5 What You Should Know About... Series

Infinite and finite series have many applications in engineering; most commonly they are used to develop simple formula for complex situations, either by allowing the sum of a large number of quantities to be written in a simple closed form, or by providing a technique to produce approximate formulas that are faster and easier to use.

5.1 Series, Sequences and Progressions

First, a bit of terminology: a sequence or progression (they both mean the same thing) is a list of numbers in a well-defined order. Usually, each term in a sequence can be calculated in a simple way from the previous terms, or from its position in the sequence, or both.

As a simple example, consider the sequence:

$$3, 4, 5, 6, 7, \ldots$$

The first term is three, and each subsequent term is one more than the previous term. A more interesting example is:

$$1, 1, 2, 3, 5, 8, 13, \ldots$$

where the first two terms are one, and each subsequent term is the sum of the previous two terms\(^1\).

Sequences can either have an infinite number of terms (they are then known as infinite sequences) or a finite number of terms (in which case they are known as finite sequences).

A series is the sum of the terms in a sequence.

5.2 Types of Sequence

In the simplest useful type of sequence, each term is just the previous term plus a constant. Let the first term be \(a\), and the constant offset be \(d\), then the sequence can be written:

$$a, a+d, a+2d, a+3d, a+4d, \ldots$$

for example, if \(a=3\) and \(d=2\), we’d get:

$$3, 5, 7, 9, 11, \ldots$$

This is known as an arithmetic sequence, or sometimes as an arithmetic progression (AP).

Next, consider a sequence in which each term is the previous term multiplied by a constant factor \(r\):

\[\ldots\]

\[3, 5, 7, 9, 11, \ldots\]

\(1\) The numbers in this sequence are known as the Fibonacci numbers, and this sequence has a lot of interesting mathematical properties – although not many find use in engineering.
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\[ a, ar, ar^2, ar^3, ar^4, \ldots \]

for example, if \( a = 3 \) and \( r = 2 \), we’d get:

\[ 3, 6, 12, 24, 48, \ldots \]

This is known as a geometric sequence, or sometimes as a geometric progression (GP).

Now for a more complex example: a sequence in which the \( n^{th} \) term is given by the formula \( nb^n \):

\[ 0, b, 2b^2, 3b^3, 4b^4, \ldots \]

5.3 Summing Arithmetic and Geometric Sequences

One of the most useful things you can do with a sequence is work out the sum of all the terms in the sequence – this is known as a series. Finite sequences always have finite sums (providing all the elements of the sequence are finite – which is usually the case). Infinite sequences may or may not have finite sums: if they do, the series is said be convergent, if not, the series is said to be divergent, and it’s not of much use in engineering other to indicate that the system you are investigating probably won’t work.

5.3.1 The Sum of Terms in an Arithmetic Sequence

In the general form, the sum of the first \( N \) terms of an arithmetic sequence can be written as:

\[
S = \sum_{n=0}^{N-1} a + nd = a + (a+d) + (a+2d) + (a+3d) + \ldots + (a+(N-1)d)
\]

and this can be simplified by writing the summation forwards and then backwards, and then adding up all the corresponding pairs of terms:

\[
S = a + (a+d) + (a+2d) + \ldots + (a+(N-1)d) \\
S = (a+(N-1)d) + (a+(N-2)d) + (a+(N-3)d) + \ldots + a \\
2S = (2a+(N-1)d) + (2a+(N-1)d) + (2a+(N-1)d) + (2a+(N-1)d) + (2a+(N-1)d)
\]

Therefore,

\[
2S = N(2a+(N-1)d) \\
S = N\left(\frac{a+a+(N-1)d}{2}\right)
\]

In other words, the sum of \( N \) terms of an arithmetic sequence is \( N \) times the average value of the first and last terms.
5.3.2 The Sum of Terms in a Geometric Sequence

