Limited Rights as Partial Veto and Sen's Impossibility Theorem

Maurice Salles

1 Introduction

The origin of the tremendous development of studies on rights and freedom within social choice theory and normative economics can be traced back to the famous short paper of Amartya Sen published in 1970 (Sen, 1970b); see also his book published the same year (Sen, 1970). In this paper, it is shown in the framework of aggregation procedures that there is a conflict between collective rationality (in terms of properties of choice functions or in terms of a transitivity-type of the social preference property – in fact, acyclicity of the asymmetric part of the social preference), Paretianism (a unanimity property) and some slight violation of neutrality (neutrality meaning that the names of options or social states are not to be taken into account) possibly combined with some slightly unequal distribution of power among individuals interpreted as an individual liberty property. Although, since then, rights have been considered within another paradigm, viz. game forms (see for instance Gärdenfors (1981, 2005), Gaertner, Pattanaik, and Suzumura (1992), Peleg (1998a,b) and Suzumura (2008)), and freedom has been mainly analyzed in the context of opportunity sets following the pioneering paper of Pattanaik and Xu (1990) (see also the survey by Barberà, Bossert, and Pattanaik (2004)), some authors (for instance Igersheim (2006) and Saari and Pétron (2006)) have recently revisited the foundational framework of Sen and ~~(Gibbard, 1974)~~ either by studying the informational structure of the aggregation procedure or by examining the consequences of taking a Cartesian structure to define the set of social states, consequences that take the form of a restriction of individual preferences. The purpose of this paper is different. I wish to formally study a weakening of the conditions associated with the notion of individual liberty. I have always considered that this condition was rather

M. Salles
CREM, Université de Caen, 14032 Caen cedex, France, and Institute for SCW, BP 1, 14111 Louvigny, France
e-mail: maurice.salles@unicaen.fr

P.K. Pattanaik et al. (eds.) *Rational Choice and Social Welfare: Theory and Applications*,
Studies in Choice and Welfare.

strong in Sen's original paper. In fact, the condition is quite strong in the mathematical framework and only the interpretation, to my view, makes it not only acceptable but obvious. In his comments to a paper by Brunel (now Pétron) and Salles (1998), Hammond (1998) writes:

> In the social choice rule approach ..., local dictatorship becomes a desideratum, provided that the 'localities' are appropriate. Our feelings of revulsion should be reserved for non-local dictatorships, or local dictatorships affecting issues that should not be treated as personal.

I entirely share this opinion, but there is nothing in the basic mathematical framework that guarantees this personal aspect (in contrast with a suitable Cartesian product structure). In this basic framework, it is, however, possible to weaken local dictatorships. Unfortunately, I will show that this weakening does not offer a very interesting escape route from Sen's negative result. From a formal point of view, I believe that there is a sort of analogy that can be made between a family of Sen's impossibility theorems and Arrovian impossibility theorems.

After introducing general definitions and recalling Sen's theorems, I will present new Sen-type impossibility theorems, then will make a comparison with Arrovian impossibility theorems, commenting on similarities and obvious differences.

2 Basic Definitions and Sen's Theorem

Let X be the set of social states. Nothing specific is assumed for this set. A binary relation, a *preference*, over X is a subset of $X \times X$. It will be denoted by $\succeq$. I will write $x \succeq y$ rather than $(x, y) \in \succeq$. All binary relations considered in this chapter are supposed to be *complete* (for all x and $y \in X$, $x \succeq y$ or $y \succeq x$) and, consequently, *reflexive* (for all $x \in X$, $x \succeq x$). The *asymmetric* part of $\succeq$, denoted $\succ$ is defined (since $\succeq$ is complete) by $x \succ y$ if $\neg y \succeq x$. The *symmetric* part of $\succeq$ is defined by $x \sim y$ if $x \succeq y$ and $y \succeq x$. Intuitively, $x \succeq y$ will mean 'x is at least as good as y', $x \succ y$ will mean 'x is preferred to y' and $x \sim y$ will mean 'there is an indifference between x and y'. A preference $\succeq$ is *transitive* if for all x, y and $z \in X$, $x \succeq y$ and $y \succeq z \Rightarrow x \succeq z$. The asymmetric part of $\succeq$, $\succ$, is transitive if for all x, y and $z \in X$, $x \succ y$ and $y \succ z \Rightarrow \ x \succ z$. The symmetric part of $\succeq$, $\sim$, is transitive if for all x, y and $z \in X$, $x \sim y$ and $y \sim z \Rightarrow x \sim z$. If $\cancel{x}$ $\succeq$ is transitive, $\succ$ and $\sim$ are transitive too. We will say that $\succeq$ is *quasi-transitive* if $\succ$ is transitive (then $\sim$ is not necessarily transitive), and that $\succ$ is *acyclic* if there is no finite subset of X, $\{x_1, \ldots, x_k\}$, for which $x_1 \succ x_2, x_2 \succ x_3, \ldots, x_{k-1} \succ x_k$ and $x_k \succ x_1$. A complete and transitive binary relation is a *complete preorder* (sometimes called 'weak ordering'). Let $\mathbb{B}$ denote the set of complete binary relations over X, $\mathbb{P}$ denote the set of complete preorders over X, $\mathbb{Q}$ denote the set of complete and quasi-transitive binary relations over X, and $\mathbb{A}$ denote the set of complete binary relations over X whose asymmetric part is acyclic.

