Ranking completely uncertain decisions by the uniform expected utility criterion*

Nicolas Gravel[†], Thierry Marchant[‡] and Arunava Sen[§]

February 28th 2008
Keywords: Complete Ignorance, uncertainy, uniform probabilities, Expected Utility.
JEL classification numbers: D80, D81.

Abstract

We provide an axiomatic characterization of a family of criteria for ranking *completely uncertain* decisions. A completely uncertain decision is described by the set of all its consequences (assumed to be finite). Every criterion characterized can be thought of as assigning to all consequences of a decision an *equal probability* of occurence and as comparing decisions on the basis of the *expected utility* of their consequences for some utility function.

Keywords: complete uncertainty, ignorance,ranking sets, probability, expected utility, consequences

JEL classification number: D81

1 Introduction

It is common to categorize decision problems by the structure of the environment that is assumed to be known to the decision maker. In situations of *certainty*, the decision maker is assumed to know the *unique consequence* of every decision which can, therefore, be usefully identified by this unique

*We are indebted, with the usual disclaiming qualification, to Peter Wakker for his helpful and encouraging comments. The first author would also like to acknowledge the hospitality of the Centre de Sciences Humaines in New Delhi (French ministry of foreign affairs) where the work leading to this paper was conducted.

[†]Université de la Méditerranée & IDEP-GREQAM, Centre de la Vieille Charité, 2, rue de la Charité, 13 002 Marseille, France, nicolas.gravel@univmed.fr

[‡]Ghent University, Department of Data analysis, H. Dunantlaan 1B-9000 Gent, Belgium, Thierry.Marchant@UGent.be

[§]Indian Statistical Institute, 7, Shaheed Jit Singh Marg, New Delhi, 11 0016 India, asen@isid.ac.in.

consequence. In situations of *risk*, studied along the lines of Neumann and Morgenstern (1947), the decision maker knows the *probability distribution* over all consequences that decisions can have so that the problem of choosing the "right" decision amounts to that of choosing the "right" probability distribution over the set of consequences. In situations of *uncertainty*, decisions are described as functions - *acts* in Savage (1954) terminology - from a set of states of nature to a set of consequences. Finally, in situations of *complete uncertainty*, or *ignorance* as these are sometimes called, a decision is described even more parsimoniously by the *set* of all its (foreseeable) consequences. The problem of ranking decisions therefore amounts to a problem of ranking sets of these consequences.[1]

An excellent discussion of the difference between uncertainty and ignorance is provided in Pattanaik and Peleg (1984). Earlier comments on the arbitrariness of identifying all contingencies of a decision by a set of states of nature, which underlies the modelling of uncertainty, can be found in Arrow and Hurwicz (1972) and Luce and Raiffa (1957).

In the last twenty years or so, there has been a sizeable literature that has analyzed the problem of ranking sets of consequences in the context of choice under ignorance. Important contributions to this literature, surveyed by Barberà, Bossert, and Pattanaik (2004), include Barberà and Pattanaik (1984), Baigent and Xu (2004), Bossert (1989), Bossert (1997), Bossert, Pattanaik, and Xu (2000), Fishburn (1984), Fishburn (1992), Heiner and Packard (1984), Kannai and Peleg (1984), Nehring and Puppe (1996), Nitzan and Pattanaik (1984) and Pattanaik and Peleg (1984). With the noticeable exception of Baigent and Xu (2004) and Nitzan and Pattanaik (1984), all rankings of decisions that have emerged in this literature are based on the *best* and the *worst* consequences of the decisions or on some lexicographic extension thereof. The limitation of these "extremist" rankings for understanding actual decision making under ignorance is clear enough. Suppose we consider an investor facing two alternative investment strategies in some completely uncertain environment. If strategy A is adopted, the investor gains (net of the cost of investing) either one or one million dollars. If strategy B is adopted, then the investor's gain is either nothing, or any (integer) amount between \$900000 and \$999999. Hence, the two investment strategies can be described by:

$A = \{1, 1000000\}$

$B = \{0, 900000, 900001, ..., 999999\}$

Under the assumption that the ranking of certain (singletons) decisions is increasing in money, most rules studied in the literature that are "monotonically increasing" with respect to the worst and the best elements would rank A above B. Yet it is not clear that an actual investor placed under that

[1] A recent alternative to this approach, examined in Ahn (2008), is to view a decision as a set of probability distributions over a more fundamental consequence space.

circumstance would make the same ranking. For instance an investor who would be somehow capable of assigning probabilities of occurrence to consequences - even without being able to identify clearly the states of nature and the mapping that associates consequences to states of nature - could very plausibly rank B above A on the basis that the "expected utility" of the consequences is higher in B than in A. The *median-based* ranking of sets characterized in Nitzan and Pattanaik (1984), and which compares sets in terms of their median consequence with respect to the underlying ranking of certain outcomes, would also consider B to be a better decision than A in a situation like this. So would the *average Borda rule* characterized in Baigent and Xu which ranks sets according to the average Borda score of their elements.

Another limitation of many rankings considered in the literature, which applies also to the median-based and the average Borda rule criteria, is that they do not allow for a diversity of attitudes toward ignorance across individuals. Consider again the case of decisions with pecuniary consequences. If all decision makers prefer more money received for sure to less and follow any particular positional rule such as the maximin, the maximax, the median or some lexicographic extension thereof, they will all rank uncertain decisions in the same fashion. This feature of positional rankings is clearly restrictive. After all, the fact that two individuals prefer more money to less and have a choice behavior that obeys the same axioms should not imply that they have the same attitude with respect to uncertainty.

The small number of criteria that have emerged for ranking decisions under ignorance is especially striking when compared with what is observed in classical (Savagian) situations of uncertainty. In the later case one finds, along with "extremist" criteria that compare acts on the basis of their worst or best consequence as characterized in Arrow and Hurwicz (1972) and Maskin (1979), the well-known Expected Utility (EU) criterion characterized in Savage (1954) as well as many other "non-additive" criteria such as "Maximin Expected Utility over a Set of Priors" (characterized in Gilboa and Schmeidler (1989) and Casadesus-Masanell, Klibanoff, and Ozdenoren (2000)) or the "Choquet Expected Utility" criterion characterized in Schmeidler (1989). Contrary to their "extremist" or positional counterparts, individuals whose behavior satisfies a particular additive or non-additive EU criterion and who have the same preferences for the consequences do not need to have the same attitude toward uncertainty.

In this paper, we provide an axiomatic characterization of a family of criteria of choice under complete uncertainty, or ignorance, that is quite close in spirit to the classical EU family. Any criterion in this family can be viewed as ranking decisions (sets) on the basis of the expected utility of their consequences for some utility function, under the assumption that the decision maker assigns to every consequence of a decision an *equal probability* of occurrence. For this reason we refer to a criterion in this family

as to a *Uniform Expected Utility* (UEU) criterion. Beside the framework of analysis, the main difference between UEU criteria and standard EU ones lies in the uniform assumption made on probabilities. In our view, the uniform assumption is not unreasonable in the context of choice under complete uncertainty. A decision maker who ignores the mechanism by which consequences are produced as a function of the states of nature, and who is only capable of identifying the set of possible consequences of a decision has *a priori* no reason to believe one consequence to be more likely than another. This principle of *insufficient reason*, renamed "principle of indifference" by Keynes (1921), has been, after all, the main justification given by early probability theorists such as Bernouilli and Laplace, to their assumption of uniform probabilities as applying to "games of chance" (see also Jaynes (2003) for a recent justification of this principle).

The framework used to characterize the family of UEU criteria is similar to that assumed in the literature on choices under ignorance in the sense that we describe decisions as *finite* sets of consequences and we propose axioms that apply to the ranking of these sets. However, we differ somewhat from most of the literature by assuming that the universe of all conceivable consequences is large and has an *Archimedean structure*, as defined in Krantz, Luce, Suppes, and Tversky (1971). While we do not, for the main results, make explicit continuity assumptions on the framework (in particular we do not endow the universe of consequences with a topological structure), it is clear that a natural context to which our abstract framework could be applied is one where the universe would be endowed with a topological structure and the ranking of sets of consequences would be assumed to be "continuous" with respect to that structure. We actually illustrate this by characterizing, in theorem 3 below, the UEU family of criteria for decisions having their consequences in $\mathbb{R}^k$. Assuming such a structure enable us to replace the Archimedean structure by a mild continuity condition.

To that extent, our framework can be usefully compared to that of Nehring and Puppe (1996) in which the universe of consequences is endowed with a topology and a continuity property is imposed on the ranking of all finite subsets of the universe. Yet continuity is not that a straightforward notion when applied to rankings of sets of objects even when these objects are coming from a set endowed with a topology. For instance, a widely used notion of continuity for sets rankings, adopted also by Nehring and Puppe (1996), is continuity with respect to the Hausdorff topology. However this notion of continuity fails to recognize as continuous a UEU ranking, even though such a ranking is continuous when characterized in a Savagian (uncertainty) framework. This remark explains the difference between our results and those of Nehring and Puppe (1996). These authors characterize rankings that compare sets on the basis of their maximal and minimal elements only using Hausdorff continuity and a mild independence condition (satisfied by UEU criteria). In contrast, we consider an abstract

setting that is compatible (as demonstrated by our theorem 3) with many topological structures. We then characterize a ranking that is continuous in a very natural sense, albeit not Hausdorff continuous, and one that is not based only on the maximal and minimal elements of the sets.

To the best of our knowledge, there has been two other papers that have provided axiomatic characterizations of UEU criteria for ranking sets of objects. The first of them is Fishburn (1972) who characterizes the UEU family of rankings of finite sets in a discrete setting without Archimedean structure. Yet the characterization provided by Fishburn is a direct adaptation of the *additivity axiom* of Scott (1964), Adams (1965) and Fishburn (1970). The unappealing nature of this axiom is well known and is especially striking when adapted to the problem of comparing sets on the basis of their average utility. This axiom involves the construction of arbitrarily long sequences of set comparisons which are both difficult to motivate as primitive axioms as well as hard to verify in practice. By contrast, the special structure of our model enables us to characterize the UEU family of rankings of sets by means of axioms that are, in our opinion, considerably easier to interpret and verify. We note that one of our axiom, "averaging", is identified by Fishburn (1972) as being satisfied by any ranking in the UEU family . We show in this paper that, along with three other axioms, averaging actually characterize the UEU family of rankings of sets if an Archimedean structure is assumed. The other paper that contains a characterization of a UEU criterion for ranking finite sets is the unpublished piece of Baigent and Xu (2004). In this paper, the authors characterize, again without an Archimedean structure, a ranking of finite sets based on the average *Borda score* of their elements. This ranking is clearly a member of the UEU family for which the utility of a consequence is defined by its Borda score. It is, here again, interesting to notice that Baigent and Xu (2004) uses, along with other axioms, the averaging axiom considered herein to obtain their characterization.

While we interpret our framework in terms of choice under complete uncertainty, this interpretation is not necessary. What we provide is an axiomatic characterization of a family of rankings of all finite subsets of some universe of objects that have the property that each of them *can* be interpreted *as if* it was assigning some utility to every object in the universe and *as if* was comparing sets on the basis of the arithmetic average of the utility of these objects. We believe that there are other contexts where such an axiomatic characterization could be useful. In this regard we note that UEU criteria have appeared in the mechanism design literature. Barberà, Dutta, and Sen (2001) consider a model where n (with $n > 1$) agents have preferences over subsets of a basic set of alternatives. A social choice function maps n-tuples of such preferences to a subset of alternatives. They characterize *strategy-proof* social choice functions when agents preferences are assumed to belong to the UEU family. Benoît (2002) studies a similar prob-

lem and recently, Ozyürt and Sanver (2006) have refined and extended this analysis. UEU criteria have also been considered by Peleg and Peters (2005) in their analysis of Nash consistent representation of effectivity functions

The plan of the rest of this paper is as follows. In the next section, we present the formal framework and the definition of the axioms and the family of rankings. The third section presents the main results and the fourth section provides an example of a plausible environment that satisfy the structural axioms and in which the four substantive axioms characterize the UEU family of rankings. The fifth section makes some brief remarks on the independence of the axioms and the sixth section concludes.

