Optimal Allocations with CARA and CRRA

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1 Introduction

In this paper we have assumed two particular forms for the utility function of the subjects: *1)* a Constant Absolute Risk Aversion (CARA) form and *2)* a Constant Relative Risk Aversion (CRRA) form.

1) We took this to be the CARA form:

$$\begin{aligned} u(x) &= \frac{1-\exp(-rx)}{1-\exp(-75r)} \text{ if } r \neq 0 \\ &= \frac{x}{75} \text{ if } r = 0 \end{aligned}$$

In this case we maximise a function of the form

$$w_j u(e_j x_j) + w_k u(e_k x_k)$$

subject to the constraint that $x_j + x_k = m$. Given the CARA form the general optimal allocations are

$$\begin{aligned} x_j^* &= \frac{e_k m + \{\ln[(w_j e_j)/(w_k e_k)]\}/r}{e_j + e_k} \\ x_k^* &= \frac{e_j m + \{\ln[(w_k e_k)/(w_j e_j)]\}/r}{e_j + e_k} \end{aligned}$$

We note that there is no guarantee that the x's are positive and less than m. In the experiment subjects were constrained to have all allocations non-negative and we took that into account in the estimation.

2)We took this to be the CRRA form:

$$\begin{aligned} u(x) &= \frac{x^{1-1/r}-1}{1-1/r} \text{ if } r \neq 1 \\ &= \ln(x) \text{ if } r = 1 \end{aligned}$$

In this case we maximise a function of the form

$$w_1 u(e_1 x_1) + w_2 u(e_2 x_2)$$

subject to the constraint that $x_1 + x_2 = 1$. Given the CRRA form the general optimal allocations are

$$x_j^* = \frac{q_j}{q_j + q_k}$$

where $q_i = e_i^{r-1} w_i^r$ for $i = 1, 2$.

In the following sections we will provide a description of the optimal allocations for the different preference functionals in both utility forms.

2 Experimental Design

In the experiment there are two types of problem: problems of Type 1 and problems of Type 2. In problems of Type 1 subjects were asked to allocate tokens between 2 colours (with the exchange rate on the third being zero). In problems of Type 2 subjects were asked to allocate tokens between one colour and the other two.

Specifically:

Problem type 11: allocation between 2 and 3

Problem type 12: allocation between 1 and 3

Problem type 13: allocation between 1 and 2

Problem type 21: allocation between 1 and (2 and 3)

Problem type 22: allocation between 2 and (1 and 3)

Problem type 23: allocation between 3 and (1 and 2)

3 Optimisations with CARA function

3.1 Optimal allocations with SEU subjects

In EU the ordering of the outcomes does not matter.

3.1.1 Type 1 problems

We need some notation. Let us say that in Problem type $1i$ the choice is between colours j and k. We note:

i	1	2	3
j	2	3	1
k	3	1	2

Then the allocation is between colours j and k. If colour i comes up the subject receives nothing. So the problem is to choose x_j and x_k to maximise $p_j u(e_j x_j) + p_k u(e_k x_k)$ st $x_j + x_k = m$

From the general results above we have:

$$\begin{aligned} x_j^* &= \frac{e_k m + \{\ln[(p_j e_j)/(p_k e_k)]\}/r}{e_j + e_k} \\ x_k^* &= \frac{e_j m + \{\ln[(p_k e_k)/(p_j e_j)]\}/r}{e_j + e_k} \end{aligned} \tag{1}$$

Here the p's are simply the probabilities of the three colours and e's are the unordered exchange rates.

3.1.2 Type 2 Problems

In Problem Type $2i$, the choice is between i and not-i, the subject allocates x_i to colour i and X_i to not-i, then if colour i is drawn the subject receives $e_i x_i$ whereas if the colour drawn is not-i then the subject receives $E_i X_i$. Here E_i denotes the exchange rate between not-i and money.

Using the above results we have that in the Problem type $2i$:

$$\begin{aligned} x_i^* &= \frac{E_i m + \{\ln[(p_i e_i)/(P_i E_i)]\}/r}{e_i + E_i} \\ X_i^* &= \frac{e_i m + \{\ln[(P_i E_i)/(p_i e_i)]\}/r}{e_i + E_i} \end{aligned} \tag{2}$$

where E_i is the exchange rate between allocations to not-i and money and where $P_1 = p_2 + p_3$, $P_2 = p_3 + p_1$, and $P_3 = p_1 + p_2$.

3.2 Optimal allocations with CEU subjects

For CEU subjects the order matters. A CEU subject is defined by six capacities. Let us denote these by the variables v and V as follows. v_1 is the capacity on colour 1, v_2 is the capacity on colour 2, v_3 is the capacity on colour 3; V_1 is the capacity on colours 2 and 3 combined, V_2 is the capacity on colours 1 and 3 combined, V_3 is the capacity on colours 1 and 2 combined.

3.2.1 Type 1 problems

Using the CEU formulation it follows that the v's in the objective function equation **??** are defined as follows:

Problem type	i	j	k	weight on j if $e_jx_j > e_kx_k$	weight on k if $e_jx_j > e_kx_k$
11	1	2	3	v_2	$V_1 - v_2$
12	2	3	1	v_3	$V_2 - v_3$
13	3	1	2	v_1	$V_3 - v_1$
1i	i	j	k	v_j	$V_i - v_j$
Problem type	i	j	k	weight on j if $e_jx_j < e_kx_k$	weight on k if $e_jx_j < e_kx_k$
11	1	2	3	$V_1 - v_3$	v_3
12	2	3	1	$V_2 - v_1$	v_1
13	3	1	2	$V_3 - v_2$	v_2
1i	i	j	k	$V_i - v_k$	v_k

We need to consider three possibilities.

Possibility 1: $e_jx_j > e_kx_k$ We apply the general result.

We have:

$$x_j^* = \frac{e_km + \{\ln[(v_je_j)/((V_i - v_j)e_k)]\}/r}{e_j + e_k} \tag{3}$$

$$x_k^* = \frac{e_jm + \{\ln[((V_i - v_j)e_k)/(v_je_j)]\}/r}{e_j + e_k} \tag{4}$$

We note that the condition for this possibility to be satisfied is that $v_je_j > (V_i - v_j)e_k$.

Possibility 2: $e_j x_j < e_k x_k$ We have:

$$x_j^* = \frac{e_k m + \{\ln[(V_i - v_k)e_j)/(v_k e_k)]\}/r}{e_j + e_k}$$
$$x_k^* = \frac{e_j m + \{\ln[(v_k e_k)/((V_i - v_k)e_j)]\}/r}{e_j + e_k}$$

We note that the condition for this possibility to be satisfied is that $(V_i - v_k)e_j > v_k e_k$. Note that if Possibility 1 and Possibility 2 are both possible, we still need to check which gives the highest utility.

Possibility 3: $e_j x_j = e_k x_k$ We must have

$$x_j^* = \frac{e_k m}{e_j + e_k} \text{ and } x_k^* = \frac{e_j m}{e_j + e_k}$$

Note that this solution in always admissible.

3.2.2 Type 2 problems

Let us consider the general Problem type 2*i*, that is, allocations between *i* and not-*i*:

There are three possibilites:

Possibility 1: $e_i x_i > E_i(1 - x_i) = E_i X_i$

Possibility 2: $e_i x_i < E_i(1 - x_i) = E_i X_i$

Possibility 3: $e_i x_i = E_i(1 - x_i) = E_i X_i$

Here the notation E_i means the exchange rate on not-*i*.

Problem type	i	not-i	weight on i if $e_i x_i > E_i X_i$	weight on not-i if $e_i x_i > E_i X_i$
21	1	2 and 3	v_1	$1 - v_1$
22	2	3 and 1	v_2	$1 - v_2$
23	3	1 and 2	v_3	$1 - v_3$
2i	i	j and k	v_i	$1 - v_i$

Problem type	i	not-i	weight on i if $e_i x_i < E_i X_i$	weight on not-i if $e_i x_i < E_i X_i$
21	1	2 and 3	$1 - V_1$	V_1
22	2	3 and 1	$1 - V_2$	V_2
23	3	1 and 2	$1 - V_3$	V_3
2i	i	j and k	$1 - V_i$	V_i

We need to consider the 3 possibilites:

Possibility 1: $e_i x_i > E_i X_i$ This has strict inequalities and we can apply general results.

We have:

$$x_i^* = \frac{E_i m + \{\ln[(v_i e_i)/((1-v_i)E_i)]\}/r}{e_i + E_i}$$
$$X_i^* = \frac{e_i m + \{\ln[(1-v_i)E_i)/(v_i e_i)]\}/r}{e_i + E_i}$$

We note that the condition for this possibility to be satisfied is that $v_i e_i > (1 - v_i)E_i$.

Possibility 2: $e_i x_i < E_i x_i$ We have:

$$x_i^* = \frac{E_i m + \{\ln[((1-V_i)e_i)/(V_i E_i)]\}/r}{e_i + E_i}$$
$$X_i^* = \frac{e_i m + \{\ln[(V_i E_i)/((1-V_i)e_i)]\}/r}{e_i + E_i}$$

We note that the condition for this possibility to be satisfied is that $(1 - V_i)e_i > V_i E_i$.

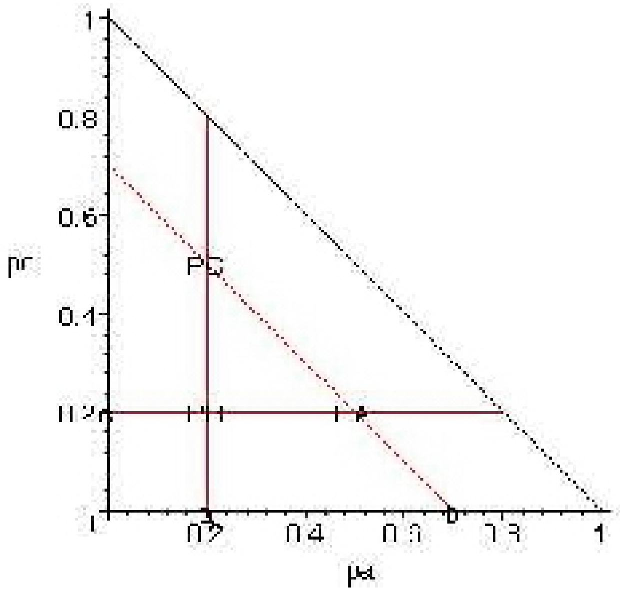
Possibility 3: $e_i x_i = E_i X_i$ We have:

$$x_i^* = \frac{E_i m}{e_i + E_i} \text{ and } X_i^* = \frac{e_i m}{e_i + E_i}$$

Note that this solution in always admissible.