The geometric sequence has the general form:

\[ S = \sum_{n=1}^{N-1} ar^n = a + ar + ar^2 + \ldots + ar^{N-1} \]

and this sum can be evaluated by multiplying each term by \( r \), and then subtracting from the initial series:

\[ S = a + ar + ar^2 + \ldots + ar^{N-1} \]
\[ rS = ar + ar^2 + \ldots + ar^{N-1} + ar^N \]
\[ (1 - r)S = a + \ldots - ar^N \]

Therefore:

\[ S = \frac{a(1 - r^N)}{1 - r} \]

In the case where \( r < 1 \), the geometric series has a limiting value of the sum when an infinite number of terms are considered:

\[ S = \frac{a}{1 - r} \]

This is a very useful result – it is worth knowing.

5.4 The Taylor Series

Many times in engineering we are interested not in the exact answer to a problem, but to an answer that is close enough, and easy to express and work with. The Taylor series is a very useful method to find approximate expressions for more complex functions, an approximation that is especially accurate around one given value of the function.

The method is to find a polynomial of the form:

\[ y(t) = A + B(t-a) + C(t-a)^2 + D(t-a)^3 + E(t-a)^4 + \ldots \]

that has the same value as the target function at the given point \( a \), as well as the same gradient, and the same gradient of the gradient (the second differential), and the same gradient of the gradient of the gradient (the third differential), etc. \( A, B, C, D, E, \) and so on are constants.

For example, assume we want to find an approximate expression for \( x(t) \) which is accurate around the point \( x = a \). We need to find values of \( A, B, C, \) etc to use so that the value, gradient, second differential etc of \( y(t) \) is the same as that of \( x(t) \). First, the value of the function at \( t = a \) has to be the same as that of \( x(t) \), so:

\[ x(a) = y(a) = A + B(a-a) + C(a-a)^2 + D(a-a)^3 + E(a-a)^4 + \ldots \]
\[ = A \]
That’s the value of \( A \) determined – it’s just equal to the value of the function \( x(t) \) at \( t = a \). Next, we want to ensure that the gradient of the curve of \( y(t) \) is the same as that of \( x(t) \) at this point, and:

\[
\left. \frac{dx(t)}{dt} \right|_{t=a} = \left. \frac{dy(t)}{dt} \right|_{t=a} = B + 2C(a-a) + 3D(a-a)^2 + 4E(a-a)^3 + ...
\]
\[
= B
\]

Continuing to match the second differential, we note that:

\[
\left. \frac{d^2 x(t)}{dt^2} \right|_{t=a} = \left. \frac{d^2 y(t)}{dt^2} \right|_{t=a} = 2C + 6D(a-a) + 12E(a-a)^2 + ...
\]
\[
= 2C
\]

and therefore:

\[
C = \frac{1}{2} \left. \frac{d^2 x(t)}{dt^2} \right|_{a}
\]

Carrying on like this, we get the general formula, known as the Taylor series:

\[
x(t-a) \approx x(a) + \left. \frac{dx(t)}{dt} \right|_{t=a}(t-a) + \frac{1}{2} \left. \frac{d^2 x(t)}{dt^2} \right|_{t=a}(t-a)^2 + \frac{1}{6} \left. \frac{d^3 x(t)}{dt^3} \right|_{t=a}(t-a)^3 + \frac{1}{24} \left. \frac{d^4 x(t)}{dt^4} \right|_{t=a}(t-a)^4 + ...
\]
\[
= \sum_{n=0}^{\infty} \frac{1}{n!} \left. \frac{d^n x(t)}{dt^n} \right|_{t=a}(t-a)^n
\]

For the special case where \( a = 0 \), in other words we want the approximation to be good for very small values of \( t \), the series is known as a Maclaurin series, and the form becomes:

\[
x(t) \approx x(0) + \left. \frac{dx(t)}{dt} \right|_{t=0} t + \frac{1}{2} \left. \frac{d^2 x(t)}{dt^2} \right|_{t=0} t^2 + \frac{1}{6} \left. \frac{d^3 x(t)}{dt^3} \right|_{t=0} t^3 + \frac{1}{24} \left. \frac{d^4 x(t)}{dt^4} \right|_{t=0} t^4 + ...
\]
\[
= \sum_{n=0}^{\infty} \frac{1}{n!} \left. \frac{d^n x(t)}{dt^n} \right|_{t=0} t^n
\]