2 Notation and basic concepts

2.1 Notation

The sets of integers, non-negative integers, real numbers and non-negative real numbers are denoted respectively by $\mathbb{N}$, $\mathbb{N}_+$, $\mathbb{R}$ and $\mathbb{R}_+$. The cardinality of any set A is denoted by $\#A$ and the k-fold Cartesian product of a set A with itself is denoted by A^k. If v is a vector in $\mathbb{R}^k$ for some strictly positive integer k and α is a real number, we denote by $\alpha.v$ the scalar product of α and v. Given a vector v in $\mathbb{R}^k$ and a positive real number ε, we denote by $N_\varepsilon(v)$ an *ε-neighborhood around v* defined by $N_\varepsilon(v) = \{x \in \mathbb{R}^k :| x_h - v_h |< \varepsilon$ for all $h = 1, ..., k\}$. Our notation for vectors inequalities is $\geqq$, $\geq$ and $>$. By a *binary relation* $\succsim$ on a set Ω, we mean a subset of $\Omega \times \Omega$. Following the convention in economics, we write $x \succsim y$ instead of $(x, y) \in R$. Given a binary relation $\succsim$, we define its *symmetric factor* $\sim$ by $x \sim y \iff x \succsim y$ and $y \succsim x$ and its *asymmetric* factor $\succ$ by $x \succ y \iff x \succsim y$ and not $(y \succsim x)$. A binary relation $\succsim$ on Ω is *reflexive* if the statement $x \succsim x$ holds for every x in Ω, is *transitive* if $x \succsim z$ always follows $x \succsim y$ and $y \succsim z$ for any x, y, $z \in \Omega$, is *symmetric* if $\succsim = \sim$ and is *complete* if $x \succsim y$ or $y \succsim x$ holds for every distinct x and y in Ω. A symmetric, reflexive and transitive binary relation is called an *equivalence relation* and a reflexive, transitive and complete binary relation is called an *ordering*. Given an equivalence relation $\sim$ on Ω, and some $\omega \in \Omega$, we denote by $E_\sim(\omega)$ the equivalence class of ω under $\sim$ defined by $E_\sim(\omega) = \{\omega' \in \Omega \mid \omega' \sim \omega\}$. It is clear that if $\sim$ is an equivalence relation, one has $E_\sim(\omega) \neq \emptyset$ for every ω, $E_\sim(\omega) = E_\sim(\omega')$ or $E_\sim(\omega) \cap E_\sim(\omega') = \emptyset$ for every two elements ω and ω' of Ω and $\bigcup_{\omega \in \Omega} E_\sim(\omega) = \Omega$ so that the equivalence class of all elements of Ω under $\sim$ forms a partition of Ω. Such a partition is called the *quotient of Ω under $\sim$*.

2.2 Basic concepts

Let X be the set of consequences. While we do not make any specific assumptions on X, it will be clear subsequently that the axioms that we impose

makes it natural to regard this set as infinite. An example of a set X, considered in some detail in section 4 below, could be $\mathbb{R}$, interpreted as the set of all conceivable financial returns (either negative or positive) of some investment decision in a highly uncertain environment.

We denote by $\mathcal{P}(X)$ the set of all non-empty *finite* subsets of X (with generic elements A, B, C, D, etc.). Any such a subset is interpreted as a description of all consequences of an *uncertain* decision or, for short, as a *decision*. A *certain* decision with consequence $x \in X$ is identified by the singleton $\{x\}$. Any set with more than one consequence is therefore interpreted as a decision involving *some* uncertainty.

Let $\succsim$ (with asymmetric and symmetric factors $\succ$ and $\sim$ respectively) be an *ordering* on $\mathcal{P}(X)$. We interpret the statement $A \succsim B$ as meaning "decision with consequences in A is weakly preferred to decision with consequences in B". A similar interpretation is given to the statements $A \succ B$ ("strictly preferred to") and $A \sim B$ ("indifference").

We want to identify the properties (axioms) of the ordering $\succsim$ that are necessary and sufficient for the existence of a function $u : X \to \mathbb{R}$ such that, for every A and B in $\mathcal{P}(X)$:

$$A \succsim B \Longleftrightarrow \sum_{a \in A} \frac{u(a)}{\#A} \geq \sum_{b \in B} \frac{u(b)}{\#B} \tag{1}$$

An ordering satisfying this property *could* therefore be thought of as resulting from the comparisons of the *expected utility* of the consequences of the decision for some utility function under the assumption that the decision maker assigns to every consequence of a decision an *equal probability* of occurrence. There are obviously many criteria like that, as many as there are logically conceivable utility functions (up to an affine transform) defined on X. We refer to any ranking that satisfies (1) for some function u as to a *Uniform Expected Utility* (UEU) criterion.

We now introduce four axioms which, as it turns out, are satisfied by any UEU criterion.

Axiom 1 *(Averaging) For all disjoint sets A and $B \in \mathcal{P}(X)$, $A \succsim B \Rightarrow A \succsim A \cup B \Rightarrow A \cup B \succsim B \Rightarrow A \succsim B$.*

Axiom 2 *(Restricted Independence) For all A, B and $C \in \mathcal{P}(X)$ satisfying $\#A = \#B$ and $A \cap C = B \cap C = \emptyset$, $A \succsim B \Leftrightarrow A \cup C \succsim B \cup C$.*

Axiom 3 *(Attenuation) For all sets A, B and $C \in \mathcal{P}(X)$ satisfying $A \sim B$, $A \cap C = B \cap C = \emptyset$ and $\#A > \#B$, $C \succ A$ implies $A \cup C \prec B \cup C$ and $A \succ C$ implies $A \cup C \succ B \cup C$.*

Axiom 4 *(Pairwise Reduction Consistency) For all distinct consequences a, b, c and d, if there are consequences e and f (not necessarily distinct) such $\{a, b\} \sim \{e\}$ and $\{c, d\} \sim \{f\}$, then we must have $\{a, b, c, d\} \sim \{e\} \cup \{f\}$.*

The *averaging* axiom asserts that enlarging a set A with a (disjoint) set of consequences B that is considered worse than A is a "worsening" of A while enlarging it with a set that is better leads to an "improvement" of A. The averaging axiom is a compact version of the four averaging conditions AC1-AC4 discussed in Fishburn (1972) and shown by him to be implied by the UEU family of criteria (as well as by a variant of the additivity axiom of Scott (1964), Adams (1965) and Fishburn (1970)). The averaging axiom has also been used by Baigent and Xu (2004) in their characterization of the average Borda ranking of sets. It is also worth mentioning that the averaging axiom implies the Gärdenfors principle discussed at length in the literature on ranking sets (see e.g. Barberà, Bossert, and Pattanaik (2004) for a recent survey). The Gärdenfors principle is stated as follows.

Condition 1 *(Gärdenfors Principle) for all $A \in \mathcal{P}(X)$, ($x \in X \backslash A$ and $\{x\} \succ \{a\}$ for all $a \in A$) $\Rightarrow A \cup \{x\} \succ A$ and ($y \in X \backslash A$ and $\{a\} \succ \{y\}$ for all $a \in A$) $\Rightarrow A \succ A \cup \{y\}$.*

This principle, first formulated in Gärdenfors (1976), says that it is always worth adding to a set a consequence which, if certain, is better than all consequence in the set. Symmetrically, it is *never* worth adding to a set, a consequence that is worse than all consequences in the set.

We now show that averaging implies the Gärdenfors principle when the ranking of sets to which the averaging axiom is applied is transitive.

Proposition 1 *Let $\succsim$ be a transitive ordering on $\mathcal{P}(X)$. If $\succsim$ satisfies averaging, then it satisfies the Gärdenfors principle*

Proof. *Assume that A is a set in $\mathcal{P}(X)$ and x is a consequence in $X \backslash A$ such that $\{x\} \succ \{a\}$ for every $a \in A$. Write $A = \{a_1, ..., a_{\#A}\}$ and without loss of generality, let $\{a_1\} \succsim ... \succsim \{a_{\#A}\}$. Since $\succsim$ satisfies Averaging, one has $\{a_1\} \succsim A$ and, by assumption, $\{x\} \succ \{a_1\}$. It follows by transitivity that $\{x\} \succ A$ and, by averaging and transitivity, that $\{x\} \succ \{x, a_1, ..., a_{\#A}\} \succ \{a_1, ..., a_{\#A}\}$ as required by the Gärdenfors principle. The case where A and x are such that $\{a\} \succ \{x\}$ for every $a \in A$ can be handled by an analogous argument.*

■

The *restricted independence* axiom requires that the ranking of sets with the same number of elements be independent of any elements that they have in common. Adding or substracting these common elements from the two sets should not affect their ranking. A weak form of the restricted independence condition, applied only to the case where A and B are singletons, plays an important role in Nehring and Puppe (1996) and Puppe (1995).

The *attenuation* axiom compares the impact of adding new consequences to two equivalent decisions that differ in terms of their "uncertainty". Consider two decisions leading to consequences in A and B between which the decision maker is indifferent. Assume that the decision leading to A is "more uncertain" than the other in the sense that its number of possible consequences is larger than those in B. The Attenuation axiom asserts that the level of uncertainty of the set "*attenuates*" the impact, positive or negative, of adding new consequences to it. Specifically, if one adds to both sets consequences in C that are better than the existing consequences, then the positive impact of the addition should be larger for the more certain set than for the less certain one. As the decision maker was originally indifferent between the two sets, this means that the more certain set B will become, when enlarged, better than the less certain set A enlarged. Of course the attenuation goes both ways so that if the added consequences are worse than the existing one, then adding them to the certain set will have a larger negative impact than adding them on the less certain one.

The *pairwise reduction consistency* axiom requires that some consistency be observed in the ranking of uncertain decisions having only two possible consequences when each of these decisions is indifferent to a certain decision. Specifically, assume that a decision having a or b as consequences can be "reduced" to a certain decision leading to e for sure in the sense that the decision maker is indifferent between the certain decision leading to e and the uncertain decision leading to either a or b. Assume also that a decision leading to either c or d can be "reduced" in a similar fashion to a decision leading for sure to f (where e or f are not necessarily distinct). The axiom requires in such a case the uncertain decision leading to consequences a, b, c or d to be indifferent to the certain decision e if e and f are the same to be indifferent to the uncertain decision leading to e or f if these two consequences are distinct. This axiom is rather weak - it only concerns specific indifference comparisons - and is only used in the characterization of UEU family on the domain restricted to pairs and singleton provided by theorem 1. It does not play any role in the characterization of the UEU family of rankings that apply to sets of larger cardinality than two.

We now formally state that the four axioms are satisfied by any UEU criterion. The straigthforward proof of the proposition is left to the reader.

Proposition 2 *Any UEU criterion satisfies averaging, restricted independence, attenuation and pairwise reduction consistency.*

As shall be seen the four axioms of averaging, restricted independence, attenuation and pairwise reduction consistency actually characterize the family of UEU rankings of sets if some structure is imposed on the environment. We provide this structure by imposing *four* other subsidiary axioms

on the pair $\langle X, \succsim \rangle$. These axioms, which we shall refer to as *structural axioms,* impose smoothness and richness on both the set X of alternatives and on the ordering $\succsim$. But they are not specifically tailored to the UEU criterion, and may even be violated by this criterion if the set X of alternatives is too "sparse". Theorem 3 actually establishes that these structural axioms can be dispensed with if X is taken to be $\mathbb{R}^k$ and if a mild continuity and monotonicity condition is imposed on $\succsim$.

Axiom 5 *(Unboundedness) For every consequence $c \in X$, there exists a consequence $c^* \in X$ such that $\{c^*\} \succ \{c\}$.*

Axiom 6 *(Certainty Equivalence) For every $A \in \mathcal{P}(X)$, there exists a consequence $a \in X$ such that $\{a\} \sim A$.*

Axiom 7 *(Richness) For every A, $B \in \mathcal{P}(X) \subset X$, if there are consequences c^* and c_* in X such that $A \cup \{c^*\} \succsim B \succsim A \cup \{c_*\}$, then there exists a consequence $c \in X$ such that $A \cup \{c\} \sim B$.*

Axiom 8 *(Archimedean) Let $\{c_i\}$, for $i = 1, 2, ...$ be a sequence where $c_i \in X$ for all i. Let a and b be consequences in X that are distinct from all elements in the sequence and be such that $\{a\} \succ \{b\}$ and $\{c_i, a\} \sim \{c_{i+1}, b\}$ for all $i, i+1$ with $i = 1, 2, ...$ If the sequence is strictly bounded by x and $y \in X$ in the sense that $\{x\} \succ \{c_i\} \succ \{y\}$ for every i, then the sequence is finite.*

The first axiom assumes that the set of all conceivable consequence is "unbounded" with respect to the ordering $\succsim$. That is given any consequence c, there is always another consequence which, if certain, would be considered better than c as per the ordering $\succsim$. This axiom is natural if X is taken to be the set of all monetary consequences of a decision, at least if preferences are increasing in money. As briefly sketched in the remark following theorem 2 below, it is actually possible to prove all the results of the paper without this axiom. Yet the complete proof involves the explicit statement and writing additional formal arguments which, in our view, are no worth the gain in generality thus obtained.