3.3 Optimal allocations with AEU subjects

Suppose now that the subject is AEU maximizer. This is defined by three probability bounds and the alpha parameter.Let us define the bounds on the convex set of possible



probabilites by v_1, v_2, v_3. These three numbers characterise the model. Assume that they add up to less than 1 (if they add up to 1 then AEU reduces to SEU). They bound a triangular area in the Marshack-Machina-Triangle.

As in the other cases the objective function is given by **??**. The crucial point is the values of the weights. Using our standard notation, where the ordered v's we have

$$\begin{aligned} AEU &= \alpha[w_1 u(e_1 x_1) + w_2 u(e_2 x_2) + (1 - w_1 - w_2) u(e_3 x_3)] + \\ &\quad (1-\alpha)[(1 - w_2 - w_3) u(e_1 x_1) + w_2 u(e_2 x_2) + w_3 u(e_3 x_3)] \end{aligned} \tag{5}$$

This can be written as

$$\begin{aligned} AEU &= [\alpha w_1 + (1-\alpha)(1 - w_2 - w_3)] u(e_1 x_1) + w_2 u(e_2 x_2) + \\ &\quad [\alpha(1 - w_1 - w_2) + (1-\alpha) w_3] u(e_3 x_3)] \end{aligned} \tag{6}$$

We note that this is exactly like the SEU case but with probabilities $[\alpha w_1 + (1-\alpha)(1 - w_2 - w_3)], w_2$ and $[\alpha(1 - w_1 - w_2) + (1-\alpha) w_3]$ on the three outcomes. Note that these add to 1, so we can apply our standard results. But note the idiosyncracy of AEU:

these 'probabilities' depend upon the ordering. Following the same notation as the CEU case, we have:

$$\varpi_1 = \alpha w_1 + (1-\alpha)(1-w_2-w_3) \quad (7)$$
$$\varpi_2 = w_2 \quad (8)$$
$$\varpi_3 = \alpha(1-w_1-w_2) + (1-\alpha)w_3$$

We need to consider all the possible cases.

3.3.1 Type 1 problems

To save some writing let us introduce the notation V_i to refer to the sum of the v's for not-i. That is, $V_1 = v_2 + v_3$, $V_2 = v_1 + v_3$ and $V_3 = v_1 + v_2$. Or more generally $V_i = v_j + v_k$.

If problem type is $1i$ then

if $e_j x_j > e_k x_k$ weight on x_j is $\alpha v_j + (1-\alpha)(1-V_j)$ and weight on x_k is v_k

if $e_j x_j < e_k x_k$ weight on x_j is v_j and weight on x_k is $\alpha v_k + (1-\alpha)(1-V_k)$

Problem type	i	j	k	weight on j if $e_j x_j > e_k x_k$	weight on k if $e_j x_j > e_k x_k$
11	1	2	3	$\alpha v_2 + (1-\alpha)(1-V_2)$	v_3
12	2	3	1	$\alpha v_3 + (1-\alpha)(1-V_3)$	v_1
13	3	1	2	$\alpha v_1 + (1-\alpha)(1-V_1)$	v_2
$1i$	i	j	k	$\alpha v_j + (1-\alpha)(1-V_j)$	v_k

Problem type	i	j	k	weight on j if $e_j x_j < e_k x_k$	weight on k if $e_j x_j < e_k x_k$
11	1	2	3	v_2	$\alpha v_3 + (1-\alpha)(1-V_3)$
12	2	3	1	v_3	$\alpha v_1 + (1-\alpha)(1-V_1)$
13	3	1	2	v_1	$\alpha v_2 + (1-\alpha)(1-V_2)$
$1i$	i	j	k	v_j	$\alpha v_k + (1-\alpha)(1-V_k)$

We need to consider the three possibilities.

Possibility 1: $e_j x_j > e_k x_k$ We have:

$$x_j^* = \frac{e_k m + \{\ln[((\alpha v_j + (1-\alpha)(1-V_j))e_j)/(v_k e_k)]\}/r}{e_j + e_k} \tag{9}$$

$$x_k^* = \frac{e_j m + \{\ln[(v_k e_k)/((\alpha v_j + (1-\alpha)(1-V_j))e_j)]\}/r}{e_j + e_k} \tag{10}$$

We note that the condition for this possibility to be satisfied is that $e_j[\alpha v_j + (1-\alpha)(1-V_j)] > e_k v_k$.

Possibility 2: $e_j x_j < e_k x_k$ We have:

$$x_j^* = \frac{e_k m + \{\ln[(v_j e_j)/((\alpha v_k + (1-\alpha)(1-V_k))e_k)]\}/r}{e_j + e_k} \tag{11}$$

$$x_k^* = \frac{e_j m + \{\ln[((\alpha v_k + (1-\alpha)(1-V_k))e_k)/(v_j e_j)]\}/r}{e_j + e_k} \tag{12}$$

We note that the condition for this possibility to be satisfied is that $e_k[\alpha v_k + (1-\alpha)(1-V_k)] > e_j v_j$. Again it does not appear that this can be simplified.

Possibility 3: $e_j x_j = e_k x_k$ We must have

$$x_j^* = \frac{e_k m}{e_j + e_k} \text{ and } x_k^* = \frac{e_j m}{e_j + e_k}$$

Note that this solution in always admissible.

3.3.2 Type 2 problems

Let us consider the general case i, that is, allocations between i and not-i.

Problem type	i	not-i	weight on i if $e_i x_i > E_i X_i$	weight on not-i if $e_i x_i > E_i X_i$
21	1	2 and 3	$\alpha v_1 + (1-\alpha)(1-V_1)$	$\alpha(1-v_1) + (1-\alpha)V_1$
22	2	3 and 1	$\alpha v_2 + (1-\alpha)(1-V_2)$	$\alpha(1-v_2) + (1-\alpha)V_2$
23	3	1 and 2	$\alpha v_3 + (1-\alpha)(1-V_3)$	$\alpha(1-v_3) + (1-\alpha)V_3$
2i	i	j and k	$\alpha v_i + (1-\alpha)(1-V_i)$	$\alpha(1-v_i) + (1-\alpha)V_i$

Problem type	i	not-i	weight on i if $e_i x_i < E_i X_i$	weight on not-i if $e_i x_i < E_i X_i$
21	1	2 and 3	$\alpha(1-V_1)+(1-\alpha)v_1$	$\alpha V_1+(1-\alpha)(1-v_1)$
22	2	3 and 1	$\alpha(1-V_2)+(1-\alpha)v_2$	$\alpha V_2+(1-\alpha)(1-v_2)$
23	3	1 and 2	$\alpha(1-V_3)+(1-\alpha)v_3$	$\alpha V_3+(1-\alpha)(1-v_3)$
2i	i	j and k	$\alpha(1-V_i)+(1-\alpha)v_i$	$\alpha V_i+(1-\alpha)(1-v_i)$

We need to consider the three possibilities.

Possibility 1: $e_i x_i > E_i X_i$**.** Here the weights on x_i is $\alpha v_i + (1-\alpha)(1-V_i)$ and the weight on X_i is $\alpha(1-v_i)+(1-\alpha)V_i$.

In this case, the optimal allocations are

$$
\begin{aligned}
x_i^* &= \frac{E_i m + \{\ln[((\alpha v_i + (1-\alpha)(1-V_i))e_i)/(\alpha(1-v_i)+(1-\alpha)V_i)E_i)]\}/r}{e_i+E_i} \\
X_i^* &= \frac{e_i m + \{\ln[(\alpha(1-v_i)+(1-\alpha)V_i)E_i)/((\alpha v_i + (1-\alpha)(1-V_i))e_i)]\}/r}{e_i+E_i}
\end{aligned}
$$

We need that $e_i x_i^* > E_i(1-x_i^*)$. This gives us the condition that $[\alpha v_i + (1-\alpha)(1-V_i)]e_i > [\alpha(1-v_i)+(1-\alpha)V_i]E_i$.

Possibility 2: $e_i x_i < E_i X_i$ Here the weight on x_i is $\alpha(1-V_i)+(1-\alpha)v_i$ and the weight on X_i is $\alpha V_i + (1-\alpha)(1-v_i)$.

In this case, the optimal allocations are

$$
\begin{aligned}
x_i^* &= \frac{E_i m + \{\ln[((\alpha(1-V_i)+(1-\alpha)v_i)e_i)/((\alpha V_i + (1-\alpha)(1-v_i))E_i)]\}/r}{e_i+E_i} \\
X_i^* &= \frac{e_i m + \{\ln[((\alpha V_i + (1-\alpha)(1-v_i))E_i)/((\alpha(1-V_i)+(1-\alpha)v_i)e_i)]\}/r}{e_i+E_i}
\end{aligned}
$$

Following the logic as above we need that $[\alpha V_i + (1-\alpha)(1-v_i)]E_i > [\alpha(1-V_i)+(1-\alpha)v_i]e_i$.

Possibility 3: $e_i x_i = E_i X_i$ We must have

$$x_i^* = \frac{E_i m}{e_i + E_i} \text{ and } X_i^* = \frac{e_i m}{e_i + E_i}$$

Note that this solution in always admissible.

3.4 Optimal allocations with VEU subjects

Suppose now that the subject is VEU maximizer. Such a subject is defined by three "adjusted" probabilities that incorporate the ambiguity about the relative number of pairs of colours (i.e., ambiguity about the relative number of colour 1 versus colour 2 balls and the ambiguity about the relative number of colour 2 versus colour 3 balls). These adjusted probabilities are defined as the baseline prior probability plus or minus the adjustment for ambiguity.

