An Existence Result for Farsighted Stable Sets of Games in Characteristic Function Form[1]

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Abstract

In this paper we show that every finite-player game in characteristic function form obeying an innocuous condition (that the set of individually rational pay-off vectors are bounded) possesses a farsighted von-Neumann-Morgenstern stable set.

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1 Introduction

In this paper we show that every finite-player game in characteristic function form obeying an innocuous condition (that the set of individually rational pay-off vectors are bounded) possesses a farsighted von-Neumann-Morgenstern stable set.

It is well-known that characteristic function form is the oldest and perhaps the most common representation of game situations for analysing group behaviour in such situations. One of the most famous results in game theory is that there exist games in characteristic function form that admit no von-Neumann-Morgenstern solution (Lucas, 1968) with respect to the classical direct dominance relation.

Studies on von-Neumann-Morgenstern solution (or von-Neumann-Morgenstern stable sets) in the environment of general coalitional games in characteristic function form (possibly without side payments) go as far back as Aumann and Peleg (1960). One significant development in the literature on coalitional games was the idea of indirect dominance introduced by Harsanyi (1974) to take into account the fact that the players may be farsighted—while making a move, they may take into account the further moves by other groups of players that may ensue. This idea was further developed in Chwe (1994) which, in turn, generated a sizeable literature on analysing game situations using indirect or farsighted dominance.

Recently Diamantoudi and Xue (2005) showed that the game in the famous Lucas counter-example admits a stable set with indirect dominance relations. This was pursued further in Beal et al. (2008) who showed that, in stark contrast to that for direct dominance relation, at least one stable set with respect to the indirect (or farsighted) dominance relation exists for every coalitional game in characteristic function form with *transferable pay-offs* (or a TU game, in short).[2] In this paper we generalize this result of Beal et al. (2008) to show that even without side payments, every game in characteristic function form obeying an innocuous condition (that the set of individually rational pay-off vectors are bounded) possesses a von-Neumann-Morgenstern stable set with respect to the farsighted dominance relation.

There exists a series of recent works studying farsighted stable sets or some analogues of them in different environments without transferable pay-offs. These include Diamantoudi and Xue (2003), Kamijo and Muto (2007), Mauleon et al. (2008), Vartiainen (2008) etc. However, the natures of the environments they studied are quite different to that for games in characteristic function form.

The next section gives the preliminary definitions and notation. The results are given in Section 3. Section 4 contains some concluding comments.

[2] Beal et al. (2008), quite naturally, worked with the set of imputations as the set of outcomes under consideration for their analysis. However, we have checked that one can get similar existence results (using their own proof-technique) as theirs even when the set of outcomes under consideration is expanded to the set of, say, efficient pay-off vectors or the set of all feasible pay-off vectors.

2 Preliminary definitions, assumptions and notation

NTU games

A finite-player game in characteristic function form (without side-payments) or in short, an NTU-game (N,V) consists of a finite player set N (with a cardinality of, say, n) and a function $V(\cdot)$ that assigns to every nonempty subset S of N, called a coalition, a set $V(S) \subseteq R^S$. It is assumed that $V(\varnothing) = \varnothing$.
For any coalition *S*, any $x \in V(S)$ specifies the payoff for every $i \in S$. We assume that for each coalition *S*, $V(S)$ is a nonempty, closed and proper subset of R^S. For each coalition S, $V(S)$ is also *comprehensive,* i.e., if $x \in V(S)$ and $y \le x$, then $y \in V(S)$. The relation $y \le x$ means that either $y_i < x_i$ or $y_i = x_i$ holds for every $i \in S$. Given $T \subseteq N$, $S \subseteq T$ and $x \in R^T$, denote the S coordinates of *x* by $x|_S$.
Let $b_i := \max\{y_i | y \in V(\{i\})\} < \infty$. For a set A, by bdA we denote the boundary of A and by $\operatorname{int} A$ we denote the interior of A.
A vector $x \in R^N$ is *efficient* if $x \in bd(V(N))$.
An NTU game is *essential* if $\exists\ y \in V(N)$ such that $y_i \ge b_i\ \forall i \in N$ and $y_i > b_i$ for at least one $i \in N$. It is inessential otherwise. We work with the class of essential games.
For any any $S \subseteq N$ and any $x, y \in V(N)$ $y \prec_S x$ means that $y_i < x_i$ for every $i \in S$.
Let $IR(N,V)$ be the set of *individually rational* pay-off vectors, that is,
$IR(N,V) = \{x \in V(N) | x_i \ge b_i\ \forall i \in N\}$.
A pay-off vector is an *imputation* if it is individually rational as well as efficient. The set of imputations is denoted by *I(N,V).*
The set of imputations is the set of outcomes or allocations that we shall study.

