

4 Calculus Solutions

1) i) Differentiate with respect to \( x \): \( y = \sin^2(x) \)

This can be done by the “function of a function” rule. Let \( u(z) = z^2 \) and \( z(x) = \sin(x) \), then:

\[
\frac{dy}{dx} = \frac{du}{dz} \frac{dz}{dx} = 2z \cos(x) = 2\sin(x)\cos(x)
\]

and this can be written in a neater form as \( \sin(2x) \).

1) ii) Differentiate with respect to \( x \): \( y = \sin^2(x^2) \)

This is another differentiation of a function of a function – but it’s bit harder, since you have to extend this method: this one is the function of a function of a function. Let \( u(z) = z^2 \) and \( z = \sin(u(x)) \), and \( u(x) = x^2 \), then:

\[
\frac{dy}{dx} = \frac{dy}{dv} \frac{dv}{du} \frac{du}{dx} = 2z \cos(u) \cdot 2x \sin(x^2) \cdot 2x = 2x \sin(2x^2)
\]

1) iii) Differentiate with respect to \( x \): \( y = x^2 \sin(x) \)

This is a differentiation of a product of \( x^2 \) and \( \sin(x) \), and therefore:

\[
\frac{dy}{dx} = 2x \sin(x) + x^2 \cos(x)
\]

1) iv) Differentiate with respect to \( x \): \( y = \frac{x^2}{\sin(x)} \)

This is a differentiation of a quotient of \( x^2 \) and \( \sin(x) \), and therefore:

\[
\frac{dy}{dx} = \frac{\sin(x)2x - x^2 \cos(x)}{\sin^2(x)} = \frac{2x}{\sin(x)} - \frac{x^2 \cos(x)}{\sin^2(x)}
\]

2) i) Integrate with respect to \( x \): \( y = \sin(x)\cos(x) \)

Substitute \( t = \sin(x) \), then \( \frac{dt}{dx} = \cos(x) \) so \( dt = \cos(x)dx \), and we have:

\[
\int \sin(x)\cos(x) \, dx = \int t \, dt = \frac{t^2}{2} + C = \frac{\sin^2(x)}{2} + C
\]

(Although it’s probably easier to note that \( y = \sin(x)\cos(x) = \sin(2x)/2 \), and integrate that.)

2) ii) Integrate with respect to \( x \): \( y = x^2 \sin(x) \)
This one can be done by parts, but we’ll have to use the integration by parts formula twice. First, let \( u(x) = x^2 \) and \( dv/dx = \sin(x) \), so that \( v(x) = -\cos(x) \). Then:

\[
\int x^2 \sin(x) \, dx = -x^2 \cos(x) + \int 2x \cos(x) \, dx
\]

and we can do the second integral by a further integration by parts, this time letting \( u(x) = x \) and \( dv/dx = \cos(x) \), so that \( v(x) = \sin(x) \):

\[
\int x^2 \sin(x) \, dx = -x^2 \cos(x) + 2x \sin(x) - \int 2 \sin(x) \, dx
\]

\[
= -x^2 \cos(x) + 2x \sin(x) + 2 \cos(x) + C
\]

3) Prove that

\[
\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \left( \frac{x}{a} \right) + C , \text{ using the substitution } t = \sin^{-1} \left( \frac{x}{a} \right). 
\]

Using this substitution \( x = a \sin(t) \), we can see that \( \frac{dx}{dt} = a \cos(t) \), and that:

\[
\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \int \frac{1}{\sqrt{a^2 - a^2 \sin^2(t)}} a \cos(t) \, dt
\]

\[
= \int \frac{1}{\sqrt{1 - \sin^2(t)}} \cos(t) \, dt
\]

\[
= \int \frac{1}{\sqrt{\cos^2(t)}} \cos(t) \, dt
\]

\[
= \int dt
\]

\[
= t
\]

which leads to the very simple result that:

\[
\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = t + C = \sin^{-1} \left( \frac{x}{a} \right) + C
\]

4) Differentiate \( y = \sin(x) \exp(-x) \).

A simpler problem – this is just another differentiation of a product:

\[
\frac{dy}{dx} = \cos(x) \exp(-x) - \sin(x) \exp(-x) = \exp(-x) \left( \cos(x) - \sin(x) \right)
\]

5) A harder one: now try to integrate \( y = \sin(x) \exp(-x) \). It can be done using integration by parts, but there’s a trick to it as well.
First, do an integration of parts, with \( u(x) = \exp(-x) \) and \( dv/dx = \sin(x) \), so that \( v(x) = -\cos(x) \). Then:

\[
\int \exp(-x) \sin(x) \, dx = -\exp(-x) \cos(x) - \int \exp(-x) \cos(x) \, dx
\]

and then another integration by parts, this time with \( u(x) = \exp(-x) \) and \( dv/dx = \cos(x) \), so that \( v(x) = \sin(x) \). This gives:

\[
\int \exp(-x) \sin(x) \, dx = -\exp(-x) \cos(x) - \int \exp(-x) \cos(x) \, dx
\]

\[
= -\exp(-x) \cos(x) - \exp(-x) \sin(x) - \int \exp(-x) \sin(x) \, dx
\]

Now the trick is to notice that the last term on the right-hand side is the same as the integral we were originally trying to solve, and so we move this over to the left-hand side, and write:

\[
2 \int \exp(-x) \sin(x) \, dx = -\exp(-x) \cos(x) - \exp(-x) \sin(x) + C
\]

\[
\int \exp(-x) \sin(x) \, dx = \frac{-\exp(-x)}{2} (\cos(x) + \sin(x)) + C'
\]

6) Evaluate: \( \int_0^1 x \exp\left(-x^2\right) \, dx \)

A definite integral. It can be readily solved by an integration by substitution using \( t = x^2 \), but the important thing is to remember about the limits. Remember that when integrating with respect to \( x \), the limits of the integration are values of \( x \) – and when integrating with respect to \( t \), they may have to be changed to the corresponding values of \( t \).

So:

\[
\int_0^a x \exp\left(-x^2\right) \, dx = \int_{x=0}^{x=a} x \exp\left(-x^2\right) \, dx
\]

and substituting for \( t \):

\[
\frac{1}{2} \int_{t=0}^{t=a} \exp(-t) \, dt = \frac{1}{2} \left[ \int_{t=0}^{t=a} \exp(-t) \, dt = \frac{1}{2} \left[ -\exp(-t) \right]_0^a \right.
\]

\[
= \frac{1}{2} \left(-\exp(-a^2) + 1\right)
\]

\[
= \frac{1}{2} \left(1 - \exp(-a^2)\right)
\]

so, if \( a = 1 \), this integral = \( \frac{1}{2} \left(1 - \exp(-1)\right) \) but when \( a = 2 \), this integral = \( \frac{1}{2} \left(1 - \exp(-4)\right) \).