Let us denote the baseline prior probabilities by v_i as follows: v_1 is the baseline probability on colour 1, v_2 is the baseline probability on colour 2, v_3 is the baseline probability on colour 3. We define by w_i the corresponding *ordered* baseline prior probabilities. So we have

$v_i = w_{bac(c,i)}$ for $i = 1, 2, 3$ or $w_i = v_{ord(c,i)}$ for $i = 1, 2, 3$ and $c = 1, ..., 6$.

Let us write eq. (**??**) in terms of the ordered baseline prior probabilities

$$VEU = w_1 u(e_1 x_1) + w_2 u(e_2 x_2) + w_3 u(e_3 x_3) - \delta(|u(e_1 x_1) - u(e_2 x_2)| + |u(e_2 x_2) - u(e_3 x_3)|) \tag{13}$$

Since we are considering an ordering, we can ignore the modulus. The (35) becomes

$$VEU = (w_1 - \delta) u(e_1 x_1) + w_2 u(e_2 x_2) + (w_3 + \delta) u(e_3 x_3) \tag{14}$$

Now we can easily define the "adjusted" probabilities and, for analogy to the AEU case, we define them by ϖ.

$$\begin{aligned} \varpi_1 &= w_1 - \delta \\ \varpi_2 &= w_2 \\ \varpi_3 &= w_3 + \delta \end{aligned} \tag{15}$$

Note that both the baseline prior probabilities and the "adjusted" probabilities sum up to one.

3.4.1 Type 1 problems

We follow the same notation of AEU. We refer to V_i as the sum of the v's for not-i. That is, $V_1 = v_2 + v_3$, $V_2 = v_1 + v_3$ and $V_3 = v_1 + v_2$. Or more generally $V_i = v_j + v_k$. If problem type is $1i$ then

Problem type	i	j	k	weight on j if $e_jx_j > e_kx_k$	weight on k if $e_jx_j > e_kx_k$
11	1	2	3	$v_2 - \delta$	v_3
12	2	3	1	$v_3 - \delta$	v_1
13	3	1	2	$v_1 - \delta$	v_2
$1i$	i	j	k	$v_j - \delta$	v_k

Problem type	i	j	k	weight on j if $e_jx_j < e_kx_k$	weight on k if $e_jx_j < e_kx_k$
11	1	2	3	v_2	$v_3 - \delta$
12	2	3	1	v_3	$v_1 - \delta$
13	3	1	2	v_1	$v_2 - \delta$
$1i$	i	j	k	v_j	$v_k - \delta$

We need to consider the three possibilities.

Possibility 1: $e_jx_j > e_kx_k$ We have

$$x_j^* = \frac{e_km + \{\ln[((v_j - \delta)e_j)/(v_ke_k)]\}/r}{e_j + e_k} \tag{16}$$

$$x_k^* = \frac{e_jm + \{\ln[(v_ke_k)/((v_j - \delta)e_j)]\}/r}{e_j + e_k} \tag{17}$$

We note that the condition for this possibility to be satisfied is that $e_j(v_j - \delta) > e_kv_k$.

Possibility 2: $e_jx_j < e_kx_k$ We have

$$x_j^* = \frac{e_km + \{\ln[(v_je_j)/((v_k - \delta)e_k)]\}/r}{e_j + e_k} \tag{18}$$

$$x_k^* = \frac{e_jm + \{\ln[((v_k - \delta)e_k)/(v_je_j)]\}/r}{e_j + e_k} \tag{19}$$

We note that the condition for this possibility to be satisfied is that $e_k(v_k - \delta) > e_j v_j$.

Possibility 3: $e_j x_j = e_k x_k$ We have

$$x_i^* = \frac{E_i m}{e_i + E_i} \text{ and } x_k^* = \frac{e_i m}{e_i + E_i}$$

Note that this solution in always admissible.

3.4.2 Type 2 problems

Let us consider the general case i, that is, allocations between i and not-i:

There are three possibilities:

Possibilty 1: $e_i x_i > E_i(1 - x_i) = E_i X_i$

Possibility 2: $e_i x_i < E_i(1 - x_i) = E_i X_i$

Possibility 3: $e_i x_i = E_i(1 - x_i) = E_i X_i$

Problem type	i	not-i	weight on i if $e_i x_i > E_i X_i$	weight on not-i if $e_i x_i > E_i X_i$
21	1	2 and 3	$\nu_1 - \delta$	$1 - \nu_1 + \delta$
22	2	3 and 1	$\nu_2 - \delta$	$1 - \nu_2 + \delta$
23	3	1 and 2	$\nu_3 - \delta$	$1 - \nu_3 + \delta$
2i	i	j and k	$\nu_i - \delta$	$1 - \nu_i + \delta$

Problem type	i	not-i	weight on i if $e_i x_i < E_i X_i$	weight on not-i if $e_i x_i < E_i X_i$
21	1	2 and 3	$1 - V_1 + \delta$	$V_1 - \delta$
22	2	3 and 1	$1 - V_2 + \delta$	$V_2 - \delta$
23	3	1 and 2	$1 - V_3 + \delta$	$V_3 - \delta$
2i	i	j and k	$1 - V_i + \delta$	$V_i - \delta$

We need to consider the three possibilities.

Possibility 1: $e_i x_i > E_i X_i$ Here the weights on x_i is $(\nu_i - \delta)$ and the weight on X_i is $(1 - \nu_i + \delta)$

The optimal allocations are:

$$\begin{aligned} x_i^* &= \frac{E_i m + \{\ln[((\nu_i - \delta)e_i)/((1-\nu_i+\delta)E_i)]\}/r}{e_i + E_i} \\ X_i^* &= \frac{e_i m + \{\ln[((1-\nu_i+\delta)E_i)/((\nu_i - \delta)e_i)]\}/r}{e_i + E_i} \end{aligned}$$

We note that the condition for this possibility to be satisfied is that $(\nu_i - \delta)e_i > (1 - \nu_i + \delta)E_i$.

Possibility 2: $e_i x_i < E_i X_i$ Here the weights on x_i is $(1 - V_i + \delta)$ and the weight on X_i is $(V_i - \delta)$

The optimal allocations are:

$$\begin{aligned} x_i^* &= \frac{E_i m + \{\ln[((1-V_i+\delta)e_i)/((V_i - \delta)E_i)]\}/r}{e_i + E_i} \\ X_i^* &= \frac{e_i m + \{\ln[((V_i - \delta)E_i)/((1-V_i+\delta)e_i)]\}/r}{e_i + E_i} \end{aligned}$$

We note that the condition for this possibility to be satisfied is that $(1 - V_i + \delta)e_i < (V_i - \delta)E_i$.

Possibility 3: $e_i x_i = E_i X_i$ We have

$$x_i^* = \frac{E_i m}{e_i + E_i} \text{ and } X_i^* = \frac{e_i m}{e_i + E_i}$$

Note that this solution in always admissible.

3.5 Optimal allocations with Contraction Model

Suppose now that a subject has preferences described by the Contraction Model. The preference functional depends crucially on the ordering between $u(e_1 x_1)$, $u(e_2 x_2)$, and $u(e_3 x_3)$.

Suppose that $u(e_1x_1) \geq u(e_2x_2) \geq u(e_3x_3)$. We have that

$$\begin{aligned} COM &= \alpha[\underline{p}_1 u(e_1x_1) + \underline{p}_2 u(e_2x_2) + (1-\underline{p}_1-\underline{p}_2)u_3] + (1-\alpha)[(\underline{p}_1 + (1-\underline{p}_1-\underline{p}_2-\underline{p}_3)/3)u(e_1x_1) \\ & (\underline{p}_2 + (1-\underline{p}_1-\underline{p}_2-\underline{p}_3)/3)u(e_2x_2) + (\underline{p}_3 + (1-\underline{p}_1-\underline{p}_2-\underline{p}_3)/3)u(e_3x_3)] \end{aligned}$$

This looks very similar to SEU with probabilities/weights which depend on the bounds and alpha and which is the bigger outcome.

This can be written as

$$\begin{aligned} COM &= [\alpha\underline{p}_1 + (1-\alpha)(\underline{p}_1 + (1-\underline{p}_1-\underline{p}_2-\underline{p}_3)/3)]u(e_1x_1) + \\ & [\alpha\underline{p}_2 + (1-\alpha)(\underline{p}_2 + (1-\underline{p}_1-\underline{p}_2-\underline{p}_3)/3)]u(e_2x_2) + \\ & [\alpha(1-\underline{p}_1-\underline{p}_2) + (1-\alpha)(\underline{p}_3 + (1-\underline{p}_1-\underline{p}_2-\underline{p}_3)/3)]u(e_3x_3) \end{aligned} \tag{20}$$

The probabilities on the three outcomes are $[\alpha\underline{p}_1 + (1-\alpha)(\underline{p}_1 + (1-\underline{p}_1-\underline{p}_2-\underline{p}_3)/3)]$, $[\alpha\underline{p}_2 + (1-\alpha)(\underline{p}_2 + (1-\underline{p}_1-\underline{p}_2-\underline{p}_3)/3)]$ and $[\alpha(1-\underline{p}_1-\underline{p}_2) + (1-\alpha)(\underline{p}_2 + (1-\underline{p}_1-\underline{p}_2-\underline{p}_3)/3)]$. Note that these add to 1, so we can apply our standard results. But note that these 'probabilities' depend upon the ordering. Following the same notation as the CEU and AEU case, we have:

$$\varpi_1 = \alpha\underline{p}_1 + (1-\alpha)(\underline{p}_1 + (1-\underline{p}_1-\underline{p}_2-\underline{p}_3)/3) \tag{21}$$

$$\varpi_2 = \alpha\underline{p}_2 + (1-\alpha)(\underline{p}_2 + (1-\underline{p}_1-\underline{p}_2-\underline{p}_3)/3) \tag{22}$$

$$\varpi_3 = \alpha(1-\underline{p}_1-\underline{p}_2) + (1-\alpha)(\underline{p}_3 + (1-\underline{p}_1-\underline{p}_2-\underline{p}_3)/3)$$

Again we need to consider all the possible cases.