The second axiom assumes the existence of a "certainty equivalent" to any set of consequences. That is, for any uncertain decision there exists a certain decision that the decision maker considers equivalent. While this axiom may look intuitively reasonable, its justification requires the assumption that there are infinitely many consequences in the set X. If X is a finite set, then an ordering satisfying averaging and certainty equivalence must rank all sets as indifferent to each other.

Proposition 3 *Suppose $\#X < \infty$ and let $\succeq$ be an ordering on $\mathcal{P}(X)$ satisfying averaging. Then $\succsim$ satisfies certainty equivalence if and only if $A \sim B$*

for all $A, B \in \mathcal{P}(X)$.

Proof. *It is clear that the ordering where $A \sim B$ for all A, $B \in \mathcal{P}(X)$ satisfies certainty equivalence. To prove the reverse implication, write the finite set X as $X = \{x_1, ..., x_{\#X}\}$ and assume without loss of generality that $x_i \succeq x_{i+1}$ for $i = 1, ..., \#X - 1$. By averaging, we must have, for every $i = 1, ..., x_{\#X}$:*

$$\{x_i\} \succsim \{x_i, x_{i+1}\} \succsim x_{i+1}$$

Certainty equivalence implies therefore that, for every i, either $\{x_i\} \sim \{x_i, x_{i+1}\}$ or $\{x_{i+1}\} \sim \{x_i, x_{i+1}\}$. In either case, averaging implies that $\{x_i\} \sim \{x_i, x_{i+1}\} \sim \{x_{i+1}\}$. Hence all pairs and singletons must be indifferent. Repeated application of averaging (adding first indifferent singletons to pairs and then indifferent singletons to triples etc.) now leads to the conclusion of universal indifference. ■

The certainty equivalence axiom precludes from consideration some "discontinuous" rankings such as the "Leximin" or the "Leximax" rules studied in Pattanaik and Peleg (1984). For instance, the Leximin rule compares sets on the basis of their worst consequences. If a tie in the worst consequence arises, then the second worst consequence is considered and so on until either a strict ranking is obtained or the consequences in at least one of the sets are exhausted. In the latter case the set which contains the most element is ranked above. It is clear that the Leximin rule violates certainty equivalence since it is impossible for any set containing more than one consequence to be indifferent to a singleton. It should be noted that reasonable as it sounds, the Certainty Equivalent axiom may be violated by a UEU criterion if the set X is not sufficiently rich. For instance if X is taken to be the set $\mathbb{N}$ of all integers, and the u function of expression (1) is the identity function, then the set $\{1, 2\}$ does not have a certainty equivalent in $\mathbb{N}$.

The richness axiom reflects the idea that the universe is sufficiently rich to enable, by the addition of single consequences to sets, various kinds of comparisons with the ordering $\succsim$. Suppose that, starting with two decisions A and B, it is possible to add consequences c^* and c_* to A in such a way that A enlarged with c^* is ranked above B and A enlarged with c_* is ranked below B. Then it must also be possible to add to A a consequence c such that the resulting set of consequences is indifferent to A. This axiom is weak since the asserted existence of the consequence c is contingent upon the existence of consequences c^* and c_* that have the required properties. As in the case of certainty equivalence, richness is violated by Leximin and Leximax rankings.

The Archimedean axiom has been widely discussed in the measurement theory literature (see e.g. Krantz, Luce, Suppes, and Tversky (1971)). It can be considered to be a mild axiom since it "bites" only when there exists

sequences of the type described by the antecedent clause of this axiom (such sequences are called "standard sequences" in the measurement theory literature). It is trivially satisfied if X is finite since all sequences of sets must be finite.

3 Main results

We first show that averaging, restricted independence and pairwise reduction consistency characterizes the family of UEU criteria in an environment where the structural axioms are satisfied when one restricts attention to subsets of X that have at most two elements. It is important to note that attenuation is trivially satisfied when only pairs and singletons are considered. In order to prove this result, we first establish the following lemma, whose proof is in the appendix.[2]

Lemma 1 *Let X be a set of consequences and let $\succsim$ be an ordering on $\mathcal{P}(X)$ that satisfies averaging, pairwise reduction consistency, unboundedness, certainty equivalence and richness. Then for all (not necessarily distinct) consequences a, b, c, d, e, f, g and $h \in X$, if $\{a\} \cup \{b\} \sim \{e\}$, $\{c\} \cup \{d\} \sim \{f\}$, $\{a\} \cup \{c\} \sim \{g\}$ and $\{b\} \cup \{d\} \sim \{h\}$, then $\{e\} \cup \{f\} \sim \{g\} \cup \{h\}$.*

Endowed with this lemma, we establish the announced characterization of the UEU family of ordering of the set of all elements of $\mathcal{P}(X)$ with cardinality no larger than 2. This result rides on the important representation theorem in Krantz, Luce, Suppes, and Tversky (1971) (Theorem 10, p. 295).

Theorem 1 *Let X be a set of consequences and let $\succsim$ be an ordering defined on the set of all subsets of X of cardinality no greater than 2 and satisfying unboundedness, certainty equivalence, richness and the Archimedean axiom. Then $\succsim$ satisfies averaging, restricted independence and pairwise reduction consistency if and only if it is a UEU criterion. Furthermore, the u function in the definition of a UEU criterion is unique up to a positive affine transformation.*

Proof. *We have seen in proposition 1 that any UEU criterion satisfies averaging, restricted independence and pairwise reduction consistency. To prove the converse implication, let $\mathcal{P}_1(X)$ be the set of all singletons in $\mathcal{P}(X)$ and let Y be the quotient of $\mathcal{P}_1(X)$ under the equivalence relation $\sim$. Hence $Y = \{\underline{a} \subset \mathcal{P}_1(X) : \{a\}, \{b\} \in \underline{a}$ if and only if $\{a\} \sim \{b\}\}$. Define the binary operation $\circ$ on Y by $\underline{a} \circ \underline{b} = \underline{c}$ iff there are $\{a\} \in \underline{a}, \{b\} \in \underline{b}$, $\{c\} \in \underline{c}$ such that $\{a, b\} \sim \{c\}$. We note that $\underline{a} \circ \underline{a} = \underline{a}$ (since $\{a\} \cup \{a\} = \{a\}$)*

[2] Proofs of lemmas are in the appendix but proofs of theorems and propositions are, when provided, kept in the main body of the text.

so that $\circ$ is an idempotent binary operation. We first show that the binary operation $\circ$ is well-defined in the sense that, for any two (not necessarily distinct) equivalence classes $\underline{a}$ and $\underline{b}$ in Y, there exists a unique $\underline{c} \in Y$ such that $\underline{a} \circ \underline{b} = \underline{c}$. Assume first that $\underline{a} \neq \underline{b}$. By certainty equivalence, there exists $c \in X$ such that, for $\{a\} \in \underline{a}$, $\{b\} \in \underline{b}$ with $\{a\} \not\sim \{b\}$ one has $\{a, b\} \sim \{c\}$. Suppose by contradiction that there are also $\{a'\} \in \underline{a}, \{b'\} \in \underline{b}$ and $\{c'\} \notin \underline{c}$ such that $\{a', b'\} \sim \{c'\}$. If this was the case, we would have $\{a, b\} \sim \{c\} \not\sim \{c'\} \sim \{a', b'\}$. By transitivity, we would have $\{a, b\} \not\sim \{a', b'\}$ but by restricted independence, we should have $\{a, b\} \sim \{a', b'\}$, a contradiction. Consider now the case where $\underline{a} = \underline{b}$. Suppose by contradiction, that there are $a, b, c \in X$ with $\{a\} \sim \{b\}$ such that $\{a, b\} \sim \{c\}$ and $\{a\} \not\sim \{c\}$. This contradicts averaging. We note also that the binary operation $\circ$ is commutative. The next step in the proof consists in verifying that the triple $\langle Y, \succsim, \circ\rangle$ (where, by a slight abuse of notation, $\succsim$ is now interpreted as the ordering of the equivalence classes rather than that of the elements of the various equivalence classes) satisfies all conditions of a bisymmetric structure as defined in Krantz, Luce, Suppes, and Tversky (1971) (p. 294, definition 10). The property of "monotonicity" of the structure is implied, given the definition of the binary operation $\circ$, by restricted independence. This structure will be "bisymmetric" if $(\underline{a} \circ \underline{b}) \circ (\underline{c} \circ \underline{d}) \sim (\underline{a} \circ \underline{c}) \circ (\underline{b} \circ \underline{d})$ for all $\underline{a}, \underline{b}, \underline{c}$ and $\underline{d} \in Y$, . Using our definition of $\circ$, the structure would therefore be bisymmetric if for all (not necessarily distinct) consequences a, b, c, d, e, f, g and h, if $\{a\} \cup \{b\} \sim \{e\}$, $\{c\} \cup \{d\} \sim \{f\}$, $\{a\} \cup \{c\} \sim \{g\}$ and $\{b\} \cup \{d\} \sim \{h\}$ imply$\{e\} \cup \{f\} \sim \{g\} \cup \{h\}$. By lemma 1, this condition is satisfied if $\succsim$ satisfies averaging, restricted independence and pairwise reduction consistency. Finally, the fact that the structure $\langle Y, \succsim, \circ\rangle$ satisfies the "restricted solvability" and "Archimedean" properties of Krantz, Luce, Suppes, and Tversky (1971) is an immediate consequences of our richness and Archimedean axioms. Hence Theorem 10 of Krantz, Luce, Suppes, and Tversky (1971) (p.295) applies to this structure so that there exists a mapping $v : Y \to \mathbb{R}$ such that $\underline{a} \succsim \underline{b}$ iff $v(\underline{a}) \geq v(\underline{b})$ and $v(\underline{a} \circ \underline{b}) = \alpha v(\underline{a}) + \beta v(\underline{b}) + \gamma$. Moreover, by clause (iii) of theorem 10 of Krantz, Luce, Suppes, and Tversky (1971), the function v is unique up to an positive affine transform. Since the binary operation $\circ$ is commutative, we must have $\alpha = \beta$. Since the binary operation is idempotent, we must have $\gamma = 0$ and $\alpha = \beta = 1/2$. Define now the function $u : X \to \mathbb{R}$ by: $u(a) = v(\underline{a})$ for all $a \in X$. The function u obviously represents $\succsim$ as per (1). ■

Our main result extends Theorem 1 to sets with an arbitrary number of consequences using the same axioms along with attenuation. As mentioned above the axiom of pairwise reduction consistency plays no role in this extension. The idea behind the proof is to show that the unique utility function whose expectation (under uniform probabilities) represents the ranking of sets containing no more than two elements exhibited in Theorem

1, also represents the ranking of sets of larger cardinality. A key step in the proof, provided by lemma 2, is the ability to approximate the arithmetic mean of a set of n numbers recursively from the arithmetic means of *pairs* of those numbers.

Theorem 2 *Let $\succsim$ be an ordering on $\mathcal{P}(X)$ satisfying unboundedness, certainty equivalence, richness and the Archimedean axiom. Then $\succsim$ satisfies averaging, restricted independence, attenuation and pairwise reduction consistency if and only if it is a UEU criterion. Furthermore, the u function in the definition of a UEU criterion is unique up to a positive affine transformation.*

The proof of this theorem requires five lemmas. The first of them is the following important one that indicates how one can approximate the arithmetic mean of n numbers by specific recursive combinations of means of two numbers.

Lemma 2 *Let $U = \{u_1, ..., u_n\}$ be a set of n numbers such that $u_1 \leq u_2 \leq ... \leq u_n$ with arithmetic mean $\overline{u}$. Define the $n-1$ sequences $\{b_h^i\}$, $i = 1, 2, ...$ and $h = 1, ..., n-1$ by:*

$$b_{n-1}^0 = (u_n + u_{n-1})/2,$$

$$b_h^0 = (u_h + b_{h+1})/2$$

for $h = 1, ..., n-2$ and for $i = 1, 2, ...$:

$$\begin{aligned} b_1^{2i-1} &= b_1^{2i-2}, \\ b_h^{2i-1} &= \frac{b_{h-1}^{2i-1} + b_h^{2i-2}}{2} \text{ for } h = 2, ..., n-1, \\ b_{n-1}^{2i} &= b_{n-1}^{2i-1} \text{ and} \\ b_h^{2i} &= \frac{b_h^{2i-1} + b_{h+1}^{2i}}{2} \text{ for } h = 1, ..., n-2. \end{aligned}$$

Then:

$$\lim_{i \to \infty} b_h^i = \overline{u} \text{ for all } h = 1, ..., n-1$$

The next lemma establishes an implication of some of the axioms that will be useful in the main proof.

