$.

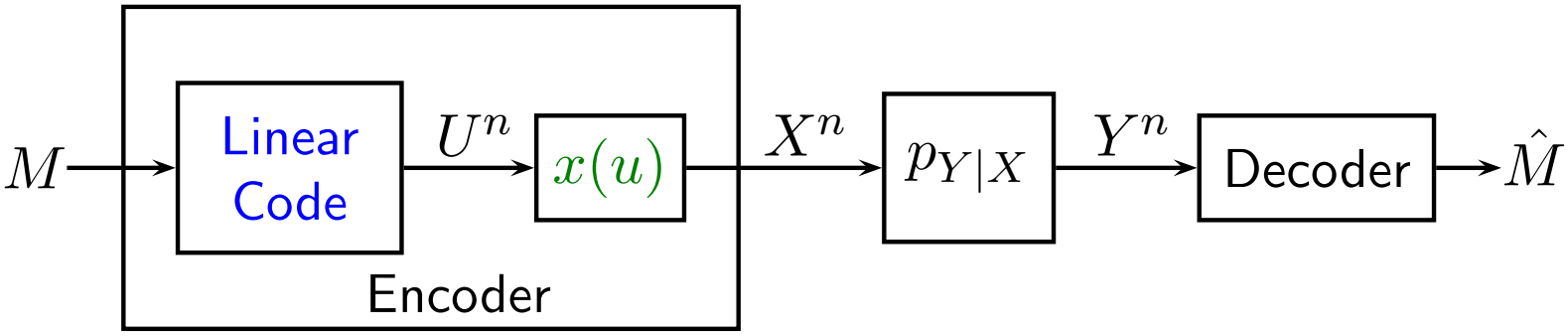
Random i.i.d. Codebooks



Random Linear Codes

- Codewords are pairwise independent of one another.
- Codewords are uniformly distributed over $\mathbb{F}_q^n$.

Point-to-Point Channels: Linear Codes



- Well known that a direct application of linear coding is not sufficient to reach the point-to-point capacity, **Ahlswede '71.**

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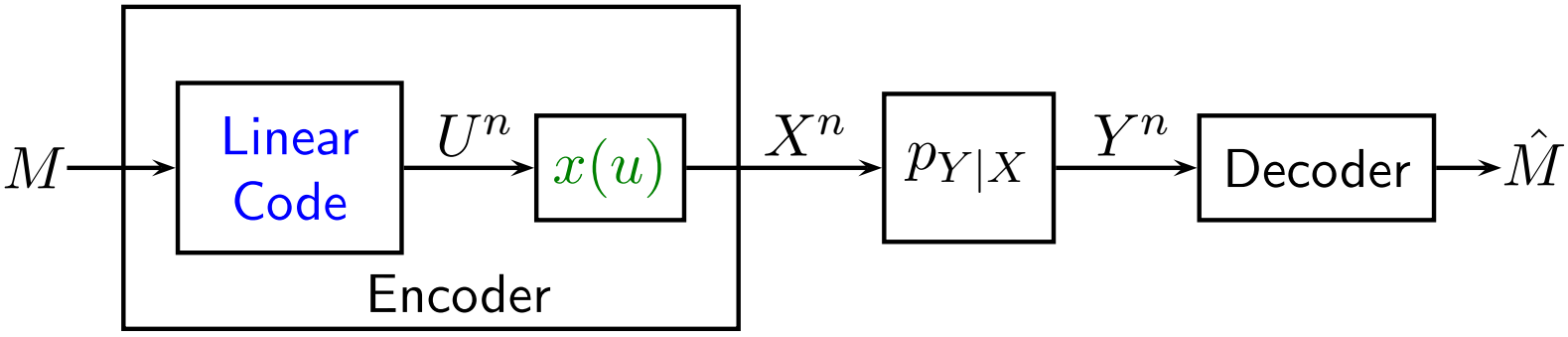

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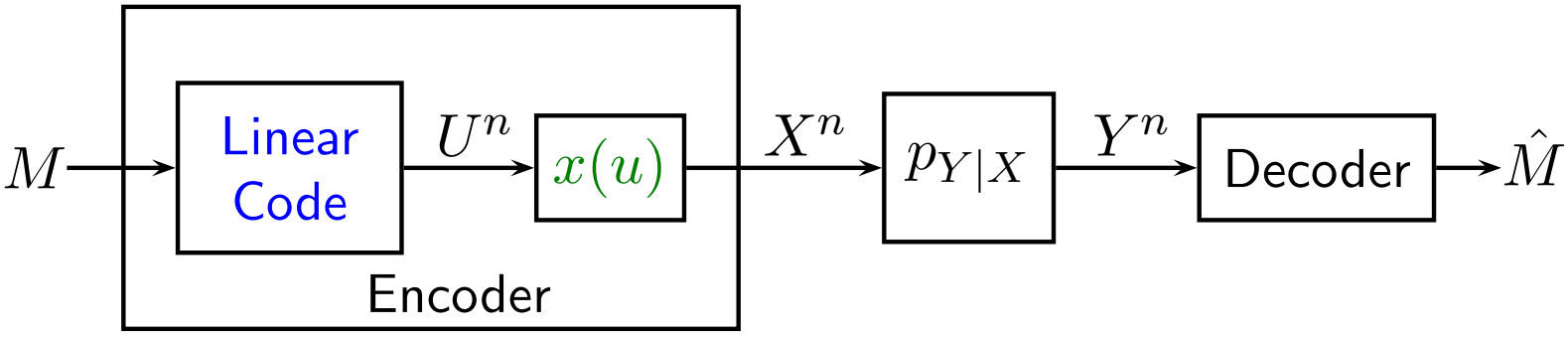

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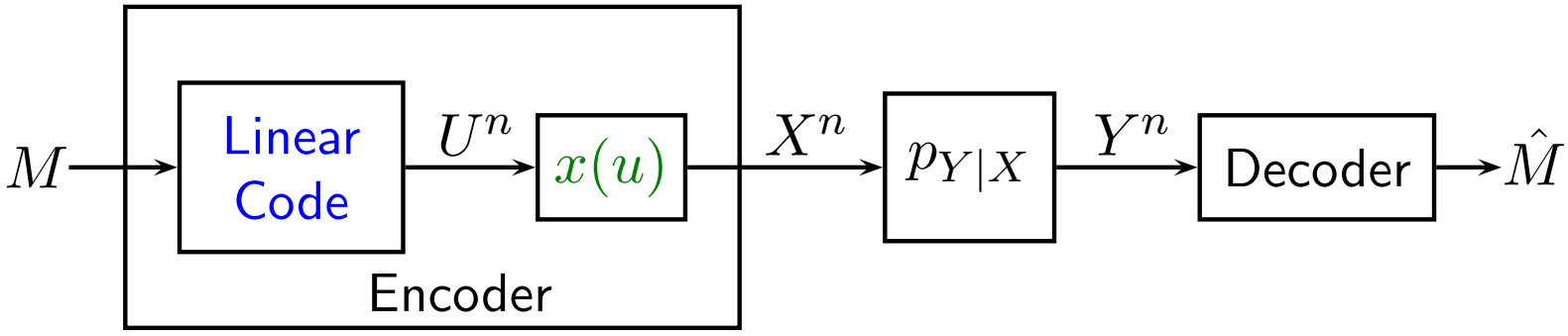

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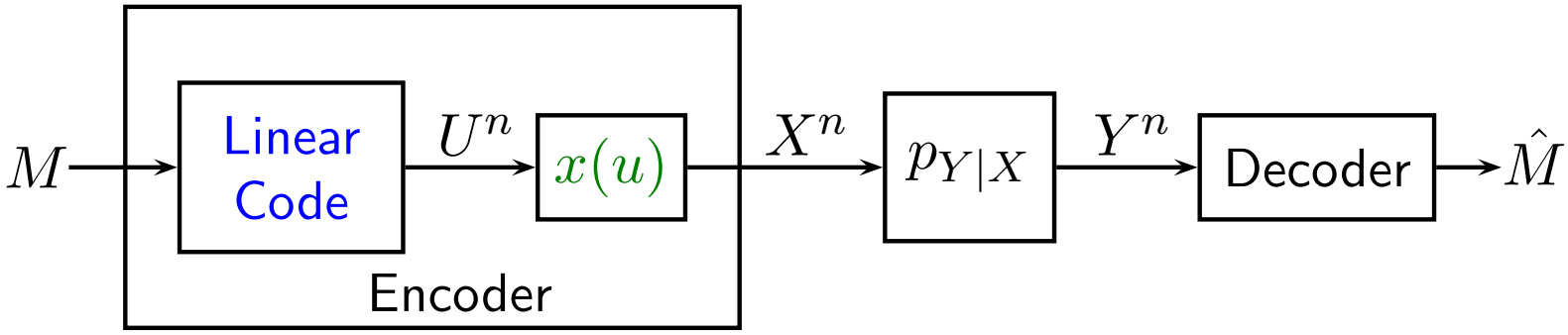

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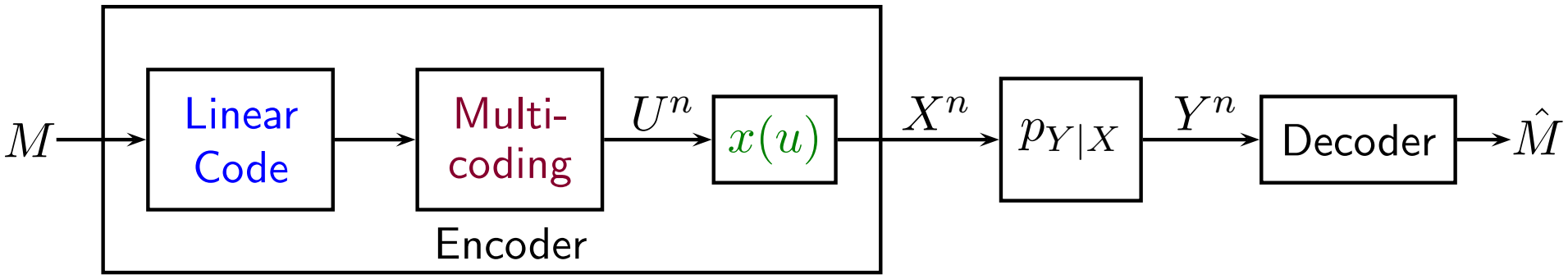
- Basic idea: Generate many codewords to represent one message. Search in this "bin" to find a codeword with the desired type, i.e., multicoding.