Farsighted dominance

A coalition S can enforce an imputation x from an imputation *z*, denoted as, $z \to_S x$, if and only if $x|_S \in V(S)$.

Remark 1: This notion of enforcement, although common in literature for analysing such environments (see Harsanyi (1974), Chwe (1994), Beal et al. (2008) for example), is not very satisfactory because this implies that a coalition can enforce pay-offs even for the members outside itself. However, we are not aware of any simple and immediate way of resolving this while remaining within this environment of characteristic functions. So, we stick to this. However, of course, the comprehensive body of literature on non-cooperative foundation of coalition and coalitional behaviour addresses this issue.
The notion of indirect or farsighted dominance we use is given below.

Definition 1 An imputation z is indirectly dominated by another imputation x, denoted as $x >> z$, if there exist $z = z^0, z^1, z^2, ..., z^m = x$ and $S_0, S_1, S_2, ..., S_{m-1}$ such that $z^i \to_{S_i} z^{i+1}$ and $z^i \prec_{S_i} x$ for $i = 0,1,2,...,m-1$.

Recall that this dominance notion captures farsightedness on the part of the players, because, when initiating a chain of enforcements, each coalition active at any outcome of the chain compares the pay-offs at the *final* outcome of the chain to the immediate pay-offs and not the very next outcome in the chain.

Stable sets with respect to farsighted dominance

Definition 2 Let (N,V) *be an NTU-game. A subset* K *of* $I(N,V)$ *is a farsighted stable set if (a) for all* $x,z \in K$ *neither* $x >> z$ *nor* $z >> x$ *(internal stability), and (b) for all* $z \in I(N,V) \setminus K$ *there exists* $x \in K$ *such that* $x >> z$ *(external stability).*

An assumption

Throughout the following analysis we assume the following.

Assumption 1: (Boundedness of Individually Rational Pay-off Vectors) The set *IR(N,V)* is a bounded set.

This assumption, of course, is quite common in the literature; some notable examples being Aumann (1985) (in his work on axiomatization of NTU value) or Peleg (1992) (in his axiomatization of the NTU core). Representation of any reasonable economy in characteristic function form should also satisfy this.

3 Farsighted stable sets for NTU games

Recall that we restrict our analysis to the set of imputations.
Below we show that for every such NTU game, a farsighted stable set exists. To show this, first we prove a proposition which is a generalization of Theorem 1 in Beal et al. (2008). Next we prove the main result for which this result is useful.

Proposition 1. *Suppose* $x \in I(N,V)$ *be such that for some* $y \in I(N,V)$ $x >> y$ *and* $z \in I(N,V)$ *is such that for some* $i(x,z) \in N$**,** $x_{i(x,z)} > z_{i(x,z)}$. *Then* $x >> z$.

The proof uses the following lemmata. The proof of this proposition uses the idea behind the proof of Theorem 1 in Beal et al. (2008).