Lemma 3 *Let $\succsim$ be an ordering on $\mathcal{P}(X)$ satisfying restricted independence, averaging, certainty equivalence and richness. Then, if A and B are subsets of X and c is a consequence in X such that $A \prec B \cup \{c\}$ and $\{d\} \prec \{c\}$ for some $d \in X$, there exists some $e \in X$ such that $\{e\} \prec \{c\}$ and $A \prec B \cup \{e\}$.*

Dually, if A and B are sets and c is a consequence in X such that $A \succ B \cup \{c\}$ and $\{d\} \succ \{c\}$ for some $d \in X$, there exists $e \in X$ such that $\{e\} \succ \{c\}$ and $A \succ B \cup \{e\}$.

Finally, the last three lemmas show, on an environment satisfying certainty equivalence and richness, that averaging, restricted independence and attenuation imply the following technical condition, referred to as *Condition C* in the proof of theorem 2 below.

Condition 2 *(C) For all distinct consequences a, b, c and $d \in X$ and every set $B \in \mathcal{P}(X)$ such that $\{b\} \sim \{c, d\}$ and $B \cap \{b, c, d\} = \emptyset$, we must have:*
(i) $\{a\} \succsim B \cup \{b\}$ and $\{b\} \succsim \{a\}$ with at least one strict ranking imply $\{a, b\} \succ B \cup \{c, d\}$, and
(ii)$\{a\} \precsim B \cup \{b\}$ and $\{b\} \prec \{a\}$ with at least one strict ranking imply $\{a, b\} \prec B \cup \{c, d\}$.

The first step in proving that, on an environment satisfying certainty equivalent and richness, averaging, restricted independence and attenuation imply condition C is the following lemma.

Lemma 4 *Let X be a set of consequences and let $\succsim$ be an ordering on $\mathcal{P}(X)$ satisfying certainty equivalence, richness, averaging and restricted independence. Let A and B be two finite subsets of X and let a, b, c and d be consequences in X satisfying $A \cup \{a\} \sim B \cup \{b\}$, $\#A = \#B$, $\{b\} \sim \{c, d\}$, $a \neq b$, $c \neq d$, $\{a, b\} \cap A = \{c, d\} \cap B = \emptyset$ and $b \notin B$. Then $A \cup \{a, b\} \sim B \cup \{c, d\}$.*

The next lemma provides the second step in the proof that averaging, restricted independence, attenuation, richness and certainty equivalence imply Condition C.

Lemma 5 *Let X be a set of consequences and let $\succsim$ be an ordering on $\mathcal{P}(X)$ satisfying certainty equivalence, richness, averaging and restricted independence and let a, b, c and d be consequences in X and B be a finite subset of X such that $\{a\} \succsim B \cup \{b\}$, $\{b\} \sim \{c, d\}$, $\{b\} \succ \{a\}$, $b \notin B$ and $\{c, d\} \cap B = \emptyset$. Then there exists a finite subset A' of X and a consequence a' in X such that $A' \cup \{a'\} \sim B \cup \{b\}$, $a' \notin A'$ and $\#A' = \#B$.*

Combining these two lemmas, we can establish the following.

Lemma 6 *Let X be a set of consequences and let $\succsim$ be an ordering on $\mathcal{P}(X)$ satisfying certainty equivalence, richness, averaging, restricted independence and attenuation. Then $\succsim$ satisfies condition* C.

Endowed with this result, we are equipped to prove theorem 2.

Proof. Using Theorem 1, we can find a function u that uniquely represents $\succsim$ as per (1) on the subset of $\mathcal{P}(X)$ containing sets of cardinality no greater than 2. We want to prove that the same function u can also be used to represent $\succsim$ on the whole set $\mathcal{P}(X)$. We must prove specifically that, for any $A \in \mathcal{P}(X)$ and $g \in X$,

$$A \succsim \{g\} \iff \sum_{a \in A} \frac{u(a)}{\#A} \geq u(g).$$

where u is the (unique) utility function identified in theorem 1. We prove $\Rightarrow$. Suppose $\#A = m \leq n$ and write $A = \{a_1, a_2, \ldots, a_m\}$ with $\{a_1\} \precsim \ldots \precsim \{a_m\}$. By certainty equivalence, there exists b^0_{m-1} in X such that $b^0_{m-1} \sim \{a_{m-1}, a_m\}$. Similarly, for $i = m-2, \ldots, 1$, we can find, by certainty equivalence, a consequence b^0_i such that $b^0_i \sim \{a_i, b^0_{i+1}\}$. Using certainty equivalence repeatedly, one can define this way for $j = 1, 2, 3, \ldots$the sequence of consequences b^j_i by:

$$b^{2j-1}_1 = b^{2j-2}_1,$$

$$b^{2j-1}_i \sim \{b^{2j-1}_{i-1}, b^{2j-2}_i\}$$

for $i = 2, \ldots, m-1$,

$$b^{2j}_{m-1} = b^{2j-1}_{m-1}$$

and

$$b^{2j}_i \sim \{b^{2j-1}_i, b^{2j}_{i+1}\}$$

for $i = m-2, \ldots, 1$. We first show that:
(i) $\{b^j_1\} \precsim \{b^j_2\} \precsim \ldots \precsim \{b^j_{m-1}\}$,
(ii) $\{a_1\} \precsim \{b^i_1\} \precsim \{b^{i+1}_1\} \precsim \{b^{i+1}_{m-1}\} \precsim \{b^i_{m-1}\} \precsim \{a_m\}$ and
(iii) $\{b^i_1\} \precsim A \precsim \{b^i_{m-1}\}$.
If $\{a_1\} \sim \{a_m\}$, then, by averaging, $\{a_1\} \sim A$, $\{b^i_j\} \sim \{a_1\} \sim A$ for all $i \in \mathbb{N}$ and all $j \in \{1, \ldots, m-1\}$ and the implications (i)-(iii) are immediately established. If $\{a_1\} \prec \{a_m\}$, let k be the largest integer such that $\{a_k\} \prec \{a_{k+1}\}$. We first prove implications (i) and (ii). By averaging, $\{a_{m-1}\} \precsim \{b^0_{m-1}\} \precsim \{a_m\}$. By transitivity, $\{a_{m-2}\} \precsim \{b^0_{m-1}\}$. By averaging again, $\{a_{m-2}\} \precsim \{b^0_{m-2}\} \precsim \{b^0_{m-1}\}$. By repeated use of transitivity and averaging, one is led to the conclusion that $\{a_{k+1}\} \precsim \{b^0_{k+1}\} \precsim \{b^0_{k+2}\}$. Now, by transitivity $\{a_k\} \prec \{b^0_{k+1}\}$ and, by averaging, $\{a_k\} \prec \{b^0_k\} \prec \{b^0_{k+1}\}$. Analogously, a repeated combination of averaging and transitivity leads to the conclusion that $\{a_1\} \prec \{b^0_1\} \prec \{b^0_2\}$. Hence, we have established that

$\{a_1\} \prec \{b_1^0\} \prec \{b_{k+1}^0\} \precsim \{b_{k+2}^0\} \precsim \ldots \precsim \{b_{m-1}^0\} \precsim \{a_m\}$. Now, by averaging, $\{b_1^0\} \prec \{b_2^1\} \prec \{b_2^0\}$ and, by transitivity, $\{b_2^1\} \prec \{b_3^0\}$. Combining in this way averaging and transitivity leads us to $\{b_{m-2}^1\} \prec \{b_{m-1}^1\} \prec \{b_{m-1}^0\}$ and, therefore, to $\{a_1\} \prec \{b_1^0\} \sim \{b_1^1\} \prec \{b_2^1\} \prec \ldots \prec \{b_{m-1}^1\} \prec \{b_{m-1}^0\} \precsim \{a_m\}$. Repeatedly using the same reasoning, we finally find that, for all $i \in \mathbb{N}$, $\{b_1^i\} \prec \{b_2^i\} \prec \ldots \prec \{b_{m-1}^i\}$ and $\{a_1\} \precsim \{b_1^i\} \precsim \{b_1^{i+1}\} \precsim \{b_{m-1}^{i+1}\} \precsim \{b_{m-1}^i\} \precsim \{a_m\}$. We now turn to implication (iii) that we prove in the following infinite number of steps.

Step 1. We notice that by virtue of the Gärdenfors principle, $\{b_{m-1}^0\} \succ A$.

Step 2. We prove that $\{b_1^0\} \prec A$. Since by assumption $a_l = a_{l+1}$ for all $l = k+1, ..., m-1$, we have by averaging that $\{b_{m-1}^0\} \sim \{a_m\} \sim \{a_{m-1}, a_m\} \sim \{a_{m-2}\} \sim \{a_{m-2}, a_{m-1}, a_m\} \sim ... \sim \{a_{k+1}, \ldots, a_{m-1}, a_m\}$. We therefore have $\{b_{k+1}^0\} \sim \{a_{k+1}, \ldots, a_{m-1}, a_m\} \sim \{a_{k+1}\}$. Now, since $\{a_k\} \prec \{b_{k+1}^0\} \sim \{a_{k+1}, \ldots, a_{m-1}, a_m\}$, it follows from the attenuation axiom that $\{a_k, b_{k+1}^0\} \prec \{a_k, a_{k+1}, \ldots, a_{m-1}, a_m\}$ and, since $\{b_k^0\} \sim \{a_k, b_{k+1}^0\}$ and $\succsim$ is transitive, that $\{b_k^0\} \prec \{a_k, \ldots, a_{m-1}, a_m\}$. Applying the same reasoning below k enables us to reach the conclusion that $\{b_1^0\} \prec \{a_1, \ldots, a_2, a_m\} = A$.

Step 3. Since $b_1^1 = b_0^1$, we trivially have that $\{b_1^1\} \prec A$.

Step 4. We prove that $\{b_{m-1}^1\} \succ A$. Notice that $\{b_1^1\} \sim \{a_1, b_2^0\}$, $\{b_2^1\} \sim \{b_1^1, b_2^0\}$, $\{b_2^0\} \sim \{a_2, b_3^0\}$, $\{b_1^1\} \prec \{b_2^0\}$ and the condition C (clause (i)) imply that $\{b_2^1\} \succ \{a_1, a_2, b_3^0\}$. Similarly, $\{b_2^1\} \sim \{b_1^1, b_2^0\}$, $\{b_3^1\} \sim \{b_2^1, b_3^0\}$ and clause (i) of the condition C imply that $\{b_3^1\} \succ \{a_1, a_2, a_3, b_4^0\}$. Repeating this reasoning, we obtain $\{b_{m-2}^1\} \succ \{a_1, \ldots, a_{m-2}, b_{m-1}^0\}$ and, finally, $\{b_{m-1}^1\} \succ \{a_1, \ldots, a_m\} = A$.

Step 5. Trivially, $\{b_{m-1}^2\} = \{b_{m-1}^1\} \succ A$.