Point-to-Point Channels: Linear Codes + Multicoding



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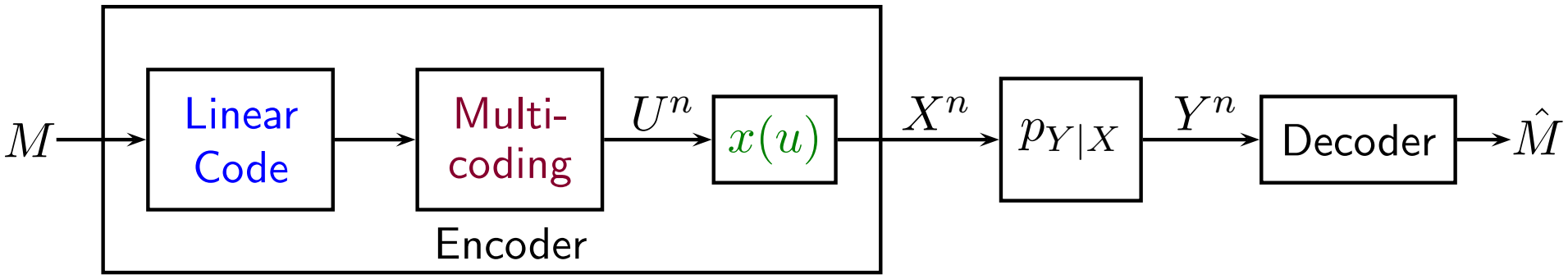

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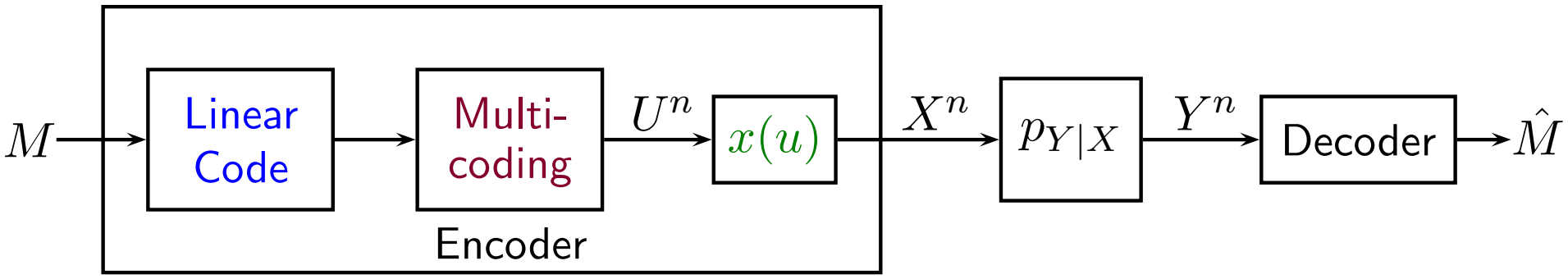

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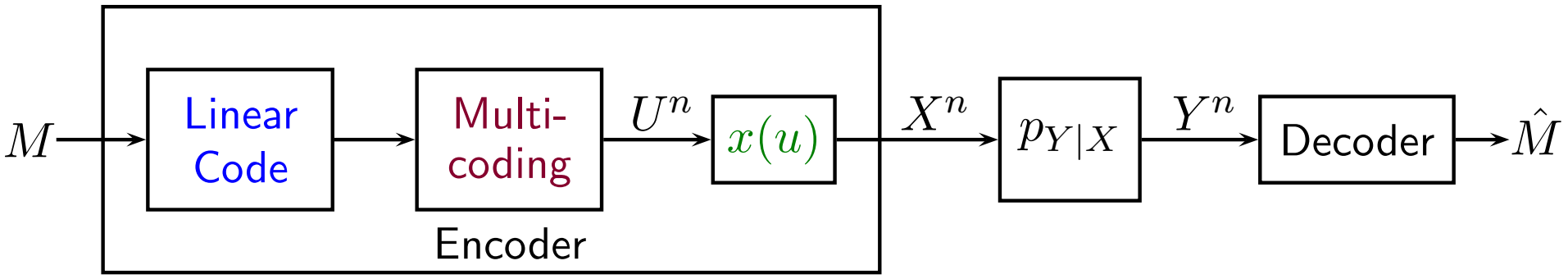

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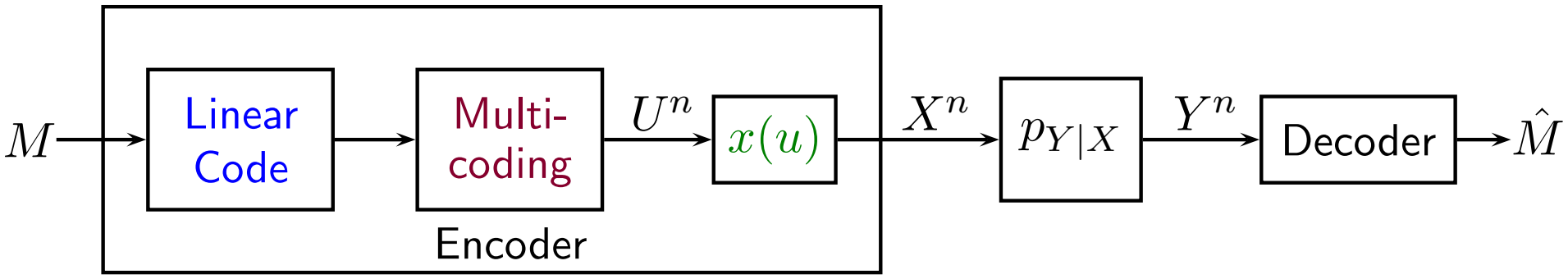

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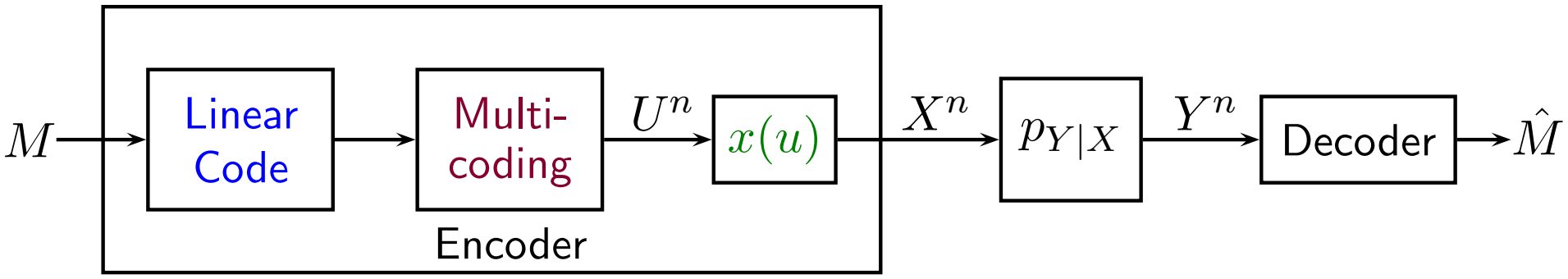

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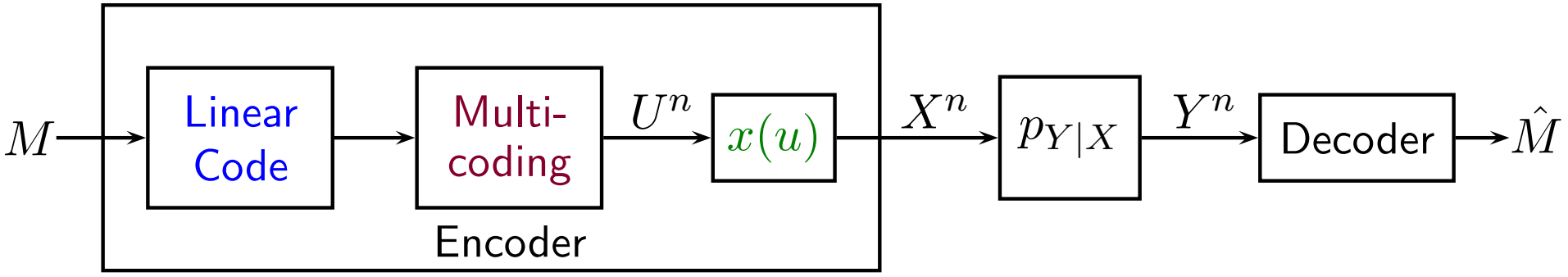

Encoding:

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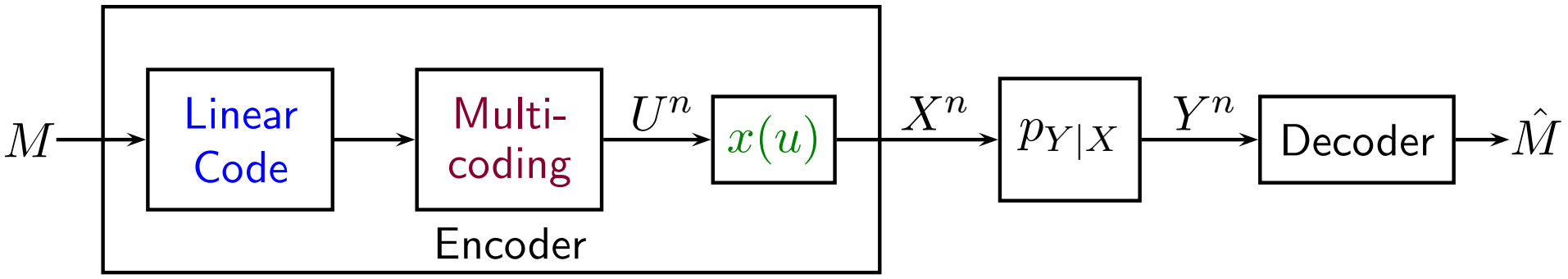