3.5.1 Type 1 problems

If problem type is $1i$ then

if $e_jx_j > e_kx_k$ weight on x_j is and weight on x_k is

if $e_jx_j < e_kx_k$ weight on x_j is and weight on x_k is.

Problem type	i	j	k	weight on j if $e_jx_j > e_kx_k$	weight on k if $e_jx_j > e_kx_k$
11	1	2	3	$\alpha\underline{p}_2 + (1-\alpha)\left(\frac{1+2\underline{p}_2-\underline{p}_3}{3}\right)$	$\alpha(1-\underline{p}_2) + (1-\alpha)\left(\frac{1+2\underline{p}_3-\underline{p}_2}{3}\right)$
12	2	3	1	$\alpha\underline{p}_3 + (1-\alpha)\left(\frac{1+2\underline{p}_3-\underline{p}_1}{3}\right)$	$\alpha(1-\underline{p}_3) + (1-\alpha)\left(\frac{1+2\underline{p}_1-\underline{p}_3}{3}\right)$
13	3	1	2	$\alpha\underline{p}_1 + (1-\alpha)\left(\frac{1+2\underline{p}_1-\underline{p}_2}{3}\right)$	$\alpha(1-\underline{p}_1) + (1-\alpha)\left(\frac{1+2\underline{p}_2-\underline{p}_1}{3}\right)$
1i	i	j	k	$\alpha\underline{p}_j + (1-\alpha)\left(\frac{1+2\underline{p}_j-\underline{p}_k}{3}\right)$	$\alpha(1-\underline{p}_j) + (1-\alpha)\left(\frac{1+2\underline{p}_k-\underline{p}_j}{3}\right)$

Problem type	i	j	k	weight on j if $e_jx_j < e_kx_k$	weight on k if $e_jx_j < e_kx_k$
11	1	2	3	$\alpha(1-\underline{p}_3) + (1-\alpha)\left(\frac{1+2\underline{p}_2-\underline{p}_3}{3}\right)$	$\alpha\underline{p}_3 + (1-\alpha)\left(\frac{1+2\underline{p}_3-\underline{p}_2}{3}\right)$
12	2	3	1	$\alpha(1-\underline{p}_1) + (1-\alpha)\left(\frac{1+2\underline{p}_3-\underline{p}_1}{3}\right)$	$\alpha\underline{p}_1 + (1-\alpha)\left(\frac{1+2\underline{p}_1-\underline{p}_3}{3}\right)$
13	3	1	2	$\alpha(1-\underline{p}_2) + (1-\alpha)\left(\frac{1+2\underline{p}_1-\underline{p}_2}{3}\right)$	$\alpha\underline{p}_2 + (1-\alpha)\left(\frac{1+2\underline{p}_2-\underline{p}_1}{3}\right)$
1i	i	j	k	$\alpha(1-\underline{p}_k) + (1-\alpha)\left(\frac{1+2\underline{p}_j-\underline{p}_k}{3}\right)$	$\alpha\underline{p}_k + (1-\alpha)\left(\frac{1+2\underline{p}_k-\underline{p}_j}{3}\right)$

We need to consider the three possibilities.

Possibility 1: $e_jx_j > e_kx_k$ We have:

$$x_j^* = \frac{e_km + \{\ln[(\alpha\underline{p}_j + (1-\alpha)\left(\frac{1+2\underline{p}_j-\underline{p}_k}{3}\right))e_j/(\alpha(1-\underline{p}_j) + (1-\alpha)\left(\frac{1+2\underline{p}_k-\underline{p}_j}{3}\right))e_k)]\}/r}{e_j+e_k}$$

$$x_k^* = \frac{e_jm + \{\ln[(\alpha(1-\underline{p}_j) + (1-\alpha)\left(\frac{1+2\underline{p}_k-\underline{p}_j}{3}\right))e_k/(\alpha\underline{p}_j + (1-\alpha)\left(\frac{1+2\underline{p}_j-\underline{p}_k}{3}\right))e_j)]\}/r}{e_j+e_k}$$

We note that the condition for this possibility to be satisfied is that $e_j[\alpha\underline{p}_j + (1-\alpha)\left(\frac{1+2\underline{p}_j-\underline{p}_k}{3}\right)] > e_k[\alpha(1-\underline{p}_j) + (1-\alpha)\left(\frac{1+2\underline{p}_k-\underline{p}_j}{3}\right))]$.

Possibility 2: $e_jx_j < e_kx_k$ We have:

$$x_j^* = \frac{e_km + \{\ln[(\alpha(1-\underline{p}_k) + (1-\alpha)\left(\frac{1+2\underline{p}_j-\underline{p}_k}{3}\right))e_j/(\alpha\underline{p}_k + (1-\alpha)\left(\frac{1+2\underline{p}_k-\underline{p}_j}{3}\right))e_k]\}/r}{e_j+e_k} \tag{23}$$

$$x_k^* = \frac{e_jm + \{\ln[(\alpha\underline{p}_k + (1-\alpha)\left(\frac{1+2\underline{p}_k-\underline{p}_j}{3}\right))e_k/(\alpha(1-\underline{p}_k) + (1-\alpha)\left(\frac{1+2\underline{p}_j-\underline{p}_k}{3}\right))e_j]\}/r}{e_j+e_k} \tag{24}$$

We note that the condition for this possibility to be satisfied is that $e_k[\alpha\underline{p}_k + (1-\alpha)\left(\frac{1+2\underline{p}_k-\underline{p}_j}{3}\right)] > e_j[\alpha(1-\underline{p}_k) + (1-\alpha)\left(\frac{1+2\underline{p}_j-\underline{p}_k}{3}\right)]$.

Possibility 3: $e_j x_j = e_k x_k$ We must have

$$x_j^* = \frac{e_k m}{e_j + e_k} \text{ and } x_k^* = \frac{e_j m}{e_j + e_k}$$

Note that this solution in always admissible.

3.5.2 Type 2 problems

Let us consider the general case i, that is, allocations between i and not-i.

To save some writing let us introduce the notation $\underline{P}_i$ to refer to the sum of the $\underline{p}$'s for not-i. That is, $\underline{P}_1 = \underline{p}_2 + \underline{p}_3$, $\underline{P}_2 = \underline{p}_1 + \underline{p}_3$ and $\underline{P}_3 = \underline{p}_1 + \underline{p}_2$. Or more generally $\underline{P}_i = \underline{p}_j + \underline{p}_k$.

There are three possibilities:

Possibilty 1: $e_i x_i > E_i(1 - x_i) = E_i X_i$

Possibility 2: $e_i x_i < E_i(1 - x_i) = E_i X_i$

Possibility 3: $e_i x_i = E_i(1 - x_i) = E_i X_i$

Problem type	i	not-i	weight on i if $e_i x_i > E_i X_i$	weight on not-i if $e_i x_i > E_i X_i$
21	1	2 and 3	$\alpha \underline{p}_1 + (1-\alpha)(\frac{1+\underline{p}_1-\underline{P}_1}{2})$	$\alpha(1-\underline{p}_1) + (1-\alpha)(\frac{1-\underline{p}_1+\underline{P}_1}{2})$
22	2	3 and 1	$\alpha \underline{p}_2 + (1-\alpha)(\frac{1+\underline{p}_2-\underline{P}_2}{2})$	$\alpha(1-\underline{p}_2) + (1-\alpha)(\frac{1-\underline{p}_2+\underline{P}_2}{2})$
23	3	1 and 2	$\alpha \underline{p}_3 + (1-\alpha)(\frac{1+\underline{p}_3-\underline{P}_3}{2})$	$\alpha(1-\underline{p}_3) + (1-\alpha)(\frac{1-\underline{p}_3+\underline{P}_3}{2})$
2i	i	j and k	$\alpha \underline{p}_i + (1-\alpha)(\frac{1+\underline{p}_i-\underline{P}_i}{2})$	$\alpha(1-\underline{p}_i) + (1-\alpha)(\frac{1-\underline{p}_i+\underline{P}_i}{2})$

Problem type	i	not-i	weight on i if $e_i x_i < E_i X_i$	weight on not-i if $e_i x_i < E_i X_i$
21	1	2 and 3	$\alpha(1-\underline{P}_1) + (1-\alpha)(\frac{1+\underline{p}_1-\underline{P}_1}{2})$	$\alpha \underline{P}_1 + (1-\alpha)(\frac{1-\underline{p}_1+\underline{P}_1}{2})$
22	2	3 and 1	$\alpha(1-\underline{P}_2) + (1-\alpha)(\frac{1+\underline{p}_2-\underline{P}_2}{2})$	$\alpha \underline{P}_2 + (1-\alpha)(\frac{1-\underline{p}_2+\underline{P}_2}{2})$
23	3	1 and 2	$\alpha(1-\underline{P}_3) + (1-\alpha)(\frac{1+\underline{p}_3-\underline{P}_3}{2})$	$\alpha \underline{P}_3 + (1-\alpha)(\frac{1-\underline{p}_3+\underline{P}_3}{2})$
2i	i	j and k	$\alpha(1-\underline{P}_i) + (1-\alpha)(\frac{1+\underline{p}_i-\underline{P}_i}{2})$	$\alpha \underline{P}_i + (1-\alpha)(\frac{1-\underline{p}_i+\underline{P}_i}{2})$

We need to consider the three possibilities

Possibility 1: $e_i x_i > E_i X_i$ Here the weights on x_i is $\alpha \underline{p}_i + (1-\alpha)(\frac{1+\underline{p}_i-\underline{P}_i}{2})$ and the weight on X_i is $\alpha(1-\underline{p}_i) + (1-\alpha)(\frac{1-\underline{p}_i+\underline{P}_i}{2})$

The optimal allocations are:

$$x_i^* = \frac{E_i m + \{\ln[(\alpha \underline{p}_i + (1-\alpha)(\frac{1+\underline{p}_i - \underline{P}_i}{2}))e_i / (\alpha(1-\underline{p}_i) + (1-\alpha)(\frac{1-\underline{p}_i+\underline{P}_i}{2}))E_i]\}/r}{e_i + E_i}$$

$$X_i^* = \frac{e_i m + \{\ln[(\alpha(1-\underline{p}_i) + (1-\alpha)(\frac{1-\underline{p}_i+\underline{P}_i}{2}))E_i / (\alpha \underline{p}_i + (1-\alpha)(\frac{1+\underline{p}_i-\underline{P}_i}{2}))e_i]\}/r}{e_i + E_i}$$

We note that the condition for this possibility to be satisfied is that $[\alpha \underline{p}_i + (1-\alpha)(\frac{1+\underline{p}_i-\underline{P}_i}{2})]e_i > [\alpha(1-\underline{p}_i) + (1-\alpha)(\frac{1-\underline{p}_i+\underline{P}_i}{2})]E_i$.