Lemma 1: *Let* $x,z \in I(N,V)$ *be such that* $x >> z$. *Then* $\exists S \subseteq N$ *such that* $1 < S < |N|$ *with* $x|_s \in V(S)$. *And for every* $i \in S$ *the relation* $x_i > b_i$ *holds.*

Proof: The relation $x|_s \in V(S)$ must hold for a coalition S to be able to enforce an allocation x.
Suppose N is the coalition that fulfills $x|_s \in V(S)$. Then N acts in the last step of the chain $z \rightarrow_{S_0} x^1 ... x^{k-1} \rightarrow_{S_{k-1}} x$. Then $x^{k-1} \prec_N x$. As $x \in I(N,V)$, x^{k-1} would have to

be in the interior of $V(N)$. But this contradicts our assumption that x^{k-1} is an imputation. So, $|S_{k-1}| < |N|$.

Suppose a singleton coalition $\{i\}$ is the coalition that fulfils $x|_s \in V(S)$. Then $\{i\}$ acts in the last step of the chain $z \to_{S_0} x^1 ... x^{k-1} \to_{S_{k-1}} x$. Then $x^{k-1} \prec_i x$. As x_{k-1} is an imputation, i.e. $x_j^{k-1} \geq b_j \quad \forall j \in N$, and as $p \to_{\{i\}} q$ only if $p_i \leq b_i$, we have a contradiction.

To prove the last part of the lemma, consider again that $x_i^{k-1} < x_i \quad \forall i \in S_{k-1}$ holds as we have $x^{k-1} \prec_{S_{k-1}} x$. As b_i is the lower bound of a player's payoff, $b_i < x_i \quad \forall i \in S_{k-1}$ holds. □

Lemma 2: For every $i \in N$, $\exists \overline{x}_i \geq b_i$ such that the vector $(b_1, b_2, ..., b_{i-1}, \overline{x}_i, b_{i+1}, ..., b_N)$, i.e. $(\overline{x}_i, (b_j)_{j \in N \setminus i})$ is in the $bdV(N)$.

Proof: By essentialness and comprehensiveness, $b = (b_1, ..., b_N) \in V(N)$. By Assumption 1, $\exists y$ large enough, such that $(y, b_{-i}) \notin V(N)$. Therefore, since $V(N)$ is closed, $\exists \overline{x}_i$ such that $(\overline{x}_i, b_{-i}) \in bdV(N)$.□

Proof of Proposition 1: We consider two cases: Case 1: $N \setminus S = \{i(x,z)\}$; Case 2: $N \setminus S \neq \{i(x,z)\}$.

Case 1: Consider the following chain of moves from z to x. First $z \to_{\{i(x,z)\}} y^1$ with $y^1_{i_{y^1}} = \overline{x}_{i_{y^1}}$ for some $i_{y^1} \in S$ and $y_i^1 = b_i$ for every $i \neq i_{y^1}$. Then $y^1 \to_{\{j\}} y^2$ with $j \in S \setminus \{i_{y^1}\}$, $y^2_{i(x,z)} = \overline{x}_{i(x,z)}$ (see Lemma 2 for the definition of $\overline{x}_{i(x,z)}$) and $y_k^2 = b_k$ for all $k \in S$. Finally, $y^2 \to_S x$. This enforcement chain is an indirect dominance as $x >_{i(x,z)} z$, $x >_j y^1$ and $x >_S y^2$.

Case 2: Consider the following chain of moves from z to x. First $z \to_{\{i(x,z)\}} y^1$ with $y_i^1 = b_i$ for every $i \neq i_{y^1} \notin S$ and $y^1_{i_{y^1}} = \overline{x}_{i_{y^1}}$. Then $y^1 \to_S x$. The enforcement chain is an indirect dominance as $x >_{i(x,z)} z$ and $x >_S y^1$.□

Now we prove the main theorem.

Theorem 1: *For any game (N,V) satisfying Assumption 1, a farsighted stable set exists.*

Proof: Suppose *I(N,V)* be such that it has no two elements *x* and *y* for which $x >> y$. Then, *I(N,V)* itself is a farsighted stable set.

Suppose otherwise, i.e., suppose that there exists an imputation x such that $x >> y$ for some $y \in I(N,V)$. Let $G(x) = \{y \in I(N,V) | y \geq x\} \setminus \{x\}$ and $D(x) = \{y \in G(x) | y >> x\}$. We consider three cases.