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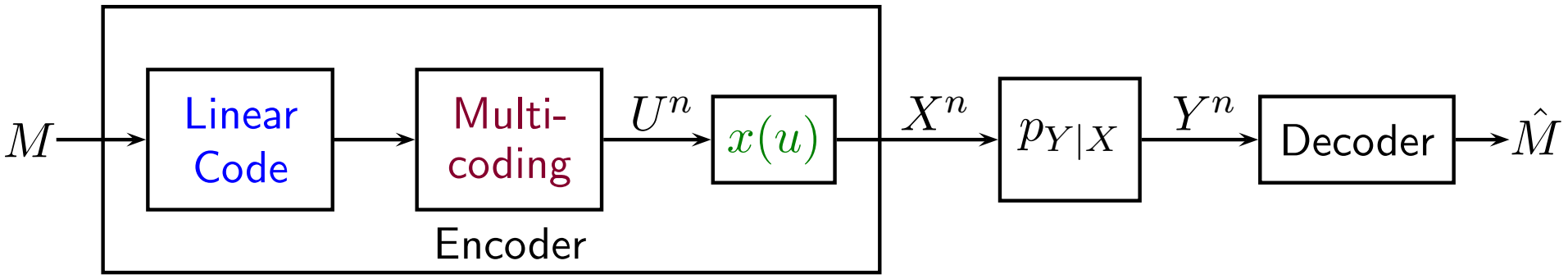

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- Fix $p(u)$ and $x(u)$.
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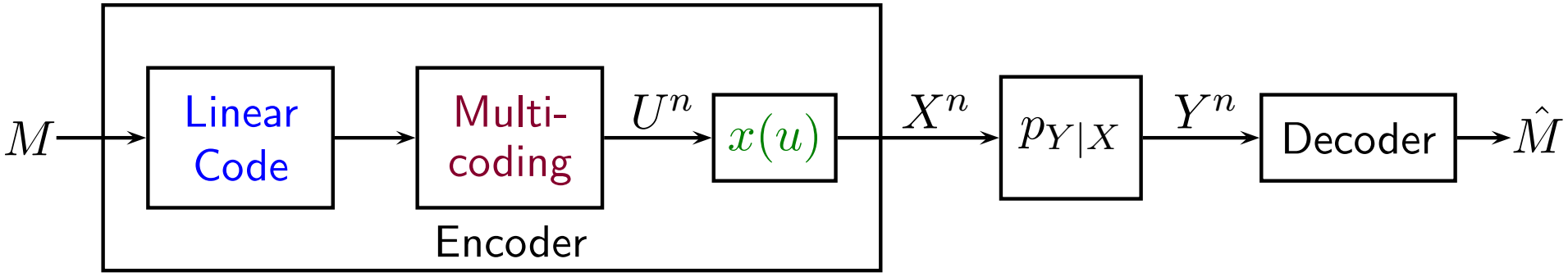
Point-to-Point Channels: Linear Codes + Multicoding



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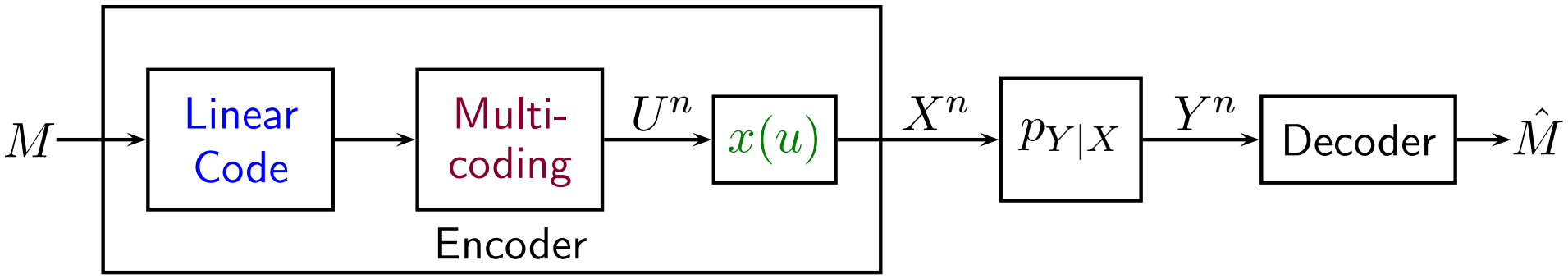

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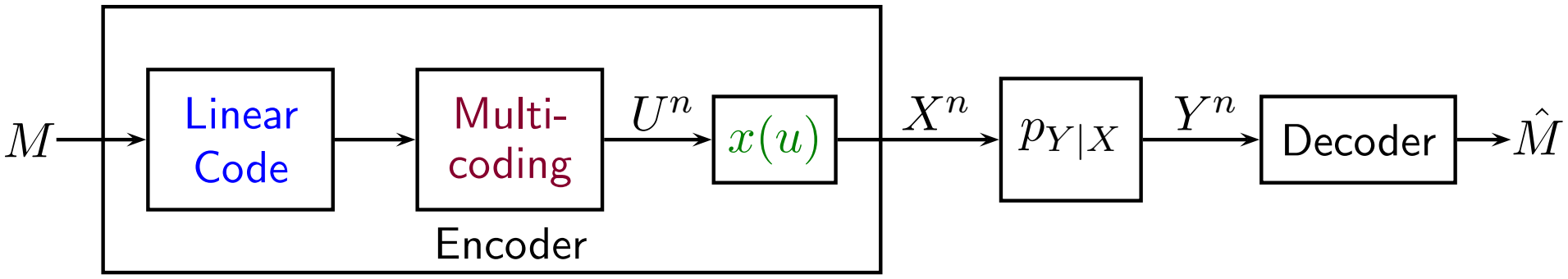

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Decoding:

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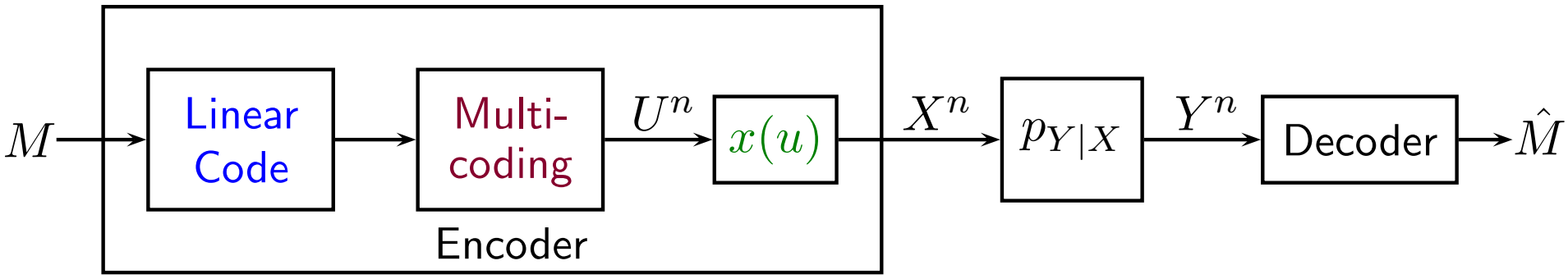

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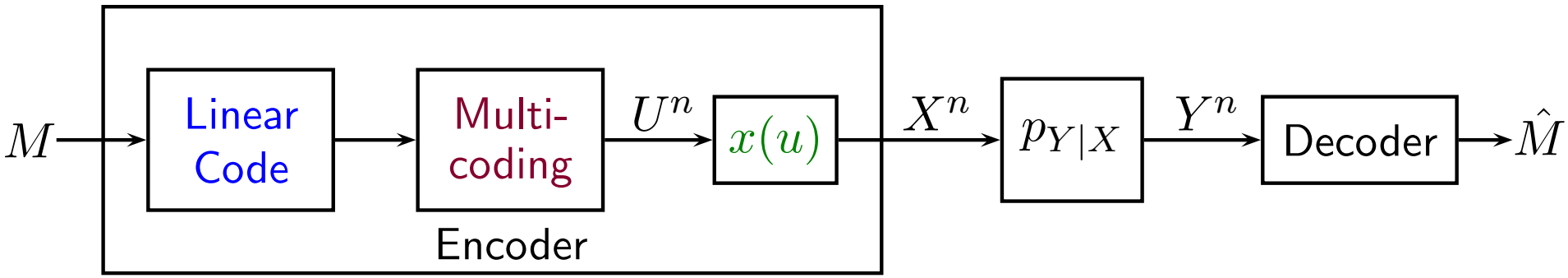
Decoding:

- Joint Typicality Decoding: Find the unique index $\hat{m}$ such that $\big(u^n(\hat{m},\hat{l}),\, y^n\big) \in \mathcal{T}_{\epsilon}^{(n)}(U,Y)$ for some index $\hat{l}$.

Point-to-Point Channels: Linear Codes + Multicoding



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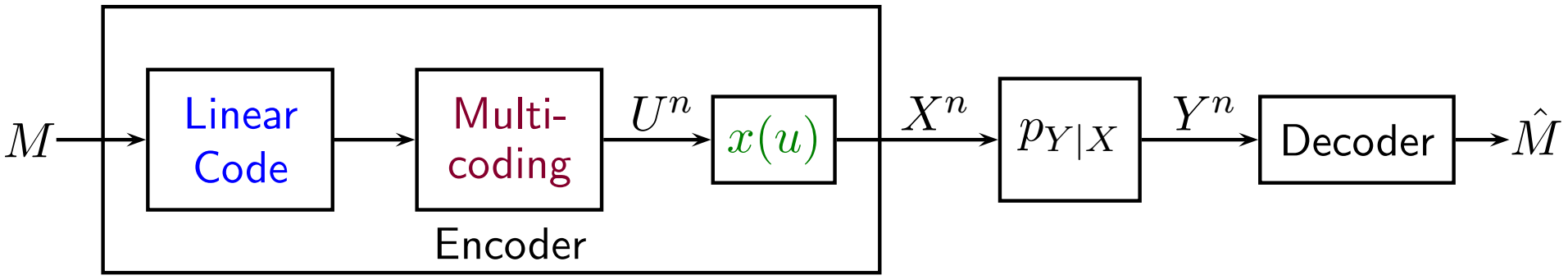
Point-to-Point Channels: Linear Codes + Multicoding



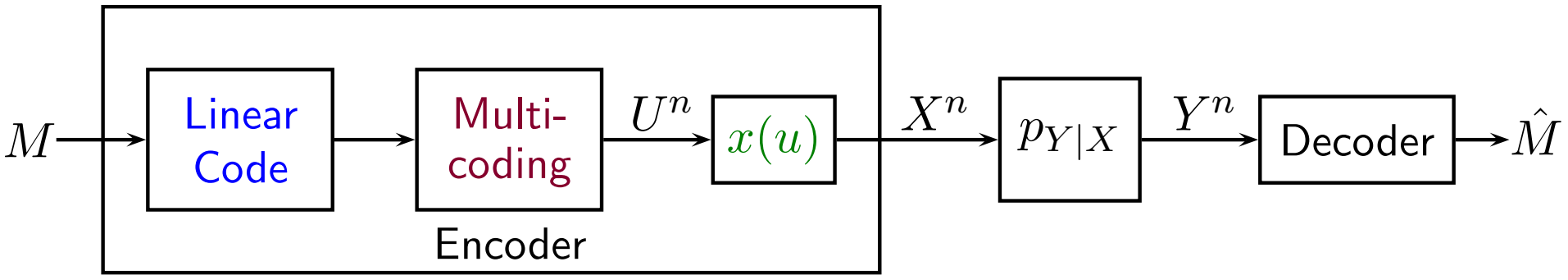
Theorem (Padakandla-Pradhan '13)

Any rate R satisfying

$$R < \max_{p(u),\, x(u)} I(U;Y)$$

is achievable. This is equal to the capacity if $\mathsf{q} \geq |\mathcal{X}|$.