Step 6. We prove that $\{b_1^2\} \prec A$. We have $\{b_{m-1}^2\} \sim \{b_{m-1}^0, b_{m-2}^0\}$, $\{b_{m-2}^2\} \sim \{b_{m-1}^2, b_{m-2}^1\}$, $\{b_{m-2}^1\} \sim \{b_{m-3}^1, b_{m-2}^0\}$ and $\{b_{m-2}^1\} \prec \{b_{m-1}^2\}$. Hence, by clause (ii) of substitution Robustness, $\{b_{m-2}^2\} \prec \{b_{m-3}^1, b_{m-2}^0, b_{m-1}^0\}$. We have also $\{b_{m-2}^2\} \prec \{b_{m-3}^1, b_{m-2}^0, b_{m-1}^0\}$, $\{b_{m-3}^2\} \sim \{b_{m-2}^2, b_{m-3}^1\}$, $\{b_{m-3}^1\} \sim \{b_{m-4}^1, b_{m-3}^0\}$ and $\{b_{m-3}^1\} \prec \{b_{m-2}^2\}$. Hence, by clause (ii) of the condition C, $\{b_{m-3}^2\} \prec \{b_{m-4}^1, b_{m-3}^0, b_{m-2}^0, b_{m-1}^0\}$. This process can be repeated until we obtain:

$$\{b_2^2\} \prec \{b_1^1, b_2^0, b_3^0, \ldots, b_{m-1}^0\} = \{b_1^0, b_2^0, b_3^0, \ldots, b_{m-1}^0\}.$$

By lemma 3, there exists $\{c_2^0\} \prec \{b_2^0\}$ such that $\{b_2^2\} \prec \{b_1^0, c_2^0, b_3^0, \ldots, b_{m-1}^0\}$. Repeatedly applying Lemma 3, we find $\{c_i^0\} \prec \{b_i^0\}$, for $i = 3 \ldots m-1$ such that $\{b_2^2\} \prec \{b_1^0, c_2^0, c_3^0, \ldots, c_{m-1}^0\}$. This, combined with $\{b_1^2\} \sim \{b_2^2, b_1^1\}$, $\{b_1^1\} \sim \{a_1, b_2^0\}$, $\{b_1^1\} \prec \{b_2^2\}$ and clause (ii) of condition C, implies $\{b_1^2\} \prec \{a_1, c_2^0, b_2^0, c_3^0, \ldots, c_{m-1}^0\}$. By averaging, we have $\{c_2^0, b_2^0\} \prec \{b_2^0\} \sim \{a_2, b_3^0\}$. By restricted independence,$\{b_1^2\} \prec \{a_1, a_2, c_3^0, b_3^0, \ldots, c_{m-1}^0\}$. By averaging, $\{c_3^0, b_3^0\} \prec \{b_3^0\} \sim \{a_3, b_4^0\}$. By restricted independence:

$$\{b_1^2\} \prec \{a_1, a_2, a_3, c_4^0, b_4^0, \ldots, c_{m-1}^0\}.$$

Repeating this process leads us to the conclusion that:

$$\{b_1^2\} \prec \{a_1, a_2, \ldots, a_{m-2}, c_{m-1}^0, b_{m-1}^0\}.$$

By averaging, $\{c_{m-1}^0, b_{m-1}^0\} \prec \{b_{m-1}^0\} \sim \{a_{m-1}, a_m\}$. By restricted independence:

$$\{b_1^2\} \prec \{a_1, a_2, \ldots, a_{m-2}, a_{m-1}, a_m\} = A.$$

Step 7. Trivially, $\{b_1^3\} = \{b_1^2\} \prec A$.

Step 8. We prove that $\{b_{m-1}^3\} \succ A$. We have $\{b_1^3\} \sim \{b_1^1, b_2^2\}$, $\{b_2^3\} \sim \{b_1^3, b_2^2\}$, $\{b_2^2\} \sim \{b_2^1, b_3^2\}$ and $\{b_2^2\} \succ \{b_1^3\}$. Hence, by clause (i) of condition C, $\{b_2^3\} \succ \{b_1^1, b_2^1, b_3^2\}$. We also have $\{b_3^3\} \sim \{b_2^3, b_3^2\}$, $\{b_3^2\} \sim \{b_3^1, b_4^2\}$ and $\{b_3^2\} \succ \{b_2^3\}$. Hence, by clause (i) of condition C, $\{b_3^3\} \succ \{b_1^1, b_2^1, b_3^1, b_4^2\}$. Continuing this process, we obtain $\{b_{m-2}^3\} \succ \{b_1^1, b_2^1, \ldots, b_{m-2}^1, b_{m-1}^2\}$. Repeatedly applying Lemma 3, we find c_i^1 such that $\{c_i^1\} \succ \{b_i^1\}$, for $i = 1 \ldots m-2$ such that $\{b_{m-2}^3\} \succ \{c_1^1, c_2^1, \ldots, c_{m-2}^1, b_{m-1}^2\}$. This, combined with $\{b_{m-1}^3\} \sim \{b_{m-1}^2, b_{m-2}^3\}$, $\{b_{m-1}^2\} \sim \{b_1^0, b_{m-2}^1\}$, $\{b_{m-1}^2\} \succ \{b_{m-2}^3\}$ and clause (i) of condition C, implies that:

$$\{b_{m-1}^3\} \succ \{c_1^1, c_2^1, \ldots, c_{m-2}^1, b_{m-2}^1, b_{m-1}^0\}.$$

By averaging and restricted independence:

$$\{b_{m-1}^3\} \succ \{c_1^1, \ldots, c_{m-3}^1, b_{m-3}^1, b_{m-2}^0, b_{m-1}^0\}.$$

By repeatedly combining averaging and restricted independence in this way, one is led to the conclusion that:

$$\{b_{m-1}^3\} \succ \{c_1^1, b_1^1, b_2^0, b_3^0, \ldots, b_{m-1}^0\}.$$

Repeatedly applying lemma 3, one finds d_i^1 such that $\{d_i^1\} \prec \{b_i^0\}$, for $i = 2 \ldots m-1$ for which:

$$\{b_{m-1}^3\} \succ \{c_1^1, b_1^1, d_2^0, d_3^0, \ldots, d_{m-1}^0\}.$$

Repeatedly applying averaging and restricted independence, we obtain $\{b^3_{m-1}\} \succ \{a_1, a_2, \ldots, a_{m-2}, \ldots, a_{m-1}, a_m\} = A$.

Step 9. Trivially, $\{b^4_{m-1}\} = \{b^3_{m-1}\} \succ A$.

Steps 6 to 9 can clearly be repeated for ever using the same argument and this remark completes the proof of (*iii*). Now, using certainty equivalence, let x be a consequence such that $A \sim \{x\}$. The mapping u that we found using Theorem 1 represents $\succsim$ and, so, $u(b^i_1) \leq u(x) \leq u(b^i_{m-1})$ for all $i \in \mathbb{N}$. Now it is easy to check that the sequence $\{u(b^i_h)\}$ for every h are just like the sequences studied in lemma 2. Because of this lemma, we have that

$$\lim_{t\to\infty} u(b^t_1) = \lim_{t\to\infty} u(b^t_{m-1}) = \sum_{a\in A} \frac{u(a)}{\#A}.$$

Hence, $u(x) = \sum_{a\in A} \frac{u(a)}{\#A}$. By transitivity, $A \succsim \{g\}$ iff $\{x\} \succsim \{g\}$ iff $\sum_{a\in A} \frac{u(a)}{\#A} \geq u(g)$. ■

Remark 1 *The strategy for proving the results of theorem 2 without the unboundedness axiom would be as follows. If X has maximal elements, we define the set X' as X without its maximal elements. Thanks to Averaging and Certainty Equivalence, X' is unbounded. We use Theorem 2 to prove the existence of a UEU criterion on X' and we then show that the mapping u, defined on X', can be extended to X by defining, for every maximal element t of X, $u(t)$ by $u(t) = \sup_{x\in X'} u(x)$.*

4 An interpretation of the structural environment

We now show that if one imposes a natural structure on the set X and the ordering $\succsim$ from the outset, there is no need to resort to unboundedness, certainty equivalence, richness and the Archimedean axiom to characterize the UEU family of orderings. Specifically, this family will then be characterized by averaging, restricted independence, attenuation and pairwise reduction consistency only.

Assume that X is an unbounded (from above)[3] closed and arc-connected[4] subset of $\mathbb{R}^k$, that we could interpret as the set of all consumption bundles that the decision maker can obtain out of an uncertain decision (taking $k = 1$ would obviously cover the case where consequences are pecuniary).

[3] Again, it is possible to avoid this unboundedness assumption.

[4] A subset A of a topological space is arc-connected if, for any two elements x and y of A, there exists a continuous function f from $[0, 1]$ to A such that $f(0) = x$ and $f(1) = y$.

Assume also that the ordering $\succsim$ is *increasing when restricted to singletons* (that is, for every x and $y \in \mathbb{R}^k$, $x \geq y$ implies $\{x\} \succsim \{y\}$ and $x > y$ implies $\{x\} \succ \{y\}$) and satisfies the following *continuity* condition.

Condition 3 (Continuity) *For every set $A \in P(X)$, and consequences y and $z \in X$, the sets $B(A) = \{x \in X : \{x\} \succsim A\}$ and $W(A) = \{x \in X : A \succsim \{x\}\}$ are both closed in X.*

This continuity condition says that a small change in a bundle should not affect drastically the ranking of a decision leading to this bundle for sure *vis-à-vis* any set. Notice that this continuity condition, which only concerns comparisons of sets vis-à-vis singletons is much weaker than the (Vietoris) continuity condition examined in Nehring and Puppe (1996) which restricts the comparisons of any two sets in a way that is even incompatible with the UEU family of set rankings.

We now establish, in the following theorem, that in this environment, the UEU family of rankings of $P(X)$ is characterized by averaging, restricted independence, attenuation and pairwise reduction consistency.

Theorem 3 *Let X be an unbounded arc-connected subset of $\mathbb{R}^k$ and let $\succsim$ be an ordering of $P(X)$ that is monotonic when restricted to singletons and which satisfies the continuity condition. Then $\succsim$ satisfies averaging, restricted independence, attenuation and pairwise combination consistency if and only if it is a UEU criterion.*

Proof. *We know from proposition 1 that a UEU criterion satisfies averaging, restricted independence, attenuation and pairwise combination consistency on any environment. Conversely, let X be an arc-connected subset of $\mathbb{R}^k$ and let $\succsim$ be an ordering of $P(X)$ satisfying the continuity condition as well as averaging, restricted independence, attenuation and pairwise reduction consistency. We will prove that under these conditions, $\succsim$ satisfies unboundednss, certainty equivalence, richness and the Archimedean axiom. Using theorems 1 and 2, the conclusion that $\succsim$ is a UEU criterion will then follow immediately. We first notice that, under averaging, if the sets $B(A) = \{x \in X : \{x\} \succsim A\}$ and $W(A) = \{x \in X : A \succsim \{x\}\}$ are closed in X for every A, then so are the sets $\widetilde{B}(A) = \{x \in X : A \cup \{x\} \succsim A\}$ and $\widetilde{W}(A) = \{x \in X : A \succsim A \cup \{x\}\}$. To see that, assume by contraposition that, say, $\widetilde{B}(A)$ is not closed (the argument for $\widetilde{W}(A)$ is similar) . Then, there exists a sequence $\{x^t\}$ $t = 1, ...$converging to some limit x such that:*

$$A \cup \{x^t\} \succsim A$$

for all t and

$$A \succ A \cup \{x\}$$

where the last strict ranking is obtained from the assumption that $\succsim$ is complete. Since $\succsim$ is also reflexive, this strict ranking implies therefore that $x \notin A$. By averaging one has therefore:

$$A \succ \{x\} \tag{2}$$

Now, since A is finite, and x^t is a sequence converging to x, either x^t is a finite sequence or x^t is infinite. If x^t is finite, then, by definition of a sequence converging to x, there exists some $s \leq t$ for which $x^s = x \notin X$. But given averaging, this is incompatible with the definition of the sequence x^t as satisfying $A \cup \{x^t\} \succsim A$ for every t. Hence we must conclude that x^t is infinite. If this is the case, there must exists, since A is finite, an infinite subsequence $\widetilde{x}^t$ of x^t converging to x and such that $\widetilde{x}^t \notin A$ for every t. Since for every t, we have

$$A \cup \{\widetilde{x}^t\} \succsim A$$

it follows from averaging that we also have:

$$\{x^t\} \succsim A$$

which, given (2), gives us the required contradiction of the closedness of the set $B(A)$. Let us now prove that $\succsim$ satisfies the four structural axioms.

