13 Systems Solutions

1) What is the convolution of a rectangular pulse (width one, height one) with itself? Assume the rectangle wave is centred at $t = 0$.

A triangular wave. You can work this out using the convolution integral:

$$\phi(\tau) = \int_{-\infty}^{\infty} x(t)x(\tau - t)\,dt$$

but if you’ve got a table of Fourier integrals to hand (as in the notes), then it’s easier to do this in the frequency domain. This is often the case: things are usually easier in the frequency domain. Here, we first take the Fourier transform of the rectangular pulse:

$$F(\omega) = T \text{sinc}(\frac{\omega T}{2})$$

and since this particular pulse has a width of $T = 1$,

$$F(\omega) = \text{sinc}\left(\frac{\omega}{2}\right)$$

Now, since a convolution in the time domain is equivalent to a multiplication in the frequency domain, we have to multiply this Fourier transform by itself, and we can write the Fourier transform of the convolution as:

$$\Phi(\omega) = \text{sinc}^2\left(\frac{\omega}{2}\right)$$

Looking back to the table of Fourier transforms, we note that there is a Fourier transform pair of:

$$\begin{cases} 1 - \frac{|\omega|}{T} & \text{for } |\omega| < T \\ 0 & \text{for } |\omega| > T \end{cases} \iff T \text{sinc}^2\left(\frac{\omega T}{2}\right)$$

and what we have here is exactly this provided $T = 1$, so the inverse Fourier must be:

$$\begin{cases} 1 - \frac{|\omega|}{T} & \text{for } |\omega| < T \\ 0 & \text{for } |\omega| > T \end{cases}$$

This is a triangle wave, with a height one, and width (at base) of two.
2) Consider a system with an impulse response of \( h(t) = \delta(t) + \delta(t - 0.5) \). Without doing any integration, determine the output of this system when given a sine wave of 1 Hz as an input. Then check you’re right by doing the calculation both in the time domain (convolving the sine wave with \( h(t) \)) and in the frequency domain (by multiplying the Fourier transform of the sine wave with the frequency response of the system).

Without any integration: the output of a system with an impulse response of \( \delta(t) \) is just the same as the input, since the impulse response is an impulse with no delay. The output of the system with an impulse response of \( \delta(t - 0.5) \) is an impulse 0.5 seconds after the input went in, so this is just a delay of 0.5 seconds.

Put a sine wave at one Hz into this system therefore, and what will come out is a copy of the sine wave you put in, together with a copy of the sine wave delayed by 0.5 seconds:

\[
\sin(2\pi t) + \sin(2\pi(t - 0.5)) = \sin(2\pi t) + \sin(2\pi t)\cos(\pi) - \sin(\pi)\cos(2\pi t) = \sin(2\pi t) - \sin(2\pi t) = 0
\]

in other words, nothing at all. This system is a filter that does not let any energy at one Hz get through.

The same result can be obtained in the time domain using the convolution integral:

\[
y(t) = \int_{-\infty}^{\infty} h(t-a) x(a) da
\]

\[
= \int_{-\infty}^{\infty} \delta(t-a) \sin(2\pi a) da + \int_{-\infty}^{\infty} \delta(t-a-0.5) \sin(2\pi a) da
\]

\[
= \int_{-\infty}^{\infty} \delta(t-a) \sin(2\pi a) da + \int_{-\infty}^{\infty} \delta(t-a-0.5) \sin(2\pi a) da
\]

\[
= \sin(2\pi t) + \sin(2\pi(t - 0.5))
\]

and from there the analysis follows the same path as above, with the same answer: zero.

This one is a bit harder in the frequency domain, but taking the Fourier transform of the input gives:

\[
\sin(\omega t) \leftrightarrow j[\pi\delta(\omega + 2\pi) - \pi\delta(\omega - 2\pi)]
\]