Point-to-Point Channels: Linear Codes + Multicoding



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Any rate R satisfying

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- This is the basic coding framework that we will use for each transmitter.

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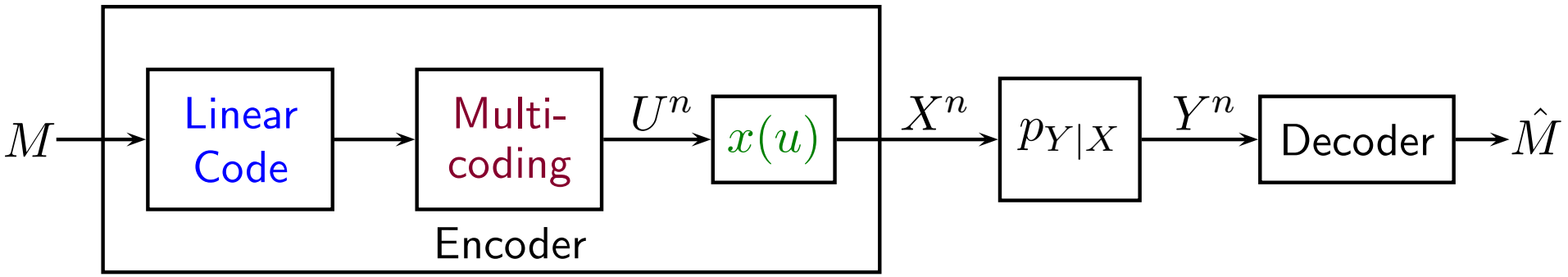

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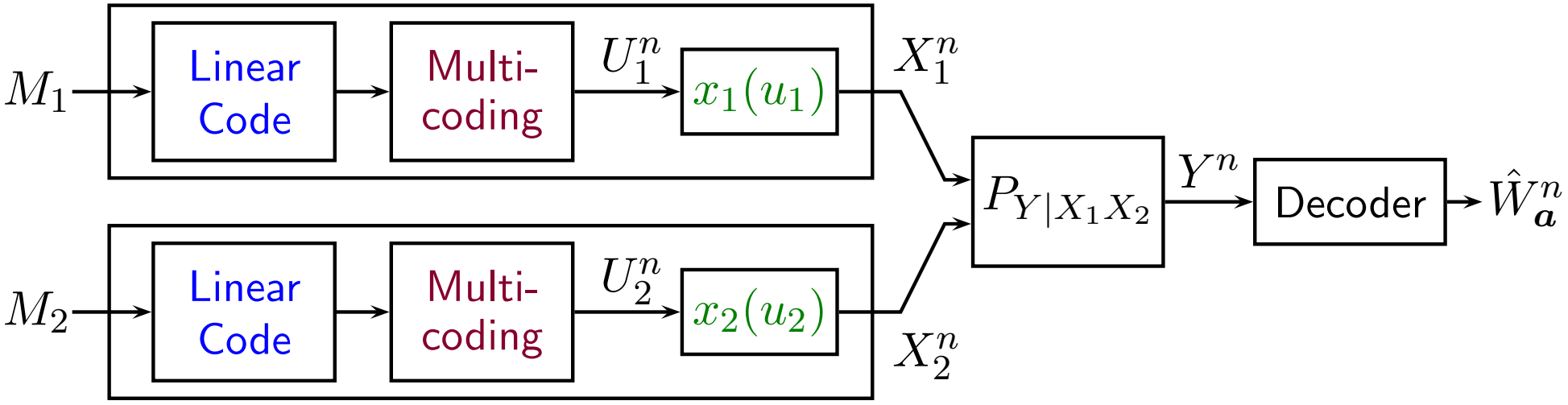
- Next, let's examine a two-transmitter, one-receiver "compute-and-forward" network.

Nested Linear Coding Architecture



Code Construction:

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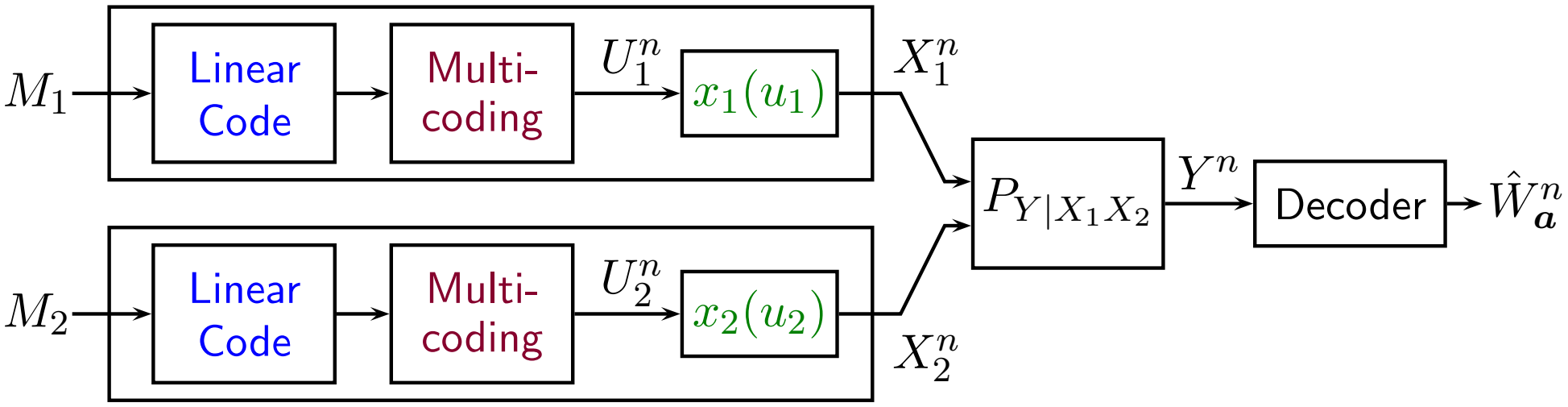

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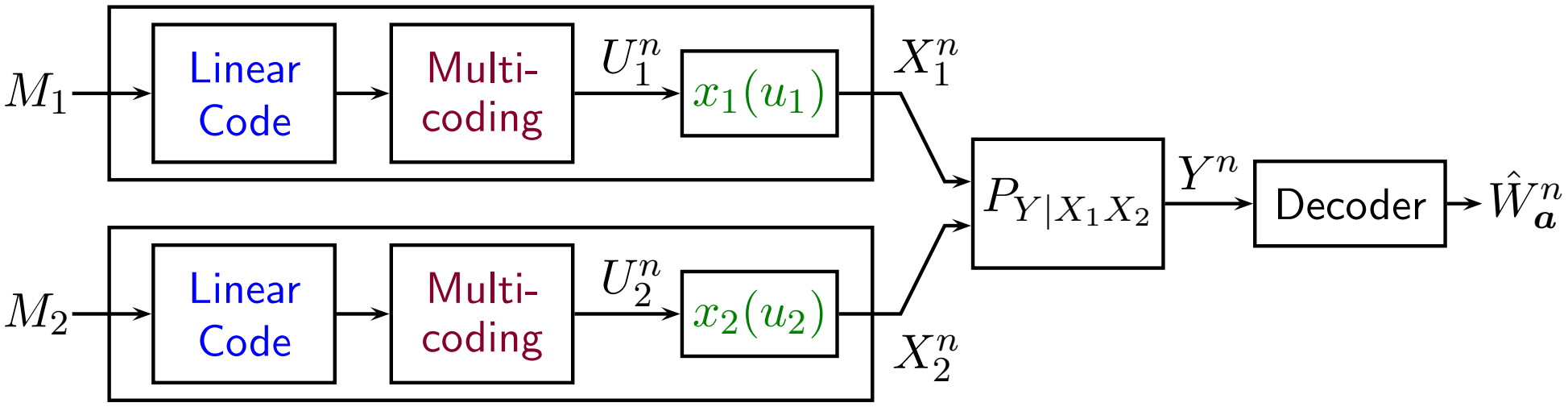

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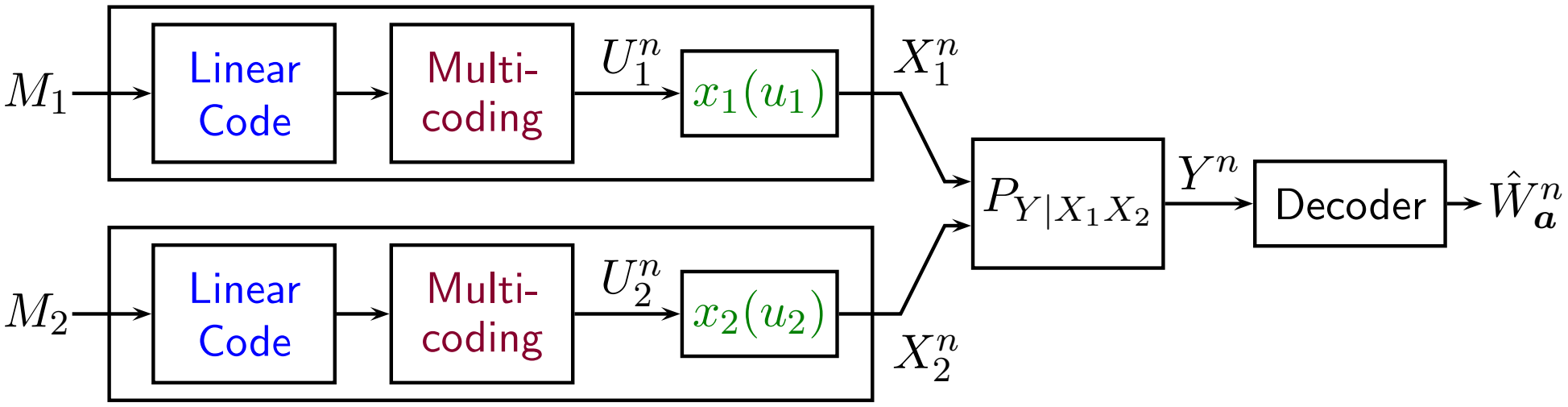