Unboundedness : *This property is clearly satisfied if X is unbounded and $\succsim$ is monotonic when restricted to singletons.*

Certainty Equivalence*: Consider any set $A \in P(X)$ and, without loss of generality, write it as $A = \{a_1, ..., a_{\#A}\}$ with $\{a_h\} \precsim \{a_{h+1}\}$ for $h = 1, ..., \#A - 1$. By averaging (and specifically the Gardenförs principle) one has that $A \succsim \{a_1\}$ and $\{a_{\#A}\} \succsim A$ so that none of the closed sets $\{x \in X : \{x\} \succsim A\}$ and $\{x \in X : A \succsim \{x\}\}$ is empty. Since $\succsim$ is complete, $X = \{x \in X : \{x\} \succsim A\} \cup \{x \in X : A \succsim \{x\}\}$. Since X is arc connected, there exists a continuous function $f : [0,1] \to X$ such that $f(0) = a_1$ and $f(1) = a_{\#A}$. By continuity, given the closedness of $\{x \in X : \{x\} \succsim A\}$ and $\{x \in X : A \succsim \{x\}\}$, there must be some $\alpha \in [0,1]$ such that $f(\alpha) \in \{x \in X : \{x\} \succsim A\} \cap \{x \in X : A \succsim \{x\}\}$. By definition $\{f(\alpha)\} \sim A$.*

Richness: *Consider sets A and B in $P(X)$ and bundles c^* and $c_* \in X$ such that $A \cup \{c^*\} \succsim B \succsim A \cup \{c_*\}$. If either $A \cup \{c^*\} \sim B$ or $B \sim A \cup \{c_*\}$, then richness is satisfied and there is nothing to be proved. Assume therefore that*

$$A \cup \{c^*\} \succ B \succ A \cup \{c_*\} \tag{3}$$

holds. let $\mathbf{1}^k$ denote the unit vector in $\mathbb{R}^k_+$. Since $\succsim$ restricted to singletons is continuous and increasing, there exists, for each $x \in X$, a real number

$v(x)$ such that $\{v(x).\mathbf{1}^k\} \sim \{x\}$. By restricted independence, one has:

$$A \cup \{v(c^*).\mathbf{1}^k\} \sim A \cup \{c^*\} \succ B \succ A \cup \{c_*\} \sim A \cup \{v(c_*).\mathbf{1}^k\} \tag{4}$$

Define then $h(x)$ by:

$$\{h(x).\mathbf{1}^k\} \sim A \cup \{x.\mathbf{1}^k\}$$

The number $h(x)$ exists by virtue of the fact that, as we have just checked, $\succsim$ satisfies certainty equivalence. Moreover $h(x)$ is clearly unique if $\succsim$ restricted to singletons is increasing. Hence h is a function from X to $\mathbb{R}$. It must also be a continuous function if, as established above, $\widetilde{B}(A)$ and $\widetilde{W}(A)$ are both closed in X. For suppose h is not continuous at some $x \in X$. This means that there exists a number $\varepsilon > 0$ for which, for every number $\delta > 0$, one can find $x' \in N_\delta(x)$ such that $h(x') \notin N_\varepsilon(h(x))$. Since $\succsim$ is complete, either, non-exclusively, $x' \in B(A)$ or $x' \in W(a)$. Since these two sets are closed, and x' is arbitrarily close to x, $h(x')$ must also be arbitrarily close to $h(x)$. Let now $v(B)$ be defined by:

$$\{v(B).\mathbf{1}^k\} \sim B$$

Because of (3) and (4), one has that $h(c^*) > v(B) > h(c_*)$. By arc connectedness, let f be a continuous function from $[0,1]$ to X satisfying $f(0) = c_*$ and $f(1) = c^*$. Consider now the function $\Psi : [0,1] \to [h(c_*), h(c^*)]$ defined by:

$$\Psi(\alpha) = h(f(\alpha))$$

This function, which composes two continuous function is continuous and satisfies $\Psi(0) = h(c_*)$ and $\Psi(1) = h(c^*)$. By the intermediate value theorem, there must be some $\alpha \in [0,1]$ such that $\Psi(\alpha) = h(f(\alpha)) = V(B)$. By definition of f and h, this implies the existence of some $x \in X$ satisfying $h(x) = V(B)$ such that

$$B \sim \{h(x).\mathbf{1}^k\} \sim A \cup \{x.\mathbf{1}^k\}$$

Archimedean axiom: *If it is impossible to construct a standard sequence as in the antecedent clause of the Archimedean axiom, then the proof is (trivially) over. Assume therefore that such a sequence exists and, therefore that a and b be two bundles of k goods such that $\{a\} \succ \{b\}$ for which one has, for a sequence of bundles $\{c_t\}_{t \in \mathbb{N}_+}$:*

$$\{c_t, a\} \sim \{c_{t+1}, b\} \tag{5}$$

for every $t = 0, \ldots$. By restricted independence, we must have $\{c_{t+1}\} \succ \{c_t\}$ for all t. By restricted independence again, one has:

$$\{b_t, a\} \sim \{b_{t+1}, b\}$$

for every consumption bundles b_t and b_{t+1} in $\mathbb{R}^k_+$ such that $\{b_t\} \sim \{c_t\}$ and $\{b_{t+1}\} \sim \{c_{t+1}\}$. Since $\succsim$ restricted to singletons induces a continuous ranking of $\mathbb{R}^k_+$, there exists, for every $t = 0, ...,$, a real number α_t such that:

$$\{\alpha_t.\mathbf{1}^k\} \sim \{c_t\}$$

Hence the existence of a sequence of bundles $\{c_t\}_{t\in\mathbb{N}_+}$ satisfying (5) implies the existence of a sequence of real numbers $\{\alpha_t\}_{t\in\mathbb{N}_+}$ such that:

$$\{\alpha_t.\mathbf{1}^k, a\} \sim \{\alpha_{t+1}.\mathbf{1}^k, b\} \tag{6}$$

Since $\{c_{t+1}\} \succ \{c_t\}$ for all t, we must have, since the ordering $\succsim$ restricted to singletons is increasing, that $\alpha_{t+1} > \alpha_t$ for all t. As the increasing sequence $\{c_t\}_{t\in\mathbb{N}_+}$ is initiated somewhere, it is bounded from below so that there exists some bundle x for which $\{c_t\} \succ \{c_0\} \succsim \{x\}$. Now, since every increasing sequence of numbers that is bounded from above is either convergent or finite, the only thing we need to check is that the sequence is not convergent. Suppose by contradiction that the sequence $\{\alpha_t\}$ is infinite and converges to some number $\overline{\alpha}$. By restricted independence (implied by proportional expansion consistency), we know that:

$$\{\overline{\alpha}.\mathbf{1}^k, a\} \succ \{\overline{\alpha}.\mathbf{1}^k, b\}$$

By continuity and restricted independence, the set $\{x : \{x, a\} \succsim \{x, b\}\}$ is closed. Because of this, there exists a number $\varepsilon > 0$ such that:

$$\{\alpha'.\mathbf{1}^k, a\} \succ \{\alpha''.\mathbf{1}^k, b\}$$

for all α' and $\alpha'' \in N_\varepsilon(\overline{\alpha})$. Assuming the sequence $\{\alpha_t\}$ to be converging to $\overline{\alpha}$ implies the existence of some positive integer s such, for all $t \geq s$, $\alpha_t \in N_\varepsilon(\overline{\alpha})$. By the continuity condition, we must therefore have:

$$\{\alpha_t.\mathbf{1}^k, a\} \succ \{\alpha_{t+1}.\mathbf{1}^k, b\}$$

for any such t, which contradicts the definition of α_t provided by (6). Hence the increasing sequence $\{\alpha_t\}$ is not convergent and must therefore be finite. ■

5 Independence of the axioms

We do not know whether our four substantive axioms (averaging, restricted independence, attenuation, and pairwise reduction consistency) are independent on an environment satisfying unboundedness, certainty equivalent, richness and the Archimedean axiom. Yet we can show on such an environment that there are orderings that satisfy averaging and pairwise combination consistency without satisfying the other axioms. Furthermore, it is easy

to exhibit, in an environment that does not satisfy the structural axioms, examples of orderings that violate averaging but satisfy all three other axioms. A good example of such an ordering is the ranking of sets based on their number of elements. This cardinality ranking of sets satisfies indeed restricted independence and, trivially, attenuation and pairwise reduction consistency but violates averaging.

A family of orderings of $P(X)$ that generalizes the UEU family is that which contains all orderings $\succsim$ of $P(X)$ that can be defined, for every sets A and B in $P(X)$, by:

$$A \succsim B \Longleftrightarrow \frac{\sum_{a\in A} p(a)u(a)}{\sum_{a\in A} p(a)} \geq \frac{\sum_{b\in B} p(b)u(b)}{\sum_{b\in B} p(b)} \tag{7}$$

for some real-valued functions u and p both having X as domain. Any UEU criterion is a member of this family that satisfies the additional property that, for all consequences $x \in X$, $p(x) = c$ for some real number c. Orderings that can be represented as per (7) for some real-valued functions u and p can be thought of as comparing sets on the basis of the expected utility of their consequence, but without imposing the requirement on the probability of all consequences to be the same. This interpretation obviously requires that we can interpret $p(x)$ as a probability, which in turns required that some measure-theoretic structure be imposed on X. But if we can provide this interpretation, any ordering of $P(X)$ that can be represented as per (7) can be viewed as comparing sets on the basis of their expected utility *conditional of being in the sets*. It is straightforward to verify that any ordering that can be represented as per (7) satisfies averaging.

Furthermore there are members of this family that satisfy pairwise reduction consistency but that violate attenuation and restricted independence. To see this, assume that $X = \mathbb{R}_{++}$ and define $\succsim$ by:

$$A \succsim B \Leftrightarrow \frac{\sum_{a\in A} a}{\sum_{a\in A} \frac{1}{a}} \geq \frac{\sum_{b\in B} b}{\sum_{b\in B} \frac{1}{b}}. \tag{8}$$

This ordering is clearly a member of the family represented as per (7) where p is defined by $p(x) = \frac{1}{x}$ and u by $u(x) = x^2$. It is straightforward to verify that this ordering satisfies pairwise combination consistency but violates attenuation and restricted independence.

6 Conclusion

This paper has characterized by four axioms the UEU ranking of completely uncertain decisions, under the assumption that the ranking of uncertain

decision is used in an Archimedean and rich environment where any set has a certainty equivalent. The axioms used in the characterization are finite and, therefore, verifiable from the mere observation of a choice behavior. Yet we have failed to prove that the axioms were "minimal" in the sense of being logically all independent.

A limitation of the UEU criterion is that it assigns to every consequence of a decision the same probability of occurrence. A next step in the research agenda is therefore to identify the properties of a more general EU criterion that does not impose this uniform assumption on the probabilities assigned to the consequences of a decision. The family of orderings that can be represented as per (7) for some functions p and u is an obvious first step into that direction. We have seen that any ordering in this family satisfies averaging but that there are orderings in that family who violate attenuation and proportional expansion consistency. It would be nice to know the axioms which, along with averaging, characterize this large family of rankings of completely uncertain decisions. A characterization of this family in the somewhat specific context in which the consequences of decisions are *lotteries* defined over a fundamental set of consequences - rather than final consequences - has been obtained recently in an interesting paper by Ahn (2008). Obtaining a characterization of this family in our context seems to be a high priority for future research.

7 Appendix

Proof of lemma 1

Suppose that consequences a, b, c, d, e, f, g and $h \in X$ are such that $\{a\} \cup \{b\} \sim \{e\}$, $\{c\} \cup \{d\} \sim \{f\}$, $\{a\} \cup \{c\} \sim \{g\}$ and $\{b\} \cup \{d\} \sim \{h\}$. We need to prove that $\{e\} \cup \{f\} \sim \{g\} \cup \{h\}$. Suppose by contradiction and without loss of generality that $\{e\} \cup \{f\} \prec \{g\} \cup \{h\}$. By certainty equivalence, there are consequences v_0 and v_1 such that $\{v_0\} \sim \{e\} \cup \{f\}$ and $\{v_1\} \sim \{g\} \cup \{h\}$. By transitivity, one has $\{v_0\} \prec \{v_1\}$ which, given the reflexivity of $\succsim$, implies $v_0 \neq v_1$. By certainty equivalence, there exists a consequence v such that $\{v\} \sim \{v_0, v_1\}$. By averaging and transitivity, one has $\{e\} \cup \{f\} \sim \{v_0\} \prec \{v\} \sim \{v_0, v_1\} \prec \{v_1\} \sim \{g\} \cup \{h\}$. Similarly, applying recursively averaging and certainty equivalence in this fashion, one can find an infinite sequence of consequences v, v', v'', etc. in X such that $\{e\} \cup \{f\} \prec \{v\} \prec \{v'\} \prec \{v''\} \prec ... \prec \{g\} \cup \{h\}$.

Choose now a'distinct from b, c and d and such that $\{a\} \prec \{a'\}$. The existence of a' is of course secured by the unboundedness axiom. Using certainty equivalence, consider e'such that $\{e'\} \sim \{a', b\}$. We can choose a' in such a way that $\{e\} \cup \{f\} \prec \{e'\} \cup \{f\} \prec \{g\} \cup \{h\}$. Indeed, given the fact that $\{a\} \cup \{b\} \sim \{e\}$, we know from restricted independence that $\{e\} \prec \{e'\}$ and $\{e\} \cup \{f\} \prec \{e'\} \cup \{f\}$. Now, if $\{e'\} \cup \{f\} \succsim \{g\} \cup \{h\}$, the richness axiom implies the existence of a consequence e'' such that $\{e''\} \cup \{f\} \sim \{\widehat{v}\}$ for some $\widehat{v}$such that $\{e\} \cup \{f\} \prec \{\widehat{v}\} \prec$

$\{g\} \cup \{h\}$. From restricted independence, we must have $\{e\} \prec \{e''\} \prec \{e'\}$. By richness again, there exists some consequence a''such that $\{e''\} \sim \{a'', b\}$. Hence, setting $a'' = a'$and $e' = e''$gives us the required conclusion that $\{e\} \cup \{f\} \prec \{e'\} \cup \{f\} \prec \{g\} \cup \{h\}$.