### 5.4.1 Common Maclaurin Series

It’s rare to have to work out the terms of a Taylor or Maclaurin series, however it is useful to know at least the first two terms of the most common ones, including:

\[
\sin \theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - ...
\]
\[
\cos \theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - ...
\]
(1 + x)^n = 1 + nx + \frac{n(n-1)}{2} x^2 + \frac{n(n-1)(n-2)}{6} x^3 + ...

\frac{1}{1-x} = 1 + x + x^2 + x^3 + ...

\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + ...

e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + ...

This is where the well-known small-angle approximations for the trigonometric functions come from: \( \sin \theta \approx \theta \), \( \tan \theta \approx \theta \), \( \cos \theta \approx 1 - \frac{\theta^2}{2!} \). (Although please note that these formula apply to angles expressed in radians only – they don’t work for angles expressed in degrees.)

Hence, for sufficiently small values of \( x \), we can approximate:

\[ \sin \theta \approx \theta \]
\[ \cos \theta \approx 1 - \frac{\theta^2}{2!} \]
\[ (1 + x)^n \approx 1 + nx \]
\[ \frac{1}{1-x} \approx 1 + x \]
\[ \ln(1 + x) \approx x \]
\[ e^x \approx 1 + x \]

What counts as a sufficiently small value of \( x \)? That depends on how accurate you need your answer to be. If in doubt, calculate the next term of the series, and see if that changes the answer significantly – if not, then it’s a good approximate to make.

**5.5 Problems**

1) An arithmetic sequence has 4 terms, first term 2 and last term 8. What is the sum of all the terms in the sequence?

2) A geometric sequence has 4 terms, first term 2 and last term 8. What is the sum of all the terms in this sequence?

3) A geometric sequence has first term one, an infinite number of terms, and each term is half of the previous term. What is the sum of all the terms in this sequence?

4) What is the sum of the sequence \( a, ar^2, ar^4, ar^6, ar^8, ... \)? For what range of values of \( r \) does this sum converge?
5) Consider the series $\sum_{n=0}^{\infty} na^n$. Find a simple expression for the sum of this series. (Hint – proceed in the same way as for the geometric series: multiply each term by $a$, and then subtract the two series). For what range of values of $a$ does this series have a finite sum?

6) When solving a problem, the formula $y = \ln(1 + x)\sin(x)$ is derived. For small values of $x$, where terms in $x^3$ and higher powers of $x$ can be neglected, find a simple approximate form of this formula.

7) What is the Maclaurin series of $\cos(x)$ if $x$ is expressed in degrees, rather than radians? Hence produce an approximate expression for $\cos(x)$ with $x$ in degrees for small values of $x$.

8) Find an approximation for $\ln(x)$ when $x$ is approximately equal to one.

9) If $x = 0.1$, what is the error (in percent) from using the small signal approximations for $\cos(x)$ and $\sin(x)$ given in the notes above, which neglect terms in $x^3$ and higher powers of $x$?

10) A right-angled triangle has two sides of lengths $a$ and $b$, which are almost the same length, although $b$ is very slightly larger than $a$. Starting with the Pythagoras theorem, derive a series formula for the length of the third side that does not require any square-roots, using the series:

$$
(1 + x)^n = 1 + nx + \frac{n(n-1)}{2} x^2 + \frac{n(n-1)(n-2)}{6} x^3 + \ldots
$$

where $x = -\left(\frac{a}{b}\right)^2$ and $n = -\frac{1}{2}$.

If $b = 21$ and $a = 20$, how accurate is your approximate formula?