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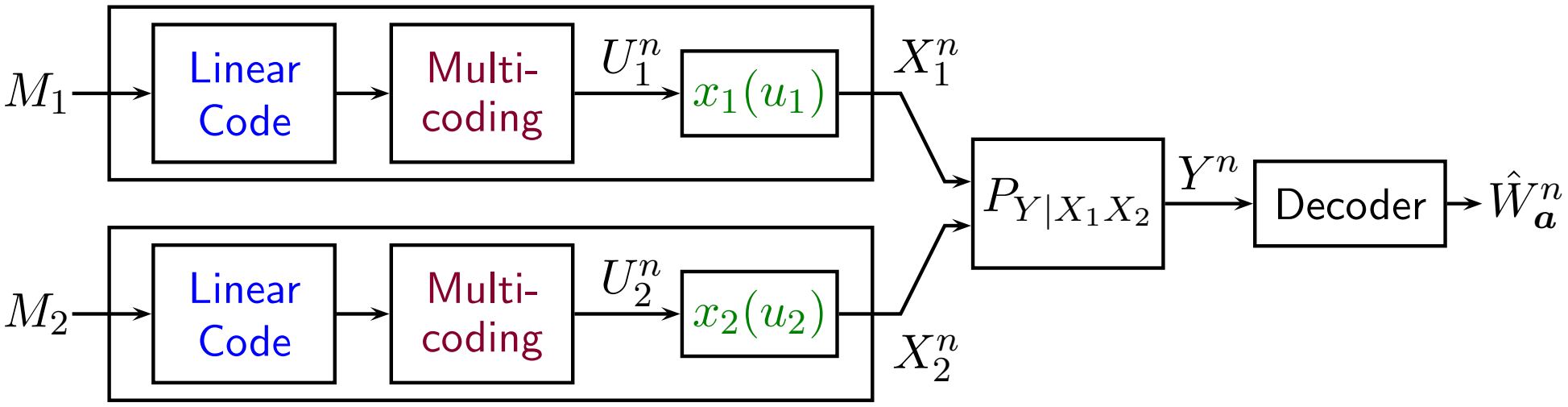

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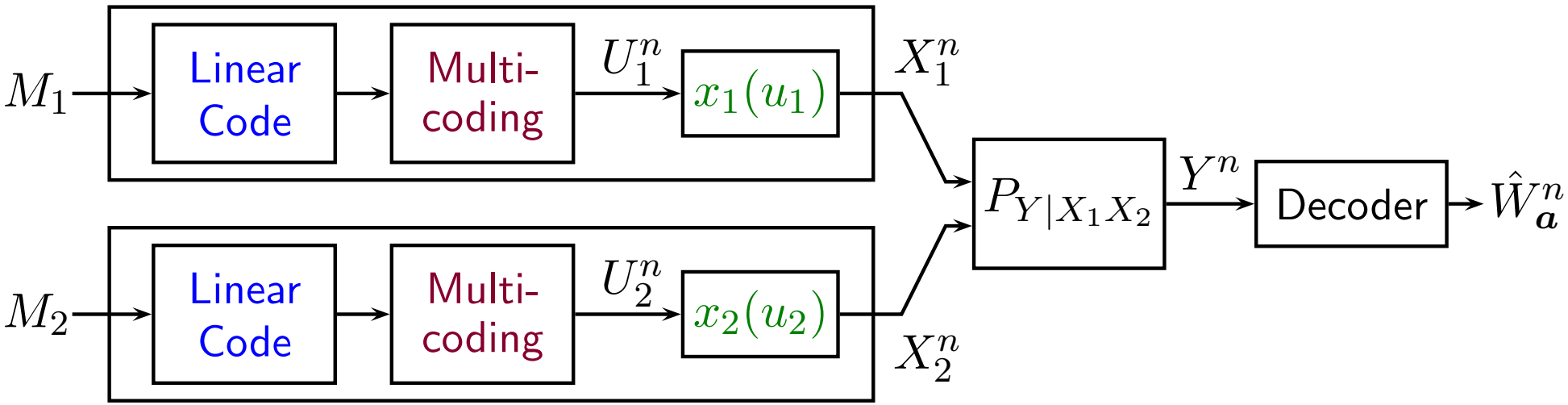

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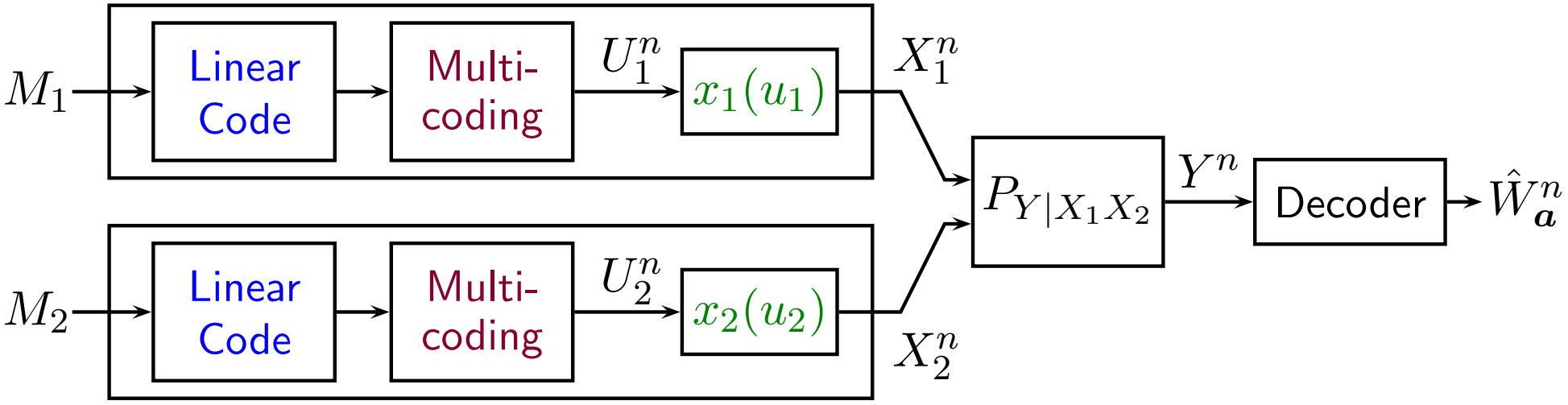

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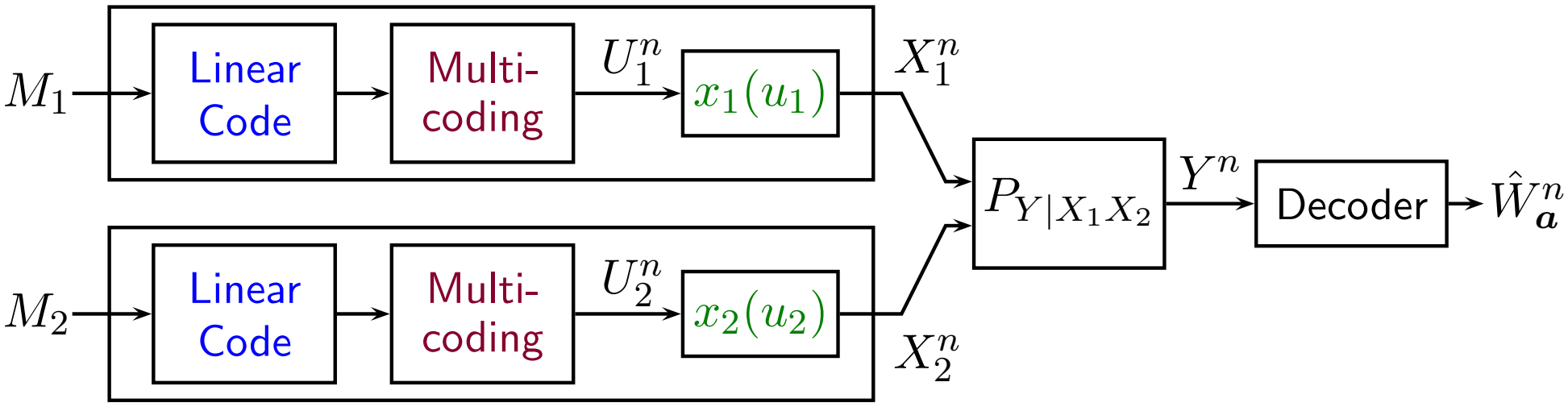

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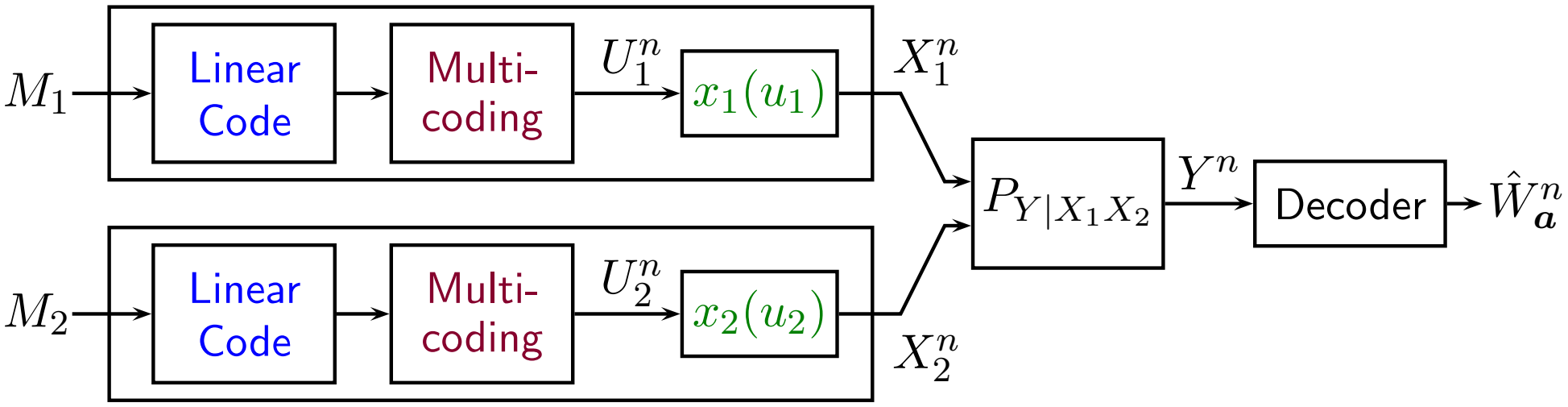