Let N be the set of individuals. Nothing specific will be assumed for this set unless it is clearly indicated that it is finite. Individual $i \in N$ has her preference given by a complete preorder $\succeq_i$ over X. A *profile* π is a function from N to $\mathbb{P}'$, $\pi : i \mapsto \succeq_i$, where $\mathbb{P}' \subseteq \mathbb{P}$ with $\mathbb{P}' \neq \emptyset$. Let Π' be the set of profiles when the $\succeq_i$'s are in $\mathbb{P}'$ and Π be the set of all profiles (when the $\succeq_i$'s are in $\mathbb{P}$). When N is finite and $\#N = n$, a profile is a n-list $(\succeq_1, \ldots, \succeq_n)$ with each $\succeq_i$ in $\mathbb{P}'$. Then $\Pi' = \mathbb{P}'^n$ and $\Pi = \mathbb{P}^n$ ($\mathbb{P}'^n$ and $\mathbb{P}^n$ are n-times Cartesian products of $\mathbb{P}'$ and $\mathbb{P}$).

Definition 1 An *aggregation function* is a function $f : \Pi' \to \mathbb{B}$.

An aggregation function associates a unique complete binary relation, a social preference, denoted by $\succeq_S$, to individual preferences (one preference for each individual).

Given an aggregation function f, and two (distinct) social states x and $y \in X$,[1] we will say that individual $i \in N$ is (x,y)-decisive if for all $\pi \in \Pi'$, $x \succ_i y \Rightarrow x \succ_S y$, where $\succ_S$ is the asymmetric part of $\succeq_S = f(\pi)$.

Definition 2 An individual who is (x,y)-decisive and (y,x)-decisive will be said to be $\{x,y\}$-*decisive* or a $\{x,y\}$-*dictator*.

I can now define Sen's two liberalism conditions. Let f be an aggregation function.

Definition 3 (Liberalism, general 2-D$^+$) For all $i \in N$, *there exist* a_i *and* $b_i \in X$ such that i is a $\{a_i, b_i\}$-dictator.

It should be noticed that $\mathbb{P}'$ must be large enough to have a non-trivial satisfaction of general 2-D$^+$: for each individual i it must be possible to have both $a_i \succ_i b_i$ and $b_i \succ_i a_i$. Also, it should be outlined that the condition is rather fair since each individual is endowed with the same kind of power. The theorem can be proved by using a weaker form of the foregoing condition.

Definition 4 (Minimal liberalism, minimal 2-D$^+$) There exist *two individuals i and* $j \in N$, and *a, b, c,* $d \in X$ such that i is a $\{a,b\}$-dictator and j is a $\{c,d\}$-dictator.

Of course, the fairness property disappeared. The options are to be 'interpreted' as being specific to the concerned individual, i.e., a and b are specific to individual i; a and b can even be 'interpreted' as perfectly identical social states except for some features that are personal to individual i. Clearly general 2-D$^+$ implies minimal 2-D$^+$.

As mentioned earlier, the domain of the aggregation function f must be rich enough. This will be taken care of (with some excess) by the following condition U.

Definition 5 (Universality, U) Let f be an aggregation function. *Universality* requires that $\mathbb{P}' = \mathbb{P}$.

This means that an individual preference can be any complete preorder. There is no restriction imposed by some kind of upper rationality or the existence of inter-individual constraints. The last condition (condition P) is a weak form of unanimity (Pareto principle).

[1] They have to be distinct so that saying it superfluous since we consider that $x \succ_i y$ and $\succ_i$ is asymmetric.

Definition 6 (Pareto principle, P) Let f be an aggregation function, $\pi \in \Pi'$ and x, $y \in X$.[2] If for all $i \in N$, $x \succ_i y$, then $x \succ_S y$ where $\succ_S$ is the asymmetric part of $\succeq_S = f(\pi)$.

Sen's theorem is obtained within a large class of aggregation functions (Sen called them *social decision functions*).

Definition 7 A $\mathbb{A}$-*valued aggregation function* (or *social decision function*) is a function $f : \Pi' \to \mathbb{A}$.

The collective rationality imposed in this case is rather weak. It has an interesting consequence on the non-emptiness of the set of maximal elements in any finite subset of X (or since we are considering complete binary relations on the non-emptiness of maximum elements or choices).

Theorem 1 *If there are at least two individuals and if* $\#X \geq 2$, *there is no* $\mathbb{A}$-*valued aggregation function satisfying minimal 2-*D^+, *U and P.*

An immediate corollary is:

Corollary 1 *If there are at least two individuals and if* $\#X \geq 2$, *there is no* $\mathbb{A}$-*valued aggregation function satisfying general 2-*D^+, *U and P.*

3 Partial Veto and Sen-Type Theorems

It is in reading Pattanaik's paper Pattanaik (1996) that I got the impetus to work on this topic. In particular in this paper, Pattanaik discusses Sen's possible views regarding a distinction between a conception of rights as the ability to prevent something and a conception of rights as the obligation to prevent something which seems to be endowed in the liberalism conditions. Although I wished to devote some time to introduce modal theoretic techniques to deal with this distinction, I will be in this chapter more modest and will consider a weakening of the liberalism conditions. It is however obvious that this weakening is not a real response to the ability-obligation problem. Nevertheless, at least from a semantical point of view, having a (partial) veto corresponds rather well to the idea of an ability to prevent something. I will then introduce the notion of partial veto and will show how robust Sen's theorem is.