(see the table of Fourier transforms in the notes), and the Fourier transform of the impulse response of this system is simple to work out:
\[ H(\omega) = \int_{-\infty}^{\infty} \delta(t) \exp(-j\omega t) \, dt + \int_{-\infty}^{\infty} \delta(t - 0.5) \exp(-j\omega t) \, dt \]

\[ = \exp(-0j\omega) + \exp(-0.5j\omega) \]

\[ = 1 + \exp(-0.5j\omega) \]

and the Fourier transform of the output can be found by multiplying the frequency response of the system with the Fourier transform of the input signal:

\[ Y(\omega) = (1 + \exp(-0.5j\omega)) \left( j\pi\delta(\omega + 2\pi) - \pi\delta(\omega - 2\pi) \right) \]

\[ = j\pi\delta(\omega + 2\pi) - j\pi\delta(\omega - 2\pi) + j\pi\delta(\omega + 2\pi)\exp(-j\omega 0.5) - j\pi\delta(\omega - 2\pi)\exp(-j\omega 0.5) \]

\[ = j\pi\delta(\omega + 2\pi) - j\pi\delta(\omega - 2\pi) + j\pi\delta(\omega + 2\pi)\exp(j2\pi 0.5) - j\pi\delta(\omega - 2\pi)\exp(j2\pi 0.5) \]

\[ = j\pi\delta(\omega + 2\pi) - j\pi\delta(\omega - 2\pi) + j\pi\delta(\omega + 2\pi) - j\pi\delta(\omega - 2\pi) \]

\[ = 0 \]

since \( \exp(j\omega) = -1 \).

3) Prove that convolution is commutative, in other words, that:

\[ y(t) = \int_{-\infty}^{\infty} x(\tau) h(t - \tau) \, d\tau = \int_{-\infty}^{\infty} x(t - \tau) h(\tau) \, d\tau \]

Quite straightforward: just substitute in \( a = t - \tau \), and since \( t \) is constant during the integration, \( da = -d\tau \), so:

\[ \int_{-\infty}^{\infty} x(\tau) h(t - \tau) \, d\tau = \int_{-\infty}^{\infty} x(t - a) h(a)(-da) \]

note the reversal of the limits of integration: when \( t \) is infinity, \( a \) is minus infinity and vice versa, and since:

\[ \int_{a}^{b} f(x) \, dx = -\int_{b}^{a} f(x) \, dx \]

we get:

\[ \int_{-\infty}^{\infty} x(\tau) h(t - \tau) \, d\tau = \int_{-\infty}^{\infty} x(t - a) h(a) \, da \]

and this proves the result. (The change in the name of the variable being integrated over doesn’t matter, this is not visible outside the integration, and we could just replace \( a \) with \( \tau \) if we wanted.)
4) A system has a frequency response of \( H(j\omega) = \delta(\omega - 1) + \delta(\omega + 1) \). What is the output if an impulse arrives at the input? How about if the input is a series of two impulses, one at time \( t = 0 \) and magnitude 1, and one at time \( t = 2\pi \) and magnitude \(-1\)? Could such a system be built?

The result when putting an impulse at time \( t = 0 \) into this system is an impulse at time \( t = 1 \), and another at time \( t = -1 \). That’s what a plot of the impulse response looks like:

Since the system is linear and time-invariant, the output from putting an impulse of magnitude \(-1\) and time \( 2\pi \) into this system is an impulse of magnitude \(-1\) at time \( (2\pi - 1) \), and another impulse of magnitude \(-1\) at time \( (2\pi + 1) \), so the total output could be written:

\[
y(t) = \delta(-1) + \delta(1) - \delta(2\pi - 1) - \delta(2\pi + 1)
\]

Could such a system be built? No. This system gives an output before the input has arrived, which is impossible.

5) A system has a frequency response of \( H(j\omega) = \frac{1}{j\omega + 2} \). It receives as an input a signal which is described in the time domain by \( x(t) = u(t)e^{-t} \) (an exponential decay starting at time \( t = 0 \)). What is the output of this system in both the frequency and the time domains?

Again, it’s easiest to work in the frequency domain, since multiplications are easier to do than convolutions, and it’s not too difficult to take Fourier transforms. We’ve already got the frequency response of the system, so all we need to do is take the Fourier transform of the input signal, and multiply them together.

Again looking up in the table, we note that there is a Fourier transform pair of:
so setting $a = 1$ gives the Fourier transform of our input signal as $1 / (j\omega + 1)$. Then multiply this with the frequency response of the system, to give:

$$Y(\omega) = \frac{1}{(j\omega + 1)(j\omega + 2)}$$

To get the output signal in the time-domain, we need the inverse Fourier transform of this expression. This one isn’t in the tables, but we can expand this expression using the method of partial fractions:

$$\frac{1}{(j\omega + 1)(j\omega + 2)} = \frac{1}{(j\omega + 1)} - \frac{1}{(j\omega + 2)}$$

and we can use the same entry in the tables to work out the inverse Fourier transform now, which gives us the output signal in the time domain:

$$y(t) = u(t)\exp(-t) - u(t)\exp(-2t)$$