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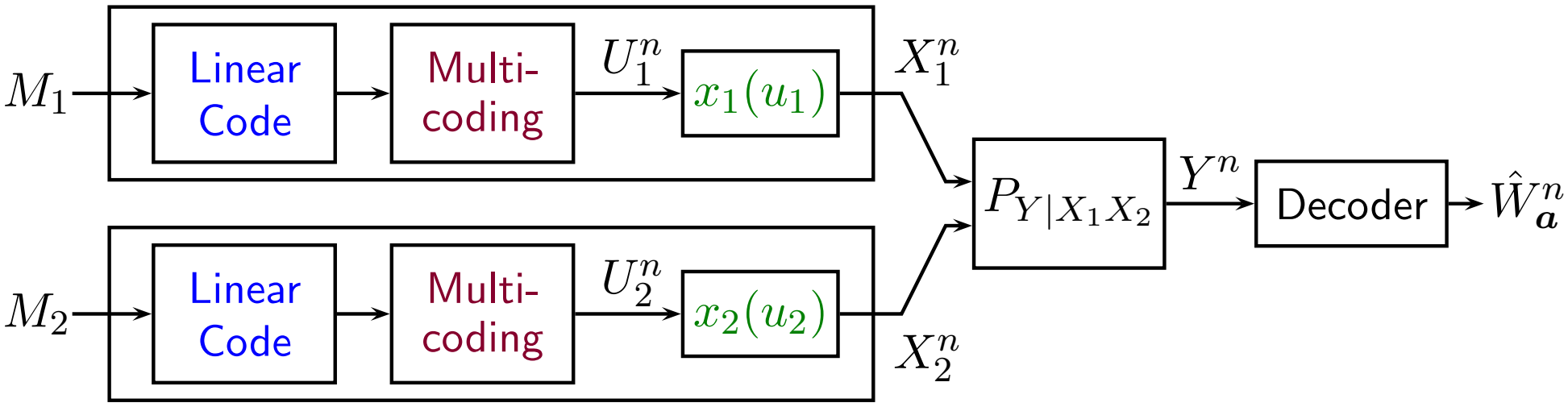

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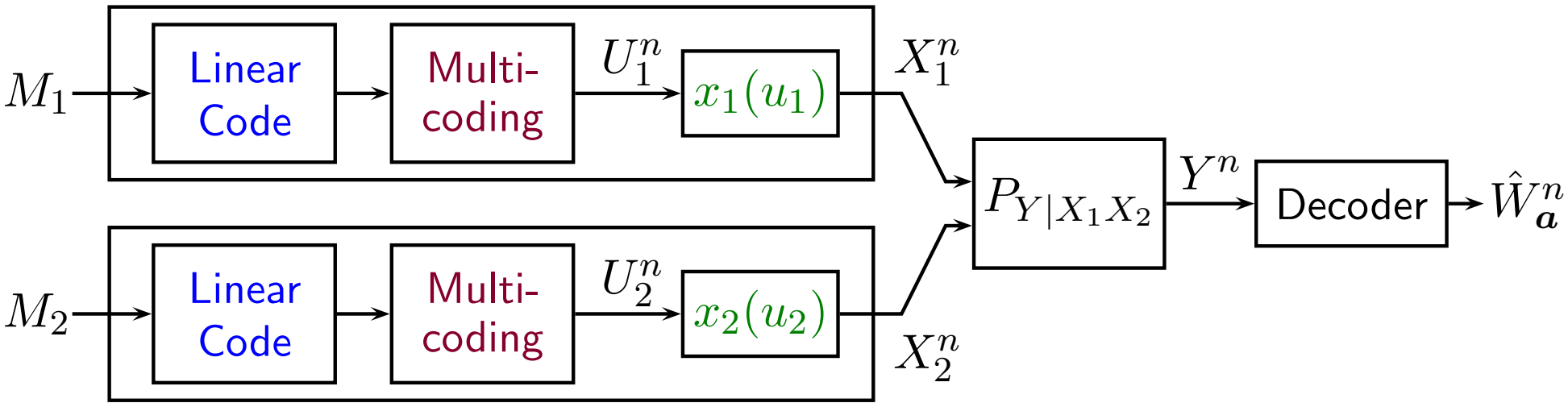

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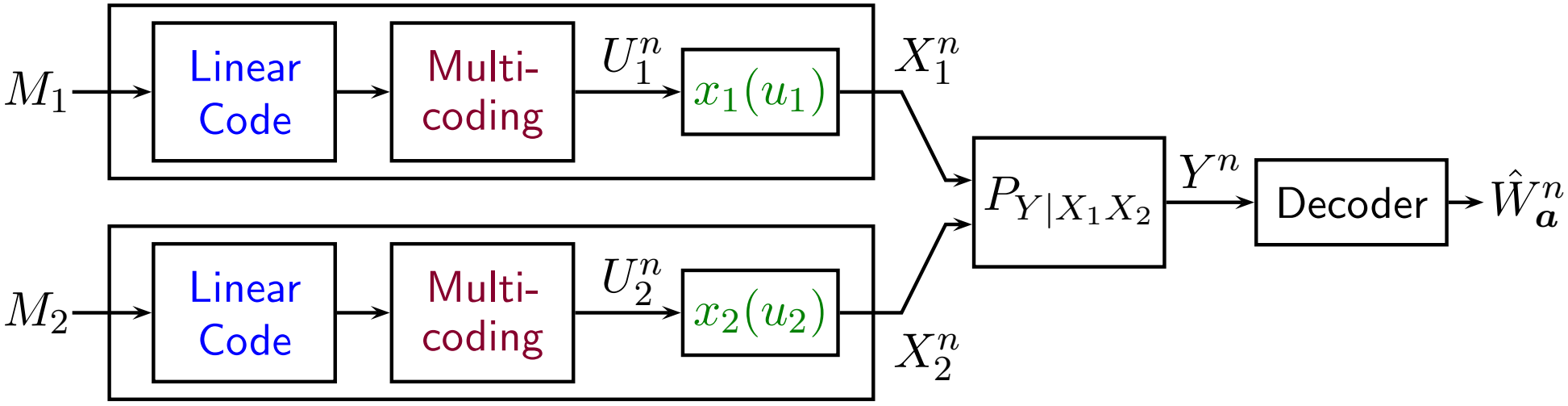
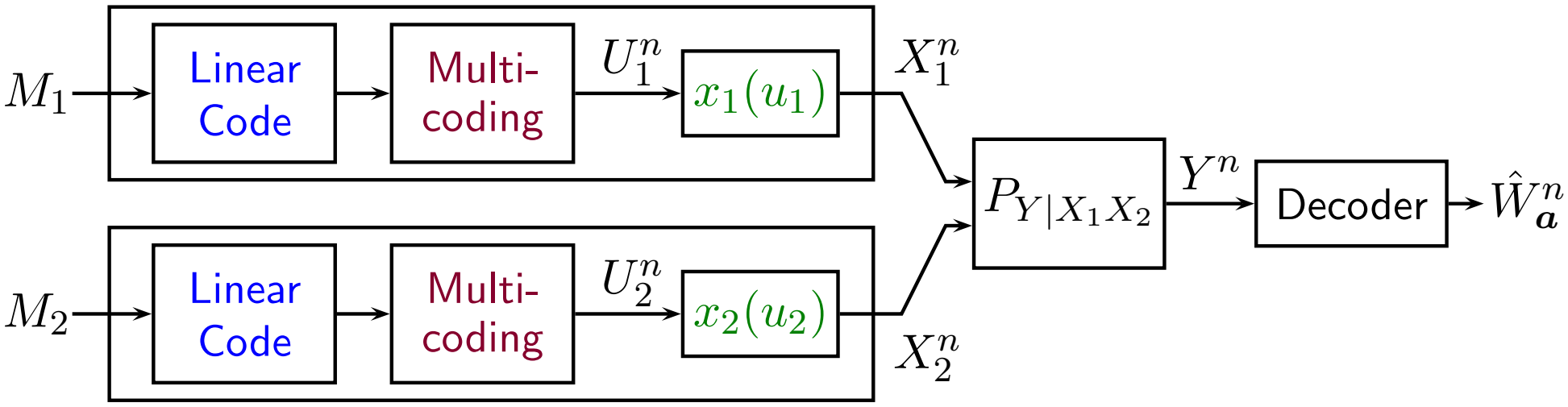


Computation Problem:

Nested Linear Coding Architecture



Computation Problem:

- For $m_k \in [2^{nR_k}]$, $l_k \in [2^{n\hat{R}_k}]$, we can express the linear combination of codewords as

$$\begin{aligned} w_{\boldsymbol{a}}^n &= a_1 u_1^n(m_1, l_1) \oplus a_2 u_2^n(m_2, l_2) \\ &= \left[a_1 \boldsymbol{\eta}(m_1, l_1) \oplus a_2 \boldsymbol{\eta}(m_2, l_2)\right] \mathsf{G} \oplus a_1 d_1^n \oplus a_2 d_2^n \\ &= \boldsymbol{\nu}(s_{\boldsymbol{a}}) \mathsf{G} \oplus a_1 d_1^n \oplus a_2 d_2^n \end{aligned}$$

where $s_{\boldsymbol{a}} \in \left[2^{n \max\{R_1+\hat{R}_1, R_2+\hat{R}_2\}}\right]$.

