12 Fourier Solutions

1) Express the co-efficients of the complex Fourier Series \( c_n \) in terms of the co-efficients of the trigonometric Fourier Series \( a_n \) and \( b_n \).

The trigonometric Fourier series is:

\[
x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi}{T} nt \right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{2\pi}{T} nt \right)
\]

and the complex exponential Fourier series is:

\[
x(t) = \sum_{n=-\infty}^{\infty} c_n \exp\left(j \frac{2\pi}{T} nt \right)
\]

The terms in \( a_n \) and \( b_n \) correspond to the terms in \( c_{-n} \) and \( c_n \). Since these must both give the component at the corresponding frequency in the signal \( x(t) \), we must have:

\[
c_n \exp\left(j \frac{2\pi}{T} nt \right) + c_{-n} \exp\left(-j \frac{2\pi}{T} nt \right) = a_n \cos\left(\frac{2\pi}{T} nt \right) + b_n \sin\left(\frac{2\pi}{T} nt \right)
\]

expanding \( \cos(x) \) in terms of \( \exp(jx) + \exp(-jx) \) and \( \sin(x) \) in terms of \( \exp(jx) - \exp(-jx) \) gives:

\[
c_n \exp\left(j \frac{2\pi}{T} nt \right) + c_{-n} \exp\left(-j \frac{2\pi}{T} nt \right) = a_n \exp\left(j \frac{2\pi}{T} nt \right) + \frac{a_n}{2} \exp\left(-j \frac{2\pi}{T} nt \right)
\]

\[
+ b_n \exp\left(j \frac{2\pi}{T} nt \right) - \frac{b_n}{2} \exp\left(-j \frac{2\pi}{T} nt \right)
\]

and collecting terms in \( \exp(j \frac{2\pi}{T} nt) \) and \( \exp(-j \frac{2\pi}{T} nt) \) gives:

\[
c_n = \frac{a_n}{2} + \frac{b_n}{2j}
\]

\[
c_{-n} = \frac{a_n}{2} - \frac{b_n}{2j}
\]

and those are the answers we want.

We could equally well do this by starting with the exponential form of the Fourier series:

\[
x(t) = \sum_{n=-\infty}^{\infty} c_n \exp\left(j \frac{2\pi}{T} nt \right)
\]

\[
= c_0 + \sum_{n=1}^{-1} c_n \exp\left(j \frac{2\pi}{T} nt \right) + \sum_{n=1}^{\infty} c_n \exp\left(j \frac{2\pi}{T} nt \right)
\]

then replacing index in the first summation \( n \) with \(-n\), we get:
Communications Engineering MSc - Preliminary Reading

\[ x(t) = c_0 + \sum_{n=1}^{\infty} c_{-n} \exp \left( -j \frac{2\pi}{T} nt \right) + \sum_{n=1}^{\infty} c_n \exp \left( j \frac{2\pi}{T} nt \right) \]

and then using the Euler formula \( \exp(x) = \cos(x) + j\sin(x) \), we get:

\[ x(t) = c_0 + \sum_{n=1}^{\infty} c_{-n} \cos \left( -j \frac{2\pi}{T} t \right) + \sum_{n=1}^{\infty} j c_{-n} \sin \left( -j \frac{2\pi}{T} t \right) \]

\[ + \sum_{n=1}^{\infty} c_n \cos \left( j \frac{2\pi}{T} nt \right) + \sum_{n=1}^{\infty} j c_n \sin \left( j \frac{2\pi}{T} nt \right) \]

\[ = c_0 + \sum_{n=1}^{\infty} (c_n + c_{-n}) \cos \left( j \frac{2\pi}{T} nt \right) \]

\[ + j c_n \sin \left( j \frac{2\pi}{T} nt \right) \]

and comparing this to the trigonometric Fourier series:

\[ x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos \left( \frac{2\pi}{T} nt \right) + \sum_{n=1}^{\infty} b_n \sin \left( \frac{2\pi}{T} nt \right) \]

confirms that:

\[ c_0 = a_0 \]
\[ c_n + c_{-n} = a_n \]
\[ j (c_n - c_{-n}) = b_n \]

from which we can easily show that:

\[ c_n = \frac{a_n}{2} + \frac{b_n}{2j} \]
\[ c_{-n} = \frac{a_n}{2} - \frac{b_n}{2j} \]

2) What is the trigonometric Fourier Series of a square wave, where the value is one for the first half of the period \( T \), and zero for the second half?