Possibility 2: $e_i x_i < E_i X_i$ Here the weights on x_i is $\alpha(1-\underline{P}_i) + (1-\alpha)(\frac{1+\underline{p}_i-\underline{P}_i}{2})$ and the weight on X_i is $\alpha \underline{P}_i + (1-\alpha)(\frac{1-\underline{p}_i+\underline{P}_i}{2})$.

The optimal allocations are:

$$x_i^* = \frac{E_i m + \{\ln[(\alpha(1-\underline{P}_i) + (1-\alpha)(\frac{1+\underline{p}_i-\underline{P}_i}{2}))e_i / (\alpha \underline{P}_i + (1-\alpha)(\frac{1-\underline{p}_i+\underline{P}_i}{2}))E_i]\}/r}{e_i + E_i}$$

$$X_i^* = \frac{e_i m + \{\ln[(\alpha \underline{P}_i + (1-\alpha)(\frac{1-\underline{p}_i+\underline{P}_i}{2}))E_i / (\alpha(1-\underline{P}_i) + (1-\alpha)(\frac{1+\underline{p}_i-\underline{P}_i}{2}))e_i]\}/r}{e_i + E_i}$$

We note that the condition for this possibility to be satisfied is that $[\alpha(1-\underline{P}_i) + (1-\alpha)(\frac{1+\underline{p}_i-\underline{P}_i}{2})]e_i < [\alpha \underline{P}_i + (1-\alpha)(\frac{1-\underline{p}_i+\underline{P}_i}{2})]E_i$.

Possibility 3: $e_i x_i = E_i X_i$ We have

$$x_i^* = \frac{E_i m}{e_i + E_i} \text{ and } X_i^* = \frac{e_i m}{e_i + E_i}$$

Note that this solution in always admissible.

4 Optimisations with CRRA function

4.1 Optimal allocations with SEU subjects

In SEU the ordering of the outcomes does not matter. We normalise the number of tokens to allocate to 1 as with the CRRA function this does not affect the *proportions* allocated to the various colours.

4.1.1 Type 1 problems

Consistently with the notation adopted with the CARA function, let us say that in Problem type 1i the choice is between colours j and k. We note:

i	1	2	3
j	2	3	1
k	3	1	2

Then the allocation is between colours j and k. If colour i comes up the subject receives nothing. So the problem is to choose x_j and x_k to maximise $p_j u(e_j x_j) + p_k u(e_k x_k)$ st $x_j + x_k = 1$

We have from our general results above:

$$x_j^* = \frac{q_j}{q_j + q_k} \text{ and } x_k^* = \frac{q_k}{q_j + q_k} \tag{25}$$

where the q's are given by $q_i = p_i^r e_i^{r-1}$ for i= 1, 2. Here the p's are simply the probabilities of the three colours and e's are the unordered exchange rates.

4.1.2 Type 2 Problems

In Problem Type 2i, the choice is between i and not-i, the subject allocates x_i to colour i and X_i to not-i, then if colour i is drawn the subject receives $e_i x_i$ whereas if the colour drawn is not-i then the subject receives $E_i X_i$. Here E_i denotes the exchange rate between not-i and money.

Using the above results we have that in the Problem type 2i:

$$x_i^* = \frac{q_i}{q_i + Q_i} \text{ and } X_i^* = \frac{Q_i}{q_i + Q_i} \tag{26}$$

where $q_i = p_i^r e_i^{r-1}$and $Q_i = P_i^r E_i^{r-1}$, $P_i = p_j + p_k$ and where E_i is the exchange rate between allocations to not-i and money. So we have that $P_1 = p_2 + p_3$, $P_2 = p_3 + p_1$, and $P_3 = p_1 + p_2$.

4.2 Optimal allocations with CEU subjects

With CEU subjects the order matters. CEU subjects are defined by six capacities. Let us denote these by the variables v and V as follows. v_1 is the capacity on colour 1, v_2 is the capacity on colour 2, v_3 is the capacity on colour 3; V_1 is the capacity on colours 2 and 3 combined, V_2 is the capacity on colours 1 and 3 combined, V_3 is the capacity on colours 1 and 2 combined. In all the problems in our experiment we effectively just have two colours in every problem.

4.2.1 Type 1 problems

Using the CEU formulation it follows that the v's in the objective function equation **??** are defined as follows:

Problem type	i	j	k	weight on j if $e_jx_j > e_kx_k$	weight on k if $e_jx_j > e_kx_k$	weight on j if e
11	1	2	3	v_2	$V_1 - v_2$	$V_1 - v_3$
12	2	3	1	v_3	$V_2 - v_3$	$V_2 - v_1$
13	3	1	2	v_1	$V_3 - v_1$	$V_3 - v_2$
1i	i	j	k	v_j	$V_i - v_j$	$V_i - v_k$

We need to consider three possibilities. We analyse Problem type 1i.

Possibility 1: $e_jx_j > e_kx_k$ We apply the general result.

We have:

$$x_j^* = \frac{q_j}{q_j + q_k} \text{ and } x_k^* = \frac{q_k}{q_j + q_k} \tag{27}$$

where the q's are given by $q_j = v_j^r e_j^{r-1}$ and $q_k = (V_i - v_j)^r e_k^{r-1}$

We note that the condition for this possibility to be satisfied is that $v_j(e_j + e_k) > V_ie_k$.

Possibility 2: $e_jx_j < e_kx_k$ This has strict inequalities and we can apply general results.

We have:

$$x_j^* = \frac{q_j}{q_j + q_k} \text{ and } x_k^* = \frac{q_k}{q_j + q_k} \tag{28}$$

where the q's are given by $q_j = (V_i - v_k)^r e_j^{r-1}$ and $q_k = v_k^r e_k^{r-1}$

We note that the condition for this possibility to be satisfied is that $v_k(e_j + e_k) > V_i e_j$.

Possibility 3: $e_j x_j = e_k x_k$ We must have

$$x_j^* = \frac{e_k}{e_j + e_k} \text{ and } x_k^* = \frac{e_j}{e_j + e_k}$$

Note that this solution in always admissible.

4.2.2 Type 2 problems

Let us consider the general Problem type 2i, that is, allocations between i and not-i:

We need to consider the 3 possibilites:

Possibility 1: $e_i x_i > E_i(1 - x_i) = E_i X_i$

Possibility 2: $e_i x_i < E_i(1 - x_i) = E_i X_i$

Possibility 3: $e_i x_i = E_i(1 - x_i) = E_i X_i$

Here the notation E_i means the exchange rate on not-i.

Problem type	i	not-i	weight on i if $e_i x_i > E_i X_i$	weight on not-i if $e_i x_i > E_i X_i$	weight o
21	1	2 and 3	v_1	$1 - v_1$	$1 - V_1$
22	2	3 and 1	v_2	$1 - v_2$	$1 - V_2$
23	3	1 and 2	v_3	$1 - v_3$	$1 - V_3$
2i	i	j and k	v_i	$1 - v_i$	$1 - V_i$

We need to consider three possibilities. We analyse Problem type 2i.

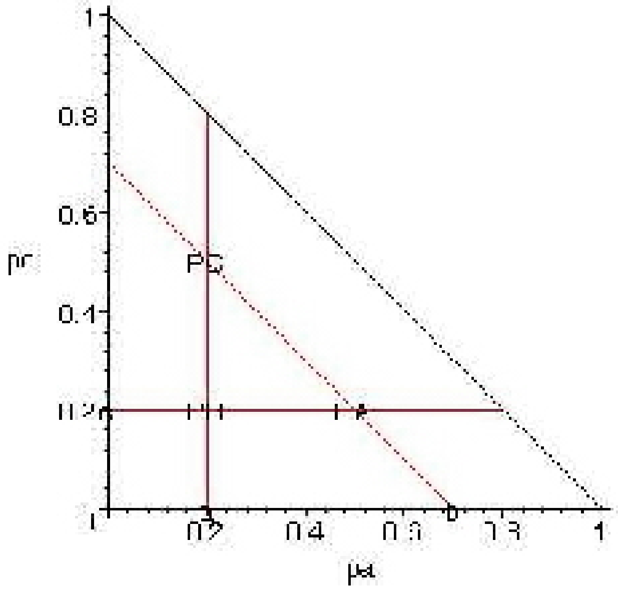
Possibility 1: $e_i x_i > E_i X_i$ This has strict inequalities and we can apply general results.

We have:

$$x_i^* = \frac{q_i}{q_i + Q_i} \text{ and } X_i^* = \frac{Q_i}{q_i + Q_i}$$

where the q's and Q's are given by $q_i = v_i^r e_i^{r-1}$ and $Q_i = (1 - v_i)^r E_i^{r-1}$

We note that the condition for this possibility to be satisfied is that $v_i(e_i + E_i) > E_i$.



Possibility 2: $e_i x_i < E_i x_i$ This has strict inequalities and we can apply general results.

We have:

$$x_i^* = \frac{q_i}{q_i + Q_i} \text{ and } X_i^* = \frac{Q_i}{q_i + Q_i} \tag{29}$$

where the q's and Q's are given by $q_i = (1 - V_i)^r e_i^{r-1}$ and $Q_i = V_i^r E_i^{r-1}$

We note that the condition for this possibility to be satisfied is that $V_i(e_i + E_i) > e_i$.

Possibility 3: $e_i x_i = E_i X_i$ We must have

$$x_i^* = \frac{E_i}{e_i + E_i} \text{ and } E_i^* = \frac{e_i}{e_i + E_i}$$

Note that this solution in always admissible.