Given an aggregation function f, and two (distinct) social states x and $y \in X$, we will say that individual $i \in N$ is (x,y)-semi-decisive if for all $\pi \in \Pi'$, $x \succ_i y \Rightarrow x \succeq_S y$, where $\succeq_S = f(\pi)$.

Definition 8 An individual who is (x,y)-semi-decisive and (y,x)-semi-decisive will be said to be $\{x,y\}$-*semi-decisive* or a $\{x,y\}$-*vetoer*.

As can be seen, the difference between a $\{x,y\}$-vetoer and a $\{x,y\}$-dictator is the difference between $x \succeq_S y$ and $x \succ_S y$. A $\{x,y\}$-vetoer's power amounts to the

[2] Again, x and y are necessarily distinct.

assurance that y will not be 'ranked' before x in the social preference.[3] I can now define weak versions of liberalism.

Definition 9 (Weak liberalism, general 2-V$^+$) For all $i \in N$, there exist a_i and $b_i \in X$ such that i is a $\{a_i, b_i\}$-vetoer.

Definition 10 (Minimal weak liberalism, minimal 2-V$^+$). There exist two individuals i and $j \in N$, and a, b, c, $d \in X$ such that i is a $\{a,b\}$-vetoer and j is a $\{c,d\}$-vetoer.[4]

One of the possible Pareto extension functions (based on the weak form of the Pareto principle of condition P) that will be discussed later, indicates that having weak liberalism will not make $\succeq_S$ non-quasi-transitive. Consequently, problems can be met only for the transitivity of social preference, or, as will be seen, for binary relations which are, in some sense, between preorders and quasi-transitive binary relations. We will define two of these 'intermediate' relations, interval orders and semiorders. These definitions can be stated as properties of the asymmetric part of the complete binary relation $\succeq$ over X.

Definition 11 A binary relation $\succeq$ on X is an *interval order* if for all w, x, y, and $z \in X$, $w \succ y$ and $x \succ z \Rightarrow w \succ z$ or $x \succ y$.

The set of interval orders over X will be denoted by $\mathbb{I}$.

Definition 12 *A binary relation $\succeq$ on X is a semiorder* if it is an interval order and if for all w, x, y, and $z \in X$, $w \succ x$ and $x \succ y \Rightarrow w \succ z$ or $z \succ y$.

The set of semiorders will be denoted by $\mathbb{S}$. These two concepts have mainly been introduced in measurement theory to deal with possible intransitive indifference. Although indifference is not necessarily transitive contrary to what is the case with preorders, it should be noted that, for both concepts, $\succ$ is transitive (see Fishburn (1985) and Suppes, Krantz, Luce, and Tversky (1989)).

My first result is for social welfare functions.

Definition 13 A $\mathbb{P}$-*valued aggregation function* (or *social welfare function*) is a function $f : \Pi' \to \mathbb{P}$.

Theorem 2 *If there are at least two individuals, there is no $\mathbb{P}$-valued aggregation function satisfying U, P and minimal 2-V$^+$, provided that, in the definition of minimal 2-V$^+$, $\{a,b\} \neq \{c,d\}$.*

---

[3] I use quotation marks for 'ranked' since a ranking can only be meaningful for X being finite with a (some) social state(s) ranked first, etc.

[4] Although I suspected that the notion of weak liberalism was not new, it is in reading (Sen, 1982) that I discovered that it had been first proposed in a different form by Karni (1974) in a working paper that is still unpublished. In Karni's paper the condition applies to subsets of alternatives. This working paper also includes important comments about the Cartesian product structure and unconditional preferences.

Proof. Let f be an aggregation function. i is a $\{a,b\}$-vetoer and j is a $\{c,d\}$-vetoer. $\{a,b\} \neq \{c,d\}$. Note that $\#X \geq 3$. Let us assume first that $\{a,b\} \cap \{c,d\} \neq \emptyset$. Without loss of generality, assume that $b = c$. Let π be a profile such that $a \succ_i b$, $b \succ_j d$, and for all $k \in N$, $d \succ_k a$. Then, we have for i, $d \succ_i a \succ_i b$ and for j, $b \succ_j d \succ_j a$. Since i is a $\{a,b\}$-vetoer, we have $a \succeq_S b$, and since j is a $\{b,d\}$-vetoer, we have $b \succeq_S d$. If f were $\mathbb{P}$-valued, by transitivity, we should have $a \succeq_S d$, but by condition P, we have $d \succ_S a$, a contradiction.