In general, we have to do three integrations to work out the trigonometric Fourier series, one for \( a_0 \), one for the \( a_n \) and one for the \( b_n \). However, here we can use a short-cut.

Firstly, it’s obvious from the form of the square wave that the average value \( a_0 \) is just 0.5.
Secondly, if you subtract this average value, the square-wave then has odd symmetry:

$$x(t) - 0.5$$

and all odd-symmetric waveforms have sine series, so we know that all the values \(a_n = 0\), we don’t need to do that integral at all. The only integral we have to do is the one to find \(b_n\):

$$b_n = \frac{2}{T} \int_0^{T/2} \sin \left(\frac{2\pi}{T} nt\right) x(t) dt$$

$$= \frac{2}{T} \int_0^{T/2} \sin \left(\frac{2\pi}{T} nt\right) dt$$

$$= \frac{2}{T} \left[ -\frac{\cos \left(\frac{2\pi}{T} nt\right)}{\frac{2\pi}{T} n} \right]_0^{T/2} = \frac{1}{\pi n} \left[ 1 - \cos \left(\pi n \right) \right]$$

You could leave the result like this, or you could carry on to notice that \(1 - \cos \left(\pi n \right)\) is zero when \(n = 2, 4, 6, 8\), or any even number, and two when \(n = 1, 3, 5, 7\) and any odd number, and you could therefore write:

$$b_n = \begin{cases} 
\frac{2}{\pi n} & \text{n even} \\
0 & \text{n odd} 
\end{cases}$$

The trigonometric Fourier series for this square wave is then:

$$x(t) = \frac{1}{2} + \sum_{n=1,3,5,7...}^{\infty} \frac{2}{n\pi} \sin \left(\frac{2\pi}{T} nt\right)$$

or any of a number of other ways of writing the result, for example:
works as well.

3) By considering the Fourier transform and Parseval’s theorem, evaluate:

\[
\frac{4}{a^2} \int_{-\infty}^{\infty} \left| \text{sinc} \left( \frac{\omega}{a} \right) \right|^2 d\omega
\]

This looks like a rather nasty integral to try and evaluate, but it’s got a very simple solution, and as the question suggests, the Fourier transform can help find it. First, consider a signal with a Fourier transform of:

\[
F(\omega) = \frac{2}{a} \text{sinc} \left( \frac{\omega}{a} \right)
\]

Looking at the table of Fourier transforms (or just recognising this one), shows that this is the Fourier transform of a rectangular pulse:

\[
f(t) = \text{rect} \left( \frac{at}{2} \right)
\]

which is a rectangle of width \(2/a\).

Applying Parseval’s theorem to this case shows that:

\[
\int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 d\omega
\]

\[
\int_{-\infty}^{\infty} \left| \text{rect} \left( \frac{at}{2} \right) \right|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left| \frac{2}{a} \text{sinc} \left( \frac{\omega}{a} \right) \right|^2 d\omega
\]

\[
\int_{-\frac{1}{a}}^{1/a} dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} \text{sinc} \left( \frac{\omega}{a} \right)^2 d\omega
\]

\[
\frac{4}{a^2} \int_{-\infty}^{\infty} \left| \text{sinc} \left( \frac{\omega}{a} \right) \right|^2 d\omega = 2\pi \int_{-\frac{1}{a}}^{1/a} dt = 2\pi \frac{2}{a} = \frac{4\pi}{a}
\]

There are other ways to do this integral, but this is a very neat way to get a simple result.

4) Prove that if \(f(t)\) has the Fourier transform \(F(\omega)\), then the Fourier transform of \(f(t - \tau)\) is \(F(\omega) \exp(-j\omega \tau)\).

