13 Getting Started With… Systems

The discussion of systems here will be restricted to the case of systems with one input signal, and one output signal. Examples of such systems are communication channels (such as radio channels, transmission lines or optical fibres) that distort the input signal to produce an output signal, and other elements in a typical communication system including filters and amplifiers.

Just as signals can be described in the time or frequency domains, so can systems. In the time domain, they take a time-domain input signal, and produce a time-domain output signal. A certain sub-class of systems, known as linear systems, can also be readily described in the frequency domain, in terms of the gain and phase shift experienced by a sinusoidal signal of any given frequency.

For example, a system that simply delays its input by one-tenth of a second could be represented in the frequency domain by a gain of 1, and a phase shift of $0.2\pi$ radians where $f$ is the frequency in Hz, since a 1 Hz oscillation would be delayed by $0.2\pi$ radians, a 2 Hz oscillation by $0.4\pi$ radians, a 0.5 Hz oscillation by $0.1\pi$ radians, and so on.

13.1 Linear and Non-Linear Systems

Most of the systems studied in communication theory are assumed to be linear. A linear system has the property that the sum of two inputs produces the sum of the two corresponding outputs. Mathematically:

If $x_1(t) \Rightarrow y_1(t)$
and $x_2(t) \Rightarrow y_2(t)$
then $x_1(t) + x_2(t) \Rightarrow y_1(t) + y_2(t)$

This implies that by setting $x_2(t)$ equal to $(a - 1)x_1(t)$, where $a$ is a constant:

If $x_1(t) \Rightarrow y_1(t)$
then $ax_1(t) \Rightarrow ay_1(t)$

so that, for example, doubling the input signal results in doubling the output signal.

A key feature of linear systems is that in the frequency domain they do not generate any new frequencies: the only frequencies present in the output signal are those already present in the input signal.

Non-linear systems do not share this property: non-linear systems generate new frequencies.

13.1.1 Linear Operators

There are only a very limited number of operations that a linear system can perform on an input: multiplication by a constant, integration, differentiation, delay and adding or subtracting integrated, differentiated or delayed versions of itself. That’s about it: if a system does anything else to an input signal, then it’s not linear.

13.1.2 Characterising Non-Linear Systems

Non-linear systems without memory (that is, systems in which the current output is a function only of the current input, and not of any previous inputs) can be characterised in terms of the
Maclaurin series of their system transfer function. The output $y(t)$ can be related to the input $x(t)$ using an expression of the form:

$$y(t) = A + Bx(t) +Cx^2(t) + Dx^3(t) + Ex^4(t) + ...$$

For a single frequency input $x(t) = \cos(\omega t)$, the output then takes the form:

$$y(t) = A + B\cos(\omega t) + C\left(\frac{1 + \cos(2\omega t)}{2}\right) + D\left(\frac{3\cos(\omega t) + \cos(3\omega t)}{4}\right) + E\left(\frac{3 + 4\cos(2\omega t) + \cos(4\omega t)}{8}\right) + ...$$

$$= \left[A + \frac{C}{2} \cos(\omega t) + \frac{E}{8} + ...\right] + \left[B + \frac{3D}{4} + ...\right] \cos(\omega t) + \left[C + \frac{E}{2} + ...\right] \cos(2\omega t) + \left[\frac{D}{4} + ...\right] \cos(3\omega t) + \left[\frac{E}{8} + ...\right] \cos(4\omega t) + ...$$

Note that as well as the input frequency, harmonics of the original input waveform and a DC component may be present in the output.