Consider now the case where $\{a,b\} \cap \{c,d\} = \emptyset$. Let π be a profile such that $a \succ_i b$, $c \succ_j d$ and for all $k \in N$, $b \succ_k c$ and $d \succ_k a$. Then, we have $d \succ_i a \succ_i b \succ_i c$ and $b \succ_j c \succ_j d \succ_j a$. By condition P, we have $b \succ_S c$ and $d \succ_S a$. Since j is a $\{c,d\}$-vetoer, we have $c \succeq_S d$. If f were $\mathbb{P}$-valued, $b \succ_S c$ and $c \succeq_S d$ and $d \succ_S a$ would imply $b \succ_S a$. But $a \succeq_S b$ since i is a $\{a,b\}$-vetoer, a contradiction. □

Although one can obtain a cycle with two options ($a \succ_S b$ and $b \succ_S a$), three options are necessary for an intransitivity. Now, I will consider the case of interval orders.

Theorem 3 *If there are at least two individuals, there is no $\mathbb{I}$-valued aggregation function satisfying U, P and minimal 2-V$^+$, provided that, in the definition of minimal 2-V$^+$, $\{a,b\} \cap \{c,d\} = \emptyset$.*

Proof. Let f be an aggregation function. Obviously, $\#X \geq 4$. Let π be a profile such that $a \succ_i b$, $c \succ_j d$, and for all $k \in N$, $b \succ_k c$ and $d \succ_k a$. Observe that $d \succ_i a \succ_i b \succ_i c$ and $b \succ_j c \succ_j d \succ_j a$. Since $b \succ_S c$ and $d \succ_S a$ by condition P, we should have if $\succeq_S$ were an interval order, i.e., if f were $\mathbb{I}$-valued, $b \succ_S a$ or $d \succ_S c$. But we have ($a \succeq_S b$ and $c \succeq_S d$) since i is a $\{a,b\}$-vetoer and j is a $\{c,d\}$-vetoer, a conjunction that is the negation of the disjunction ($b \succ_S a$ or $d \succ_S c$) (given completeness of $\succeq_S$). □

If $\#X = 3$, the condition given in Definition 11 is reduced to the transitivity of $\succ_S$. Then, the Pareto extension function based on the weak form of the Pareto principle given in condition P is a counter-example. Let me define this Pareto extension function.

Definition 14 Let $\pi \in \Pi'$ and $x, y \in X$. f is the *weak Pareto extension function* if $x \succ_S y \Leftrightarrow \forall k \in N\ x \succ_k y$, and $y \succeq_S x$ otherwise.

One can easily see that $\succ_S$ is transitive and that each individual i is a $\{x,y\}$-vetoer for all $\{x,y\} \subseteq X$.

Semiorders are 'between' preorders and interval orders. Can we expect to have some progress? In fact, one obtains, as could be expected, a theorem 'between' Theorem 2 and 3, although the refinement is quite modest.

Theorem 4 *If there are at least two individuals, there is no $\mathbb{S}$-valued aggregation function satisfying U, P and minimal 2-V$^+$, provided that, in the definition of minimal 2-V$^+$, $\{a,b\} \neq \{c,d\}$ and provided that $\#X \geq 4$.*

Proof. Let f be an aggregation function. Suppose first that $\{a,b\} \neq \{c,d\}$, but $\{a,b\} \cap \{c,d\} \neq \emptyset$. Without loss of generality, assume that $a = d$. Consider a profile

π such that $a \succ_i b$, $c \succ_j a$, and for all $k \in N$, $b \succ_k e$ and $e \succ_k c$, where e is a fourth social state. Note that $a \succ_i b \succ_i e \succ_i c$ and $b \succ_j e \succ_j c \succ_j a$. Then, by condition P, $b \succ_S e$ and $e \succ_S c$. If $\succ_S$ were a semiorder, i.e., if f were $\mathbb{S}$-valued, we should have ($b \succ_S a$ or $a \succ_S c$). But we have ($a \succeq_S b$ and $c \succeq_S a$), since i is a $\{a,b\}$-vetoer and j is a $\{a,c\}$-vetoer, a conjunction that is the negation of the disjunction ($b \succ_S a$ or $a \succ_S c$), a contradiction.

If $\{a,b\} \cap \{c,d\} = \emptyset$, the proof is of course similar to the proof of Theorem 3. □

Theorems 2–4 have obvious corollaries (omitted) when minimal weak liberalism is replaced by weak liberalism.

This hierarchy of results is reminiscent of the family of Arrovian impossibility theorems. In Sect. 4, I will present a parallel between these two families of impossibility results.

4 Comparing Sen-Type Impossibilities with Arrovian Impossibilities

I will very briefly state Arrovian theorems with their necessary supplementary definitions. For all these theorems N is supposed to be finite with $\#N = n$. In all the definitions of this section, we suppose that f is an aggregation function.

Definition 15 (Independence-binary form, I) Let π and $\pi' \in \Pi'$ with $\pi : i \mapsto \succeq_i$ and $\pi' : i \mapsto \succeq'_i$. Consider any $x, y \in X$. If $\succeq_i |\{x,y\} = \succeq'_i |\{x,y\}$ for all $i \in N$, then $\succeq_S |\{x,y\} = \succeq'_S |\{x,y\}$ where $\succeq_S = f(\pi)$ and $\succeq'_S = f(\pi')$. ($\succeq_S |\{x,y\}$ is the restriction of $\succeq_S$ to $\{x,y\}$.)