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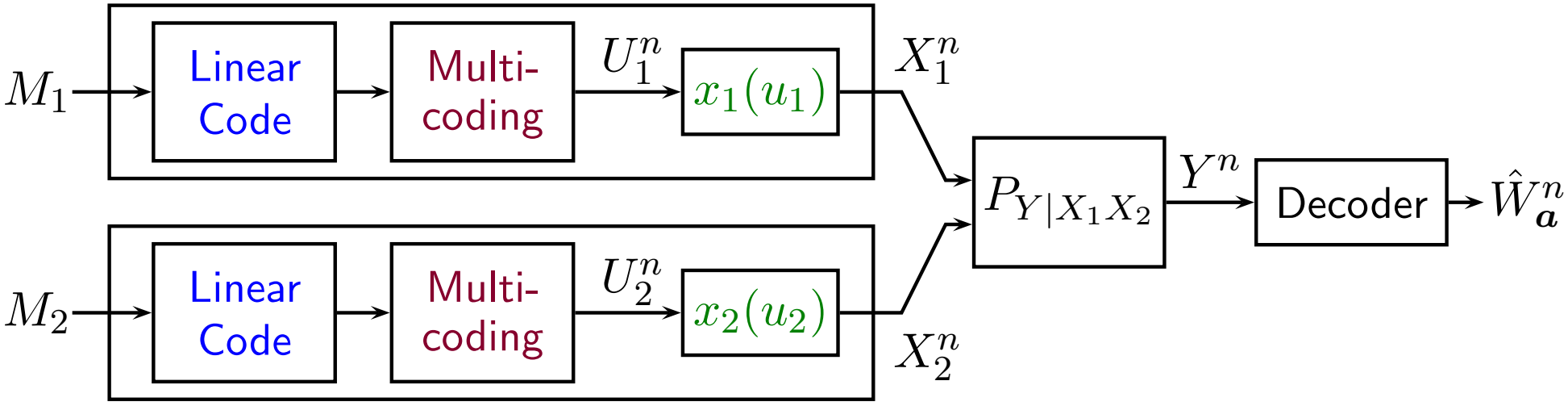

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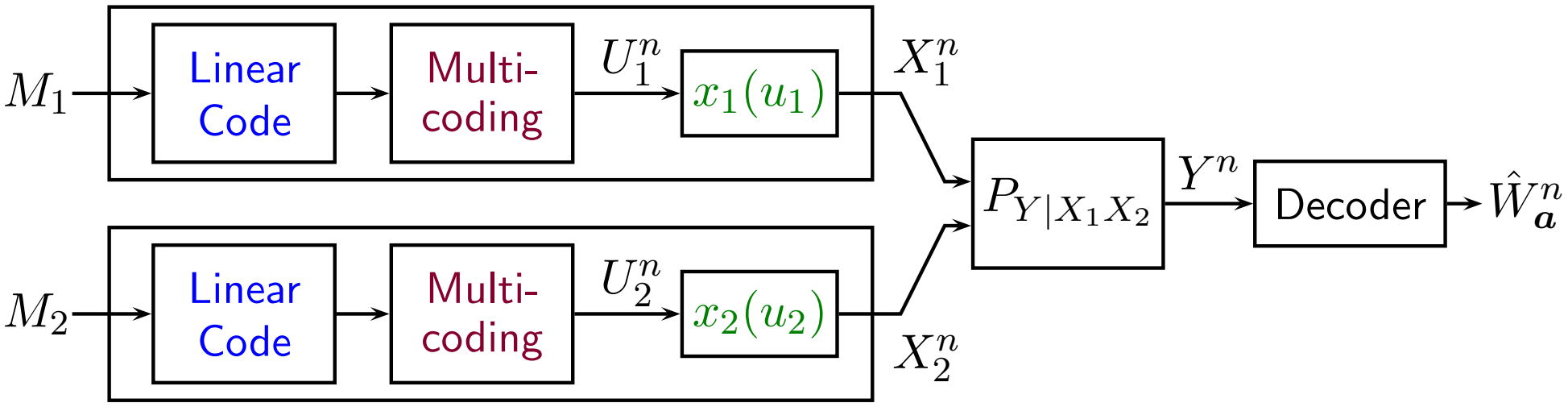

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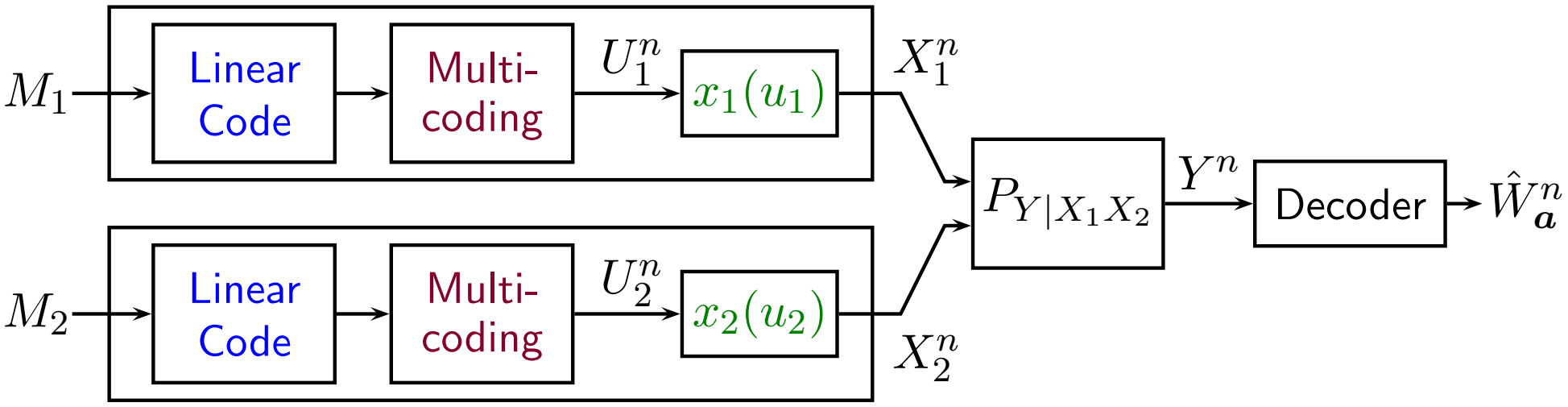

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- If there is no such index, or more than one, the decoder declares an error.

Error Analysis

An error occurs only if one or more of the following events occur,

- For some message, we cannot find a typical linear codeword:

$$\mathcal{E}_1 = \{U_k^n(m_k, l_k) \notin \mathcal{T}_{\epsilon'}^{(n)} \text{ for all } l_k, \text{ for some } m_k, k = 1, 2\}.$$

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Then, by the union of events bound,

$$\mathsf{P}\{\hat{W}_{\boldsymbol{a}}^n \neq W_{\boldsymbol{a}}^n\} \leq \mathsf{P}\{\mathcal{E}_1\} + \mathsf{P}\{\mathcal{E}_2 \cap \mathcal{E}_1^c\} + \mathsf{P}\{\mathcal{E}_3 \cap \mathcal{E}_1^c\}.$$

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- Proof just requires second moment method.

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- Our proof handles these statistical dependencies by breaking up the possible error events according to the underlying rank of the selected linear codewords. (Markov Lemma for Nested Linear Codes.)
- Prior work by **Padakandla-Pradhan '13** developed a bound that also requires $\hat{R}_k < D(p_{U_k} \| p_{\mathsf{q}}) + 3\delta(\epsilon)$.

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$$\mathcal{E}_3 = \{(U_1^n(m_1, l_1), U_2^n(m_2, l_2), Y^n) \in \mathcal{T}_\epsilon^{(n)} \text{ for some } (m_1, l_1, m_2, l_2) \text{ such that } \boldsymbol{\nu}(S_{\boldsymbol{a}}) \neq a_1 \boldsymbol{\eta}(m_1, l_1) \oplus a_2 \boldsymbol{\eta}(m_2, l_2)\}.$$

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- Can show that $\lim_{n\to\infty} \mathsf{P}\{\mathcal{E}_3 \cap \mathcal{E}_1^c\} = 0$ if

$$R_1 < I_{\mathsf{CF},1}(\boldsymbol{a}) \triangleq H(U_1) - H(W_{\boldsymbol{a}}|Y)$$
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  which matches our intuition from earlier.

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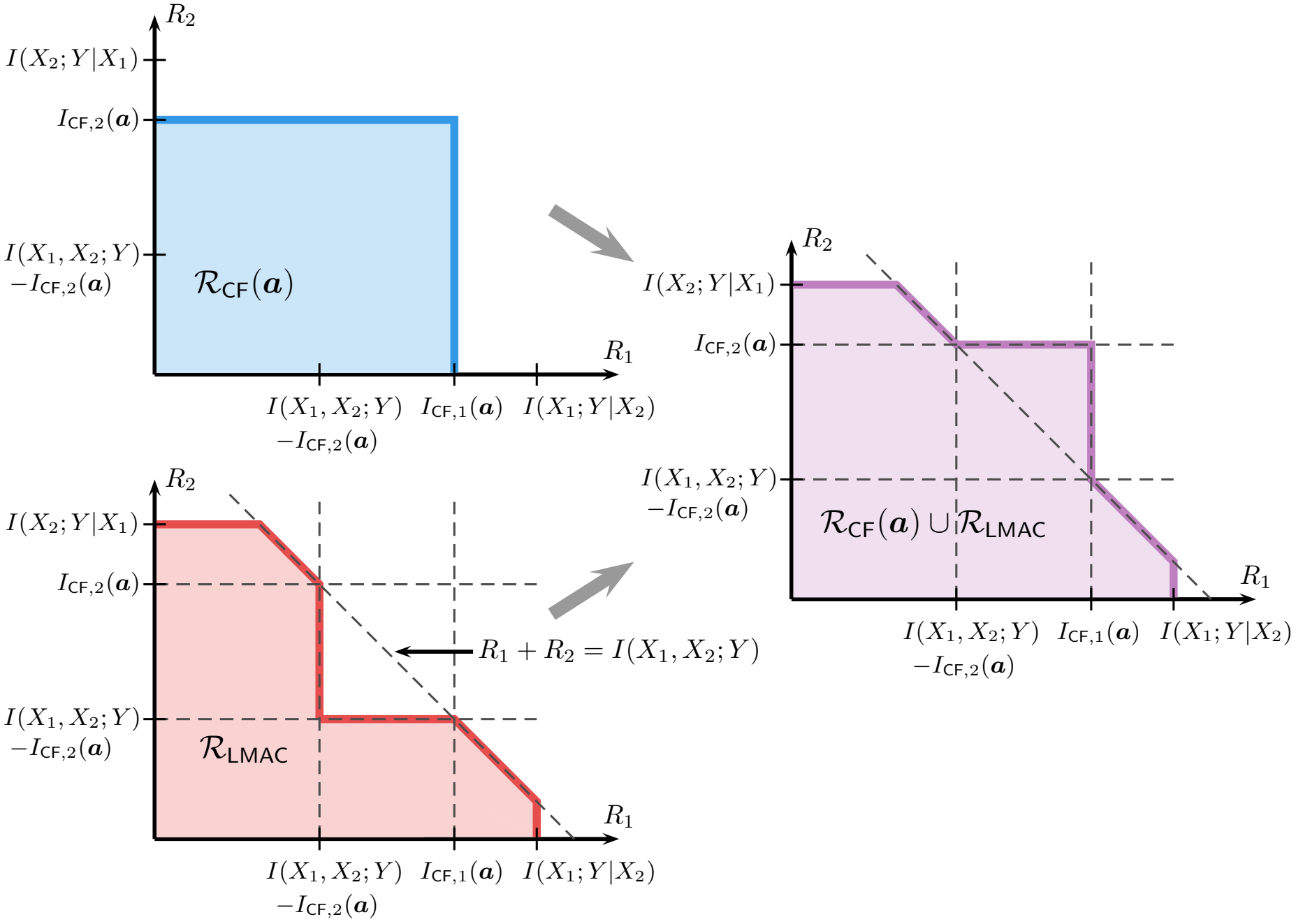
- The $I_{\mathsf{CF},2}(\boldsymbol{b})$ term plays a key role in handling the dependencies between competing pairs of linear codewords.