It’s good practice to try and derive all the expressions in the table of functions of Fourier transforms. For this one, we start with the Fourier transform equation:
\[ F(\omega) = \int_{-\infty}^{\infty} f(t) \exp(-j\omega t) \, dt \]

and then replace \( t \) with \((t - \tau)\). If \( t \) is a constant, the \( d(t - \tau) \) is just \( dt \), and

\[ F(\omega) = \int_{-\infty}^{\infty} f(t-\tau) \exp(-j\omega(t-\tau)) \, dt \]
\[ = \int_{-\infty}^{\infty} f(t-\tau) \exp(-j\omega t) \exp(j\omega\tau) \, dt \]

and since \( \tau \) remains constant for all values of \( t \) we can take it outside the integral:

\[ Y(\omega) = \exp(j\omega\tau) \int_{-\infty}^{\infty} f(t-\tau) \exp(-j\omega t) \, dt = \exp(j\omega\tau) F(\omega) \]

or in words, the Fourier transform of a delayed version of a signal is equal to the Fourier transform of the original signal multiplied by \( \exp(j\omega\tau) \), where \( \tau \) is the delay.

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5) Prove that if \( f(t) \) has the Fourier transform \( F(\omega) \), then the Fourier transform of \( \frac{df(t)}{dt} \) is \( (j\omega) F(\omega) \).

You could start with the Fourier transform:

\[ F(\omega) = \int_{-\infty}^{\infty} f(t) \exp(-j\omega t) \, dt \]

and differentiate with respect to \( t \), but that rapidly gets quite complicated. It’s much easier to start with the inverse Fourier transform:

\[ f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) \exp(j\omega t) \, d\omega \]

and differentiate this:

\[ \frac{df(t)}{dt} = \frac{d}{dt} \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) \exp(j\omega t) \, d\omega \]

The only term on the right-hand side that’s a function of \( t \) is the \( \exp(j\omega t) \), so we can swop the order of differentiation (with respect to \( t \)) and integration (with respect to \( \omega \)), and get:
\[
\frac{df(t)}{dt} = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) \frac{d}{dt}(j\omega t) d\omega
\]

\[
= \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) j\omega \exp(j\omega t) d\omega
\]

\[
= \frac{1}{2\pi} \int_{-\infty}^{\infty} (j\omega F(\omega)) \exp(j\omega t) d\omega
\]

and comparing this to the form of the inverse Fourier transform shows that the Fourier transform of \( \frac{df(t)}{dt} \) is \( j\omega F(\omega) \).

6) Derive the result for the Fourier transform of the rectangular pulse given in table above.

This is just good practice in integration, there’s no trick to this.

\[
F(\omega) = \int_{-\infty}^{\infty} \text{rect}\left(\frac{t}{T}\right) \exp(-j\omega t) dt
\]

First, the rectangle function \( \text{rect}(t / T) \) has a value of one from \( t = -T/2 \) to \( t = +T/2 \), and zero elsewhere, so we can write the integral as:

\[
F(\omega) = \int_{-T/2}^{T/2} \exp(-j\omega t) d\omega
\]

\[
= \left[ \frac{\exp(-j\omega t)}{-j\omega} \right]_{-T/2}^{T/2}
\]

\[
= \frac{j}{\omega} \left[ \exp\left(\frac{-j\omega T}{2}\right) - \exp\left(\frac{j\omega T}{2}\right) \right]
\]

and using the identity \( \sin(x) = 1/2j (\exp(jx) - \exp(-jx)) \) gives:

\[
F(\omega) = \frac{-j}{\omega} \left( 2j \sin\left(\frac{\omega T}{2}\right) \right)
\]

\[
= T \frac{2}{\omega T} \sin\left(\frac{\omega T}{2}\right)
\]

\[
= T \text{sinc}\left(\frac{\omega T}{2}\right)
\]

noting that \( \text{sinc}(x) = \sin(x) / x \). That’s it.