Definition 16 A *dictator* is an individual who is a $\{x,y\}$-dictator for all $\{x,y\} \subseteq X$.

Definition 17 (Condition D$^-$, non-dictatorship) There is no dictator.

AQ: The reference "Arrow (1951)" is cited but not listed. Please check and provide in the list.

Theorem 5 (Arrow, 1950, 1951, 1963) If $n \geq 2$ and $\#X \geq 3$, there is no $\mathbb{P}$-valued aggregation function (social welfare function) satisfying U, P, I and D$^-$.

The Pareto extension function is a counter-example to a theorem which would be similar to Arrow's theorem except that $\mathbb{P}$-valuedness would be replaced by $\mathbb{Q}$-valuedness. However, if non-dictatorship is replaced by a no-vetoer condition, the result is restaured.

Definition 18 A *vetoer* is an individual who is a $\{x,y\}$-vetoer for all $\{x,y\} \subseteq X$.

Definition 19 (Condition V$^-$, no-vetoer) There is no vetoer.

Theorem 6 (Gibbard, 1969) *If $n \geq 2$ and $\#X \geq 3$, there is no $\mathbb{Q}$-valued aggregation function satisfying U, P, I and V$^-$.*

There is more in the original Gibbard's paper, since Gibbard shows that if f is a $\mathbb{Q}$-valued aggregation function satisfying U, P, I, there exists an oligarchy, a group of individuals having full power if they act unanimously and whose members are all vetoers. For $n = 2$, majority rule gives a quasi-transitive social preference, but, in this case, each of the two individuals is a vetoer.

If one considers $\mathbb{A}$-valued aggregation function (social decision function), one can still get an impossibility provided that the aggregation function is increasing (this property is often called strict monotonicity or positive responsiveness).

Definition 20 (Increasing aggregation function, IF) An aggregation function f is an increasing aggregation function if for all π, $\pi' \in \Pi'$, and all $x, y \in X$, if for all $i \in N$, $(x \succ_i y \Rightarrow x \succ'_i y$ and $x \sim_i y \Rightarrow x \succeq'_i y)$, and there exists $j \in N$ such that $(y \succ_j x$ and $x \succeq'_j y)$ or $(x \sim_j y$ and $x \succ'_j y)$, then $x \succeq_S y \Rightarrow x \succ'_S y$, where $\succeq_S = f(\pi)$ and $\succ'_S$ is the asymmetric part of $\succeq'_S = f(\pi')$.

Intuitively this condition means that if option x does not decrease vis-à-vis option y in all individual preferences, and if x increases vis-à-vis y in at least one individual preference, then, this increase must be reflected at the social level, when possible.

Theorem 7 (Mas-Colell & Sonnenschein, 1972) *If $n \geq 4$ and $\#X \geq 3$, there is no $\mathbb{A}$-valued increasing aggregation function (increasing social decision function) satisfying U, P, I,and V^-.*

A rather confidential result offers a refinement of this theorem.

Definition 21 A *quasi-dictator* is an individual i who is a vetoer such that for all $\pi \in \Pi'$, and all $x, y \in X$, $x \succ_i y$ and $x \sim_S y \Rightarrow$ for all $j \neq i$, $y \succ_j x$, where $\sim_S$ is the symmetric part of $\succeq_S = f(\pi)$.

A quasi-dictator is then nearly exactly similar to the Arrovian dictator except in the case where all other individuals have a strict preference that is the inverse of his strict preference.[5]

Definition 22 (Condition Q-D$^-$, non-quasi-dictatorship) There is no quasi-dictator.

Theorem 8 (Bordes & Salles, 1978) *If $n \geq 4$ and $\#X \geq 3$, there is no $\mathbb{A}$-valued increasing aggregation function (increasing social decision function) satisfying U, P, I, and Q-D$^-$.*

Surprisingly, Arrovian theorems regarding semiorder-valued (or interval order-valued) aggregation functions appeared later. It was, however, quite important to know that Arrow's theorem could be obtained without postulating that the social preference be a complete preorder. These important and somewhat neglected results are due to Blair and Pollack (1979) and Blau (1979) (in fact their papers appeared in the same issue of the *Journal of Economic Theory*).

---

[5] I prefer to use 'his' rather than 'her' for this sort of people! I hope that this is not politically incorrect.

Table 1 Comparing Arrovian and Sen-type impossibility theorems

	Impossibility theorems	
Aggregation function	Arrovian (with U+I+P)	Sen-Type (with U+P)
$\mathbb{P}$-valued	D^-	Minimal 2-V^+
$\mathbb{S}$-valued	D^-	Minimal 2-V^+
$\mathbb{I}$-valued	D^-	Minimal 2-V^+
$\mathbb{Q}$-valued	V^-	Minimal 2-D^+
$\mathbb{A}$-valued	(V^- or Q-D^-)+IF	minimal 2-D^+

Theorem 9 (Blair & Pollack, 1979; Blau, 1979) *If $n \geq 4$ and $\#X \geq 4$, there is no $\mathbb{S}$-valued aggregation function or no $\mathbb{I}$-valued aggregation function satisfying U, P, I and D^-.*

To the best of my knowledge and contrary to the case of Sen-type impossibility theorems, there is no way to make a distinction between $\mathbb{S}$-valued and $\mathbb{I}$-valued aggregation function. Of course, for three options, the properties of interval orders and of semiorders are both reduced to quasi-transitivity and then the Pareto extension function gives an appropriate counter-example.