Rate Region


R_2
$I(X_2;Y|X_1)$
$I_{\mathsf{CF},2}(\boldsymbol{a})$
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$\mathcal{R}_{\mathsf{CF}}(\boldsymbol{a})$
R_1
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Theorem (Nazer-Gastpar '11)

For any channel vector $\mathbf{h}$ *and integer coefficient vector* $\mathbf{a}$*, any rate tuple satisfying* $R_k < R_{comp}(\mathbf{h}, \mathbf{a})$ *for* k *s.t.* $a_k \neq 0$ *is achievable where*

$$R_{comp}(\mathbf{h}, \mathbf{a}) = \frac{1}{2} \log^+ \left(\frac{P}{\mathbf{a}^\mathsf{T} \left(P^{-1} \mathbf{I} + \mathbf{h}\mathbf{h}^\mathsf{T} \right)^{-1} \mathbf{a}} \right)$$

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- **Ordentlich-Erez '13** derived bounds for lattice-based codes.
- **This talk:** We can analyze this via joint typicality decoding to get an achievable rate region.

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- **Key Technical Issue:** Random linear codewords are pairwise independent, but not 4-wise independent!

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Theorem

A rate tuple $(R_1, \ldots, R_K)$ is achievable for computing two linear combinations if

$$R_k < \max_{\boldsymbol{b} \in \mathbb{F}_q^2 \setminus \{\mathbf{0}\}} \min\{H(U_k) - H(V_{\boldsymbol{b}}|Y), H(U_k) - H(W_{\boldsymbol{a}_1}, W_{\boldsymbol{a}_2}|Y, V_{\boldsymbol{b}})\}$$

$$R_j < I(W_{\boldsymbol{a}_2}; Y, W_{\boldsymbol{a}_1}) - H(W_{\boldsymbol{a}_2}) + H(U_j)$$

$$R_k + R_j < I(W_{\boldsymbol{a}_1}, W_{\boldsymbol{a}_2}; Y) - H(W_{\boldsymbol{a}_1}, W_{\boldsymbol{a}_2}) + H(U_k) + H(U_j)$$

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for all j, k such that $a_{1,k} \neq 0$, $a_{2,j} \neq 0$ for some input pmf $\prod_{k=1}^{K} p(u_k)$, mappings $x_k(u_k)$, and $V_{\boldsymbol{b}} = b_1 W_{\boldsymbol{a}_1} \oplus b_2 W_{\boldsymbol{a}_2}$.

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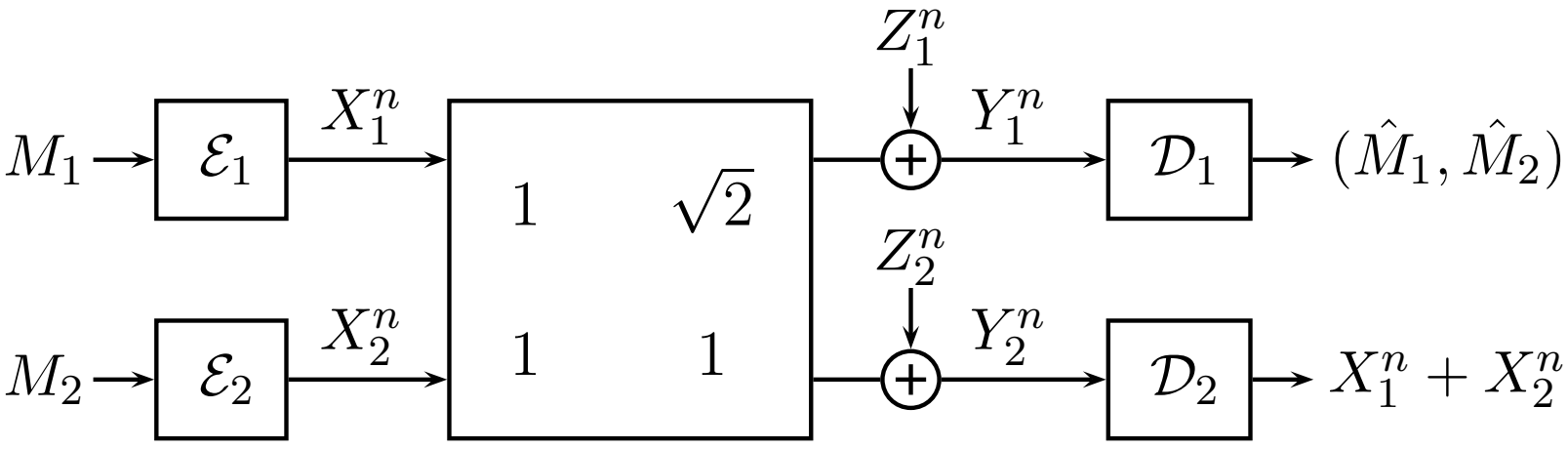
- The auxiliary linear combination $V_{\boldsymbol{b}}$ plays a key role in classifying dependent competing pairs in the error analysis.

Case Study: Two-Sender, Two-Receiver Network



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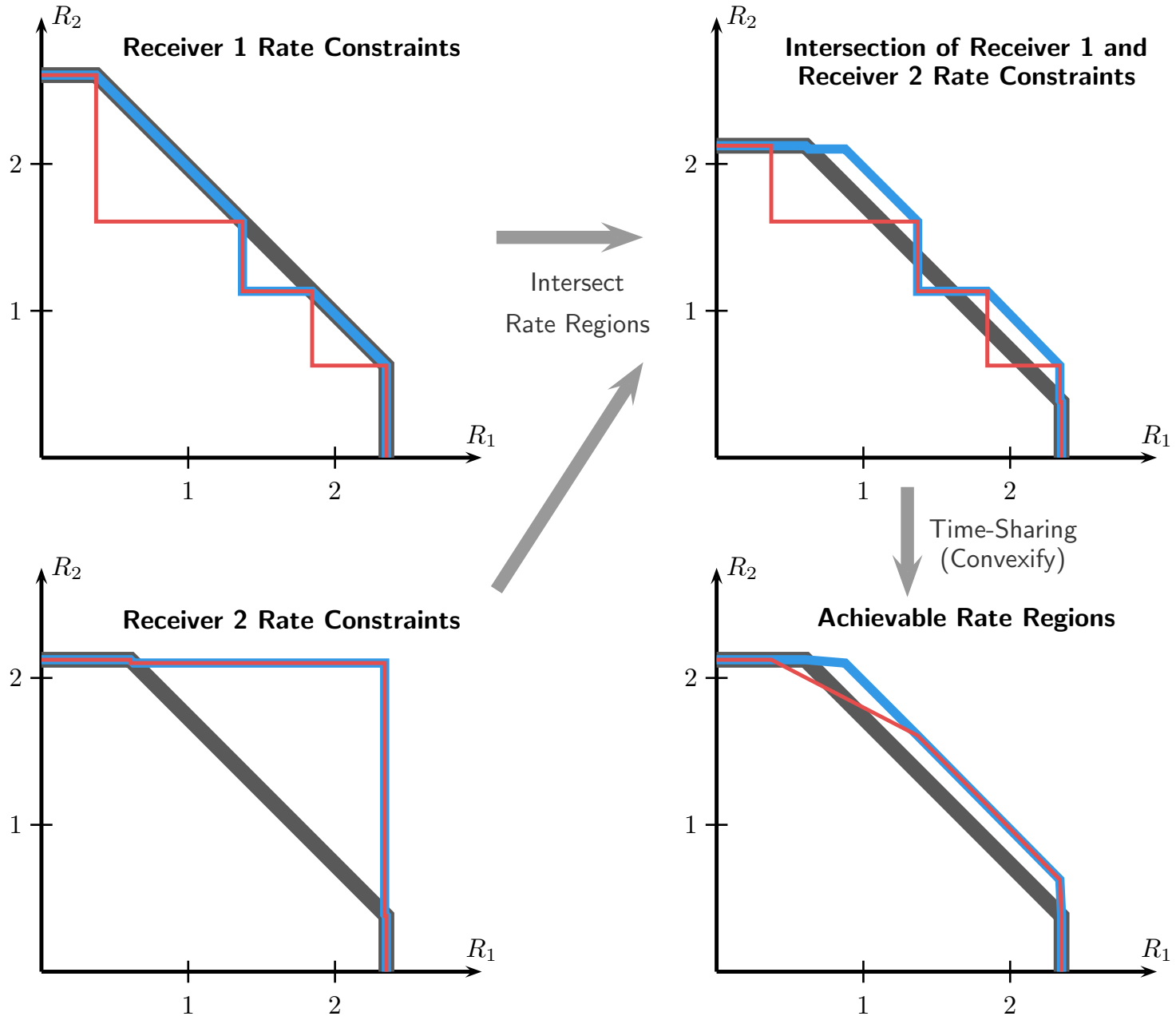

Concluding Remarks

- First steps towards bringing algebraic network information theory back into the realm of joint typicality.

- Joint decoding rate region for compute-and-forward that outperforms parallel and successive decoding.