Using a similar reasoning, one can find consequences b', $c'$and d'all pairwise distinct from each others and from a'such that:

(i)$\{b\} \prec \{b'\}$ and $\{e\} \cup \{f\} \prec \{e''\} \cup \{f\} \prec \{g\} \cup \{h\}$ for some consequence e''such that $e'' \sim \{a', b'\}$.

(ii) $\{c\} \prec \{c'\}$ and $\{e\} \cup \{f\} \prec \{e''\} \cup \{f'\} \prec \{g\} \cup \{h\}$ for some consequence f'such that $f' \sim \{c', d\}$.

(iii).$\{d\} \prec \{d'\}$ and $\{e\} \cup \{f\} \prec \{e''\} \cup \{f''\} \prec \{g\} \cup \{h\}$ for some consequence f''such that $f'' \sim \{c', d'\}$.

The consequences a', b', $c'$and d'can all be chosen to be distinct because there are infinitely many $v\prime$ such that $\{e\} \cup \{f\} \prec \{v'\} \prec \{g\} \cup \{h\}$. Using certainty equivalence, define $g'$and $h'$by $\{g'\} \sim \{a', c'\}$ and $\{h'\} \sim \{b', d'\}$. By restricted independence, one has $\{g\} \prec \{g'\}$, $\{h\} \prec \{h'\}$ and $\{g\} \cup \{h\} \prec \{g'\} \cup \{h'\}$. We therefore have $\{e\} \cup \{f\} \prec \{e''\} \cup \{f''\} \prec \{g\} \cup \{h\} \prec \{g'\} \cup \{h'\}$. Yet the axiom of pairwise reduction consistency implies, in view of (i) and (ii) that $\{e''\} \cup \{f''\} \sim \{a', b', c', d'\} \sim \{g'\} \cup \{h'\}$, a contradiction.

Proof of lemma 2.

We find useful to represent the sequence defined in this lemma in the following array, with $n-1$ columns and an infinite number of rows:

	1	2	...	$n-2$	$n-1$
$\mathbf{b}^0$	$(u_1 + b_2^0)/2$	$(u_2 + b_3^0)/2$	...$\leftarrow$	$(u_{n-2} + b_{n-1}^0)/2$	$(u_{n-1} + u_n)/2$
$\mathbf{b}^1$	$(u_1 + b_2^0)/2$	$(b_1^1 + b_2^0)/2$	$\rightarrow$...	$(b_{n-3}^1 + b_{n-2}^0)/2$	$(b_{n-2}^1 + b_{n-1}^0)/2$
$\mathbf{b}^2$	$(b_2^2 + b_1^1)/2$	$(b_3^2 + b_2^1)/2$	...$\leftarrow$	$(b_{n-1}^2 + b_{n-2}^1)/2$	$(b_{n-2}^1 + b_{n-1}^0)/2$
$\mathbf{b}^3$	$(b_2^2 + b_1^1)/2$	$(b_1^3 + b_2^2)/2$	$\rightarrow$...	$(b_{n-3}^3 + b_{n-2}^2)/2$	$(b_{n-2}^3 + b_{n-1}^2)/2$
...	...	...	...	...	...

We are going to show that the "grand" sequence that starts from the "north-east" of the array and follows the arrows up to infinity, converges to $\overline{u}$. Since the sequence $\{b_h^i\}$ is the h^{th}column of this array and therefore, a subsequence of the grand sequence, the conclusion of the lemma would follow immediately. Define accordingly the grand sequence $\{\widehat{b}^t\}$, for $t = i(n-1)+1, ..., (i+1)(n-1)$, and $i = 0, 1, 2, ...$by:

$$\widehat{b}^t = b^i_{n-(t+1-(i(n-1)+1))} \text{ if } i \text{ is even and}$$
$$\widehat{b}^t = b^i_{t+1-(i(n-1)+1)} \text{ if } i \text{ is odd}$$

Any element of the grand sequence can be written as a weighted average of $\{u_1, ..., u_n\}$. In particular, for all $t = 1, ...,$ there exists $n-1$ real numbers

$\beta_1^t, ..., \beta_{n-1}^t$ such that:

$$\widehat{b}^t = \beta_1^t u_1 + \beta_2^t u_2 + ... + \beta_{n-1}^t u_{n-1} + \beta_{n-1}^t u_n$$

Moreover inspection reveals that β_h^t is defined by the following recursive formula:

$$\begin{aligned} \beta_h^t &= 0 \text{ if } t \in \{1, ..., n-h-1\} \\ \beta_h^{n-h} &= \frac{1}{2} \text{ and} \\ \beta_h^t &= \frac{1}{2}(\beta_h^{t-1} + \beta_h^{2m(t)-t+1} \text{ if } t \geq n-h+1 \end{aligned} \tag{9}$$

where $m(t)$ is defined as the largest integer strictly smaller than t that is divisible by $n-1$. In order to prove the lemma, it suffices to prove that $\lim_{t\to\infty} \beta_h^t = \frac{1}{n}$for all h. In what follows we will fix $h \in \{1, .., n-1\}$ and drop the subscript h from the sequence $\{\beta_h^t\}$ for notational convenience.

Once again, it is convenient to refer to the aforementioned representation of the sequence $\{\beta^t\}$, $t = 1, ..$as an array with $n-1$ columns and an infinite number of rows. We start from the first row with $\beta^1$and move left until we reach β^{n-1}. We then move down to the second row where the first element from the left is β^n. The sequence then increases from left right and the right-most element in the this row is β^{2n-2}. The right-most element in the third row is then β^{2n-1} and the sequence increases as it moves left (like in the first row) so that the left-most element is β^{3n-3} and so on. Let t be an arbitrary integer. If we write $t = m(t) + s$, it follows that β^t lies in the $(m(t)+1)^{th}$ row of this array. If $m(t)$ is even then, the $(m(t)+1)^{th}$ row is increasing from right to left so that β^t is the $(s+1)^{th}$ element from the right in this row. If $m(t)$ is odd, then β^t is the $(s+1)^{th}$ element from the left in the $(m(t)+1)^{th}$ row which increases from left to right. It follows that in this array, β^t for $t > n-1$ is the arithmetic mean of the element which immediately precedes it and the element directly in the row above.

The proof proceeds in two steps. The first is to show that the sequence $\{\beta^t\}$, $t = 1, ...$is convergent and the second is to show that the limit of the sequence is, in fact $\frac{1}{n}$. In order to establish the first step, we first record the two following properties $P1$and $P2$ of the sequence which can be easily verified.

$P1$. Let $r > 1$ be an odd integer. The sequence strictly increases from $\beta^{(r-1)(n-1)+1}$ to $\beta^{(r-1)(n-1)+h}$ and then strictly decreases from $\beta^{(r-1)(n-1)+h}$ to $\beta^{r(n-1)}$. If r is an even integer, then the sequence strictly increases from $\beta^{(r-1)(n-1)+1}$ to $\beta^{(r-1)(n-1)+n-h}$ and strictly decreases from $\beta^{(r-1)(n-1)+n-h}$ to $\beta^{r(n-1)}$. Thus for every row r in the array, the sequence increases from the right as we move left for h terms and then decreases for the remaining $n-h-1$ terms. Clearly $\beta^{(r-1)(n-1)+h}$ is the largest element of the rth row if r is odd and $\beta^{(r-1)(n-1)+n-h}$ if r is even. Note that the maximal element of any row is in the h^{th}column from the right.

$P2$. Let $t = (n-1)r + s$ where $m(t) = r$ (note that $1 \leq s \leq n-1$). Then $\beta^t = \frac{1}{2}\beta^{(n-1)(r-1)+(n-s)} + \frac{1}{2^2}\beta^{(n-1)(r-1)+(n-s-1)} + ... + \frac{1}{2^{s-1}}\beta^{(n-1)(r-1)+1} + \frac{1}{2^{s-1}}\beta^{(n-1)(r-1)}$. Thus each term of the sequence can be expressed as the weighted sum of the terms of the sequence in the row above.

CLAIM: Let $r > 1$ be an integer. Then:
(i) $\beta^{(r-1)(n-1)+h} - \beta^{r(n-1)} < \gamma_1(\beta^{(r-2)(n-1)+n-h} - \beta^{(r-2)(n-1)+1})$ and
(ii) $\beta^{(r-1)(n-1)+h} - \beta^{(r-1)(n-1)+1} < \gamma_2(\beta^{(r-2)(n-1)+n-h} - \beta^{(r-1)(n-1)})$
where $\gamma_1 = \frac{2^{n-h-1}-1}{2^{n-h-1}}$ and $\gamma_2 = \frac{2^{h-1}-1}{2^{h-1}}$ if r is odd and:
(iii) $\beta^{(r-1)(n-1)+n-h} - \beta^{(r-1)(n-1)+1} < \gamma_1(\beta^{(r-2)(n-1)+h} - \beta^{(r-1)(n-1)})$ and,
(iv) $\beta^{(r-1)(n-1)+n-h} - \beta^{r(n-1)} < \gamma_2(\beta^{(r-2)(n-1)+h} - \beta^{(r-2)(n-1)+1})$
if r is even.

Proof of the Claim: We first prove (ii). We do that by first noting that, according to $P2$:

$$\beta^{(r-1)(n-1)+h} = \frac{1}{2}\beta^{(r-2)(n-1)+n-h} + ... + \frac{1}{2^{h-1}}\beta^{(r-1)(n-1)+1} + \frac{1}{2^{h-1}}\beta^{(r-1)(n-1)}$$

Since $\beta^{(r-1)(n-1)+1} = \beta^{(r-1)(n-1)}$ and $\beta^{(r-2)(n-1)+n-h}$ is the largest term in the $(r-1)^{th}$ row according to $P1$, we conclude that :

$$\begin{aligned}\beta^{(r-1)(n-1)+h} - \beta^{(r-1)(n-1)+1} &< \beta^{(r-2)(n-1)+h}(\frac{1}{2} + ... + \frac{1}{2^{h-1}}) \\ &-(1 - \frac{1}{2^{h-1}})\beta^{(r-1)(n-1)} \\ &= (1 - \frac{1}{2^{h-1}})(\beta^{(r-2)(n-1)+n+h} - \beta^{(r-1)(n-1)})\end{aligned}$$

Since $\gamma_2 = (1 - \frac{1}{2^{h-1}})$, this establishes (ii).

We now prove (iii). According to $P2$:

$$\beta^{(r-1)(n-1)+n-h} = \frac{1}{2}\beta^{(r-2)(n-1)+h} + ... + \frac{1}{2^{n-h-1}}[\beta^{(r-1)(n-1)+1} + \beta^{(r-1)(n-1)}]$$

Since $\beta^{(r-1)(n-1)+1} = \beta^{(r-1)(n-1)}$ and since, from $P1$, we know that $\beta^{(r-2)(n-1)+h}$ is the largest term in the $(r-1)^{th}$ row, we obtain:

$$\begin{aligned}\beta^{(r-1)(n-1)+n-h} - \beta^{(r-1)(n-1)+1} &< (\frac{1}{2} + ... + \frac{1}{2^{n-h-1}})\beta^{(r-2)(n-1)+n-h} \\ &- (1 - \frac{1}{2^{n-h-1}})\beta^{(r-1)(n-1)+1} \\ &= (1 - \frac{1}{2^{n-h-1}})(\beta^{(r-2)(n-1)+n-h} - \beta^{(r-1)(n-1)+1})\end{aligned}$$

Since $\gamma_2 = (1 - \frac{1}{2^{n-h-1}})$, this establishes (iii).