4.3 Optimal allocations with AEU subjects

Suppose now that the subject is AEU. This is defined by three probability bounds and the alpha parameter.

Let us define the bounds on the convex set of possible probabilites by v_1, v_2, v_3. These three numbers characterise the model. Assume that they add up to less than 1 (if they add up to 1 then AEU reduces to SEU). They bound a triangular area in the Mashack-Machina Triangle.

As in the other cases the objective function is given by **??**. The crucial point is the values of the weights. Using our standard notation, where the ordered v's we have

$$\begin{aligned} AEU &= \alpha[w_1 u(e_1 x_1) + w_2 u(e_2 x_2) + (1 - w_1 - w_2) u(e_3 x_3)] + \\ & (1-\alpha)[(1 - w_2 - w_3) u(e_1 x_1) + w_2 u(e_2 x_2) + w_3 u(e_3 x_3)] \end{aligned} \tag{30}$$

This can be written as

$$\begin{aligned} AEU &= [\alpha w_1 + (1-\alpha)(1 - w_2 - w_3)] u(e_1 x_1) + w_2 u(e_2 x_2) + \\ & [\alpha(1 - w_1 - w_2) + (1-\alpha) w_3] u(e_3 x_3)] \end{aligned} \tag{31}$$

We note that this is exactly like the SEU case but with probabilities $[\alpha w_1 + (1 - \alpha)(1 - w_2 - w_3)], w_2$ and $[\alpha(1 - w_1 - w_2) + (1 - \alpha) w_3]$ on the three outcomes. Note that these add to 1, so we can apply our standard results. But note the idiosyncracy of AEU: these 'probabilities' depend upon the ordering. Following the same notation as the CEU case, we have:

$$\begin{aligned} \varpi_1 &= \alpha w_1 + (1-\alpha)(1 - w_2 - w_3) \\ \varpi_2 &= w_2 \\ \varpi_3 &= \alpha(1 - w_1 - w_2) + (1-\alpha) w_3 \end{aligned} \tag{32}$$

Then we can write the AEU objective function in the standard format of equation **??**. Hence the standard results hold.

4.3.1 Type 1 problems

To save some writing let us introduce the notation V_i to refer to the sum of the v's for not-i. That is, $V_1 = v_2 + v_3$, $V_2 = v_1 + v_3$ and $V_3 = v_1 + v_2$. Or more generally

$V_i = v_j + v_k$.

If problem type is $1i$ then

if $e_j x_j > e_k x_k$ weight on x_j is $\alpha v_j + (1-\alpha)(1-V_j)$ and weight on x_k is v_k

if $e_j x_j < e_k x_k$ weight on x_j is v_j and weight on x_k is $\alpha v_k + (1-\alpha)(1-V_k)$

We need to consider three possibilities. We analyse Problem type $1i$.

Possibility 1: $e_j x_j > e_k x_k$ We have:

$$x_j^* = \frac{q_j}{q_j + q_k} \text{ and } x_k^* = \frac{q_k}{q_j + q_k} \tag{33}$$

where the q's are given by $q_j = [\alpha v_j + (1-\alpha)(1-V_j)]^r e_j^{r-1}$ and $q_k = v_k^r e_k^{r-1}$

We note that the condition for this possibility to be satisfied is that $e_j[\alpha v_j + (1-\alpha)(1-V_j)] > e_k v_k$. It does not appear that this can be simplified .

Possibility 2: $e_j x_j < e_k x_k$ This has strict inequalities and we can apply general results.

We have:

$$x_j^* = \frac{q_j}{q_j + q_k} \text{ and } x_k^* = \frac{q_k}{q_j + q_k} \tag{34}$$

where the q's are given by $q_j = v_j^r e_j^{r-1}$ and $q_k = [\alpha v_k + (1-\alpha)(1-V_k)]^r e_k^{r-1}$

We note that the condition for this possibility to be satisfied is that $e_k[\alpha v_k + (1-\alpha)(1-V_k)] > e_j v_j$. Again it does not appear that this can be simplified.

Possibility 3: $e_j x_j = e_k x_k$ We must have

$$x_j^* = \frac{e_k}{e_j + e_k} \text{ and } x_k^* = \frac{e_j}{e_j + e_k}$$

Note that this solution in always admissible.

4.3.2 Type 2 problems

If problem type is $2i$ then

if $e_i x_i > E_i X_i$ weight on x_i is $\alpha v_i + (1-\alpha)(1-V_i)$ and weight on X_i is $\alpha(1-v_i) + (1-\alpha)V_i$.

if $e_i x_i < E_i X_i$ weight on x_i is x_i is $\alpha(1-V_i) + (1-\alpha)v_i$ and the weight on X_i is $\alpha V_i + (1-\alpha)(1-v_i)$.

Again we can work with the unordered v's. Let us introduce the notation V_i to refer to the sum of the v's for not-i. That is, $V_1 = v_2 + v_3$, $V_2 = v_1 + v_3$ and $V_3 = v_1 + v_2$.

Possibility 1: $e_i x_i > E_i X_i$ Here we use again the notation X_i to refer to the allocation to not-i.

Here the weights on x_i is $\alpha v_i + (1-\alpha)(1-V_i)$ and the weight on X_i is $\alpha(1-v_i) + (1-\alpha)V_i$.

In this case, the optimal allocations are

$$\begin{aligned} x_i^* &= \frac{[\alpha v_i + (1-\alpha)(1-V_i)]^r e_i^{r-1}}{[\alpha v_i + (1-\alpha)(1-V_i)]^r e_i^{r-1} + [\alpha(1-v_i) + (1-\alpha)V_i]^r E_i^{r-1}} \\ X_i^* &= \frac{[\alpha(1-v_i) + (1-\alpha)V_i]^r E_i^{r-1}}{[\alpha v_i + (1-\alpha)(1-V_i)]^r e_i^{r-1} + [\alpha(1-v_i) + (1-\alpha)V_i]^r E_i^{r-1}} \end{aligned}$$

Let us ask ourselves what is the condition such that the ranking is satisfied. We need that $e_i x_i^* > E_i(1-x_i^*)$. This gives us the condition that $[\alpha v_i + (1-\alpha)(1-V_i)]e_i > [\alpha(1-v_i) + (1-\alpha)V_i]E_i$.

Possibility 2: $e_i x_i < E_i X_i$ Here the weight on x_i is $\alpha(1-V_i) + (1-\alpha)v_i$ and the weight on X_i is $\alpha V_i + (1-\alpha)(1-v_i)$.

In this case, the optimal allocations are

$$\begin{aligned} x_i^* &= \frac{[\alpha(1-V_i) + (1-\alpha)v_i]^r e_i^{r-1}}{[\alpha(1-V_i) + (1-\alpha)v_i]^r e_i^{r-1} + [\alpha V_i + (1-\alpha)(1-v_i)]^r E_i^{r-1}} \\ X_i^* &= \frac{[\alpha V_i + (1-\alpha)(1-v_i)]^r E_i^{r-1}}{[\alpha(1-V_i) + (1-\alpha)v_i]^r e_i^{r-1} + [\alpha V_i + (1-\alpha)(1-v_i)]^r E_i^{r-1}} \end{aligned}$$

Again let us ask ourselves what is the condition that the ranking is satisfied. Following the logic as above we need that $[\alpha V_i + (1-\alpha)(1-v_i)]E_i > [\alpha(1-V_i) + (1-\alpha)v_i]e_i$.

Possibility 3: $e_i x_i = E_i X_i$ We must have

$$x_i^* = \frac{E_i}{e_i + E_i} \text{ and } E_i^* = \frac{e_i}{e_i + E_i}$$

Note that this solution in always admissible.

4.4 Optimal allocations with VEU subjects

A VEU maximizer is defined by three "adjusted" probabilities that incorporate the ambiguity about the relative number of pairs of colours (i.e., ambiguity about the relative number of colour 1 versus colour 2 balls and the ambiguity about the relative number of colour 2 versus colour 3 balls). These adjusted probabilities are defined as the baseline prior probability plus or minus the adjustment for ambiguity.

Let us denote the baseline prior probabilities by v_i as follows: v_1 is the baseline probability on colour 1, v_2 is the baseline probability on colour 2, v_3 is the baseline probability on colour 3. We define by w_i the corresponding *ordered* baseline prior probabilities. So we have

$v_i = w_{bac(c,i)}$ for $i = 1, 2, 3$ or $w_i = v_{ord(c,i)}$ for $i = 1, 2, 3$ and $c = 1, ..., 6$.

Let us write eq. (**??**) in terms of the ordered baseline prior probabilities

$$VEU = w_1 u(e_1 x_1) + w_2 u(e_2 x_2) + w_3 u(e_3 x_3) - \delta(|u(e_1 x_1) - u(e_2 x_2)| + |u(e_2 x_2) - u(e_3 x_3)|) \tag{35}$$

Since we are considering an ordering, we can ignore the modulus. The (35) becomes

$$VEU = (w_1 - \delta) u(e_1 x_1) + w_2 u(e_2 x_2) + (w_3 + \delta) u(e_3 x_3) \tag{36}$$

Now we can easily define the "adjusted" probabilities and, for analogy to the AEU case, we define them by ϖ.

$$\begin{aligned} \varpi_1 &= w_1 - \delta \\ \varpi_2 &= w_2 \\ \varpi_3 &= w_3 + \delta \end{aligned} \tag{37}$$

Note that both the baseline prior probabilities and the "adjusted" probabilities sum up to one.

4.4.1 Type 1 problems

We follow the same notation of AEU. We refer to V_i as the sum of the v's for not-i. That is, $V_1 = v_2 + v_3$, $V_2 = v_1 + v_3$ and $V_3 = v_1 + v_2$. Or more generally $V_i = v_j + v_k$. If problem type is $1i$ then

Problem type	i	j	k	weight on j if $e_j x_j > e_k x_k$	weight on k if $e_j x_j > e_k x_k$
11	1	2	3	$v_2 - \delta$	v_3
12	2	3	1	$v_3 - \delta$	v_1
13	3	1	2	$v_1 - \delta$	v_2
$1i$	i	j	k	$v_j - \delta$	v_k

Problem type	i	j	k	weight on j if $e_j x_j < e_k x_k$	weight on k if $e_j x_j < e_k x_k$
11	1	2	3	v_2	$v_3 - \delta$
12	2	3	1	v_3	$v_1 - \delta$
13	3	1	2	v_1	$v_2 - \delta$
$1i$	i	j	k	v_j	$v_k - \delta$

We need to consider the three possibilities.