Table 1 provides a useful summary of the preceding results and establishes a sort of parallel.

What this table shows is the connection between D^- and 2-V^+, and between V^- and 2-D^+. However, as shown in the next section, this parallel should not be taken too seriously.

5 Discussion

In this section, I will outline the major differences between the two categories of impossibility theorems. The first one concerns the set of individuals *N*. As I mentioned at the beginning of the preceding section, for Arrovian impossibility theorems we assume that *N* is finite. From a historical point of view, after the publication of Arrow's papers and book, a question was whether this assumption was only there to make the proof of the theorem easier (and, after all, *N* finite is a rather easily justifiable property) or whether this was a necessity. Fishburn (1970) was the first to show that it was a necessary assumption. With *N* infinite, Fishburn provided a counter-example. Incidentally, Fishburn's short paper was the starting point of an active research with possibly metaphysical implications.

As clearly stated in the Table 1, independence (of irrelevant alternatives–different from the Chernoff variety) is a very important feature of Arrovian theorems. This property is not used at all in Sen-type theorems. The interplay between the various conditions of social rationality, the different Pareto principles and the properties of decisiveness and semi-decisiveness in the light of Sen's Paretian epidemic deserves to be scrutinized in a future work.[6]

Both categories of impossibility theorems are stated with condition U. It is, however, possible to define smaller domains so that we could obtain impossibilities (see Kalai and Muller (1977) for Arrovian social welfare functions). For Sen-type theorems, one only need a domain rich enough to include the profiles leading to the impossibilities.

Sen's theorem is often considered with social choice functions rather than $\mathbb{A}$-valued functions (social decision functions). A *social choice function* is a function $f^\star : 2^X - \emptyset \times \Pi' \to 2^X - \emptyset$ such that for all $S \in 2^X - \emptyset$ and all $\pi \in \Pi'$, $f^\star(S,\pi) \subseteq S$. This function selects social states in each non-empty subset of the set of social states. To define *liberalism*, one can say that for all $i \in N$, there exist two social states $\{a_i, b_i\}$ such that for all non-empty $S \subseteq X$ and all $\pi \in \Pi'$, if $a_i \in S$ and $a_i \succ_i b_i$, then $b_i \notin f^\star(S,\pi)$ and if $b_i \in S$ and $b_i \succ_i a_i$, then $a_i \notin f^\star(S,\pi)$. A similar definition can be given for *minimal liberalism* by restricting the definition to only two individuals in a way similar to what was done previously. Using this framework, the proof of the corresponding theorem consists in emptying $f^\star(S,\pi)$ for specific S and π. This proof is as easy as the proof for $\mathbb{A}$-valued functions, and it is not surprising given the strong relations between $\succ$-cycles and the absence of maximal elements (or between acyclicity and the existence of maximal elements, as previously mentioned). Things are less simple with weak liberalism. With liberalism, if $a_i \succ_i b_i$, then b_i is rejected from all choice sets $f^\star(S,\pi)$ such that $a_i \in S$. This means that $f^\star(\{a_i, b_i\},\pi) = \{a_i\}$. This corresponds intuitively well to $a_i \succ_S b_i$. Weak liberalism only tells us that $a_i \succeq_S b_i$. Intuitively but also in the standard choice literature, this corresponds to $a_i \in f^\star(\{a_i, b_i\},\pi)$. This means that a_i must be selected but it does not say that b_i is rejected, and, furthermore, we cannot say anything about the selection from the other sets to which a_i belongs. Of course this difficulty can be probably taken care of by imposing to $f^\star$ properties borrowed from the revealed preference and rationalizability literature (this will be the subject of another paper).

Finally and this is the main difference, difference which is at the origin of recent major developments on non-welfaristic issues in normative economics, Sen-type theorems are non-welfaristic.[7] The word *welfarism* is associated with the idea that the goodness of social states are evaluated only on the basis of individual utilities attached to these social states. This leads to the following observation. If we have four social states w, x, y and z and if each individual i attributes the same utility to w and to x, and the same utility to y and to z, then, the social ranking of w and y must be the same as the social ranking of x and z. This can lead to various properties of neutrality for functions defined on profiles of utility functions and this can be extended to profiles of individual preferences in which case one obtains intra or inter profiles neutrality (for an introduction to the non-welfaristic literature, I rec-

[6] Paretian epidemic was first described and analyzed in Sen (1976, 1982). This remark was prompted by a comment from the referee, a comment that is very gratefully acknowledged.

[7] The following remarks owe much to Kotaro Suzumura. I am very grateful to him for calling my attention to this crucial aspect.

ommend the remarkable article by Pattanaik published in a too confidential book, see ~~Pattanaik (1994))~~. Intuitively, neutrality means that names of social states do not matter. The liberalism conditions obviously violate neutrality since specific social states are attached to specific individuals.