When two frequencies are present in the input to a non-linear system, the output takes a much more complex form. Consider just the results of the term $Cx^2(t)$, and $Dx^3(t)$. First, the term $Cx^2(t)$, when the input takes the form $x(t) = \cos(\omega_1 t) + \cos(\omega_2 t)$ gives as the output:

$$y = C\left[\cos(\omega_1 t) + \cos(\omega_2 t)\right]^2$$

$$= C\cos^2(\omega_1 t) + C\cos^2(\omega_2 t) + 2C\cos(\omega_1 t)\cos(\omega_2 t)$$

$$= \frac{C}{2} \left(1 + \cos(2\omega_1 t)\right) + \frac{C}{2} \left(1 + \cos(2\omega_2 t)\right) + C\cos(\omega_1 - \omega_2) t + C\cos(\omega_1 + \omega_2) t$$

$$= C + \frac{C}{2} \cos(2\omega_1 t) + \frac{C}{2} \cos(2\omega_2 t) + C\cos(\omega_1 - \omega_2) t + C\cos(\omega_1 + \omega_2) t$$

Note that the output contains terms in the sum and difference of the two input frequencies. In many radio communications systems this is not a significant problem, since these frequencies are a long way from the frequencies of interest in the signal, and can be easily filtered out.

However, for the term $Dx^3(t)$ when the input takes the form $\cos(\omega_1 t) + \cos(\omega_2 t)$, the output has the form:

$$y = D\left[\cos(\omega_1 t) + \cos(\omega_2 t)\right]^3$$

$$= D\left[\cos^3(\omega_1 t) + \cos^2(\omega_2 t) + 2\cos(\omega_1 t)\cos(\omega_2 t)\right] \times \left[\cos(\omega_1 t) + \cos(\omega_2 t)\right]$$

$$= D\cos^3(\omega_1 t) + D\cos(\omega_1 t)\cos^2(\omega_2 t) + 2D\cos^2(\omega_1 t)\cos(\omega_2 t)$$

$$D\cos^3(\omega_2 t) + D\cos^2(\omega_1 t)\cos(\omega_2 t) + 2D\cos(\omega_1 t)\cos^2(\omega_2 t)$$
Here there are eight component frequencies in the output, two of which are of particular concern. When \( \omega_1 \) is close to \( \omega_2 \), the components at \( (2\omega_1 - \omega_2) \) and \( (2\omega_2 - \omega_1) \) can be very close in frequency to the two input signals, and can therefore be very difficult (and sometimes impossible) to filter out. This “third-order harmonic” distortion is a critical performance parameter of many communication systems.

All real analogue systems are non-linear to some extent; so all radio systems will experience some of this type of distortion.

### 13.2 System Frequency Response

Since linear systems do not generate any new frequencies, the only thing a linear system can do when presented with an input consisting of a single frequency is to change the amplitude and phase of that single frequency. This allows the behaviour of a system to be characterised in terms of a frequency response: the amount by which every frequency is amplified and phase shifted.

Since the Fourier transform allows us to express any finite-energy waveform\(^1\) in terms of a sum of complex frequencies, a useful form of the system frequency response is:

\[
y = \frac{D}{4}(3\cos(\omega_1 t) + \cos(3\omega_1 t)) + \frac{D}{4}(3\cos(\omega_2 t) + \cos(3\omega_2 t)) \]
\[
+ \frac{3D}{2}\cos(\omega_1 t)[1 + \cos(2\omega_2 t)] + \frac{3D}{2}\cos(\omega_2 t)[1 + \cos(2\omega_1 t)]
\]
\[
= \frac{3D}{4}\cos(\omega_1 t) + \frac{3D}{4}\cos(\omega_2 t) + \frac{D}{4}\cos(3\omega_1 t) + \frac{D}{4}\cos(3\omega_2 t) + \frac{3D}{2}\cos(\omega_1 t) + \frac{3D}{2}\cos(\omega_2 t) \]
\[
+ \frac{3D}{2}\cos(\omega_1 t)\cos(2\omega_2 t) + \frac{3D}{2}\cos(\omega_2 t)\cos(2\omega_1 t)
\]

\[
y = \frac{9D}{4}\cos(\omega_1 t) + \frac{9D}{4}\cos(\omega_2 t) \]
\[
+ \frac{D}{4}\cos(3\omega_1 t) + \frac{D}{4}\cos(3\omega_2 t) \]
\[
+ \frac{3D}{4}[\cos(2\omega_1 + \omega_2)t + \cos(2\omega_2 - \omega_1)t] + \frac{3D}{4}[\cos(2\omega_1 + \omega_2)t + \cos(2\omega_1 - \omega_2)t]
\]

