Baker-Schmidt Problem (1970)

The Generalised Baker-Schmidt Problem, inspired by the pioneering work of A. Baker and W.M. Schmidt [3], is at the heart of Diophantine approximation on manifolds. This area of number theory has seen an explosion of activity in the last decade. Without doubt, the catalyst for this was the spectacular work of Kleinbock and Margulis [27] proving the profound conjectures of Baker and Sprindzuk dating back to the seventies. In short, Kleinbock and Margulis showed that for any non-degenerate manifold $\mathcal{M}$ in $\mathbb{R}^n$ and any $\varepsilon > 0$, almost all $(\alpha_1, \ldots, \alpha_n) \in \mathcal{M}$ satisfy the inequality

$$\liminf_{q\to\infty} q^{1+\varepsilon} \prod_{i=1}^{n} \|q\alpha_i\| > 0. \qquad (1)$$

This property of manifolds, often referred to as (strong) extremality, has subsequently been verified within the context of affine subspaces and their submanifolds [24], complex analytic manifolds [25], p-adic approximation [28] and friendly measures as introduced by Kleinbock, Lindenstrauss and Weiss [26]. Most recently a comparable and coherent 'extremal' inhomogeneous theory for manifolds has been developed. The key to this is the discovery of the Inhomogeneous Transference Principle [11] which actually unifies the 'extremal' homogeneous and inhomogeneous theories. On another front, there has been substantial progress regarding the more delicate and precise Groshev theory [3,6,7,14,20] and Khintchine theory [4,5,8,9,10,12,21,29] for manifolds with respect to both Lebesgue and Hausdorff measures. In particular, the Groshev type theorem proved in [3] provides a 'classical' proof of Sprindzuk's Conjecture and gives an alternative to the dynamical proof of Kleinbock and Margulis.

The generalised Baker-Schmidt problem corresponds to determining the Hausdorff dimension (and more generally Hausdorff measures) of Diophantine subsets of $\mathcal{M}$ arising from extremality. This deep problem takes on different forms depending on the type of approximation under consideration (e.g. simultaneous - approximation by rational points and dual - approximation by rational hyperplanes). The original Baker-Schmidt Conjecture was motivated by the development of transcendental number theory and was stated purely in terms of integral polynomials. It can naturally be interpreted in terms of approximating points on the Veronese curves $\{(x, \ldots, x^n) : x \in \mathbb{R}\}$ and was solved by Bernik [13]. The generalised problem in which the Veronese curves are replaced by non-degenerate manifolds is a major research challenge. Even with the dramatic progress of the aforementioned Khintchine-Groshev theory and other deep dimension results [1,2,17,18], still the situation of a curve in $\mathbb{R}^3$ remains completely open. The simultaneous form of the problem is intimately linked to fundamental questions regarding the distribution of rational points near manifolds. The latter represents an area of outright independent interest which in turn has strong links with lattice point problems and algebraic geometry. In particular, for algebraic curves and surfaces the rational points of interest must lie on the manifold itself when ε in (1) is large enough – see for example [15,16,19]. Regarding the general situation, the series of papers [4,8,9,10,12,29] have made a significant contribution to the understanding of the distribution of rational points near manifolds. In particular, they contain the best and essentially non-improvable bounds for planar curves (the previous record was due to Huxley [22,23]) and sharp lower bounds for non-degenerate analytic manifolds in $\mathbb{R}^n$.

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