6 Conclusion

In this chapter, Sen's liberalism conditions have been weakened. Partial dictatorship has been replaced by partial veto. This weakening could be justified to some extent by a wish to consider rights as the ability to prevent something to happen rather than the obligation to prevent something to happen. Unfortunately, this weakening does not take us very far since impossibilities will occur if we replace *social decision functions* by *social welfare functions* or other aggregation functions 'between' social welfare functions and social decision functions. Considering a kind of hierarchy of aggregation functions on the basis of the collective rationality of the associated social preference, it was natural to compare Sen-type impossibility theorems with Arrovian impossibility theorems. Although this comparison shows that there is some interesting relations from a mathematical point of view, the discrepancies are probably more important from an interpretative point of view. In particular, the discrepancy between welfaristic aspects of the Arrovian theorems and the non-welfaristic aspects of Sen-type theorems is a very important one that has been outlined. Sen's theorem has been justly considered as the foundational result of non-welfaristic normative economics. ~~In,~~ forthcoming papers, I will consider the extension of the results of the present paper to a choice-theoretic framework. I will also come back to the debate between ability (possibility) and obligation by using modal logic.

Appendix 1

Blau (1979) and to some extent (Blair & Pollack, 1979) (and probably others) define interval orders and semiorders differently from Definitions 11 and 12. Given that $\succeq_S$ is complete, the following propositions show the equivalence between the definitions of the present chapter which are borrowed from Fishburn (1985) and Suppes et al. (1989) and the definitions used by Blau (these results are probably already known, but I have been unable to find where it could be).

Proposition 1 *Let $\succeq$ be complete. Then the following two statements are equivalent.*

(i) *For all $w, x, y,$ and $z \in X$, $w \succ y$ and $x \succ z \Rightarrow w \succ z$ or $x \succ y$.*
(ii) *For all $w, x, y,$ and $z \in X$, $w \succ x$ and $x \sim y$ and $y \succ z \rightarrow w \succ z$.*

Proof. (i) $\Rightarrow$ (ii). Suppose (i) and $w \succ x$ and $x \sim y$ and $y \succ z$. Since $w \succ x$ and $y \succ z$, then by (i), $w \succ z$ or $y \succ x$. But $\neg y \succ x$ since $x \sim y$, and then $w \succ z$.
(ii) $\Rightarrow$ (i). Suppose (ii) but not (i). Then there exist a, b, c and $d \in X$ such that $a \succ c$ and $b \succ d$ and $\neg(a \succ d$ or $b \succ c)$. But $\neg(a \succ d$ or $b \succ c)$ is equivalent to $(d \succeq a$ and $c \succeq b)$ since $\succeq$ is complete. $c \succeq b$ is either $c \succ b$ or $c \sim b$. If $c \succ b$, $a \succ c$ and $c \succ b$ and $b \succ d$ imply $a \succ d$ since $\succ$ is transitive by (ii) (take $x = y$ and note that $\sim$ is reflexive). This contradicts $d \succeq a$. If $c \sim b$, $a \succ c$ and $c \sim b$ and $b \succ d$ imply $a \succ d$ by (ii), contradicting $d \succeq a$ again. □

The next result concerns Definition 12.

Proposition 2 *Let $\succeq$ be complete. Then the following two statements are equivalent.*

(i) For all w, x, y, and $z \in X$, $w \succ x$ and $x \succ y \Rightarrow w \succ z$ or $z \succ y$.
(ii) For all w, x, y, and $z \in X$, $w \succ x$ and $x \succ y$ and $y \sim z \Rightarrow w \succ z$.

Proof. (i) $\Rightarrow$ (ii). Suppose we have (i) and that $w \succ x$ and $x \succ y$ and $y \sim z$. But $w \succ x$ and $x \succ y$ imply by (i) ($w \succ z$ or $z \succ y$). Since $y \sim z$, then $\neg z \succ y$, and $w \succ z$.
(ii) $\Rightarrow$ (i). Suppose we have (ii) and not (i). Then there exist a, b, c and $d \in X$ such that $a \succ b$ and $b \succ c$ and $\neg(a \succ d$ or $d \succ c)$, i.e., $(d \succeq a$ and $c \succeq d)$ by completeness of $\succeq$. If $c \succ d$, $a \succ b$ and $b \succ c$ and $c \succ d$ imply $a \succ d$ since $\succ$ is transitive by (ii) (take $y = z$ and note that $\sim$ is reflexive). But this contradicts $d \succeq a$. If $c \sim d$, $a \succ b$ and $b \succ c$ and $c \sim d$ imply $a \succ d$ by (ii), which contradicts $d \succeq a$. □

Acknowledgments This paper is dedicated to Kotaro Suzumura. It is a variation on and a tribute to Amartya Sen's 1970 landmark paper. I am grateful to Feng Zhang for the discussions on this topic and to Nick Baigent and other participants at the International Conference on Rational Choice, Individual Rights and Non-Welfaristic Normative Economics, Hitotsubashi University, Tokyo, 11–13 March 2006, for their remarks and comments. The conversation I had with Kotaro Suzumura after my talk there was crucial for the comparison section. The comments on previous drafts of the paper from an anonymous referee, Edi Karni and Prasanta Pattanaik were very helpful for the preparation of the final version. Financial support from the French ANR through contract NT05-1_42582 (3LB) is gratefully acknowledged.