Case 1: Suppose $G(x) = \phi$. Then, by Proposition 1 the singleton set $\{x\}$ is a farsighted stable set.

Case 2: Suppose $G(x) \neq \phi$, but $D(x) = \phi$.Then, we show below that $\{x\} \cup G(x)$ is a farsighted stable set.
External stability: For every $z \in I(N,V) \setminus G(x)$, it is not the case that $z \geq x$ holds. Therefore, $\forall z \in I(N,V) \setminus G(x)$ there is some $i(x,z) \in N$ such that $x_{i(x,z)} > z_{i(x,z)}$. Then, by Proposition 1, external stability of $\{x\} \cup G(x)$ is given.
Internal stability: To check that $G(x)$ is internally stable take $y^1, y^2 \in G(x)$ and suppose $y^2 >> y^1$. By the definition of indirect dominance there must be a chain $y^1 \rightarrow_{S_0} z^1 ... z^{k-1} \rightarrow_{S_{k-1}} y^2$ and $y_j^2 > y_j^1$ for every $j \in S_0$. Since $y^1 \in G(x)$, for every $j \in S_0$, $y_j^2 > x_j$. But then $y^2 >> x$ which is a contradiction as $D(x) = \phi$.

Case 3: Suppose $D(x) \neq \phi$.
Pick some $z \in D(x)$ as follows:
$z_1 \geq y_1 \forall y \in D(x)$
$z_2 \geq y_2 \forall y \in \{w \in D(x) | w_1 = z_1\}$
$z_3 \geq y_3 \forall y \in \{w \in D(x) | w_1 = z_1; w_2 = z_2\}$
And in general, for any k less than or equal to n,
$z_k \geq y_k \forall y \in \{w \in D(x) | w_l = z_l \forall l < k\}$
Note that by construction $z \in I(N,V)$ and $z >> x$. We show below that $K = \{z\} \cup \{y \in G(x) \setminus D(x) | y \geq z\}$ is a farsighted stable set.
Internal stability: Of course, by the definition of the set K, $z >/> y$ since $y \geq z$ holds for every $y \in K$. Now suppose, if possible, $y^1, y^2 \in \{y \in G(x) \setminus D(x) | y \geq z\}$ such that $y^2 >> y^1$. By the definition of indirect dominance there must be a chain $y^1 \rightarrow_{S_0} z^1 ... z^{k-1} \rightarrow_{S_{k-1}} y^2$ and $y_j^1 < y_j^2$ for every $j \in S_0$. Since $y^1 \in \{y \in G(x) \setminus D(x) | y \geq z\}$, $y^1 \geq x$. Then $y_j^1 \geq x_j$ for every $j \in S_0$ and $y_j^2 > x_j$ for every $j \in S_0$. But then $y^2 >> x$ which is a contradiction as $y^2 \in \{y \in G(x) \setminus D(x) | y \geq z\}$ and therefore $y^2 >/> y^1$.
External stability: Subcase 1: Suppose $y \in I(N,V) \setminus G(x)$. Then there exists some $i(x,y) \in N$ such that $z_{i(x,y)} \geq x_{i(x,y)} > y_{i(x,y)}$. Then, $z >> y$ by Proposition 1.
Subcase 2: Suppose $y \in D(x)$. Then by the construction of z there exists some $i(z,y) \in N$ such that $z_{i(z,y)} > y_{i(z,y)}$. Then, by Proposition 1, again we are done.

Subcase 3: Suppose $y \in \{G(x) \setminus D(x)\}$ but it is not the case that $y \geq z$. Then there exists some $i(z, y) \in N$ such that $z_{i(z,y)} > y_{i(z,y)}$. Then, by Proposition 1 we are again done. □

Concluding Remarks

Since a farsighted stable set is contained in the largest consistent set (Chwe (1994)), the result above also shows that the largest consistent set is non-empty for games in characteristic function form which is an interesting fact given the importance of this solution concept in analysing farsighted behaviour.

In this work we have only provided an existence theorem; we have not attempted to study in any detail the *structure* of the farsighted stable sets. This still remains to be explored.

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