\(^\text{1}\) Well, not quite any. Similar conditions apply as for the Fourier series: no sudden changes in value or slope, and of course it has to have a finite amount of energy.
since $X(\omega)$ indicates how much, and of what phase, complex exponential oscillations are required when added together, to produce $x(t)$. Each one is amplified by the magnitude of $H(\omega)$ and rotated in phase by the argument of $H(\omega)$ at the corresponding frequency $\omega$. This is equivalent to multiplying by $H(\omega)$ – this can most easily be seen by expressing $X(\omega)$ and $H(\omega)$ in polar form:

$$H(\omega)X(\omega) = R(\omega)|X(\omega)|\exp(j\arg(X(\omega)))$$

$$= R(\omega)|X(\omega)|\exp(j(\theta + \arg(X(\omega))))$$

so the result has increased the amplitude of $X(\omega)$ by $R(\omega)$, and added a phase of $\theta$. Back in the time domain, the output can be expressed as:

$$y(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} H(\omega)X(\omega)\exp(j\omega t) d\omega$$

To determine the output of a system with a known frequency response $H(\omega)$ is then a matter of taking the Fourier transform of the input waveform producing $X(\omega)$, multiplying by the frequency response of the system $H(\omega)$, and then taking the inverse Fourier transform of the result $Y(\omega)$, giving the output waveform $y(t)$.

### 13.2.1 Example of Frequency Responses

For example: put a signal $x(t) = u(t)\exp(-at)$ through a filter with a frequency response of

$$H(\omega) = \frac{5}{(j\omega + a)}$$

and the output can be calculated as:

$$X(\omega) = \frac{1}{(j\omega + a)}$$

$$Y(\omega) = H(\omega)X(\omega)$$

$$= \frac{5}{(j\omega + a)^2} \frac{1}{(j\omega + a)} = \frac{5}{(j\omega + a)^2}$$

$$y(t) = 5t u(t) \exp(-at)$$

### 13.3 System Impulse Response

Another way of specifying the response of a system is in terms of the output when the input is an impulse – an impulse being a pulse of negligible duration, infinite amplitude, and unit area. Impulses can be considered as the limiting case of a rectangular pulse with width $dx$ and height $1/dx$ as $dx$ approaches zero:
An impulse at time $t = 0$ is written as $\delta(t)$, so an impulse at time $\tau$ (as shown) is $\delta(t-\tau)$.

Since the Fourier transform of an impulse is just a constant value of one for all frequencies, any system with a delta function as the input produces an output that depends on the response of the system to all possible frequencies – in other words it will completely specify the system.

There is a very simple relationship between the impulse response of a system and the frequency response. If a system of frequency response $H(\omega)$ is provided with an input of a delta function at time $t = 0$, then the output in the frequency domain is just:

$$Y(\omega) = H(\omega) X(\omega)$$

since the Fourier transform of an impulse is just $X(\omega) = 1$. This means that $H(\omega)$ must be the Fourier transform of the impulse response, since the output when the input is an impulse at time $t = 0$ is, by definition, the impulse response.

### 13.4 Time Variant and Time-Invariant Systems

It is important to consider whether a system’s behaviour does not change with time (time-invariant, e.g. a filter), or whether it does change with time (time-variant, e.g. a mobile radio channel).

Time-variant systems can be represented by a time-dependent impulse response $h(\tau, t)$, or by a time-dependent frequency response $H(\omega, t)$. For the rest of this chapter we will consider time-invariant systems only: this is an important consideration when deriving the form of the convolution integral.