Possibility 1: $e_j x_j > e_k x_k$ We have

$$x_j^* = \frac{(v_j - \delta)^r e_j^{r-1}}{(v_j - \delta)^r e_j^{r-1} + v_k^r e_k^{r-1}} \tag{38}$$
$$x_k^* = \frac{v_k^r e_k^{r-1}}{(v_j - \delta)^r e_j^{r-1} + v_k^r e_k^{r-1}}$$

We note that the condition for this possibility to be satisfied is that $e_j(v_j - \delta) > e_k v_k$.

Possibility 2: $e_j x_j < e_k x_k$ We have

$$x_j^* = \frac{{v_j}^r e_j^{r-1}}{{v_j}^r e_j^{r-1} + (v_k - \delta)^r e_j^{r-1}} \quad (39)$$

$$x_k^* = \frac{(v_k - \delta)^r e_j^{r-1}}{{v_j}^r e_j^{r-1} + (v_k - \delta)^r e_j^{r-1}} \quad (40)$$

We note that the condition for this possibility to be satisfied is that $e_k(v_k - \delta) > e_j v_j$.

Possibility 3: $e_j x_j = e_k x_k$ We have

$$x_i^* = \frac{E_i}{e_i + E_i} \text{ and } x_k^* = \frac{e_i}{e_i + E_i}$$

Note that this solution in always admissible.

4.4.2 Type 2 problems

Let us consider the general case i, that is, allocations between i and not-i:

There are three possibilities:

Possibilty 1: $e_i x_i > E_i(1 - x_i) = E_i X_i$

Possibility 2: $e_i x_i < E_i(1 - x_i) = E_i X_i$

Possibility 3: $e_i x_i = E_i(1 - x_i) = E_i X_i$

Problem type	i	not-i	weight on i if $e_i x_i > E_i X_i$	weight on not-i if $e_i x_i > E_i X_i$
21	1	2 and 3	$v_1 - \delta$	$1 - v_1 + \delta$
22	2	3 and 1	$v_2 - \delta$	$1 - v_2 + \delta$
23	3	1 and 2	$v_3 - \delta$	$1 - v_3 + \delta$
2i	i	j and k	$v_i - \delta$	$1 - v_i + \delta$
Problem type	i	not-i	weight on i if $e_i x_i < E_i X_i$	weight on not-i if $e_i x_i < E_i X_i$
21	1	2 and 3	$1 - V_1 + \delta$	$V_1 - \delta$
22	2	3 and 1	$1 - V_2 + \delta$	$V_2 - \delta$
23	3	1 and 2	$1 - V_3 + \delta$	$V_3 - \delta$
2i	i	j and k	$1 - V_i + \delta$	$V_i - \delta$

We need to consider the three possibilities.

Possibility 1: $e_i x_i > E_i X_i$ Here the weights on x_i is $(\nu_i - \delta)$ and the weight on X_i is $(1 - \nu_i + \delta)$

The optimal allocations are:

$$\begin{aligned} x_i^* &= \frac{(v_i - \delta)^r e_i^{r-1}}{(v_i - \delta)^r e_i^{r-1} + (1 - v_i + \delta)^r E_i^{r-1}} \\ X_i^* &= \frac{(1 - v_i + \delta)^r E_i^{r-1}}{(v_i - \delta)^r e_i^{r-1} + (1 - v_i + \delta)^r E_i^{r-1}} \end{aligned}$$

We note that the condition for this possibility to be satisfied is that $(v_i - \delta)e_i > (1 - v_i + \delta)E_i$.

Possibility 2: $e_i x_i < E_i X_i$ Here the weights on x_i is $(1 - V_i + \delta)$ and the weight on X_i is $(V_i - \delta)$

The optimal allocations are:

$$\begin{aligned} x_i^* &= \frac{(1 - V_i + \delta)^r e_i^{r-1}}{(1 - V_i + \delta)^r e_i^{r-1} + (V_i - \delta)^r E_i^{r-1}} \\ X_i^* &= \frac{(V_i - \delta)^r E_i^{r-1}}{(1 - V_i + \delta)^r e_i^{r-1} + (V_i - \delta)^r E_i^{r-1}} \end{aligned}$$

We note that the condition for this possibility to be satisfied is that $(1 - V_i + \delta)e_i < (V_i - \delta)E_i$.

Possibility 3: $e_i x_i = E_i X_i$ We have

$$x_i^* = \frac{E_i}{e_i + E_i} \text{ and } X_i^* = \frac{e_i}{e_i + E_i}$$

Note that this solution in always admissible.

4.5 Optimal allocations with COM subjects

The preference functional for COM can be written as

$$\begin{aligned} COM &= [\alpha \underline{p}_1 + (1-\alpha)(\underline{p}_1 + (1-\underline{p}_1-\underline{p}_2-\underline{p}_3)/3)]u(e_1x_1) + \\ & [\alpha \underline{p}_2 + (1-\alpha)(\underline{p}_2 + (1-\underline{p}_1-\underline{p}_2-\underline{p}_3)/3)]u(e_2x_2) + \\ & [\alpha(1-\underline{p}_1-\underline{p}_2) + (1-\alpha)(\underline{p}_3 + (1-\underline{p}_1-\underline{p}_2-\underline{p}_3)/3)]u(e_3x_3) \end{aligned} \tag{41}$$

The probabilities on the three outcomes are $[\alpha \underline{p}_1 + (1-\alpha)(\underline{p}_1 + (1-\underline{p}_1-\underline{p}_2-\underline{p}_3)/3)]$, $[\alpha \underline{p}_2 + (1-\alpha)(\underline{p}_2 + (1-\underline{p}_1-\underline{p}_2-\underline{p}_3)/3)]$ and $[\alpha(1-\underline{p}_1-\underline{p}_2) + (1-\alpha)(\underline{p}_2 + (1-\underline{p}_1-\underline{p}_2-\underline{p}_3)/3)]$.

Again we need to consider all the possible cases.

4.5.1 Type 1 problems

If problem type is $1i$ then

if $e_jx_j > e_kx_k$ weight on x_j is and weight on x_k is

if $e_jx_j < e_kx_k$ weight on x_j is and weight on x_k is.

Problem type	i	j	k	weight on j if $e_jx_j > e_kx_k$	weight on k if $e_jx_j > e_kx_k$
11	1	2	3	$\alpha \underline{p}_2 + (1-\alpha)\left(\frac{1+2\underline{p}_2-\underline{p}_3}{3}\right)$	$\alpha(1-\underline{p}_2) + (1-\alpha)\left(\frac{1+2\underline{p}_3-\underline{p}_2}{3}\right)$
12	2	3	1	$\alpha \underline{p}_3 + (1-\alpha)\left(\frac{1+2\underline{p}_3-\underline{p}_1}{3}\right)$	$\alpha(1-\underline{p}_3) + (1-\alpha)\left(\frac{1+2\underline{p}_1-\underline{p}_3}{3}\right)$
13	3	1	2	$\alpha \underline{p}_1 + (1-\alpha)\left(\frac{1+2\underline{p}_1-\underline{p}_2}{3}\right)$	$\alpha(1-\underline{p}_1) + (1-\alpha)\left(\frac{1+2\underline{p}_2-\underline{p}_1}{3}\right)$
$1i$	i	j	k	$\alpha \underline{p}_j + (1-\alpha)\left(\frac{1+2\underline{p}_j-\underline{p}_k}{3}\right)$	$\alpha(1-\underline{p}_j) + (1-\alpha)\left(\frac{1+2\underline{p}_k-\underline{p}_j}{3}\right)$

Problem type	i	j	k	weight on j if $e_jx_j < e_kx_k$	weight on k if $e_jx_j < e_kx_k$
11	1	2	3	$\alpha(1-\underline{p}_3) + (1-\alpha)\left(\frac{1+2\underline{p}_2-\underline{p}_3}{3}\right)$	$\alpha \underline{p}_3 + (1-\alpha)\left(\frac{1+2\underline{p}_3-\underline{p}_2}{3}\right)$
12	2	3	1	$\alpha(1-\underline{p}_1) + (1-\alpha)\left(\frac{1+2\underline{p}_3-\underline{p}_1}{3}\right)$	$\alpha \underline{p}_1 + (1-\alpha)\left(\frac{1+2\underline{p}_1-\underline{p}_3}{3}\right)$
13	3	1	2	$\alpha(1-\underline{p}_2) + (1-\alpha)\left(\frac{1+2\underline{p}_1-\underline{p}_2}{3}\right)$	$\alpha \underline{p}_2 + (1-\alpha)\left(\frac{1+2\underline{p}_2-\underline{p}_1}{3}\right)$
$1i$	i	j	k	$\alpha(1-\underline{p}_k) + (1-\alpha)\left(\frac{1+2\underline{p}_j-\underline{p}_k}{3}\right)$	$\alpha \underline{p}_k + (1-\alpha)\left(\frac{1+2\underline{p}_k-\underline{p}_j}{3}\right)$

We need to consider the three possibilities.