References

Arrow, K. J. (1950). A difficulty in the concept of social welfare. *Journal of Political Economy*, *58*, 328–346
Arrow, K. J. (1963). *Social choice and individual values* (2nd ed.). New York (1st ed. 1951): Wiley
Barberà, S., Bossert, W., & Pattanaik, P. K. (2004). Ranking sets of objects. In S. Barberà, P. J. Hammond—, & C. Seidl (Eds.), *Handbook of utility theory, volume 2* (pp. 893–977). Dordrecht: Kluwer
Blair, D. H., & Pollack, R. A. (1979). Collective rationality and dictatorship: The scope of the Arrow theorem. *Journal of Economic Theory*, *21*, 186–194
Blau, J. H. (1979). Semiorders and collective choice. *Journal of Economic Theory*, *21*, 195–206
Bordes, G., & Salles, M. (1978). Sur l'impossibilité des fonctions de décision collective: Un commentaire et un résultat. *Revue d'Economie Politique*, *88*, 442–448

Brunel, A., & Salles, M. (1998). Interpretative, semantic and formal difficulties of the social choice approach to rights. In J.-F. Laslier, M. Fleurbaey, & A. Trannoy (Eds.), *Freedom in economics: New perspectives in normative economics* (pp. 101–111). London: Routledge

Fishburn, P. C. (1970). Arrow's impossibility theorem: Concise proof and infinite voters. *Journal of Economic Theory*, *2*, 103–106

Fishburn, P. C. (1985). *Interval orders and interval graphs*. New York: Wiley

Gaertner, W., Pattanaik, P. K., & Suzumura, K. (1992). Individual rights revisited. *Economica*, *59*, 161–177

Gärdenfors, P. (1981). Rights, games and social choice. *Nous*, *15*, 341–356

Gärdenfors, P. (2005). *The dynamics of thought*. Heidelberg: Springer

Gibbard, A. (1969). *Social choice and Arrow's conditions* (mimeo). University of Michigan

Gibbard, A. (1974). A Pareto consistent libertarian claim. *Journal of Economic Theory*, *7*, 388–410

Hammond, P. J. (1998). Some comments on Brunel and Salles. In J.-F. Laslier, M. Fleurbaey, & A. Trannoy (Eds.), *Freedom in economics: New perspectives in normative economics* (pp. 112–115). London: Routledge

Igersheim, H. (2006). *Invoking the Cartesian product structure on social states: New resolutions of Sen's and Gibbard's impossibility theorems* (mimeo). Universitat Autonoma de Barcelona

Kalai, E., & Muller, E. (1977). Characterizations of domains admitting nondictatorial social welfare functions and nonmanipulable voting procedures. *Journal of Economic Theory*, *16*, 457–469

Karni, E. (1974). *Individual liberty, the Pareto principle and the possibility of social decision functions* (mimeo). The Foerder Institute for Economic Research, Tel-Aviv University

Mas-Colell, A., & Sonnenschein, H. (1972). General possibility theorems for group decision functions. *Review of Economic Studies*, *39*, 185–192

Pattanaik, P. K. (1994). Some non-welfaristic issues in welfare economics. In B. Dutta (Ed.), *Welfare economics* (pp. 197–248). Delhi: Oxford University Press

Pattanaik, P. K. (1996). On modelling individual rights: Some conceptual issues. In K. J. Arrow, A. K. Sen, & K. Suzumura (Eds.), *Social choice re-examined, volume 2* (pp. 100–128). London: Macmillan

Pattanaik, P. K., & Xu, Y. (1990). On ranking opportunity sets in terms of freedom of choice. *Recherches Economiques de Louvain*, *56*, 383–390

Peleg, B. (1998a). Effectivity functions, game forms, games, and rights. *Social Choice and Welfare*, *15*, 67–80

Peleg, B. (1998b). Effectivity functions, game forms, games, and rights. In J.-F. Laslier, M. Fleurbaey, & A. Trannoy (Eds.), *Freedom in economics: New perspectives in normative economics* (pp. 116–132). London: Routledge

Saari, D. G. , & Pétron, A. (2006). Negative externalities and Sen's liberalism theorem. *Economic Theory*, *28*, 265–281

Sen, A. K. (1970a). *Collective choice and social welfare*. San Francisco: Holden-Day

Sen, A. K. (1970b). The impossibility of a Paretian liberal. *Journal of Political Economy*, *78*, 152–157

Sen, A. K. (1976). Liberty, unanimity, and rights. *Economics*, *49*, 217–245

Sen, A. K. (1982). *Choice, welfare and measurement*. Oxford: Blackwell

Suppes, P., Krantz, D. H., Luce, R. D., & Tversky, A. (1989). *Foundations of measurement, volume 2*. New York: Academic Press

Suzumura, K. (2008). Rights, opportunities, and social choice procedures. In K. J. Arrow, A. K. Sen, & K. Suzumura (Eds.), *Handbook of social choice and welfare, volume 2*. Amsterdam: Elsevier

AQ: Please provide page range for the reference "Suzumura (2008)."