### 13.5 Convolution

It is possible to derive the output of a time-invariant linear system knowing its impulse response and the input waveform in two ways. Firstly, as we have seen, we can convert the impulse response into the frequency response of the system (by taking the Fourier transform), and convert the input waveform to a weighted sum of complex oscillations (again by taking the Fourier transform). Multiplying these Fourier transforms together gives the Fourier transform of the output waveform, which can be converted back into a time-domain description of the output waveform by taking the inverse Fourier transform.

Alternatively, we can do the whole calculation in the time domain. Consider the input waveform as the sum of a very large number of delta functions, each of amplitude $x(\tau)\delta\tau$, at all values of time $\tau$ (see figure below).
Expressing this in terms of the integral (as the limit of sum) gives:

\[ x(t) = \int_{-\infty}^{\infty} x(\tau) \delta(t - \tau) d\tau \]

Since the impulse response is the output when an impulse is provided as the input, then provided the system is linear and time-invariant, the output for an input of a delta function of magnitude \( x(\tau) \delta \tau \), at time \( t - \tau \) must be equal to \( h(t - \tau) x(\tau) \delta \tau \), and the total output waveform is the sum of all these impulse responses summed over all time:

\[ y(t) = \int_{-\infty}^{\infty} x(\tau) h(t - \tau) d\tau \]

This is known as the convolution integral, and it relates the output signal to the input signal and the impulse response of a channel, provided the channel is time-invariant and linear.

Notice that the convolution integral is commutative, in the sense that:

\[ y(t) = \int_{-\infty}^{\infty} x(\tau) h(t - \tau) d\tau = \int_{-\infty}^{\infty} x(t - \tau) h(\tau) d\tau \]

(this is straightforward to prove by a simple substitution of \( t - \tau = t' \)).

Usually it is more computationally efficient to determine the output waveform in the frequency domain, so most calculations are done by taking the Fourier transform of the input waveform, and the Fourier transform of the impulse response (if the system is specified in the time domain), doing the multiplication, then taking the inverse Fourier transform of the output. Algorithms including the Fast Fourier Transform make this calculation more efficient than working out the convolution integral.

### 13.6 Systems in the Time and Frequency Domain

Both signals and systems can be expressed in the time domain (as waveforms and impulse responses) and the frequency domain (as Fourier transforms and frequency responses). These relationships can be summarised in the diagram below.

In all cases, moving from the time domain to the frequency domain involves taking the Fourier transform, moving back from the frequency domain to the time domain involves taking the inverse Fourier transform.
13.7 Problems

(Note – you might need to refer to the tables of Fourier transforms in the section on Fourier Techniques to complete some of these.)

1) What is the convolution of a rectangular pulse (width one, height one) with itself? Assume the rectangle wave is centred at \( t = 0 \).

2) Consider a system with an impulse response of \( h(t) = \delta(t) + \delta(t - 0.5) \). Without doing any integration, determine the output of this system when given a sine wave of 1 Hz as an input. Then check you’re right by doing the calculation both in the time domain (convolving the sine wave with \( h(t) \)) and in the frequency domain (by multiplying the Fourier transform of the sine wave with the frequency response of the system).

3) Prove that convolution is commutative, in other words, that:

\[
y(t) = \int_{-\infty}^{\infty} x(\tau) h(t - \tau) \, d\tau = \int_{-\infty}^{\infty} x(t - \tau) h(\tau) \, d\tau
\]

4) A system has a frequency response of \( H(j\omega) = \delta(\omega - 1) + \delta(\omega + 1) \). What is the output if an impulse arrives at the input? How about if the input is a series of two impulses, one at time \( t = 0 \) and magnitude 1, and one at time \( t = 2\pi \) and magnitude \(-1\)? Could such a system be built?

5) A system has a frequency response of \( H(j\omega) = \frac{1}{j\omega + 2} \). It receives as an input a signal which is described in the time domain by \( x(t) = u(t) \exp(-t) \) (an exponential decay starting at time \( t = 0 \)). What is the output of this system in both the frequency and the time domains?