Possibility 1: $e_j x_j > e_k x_k$ We have:

$$x_j^* = \frac{\left(\alpha \underline{p}_j + (1-\alpha)\left(\frac{1+2\underline{p}_j - \underline{p}_k}{3}\right)\right)^r e_j^{r-1}}{\left(\alpha \underline{p}_j + (1-\alpha)\left(\frac{1+2\underline{p}_j - \underline{p}_k}{3}\right)\right)^r e_j^{r-1} + \left(\alpha(1-\underline{p}_j) + (1-\alpha)\left(\frac{1+2\underline{p}_k - \underline{p}_j}{3}\right)\right)^r e_k^{r-1}}$$

$$x_k^* = \frac{\left(\alpha(1-\underline{p}_j) + (1-\alpha)\left(\frac{1+2\underline{p}_k - \underline{p}_j}{3}\right)\right)^r e_k^{r-1}}{\left(\alpha \underline{p}_j + (1-\alpha)\left(\frac{1+2\underline{p}_j - \underline{p}_k}{3}\right)\right)^r e_j^{r-1} + \left(\alpha(1-\underline{p}_j) + (1-\alpha)\left(\frac{1+2\underline{p}_k - \underline{p}_j}{3}\right)\right)^r e_k^{r-1}}$$

We note that the condition for this possibility to be satisfied is that $e_j[\alpha \underline{p}_j + (1-\alpha)\left(\frac{1+2\underline{p}_j - \underline{p}_k}{3}\right)] > e_k[\alpha(1-\underline{p}_j) + (1-\alpha)\left(\frac{1+2\underline{p}_k - \underline{p}_j}{3}\right)]$.

Possibility 2: $e_j x_j < e_k x_k$ We have:

$$x_j^* = \frac{\left(\alpha(1-\underline{p}_k) + (1-\alpha)\left(\frac{1+2\underline{p}_j - \underline{p}_k}{3}\right)\right)^r e_j^{r-1}}{\left(\alpha(1-\underline{p}_k) + (1-\alpha)\left(\frac{1+2\underline{p}_j - \underline{p}_k}{3}\right)\right)^r e_j^{r-1} + \left(\alpha \underline{p}_k + (1-\alpha)\left(\frac{1+2\underline{p}_k - \underline{p}_j}{3}\right)\right)^r e_k^{r-1}} \tag{42}$$

$$x_k^* = \frac{\left(\alpha \underline{p}_k + (1-\alpha)\left(\frac{1+2\underline{p}_k - \underline{p}_j}{3}\right)\right)^r e_k^{r-1}}{\left(\alpha(1-\underline{p}_k) + (1-\alpha)\left(\frac{1+2\underline{p}_j - \underline{p}_k}{3}\right)\right)^r e_j^{r-1} + \left(\alpha \underline{p}_k + (1-\alpha)\left(\frac{1+2\underline{p}_k - \underline{p}_j}{3}\right)\right)^r e_k^{r-1}}$$

We note that the condition for this possibility to be satisfied is that $e_k[\alpha \underline{p}_k + (1-\alpha)\left(\frac{1+2\underline{p}_k - \underline{p}_j}{3}\right)] > e_j[\alpha(1-\underline{p}_k) + (1-\alpha)\left(\frac{1+2\underline{p}_j - \underline{p}_k}{3}\right)]$.

Possibility 3: $e_j x_j = e_k x_k$ We must have

$$x_j^* = \frac{e_k}{e_j + e_k} \text{ and } x_k^* = \frac{e_j}{e_j + e_k}$$

Note that this solution in always admissible.

4.5.2 Type 2 problems

As for the CARA case, let us use the notation $\underline{P}_i$ to refer to the sum of the $\underline{p}$'s for not-i. That is, $\underline{P}_1 = \underline{p}_2 + \underline{p}_3$, $\underline{P}_2 = \underline{p}_1 + \underline{p}_3$ and $\underline{P}_3 = \underline{p}_1 + \underline{p}_2$. Or more generally $\underline{P}_i = \underline{p}_j + \underline{p}_k$.

There are three possibilities:

Possibilty 1: $e_i x_i > E_i(1-x_i) = E_i X_i$

Possibility 2: $e_i x_i < E_i(1-x_i) = E_i X_i$

Possibility 3: $e_i x_i = E_i(1-x_i) = E_i X_i$

Problem type	i	not-i	weight on i if $e_i x_i > E_i X_i$	weight on not-i if $e_i x_i > E_i X_i$
21	1	2 and 3	$\alpha \underline{p}_1 + (1-\alpha)(\frac{1+\underline{p}_1-\underline{P}_1}{2})$	$\alpha(1-\underline{p}_1) + (1-\alpha)(\frac{1-\underline{p}_1+\underline{P}_1}{2})$
22	2	3 and 1	$\alpha \underline{p}_2 + (1-\alpha)(\frac{1+\underline{p}_2-\underline{P}_2}{2})$	$\alpha(1-\underline{p}_2) + (1-\alpha)(\frac{1-\underline{p}_2+\underline{P}_2}{2})$
23	3	1 and 2	$\alpha \underline{p}_3 + (1-\alpha)(\frac{1+\underline{p}_3-\underline{P}_3}{2})$	$\alpha(1-\underline{p}_3) + (1-\alpha)(\frac{1-\underline{p}_3+\underline{P}_3}{2})$
2i	i	j and k	$\alpha \underline{p}_i + (1-\alpha)(\frac{1+\underline{p}_i-\underline{P}_i}{2})$	$\alpha(1-\underline{p}_i) + (1-\alpha)(\frac{1-\underline{p}_i+\underline{P}_i}{2})$

Problem type	i	not-i	weight on i if $e_i x_i < E_i X_i$	weight on not-i if $e_i x_i < E_i X_i$
21	1	2 and 3	$\alpha(1-\underline{P}_1) + (1-\alpha)(\frac{1+\underline{p}_1-\underline{P}_1}{2})$	$\alpha \underline{P}_1 + (1-\alpha)(\frac{1-\underline{p}_1+\underline{P}_1}{2})$
22	2	3 and 1	$\alpha(1-\underline{P}_2) + (1-\alpha)(\frac{1+\underline{p}_2-\underline{P}_2}{2})$	$\alpha \underline{P}_2 + (1-\alpha)(\frac{1-\underline{p}_2+\underline{P}_2}{2})$
23	3	1 and 2	$\alpha(1-\underline{P}_3) + (1-\alpha)(\frac{1+\underline{p}_3-\underline{P}_3}{2})$	$\alpha \underline{P}_3 + (1-\alpha)(\frac{1-\underline{p}_3+\underline{P}_3}{2})$
2i	i	j and k	$\alpha(1-\underline{P}_i) + (1-\alpha)(\frac{1+\underline{p}_i-\underline{P}_i}{2})$	$\alpha \underline{P}_i + (1-\alpha)(\frac{1-\underline{p}_i+\underline{P}_i}{2})$

We need to consider the three possibilities

Possibility 1: $e_i x_i > E_i X_i$ Here the weights on x_i is $\alpha \underline{p}_i + (1-\alpha)(\frac{1+\underline{p}_i-\underline{P}_i}{2})$ and the weight on X_i is $\alpha(1-\underline{p}_i) + (1-\alpha)(\frac{1-\underline{p}_i+\underline{P}_i}{2})$

The optimal allocations are:

$$x_i^* = \frac{\left(\alpha \underline{p}_i + (1-\alpha)(\frac{1+\underline{p}_i-\underline{P}_i}{2})\right)^r e_i^{r-1}}{\left(\alpha \underline{p}_i + (1-\alpha)(\frac{1+\underline{p}_i-\underline{P}_i}{2})\right)^r e_i^{r-1} + \left(\alpha(1-\underline{p}_i) + (1-\alpha)(\frac{1-\underline{p}_i+\underline{P}_i}{2})\right)^r E_i^{r-1}}$$
$$X_i^* = \frac{\left(\alpha(1-\underline{p}_i) + (1-\alpha)(\frac{1-\underline{p}_i+\underline{P}_i}{2})\right)^r E_i^{r-1}}{\left(\alpha \underline{p}_i + (1-\alpha)(\frac{1+\underline{p}_i-\underline{P}_i}{2})\right)^r e_i^{r-1} + \left(\alpha(1-\underline{p}_i) + (1-\alpha)(\frac{1-\underline{p}_i+\underline{P}_i}{2})\right)^r E_i^{r-1}}$$

We note that the condition for this possibility to be satisfied is that $[\alpha \underline{p}_i + (1-\alpha)(\frac{1+\underline{p}_i-\underline{P}_i}{2})]e_i > [\alpha(1-\underline{p}_i) + (1-\alpha)(\frac{1-\underline{p}_i+\underline{P}_i}{2})]E_i$.

Possibility 2: $e_i x_i < E_i X_i$ Here the weights on x_i is $\alpha(1-\underline{P}_i) + (1-\alpha)(\frac{1+\underline{p}_i-\underline{P}_i}{2})$ and the weight on X_i is $\alpha \underline{P}_i + (1-\alpha)(\frac{1-\underline{p}_i+\underline{P}_i}{2})$.

The optimal allocations are:

$$x_i^* = \frac{\left(\alpha(1-\underline{P}_i)+(1-\alpha)(\frac{1+\underline{p}_i-\underline{P}_i}{2})\right)^r e_i^{r-1}}{\left(\alpha(1-\underline{P}_i)+(1-\alpha)(\frac{1+\underline{p}_i-\underline{P}_i}{2})\right)^r e_i^{r-1}+\left(\alpha\underline{P}_i+(1-\alpha)(\frac{1-\underline{p}_i+\underline{P}_i}{2})\right)^r E_i^{r-1}}$$

$$X_i^* = \frac{\left(\alpha\underline{P}_i+(1-\alpha)(\frac{1-\underline{p}_i+\underline{P}_i}{2})\right)^r E_i^{r-1}}{\left(\alpha(1-\underline{P}_i)+(1-\alpha)(\frac{1+\underline{p}_i-\underline{P}_i}{2})\right)^r e_i^{r-1}+\left(\alpha\underline{P}_i+(1-\alpha)(\frac{1-\underline{p}_i+\underline{P}_i}{2})\right)^r E_i^{r-1}}$$

We note that the condition for this possibility to be satisfied is that $[\alpha(1-\underline{P}_i)+(1-\alpha)(\frac{1+\underline{p}_i-\underline{P}_i}{2})]e_i < [\alpha\underline{P}_i+(1-\alpha)(\frac{1-\underline{p}_i+\underline{P}_i}{2})]E_i$.

Possibility 3: $e_i x_i = E_i X_i$ We have

$$x_i^* = \frac{E_i}{e_i+E_i} \text{ and } X_i^* = \frac{e_i}{e_i+E_i}$$

Note that this solution in always admissible.