We now prove (i). Applying $P2$, we have:

$$\beta^{(r-1)(n-1)+h} = \frac{1}{2}\beta^{(r-2)(n-1)+n-h} + ... + \frac{1}{2^{h-1}}\beta^{(r-1)(n-1)+1} + \frac{1}{2^{h-1}}\beta^{(r-1)(n-1)}$$

and:

$$\beta^{r(n-1)} = \frac{1}{2}\beta^{(r-2)(n-1)+1} + ... + \frac{1}{2^{n-h}}\beta^{(r-1)(n-1)+n-h} + ... + \frac{1}{2^{n-2}}\beta^{(r-1)(n-1)+1} + \frac{1}{2^{n-2}}\beta^{(r-1)(n-1)}$$

We thus have:

$$\Delta = \beta^{(r-1)(n-1)+h} - \beta^{r(n-1)}$$

$$= (\frac{1}{2} - \frac{1}{2^{n-h}})\beta^{(r-1)(n-1)+n-h} + ... + (\frac{1}{2^{h-1}} - \frac{1}{2^{n-2}})\beta^{(r-1)(n-1)+1}$$
$$+ (\frac{1}{2^{h-1}} - \frac{1}{2^{n-2}})\beta^{(r-1)(n-1)}$$
$$- \frac{1}{2}\beta^{(r-1)(n-1)+1}..... - \frac{1}{2^{n-h-1}}\beta^{(r-1)(n-1)+n-h-1}$$

Note that, according to $P1$, $\beta^{(r-1)(n-1)+n-h}$ is the largest element in its row. This, combined to the fact that:

$$\beta^{(r-1)(n-1)+1} < ... < \beta^{(r-2)(n-1)+n-h-1}$$

implies:

$$\Delta < ((\frac{1}{2} - \frac{1}{2^{n-h}}) + ... + (\frac{1}{2^{h-1}} - \frac{1}{2^{n-2}})$$
$$+ (\frac{1}{2^{h-1}} - \frac{1}{2^{n-2}}))\beta^{(r-1)(n-1)+n-h}$$
$$- (\frac{1}{2} + ... + \frac{1}{2^{n-h-1}})\beta^{(r-2)(n-1)+1}$$
$$= (\frac{1}{2} - \frac{1}{2^{n-h}})(1 + ... + \frac{1}{2^{h-2}}) + (\frac{1}{2^{h-1}} - \frac{1}{2^{n-2}}))\beta^{(r-2)(n-1)+n-h}$$
$$+ (1 - \frac{1}{2^{n-h-1}})\beta^{(r-1)(n-1)+1}$$
$$= (\frac{1}{2} - \frac{1}{2^{n-h}})(2 - \frac{1}{2^{n-h}}) + (\frac{1}{2^{h-1}} - \frac{1}{2^{n-2}}))\beta^{(r-2)(n-1)+1}$$
$$+ (1 - \frac{1}{2^{n-h-1}})\beta^{(r-1)(n-1)+1}$$
$$= (1 - \frac{1}{2^{n-h-1}})(\beta^{(r-2)(n-1)+n-h} - \beta^{(r-1)(n-1)+1})$$
$$= \gamma_1(\beta^{(r-2)(n-1)+n-h} - \beta^{(r-1)(n-1)+1})$$

which proves (i).

The proof of (iv) is symmetric to that of (i) and we omit the details.

We will use the inequalities in the Claim to put an upper bound on the distance between terms in the same row of the array. Let $r > 1$be an odd integer. Applying (i) in the Claim, we have:

$$\beta^{(r-1)(n-1)+h} - \beta^{r(n-1)} < \gamma_1(\beta^{(r-2)(n-1)+n-h} - \beta^{(r-2)(n-1)+1})$$

Observe that $\beta^{(r-2)(n-1)+n-h} - \beta^{(r-2)(n-1)+1})$ can be written as $\beta^{(r'-1)(n-1)+n-h} - \beta^{(r'-1)(n-1)+1})$ where $r' = r-1$. Since r' is an even integer, we can apply (iii) to obtain:

$$\beta^{(r-1)(n-1)+h} - \beta^{r(n-1)} < \gamma_1^2(\beta^{(r-3)(n-1)+n-h} - \beta^{(r-2)(n-1)+1}).$$

Hence applying (i) and (iii) repeatedly, we conclude that:

$$\begin{aligned}\beta^{(r-1)(n-1)+h} - \beta^{r(n-1)} &< \gamma_1^{r-1}(\beta^h - \beta^{n-1}) \\ &= \gamma_1^{r-1}(\frac{1}{2} - \frac{1}{2^{n-h-1}}) \\ &< \gamma_1^{r-1}(\frac{1}{2}).\end{aligned}$$

By the same argument $\beta^{(r-1)(n-1)+n-h} - \beta^{(r-1)(n-1)+1} < \gamma_1^{r-1}(\frac{1}{2})$ when r is even. Moreover, from analogous arguments, we obtain that:

$$\beta^{(r-1)(n-1)+h} - \beta^{(r-1)(n-1)+1} < \gamma_2^{r-1}(\frac{1}{2})$$

when r is odd and:

$$\beta^{(r-1)(n-1)+n-h} - \beta^{r(n-1)} < \gamma_2^{r-1}(\frac{1}{2})$$

when r is even.

Let r be an odd integer. The left-most and right-most terms in row r are $\beta^{(r(n-1)}$and $\beta^{(r-1)(n-1)+1}$respectively. Using the triangle inequality and the bounds derived in the previous paragraph, it follows that:

$$\begin{aligned}||\beta^{r(n-1)} - \beta^{(r-1)(n-1)+1}|| &\leq ||\beta^{r(n-1)} - \beta^{r(n-1)+h}|| + ||\beta^{r(n-1)+h} - \beta^{(r-1)(n-1)+1}|| \\ &< \frac{1}{2}(\gamma_1^{r-1} + \gamma_2^{r-1}).\end{aligned}$$

If r is an even integer, and the left-most and right-most terms in row r are $\beta^{(r-1)(n-1)+1}$ and $\beta^{(r(n-1)}$ respectively, one has:

$$\begin{aligned}||\beta^{r(n-1)} - \beta^{(r-1)(n-1)+1}|| &\leq ||\beta^{r(n-1)} - \beta^{r(n-1)+n-h}|| + ||\beta^{(r(n-1)+n-h} - \beta^{(r-1)(n-1)+1}|| \\ &< \frac{1}{2}(\gamma_1^{r-1} + \gamma_2^{r-1}).\end{aligned}$$

Note that the maximal difference of terms in row r is strictly less than $\frac{1}{2}\max[\gamma_1,\gamma_2]^{r-1}$.

Pick an integer t such that $t = r(n-1)$ where r is an odd integer i.e. β^t is the left-most term in row r and $m(t) = r-1$. Let $q = r'(n-1)$ where $r' > r$. Note that, by repeated application of the triangle inequality, it follows that $||\beta^t - \beta^q||$ is less than the sum of the differences between the left-most and right-most terms of all rows starting from $r+1$. Hence:

$$\begin{aligned}
||\beta^t - \beta^q|| &< \frac{1}{2}(\gamma_1^r + \gamma_1^{r+1} + ... + \gamma_2^r + \gamma_2^{r+1}.....) \\
&= \frac{1}{2}(\frac{{\gamma_1}^r}{1-\gamma_1} + \frac{{\gamma_2}^r}{1-\gamma_2}) \\
&\equiv \lambda(r) \\
&\equiv \lambda(m(t))
\end{aligned}$$

(note that we critically use the fact that γ_1 and γ_2 are strictly less than 1). Now let β^q be a term in row r' where $r' > r$. Applying the triangle inequality again, we have:

$$\begin{aligned}
||\beta^t - \beta^q|| &< \lambda(m(t)) + \frac{1}{2}\max[\gamma_1,\gamma_2]^{r-1} \\
&< \lambda(m(t)) + \frac{1}{2}\max[\gamma_1,\gamma_2]^{m(t)} \\
&\equiv \hat{\lambda}(t).
\end{aligned}$$

Observe that $\hat{\lambda}(t) \to 0$ as $t \to \infty$. Pick $\varepsilon > 0$ and let T be such that $\hat{\lambda}(t) < \varepsilon$ for all $t > T$. We have shown that $||\beta^T - \beta^q|| < \varepsilon$ for all $q > T$. Hence the sequence β^t is a Cauchy sequence and is convergent.

We now show that the sequence converges to $\frac{1}{n}$. Suppose it converges to α. Let $t$and k be positive integers such that $t+1 = k(n-1)$ and consider the following sequence of differences.

$$\beta^{t+1} - \beta^t = \frac{1}{2}(\beta^{(k-2)(n-1)+1} - \beta^t) \tag{10}$$

$$\beta^t - \beta^{t-1} = \frac{1}{2}(\beta^{(k-2)(n-1)+2} - \beta^{t-1}) \tag{11}$$

$$... = ...$$

$$\beta^{t-(n-3)} - \beta^{t-(n-2)} = \frac{1}{2}(\beta^{(k-1)(n-1)} - \beta^{(k-1)(n-1)}) \tag{12}$$

$$\beta^{t-(n-2)} - \beta^{t-(n-1)} = \frac{1}{2}(\beta^{(k-2)(n-1)+1} - \beta^{(k-1)(n-1)-1}) \tag{13}$$

$$... = ...$$

$$\beta^{n-h+1} - \beta^{n-h} = \frac{1}{2}(\beta^0 - \beta^{n-h}) \tag{14}$$

It is clear from these $(t-(n-h))$ equalities that except for the first $n-2$ negative terms of the right hand sides, every positive term of the first $n-1$ lines has an identical negative term in one of the lines $n+1$,...,$2n$. Hence, if we sum the equalities (10)-(14), we get:

$$\beta^{t+1} - \beta^{n-h} = \frac{1}{2}(\sum_{i=1}^{n-2} \beta^{k(n-1)+i})$$

Observe that $\beta^{n-h} = 1/2$. Also, $\{\beta^{k(n-1)+i}\}$, for $k = 1, ...,$ is a subsequence of the original sequence for all $i = 1, ..., n-2$. Since the original sequence converges to α, these subsequences must also converge to α. Therefore by taking limits on both sides of the equation above, we obtain $\alpha - 1/2 = -1/2(n-2)\alpha$, so that $\alpha = \frac{1}{n}$, as required.

Proof of lemma 3.

We only prove the first statement and distinguish three cases.

(a) $A \prec B \cup \{d\}$, in which case the proof is done.

(b) $A \sim B \cup \{d\}$. Then, by certainty equivalence, there exists e such that $\{e\} \sim \{d, c\}$. By averaging, $\{d\} \prec \{e\} \prec \{c\}$. By restricted independence, $B \cup \{d\} \prec B \cup \{e\}$ so that the statement $A \prec B \cup \{e\}$ follows.

(c) $A \succ B \cup \{d\}$. In that case the richness axiom applies and there is a consequence f such that $A \sim B \cup \{f\}$ and we proceed as in case (b).

Proof of lemma 4

Suppose first $\{c\} \sim \{d\}$. By averaging, $\{b\} \sim \{c\} \sim \{d\}$. Since $c \neq d$, we have $c \neq b$ or $d \neq b$. Assume without loss of generality that $c \neq b$. By restricted independence, $B \cup \{b\} \sim B \cup \{c\}$. Therefore $A \cup \{a\} \sim B \cup \{b\} \sim B \cup \{c\}$ and, by restricted independence, $A \cup \{a, b\} \sim B \cup \{c, b\}$. By restricted independence again, $B \cup \{c, b\} \sim B \cup \{c, d\}$. Finally, by transitivity, $A \cup \{a, b\} \sim B \cup \{c, d\}$.

Suppose now $\{c\} \not\sim \{d\}$ and assume, without loss of generality, that $\{c\} \prec \{d\}$. Two cases need to be considered.

1. Assume by contradiction that $A \cup \{a, b\} \prec B \cup \{c, d\}$. Let us show that there is a consequence $\underline{d}$such that $A \cup \{a, b\} \prec B \cup \{c, \underline{d}\} \prec B \cup \{c, d\}$. Choose a consequence u distinct from c such that $\{u\} \prec \{d\}$. The existence of such a consequence is guaranteed by the fact that $\{c\} \prec \{d\}$ and, using certainty equivalence, that one can always define u by $\{u\} \sim \{c, d\}$. By averaging, one must have $\{c\} \prec \{u\} \prec \{d\}$ which, given the reflexivity of $\succsim$, implies that u is distinct from both c and d. By restricted independence, one has $B \cup \{c, u\} \prec B \cup \{c, d\}$. Two mutually exclusive cases can occur.

   - $B \cup \{c, u\} \precsim A \cup \{a, b\}$. By averaging and certainty equivalence, one can find a consequence e such that $A \cup \{a, b\} \prec e \prec B \cup \{c, d\}$. By Richness, there is $\underline{d} : B \cup \{c, d\} \sim \{e\}$. Hence $A \cup \{a, b\} \prec B \cup \{c, \underline{d}\} \prec B \cup \{c, d\}$
   - $A \cup \{a, b\} \prec B \cup \{c, u\}$. In this case, let $\underline{d} = u$.

By certainty equivalence, there is a consequence $\underline{b}$ such that $\{\underline{b}\} \sim \{c, \underline{d}\}$. Notice that we can always choose $\underline{d}$ so that $\underline{d}$ and $\underline{b}$ do not belong to $B \cup \{c\} \cup A \cup \{a\}$. By restricted independence, $\{\underline{b}\} \prec \{b\}$. By averaging, $\{\underline{b}\} \sim \{c, \underline{d}\} \prec \{b, \underline{b}\} \prec \{b\}$. By restricted independence, $B \cup \{c, \underline{d}\} \prec B \cup \{b, \underline{b}\}$. By restricted independence, $A \cup \{a, \underline{b}\} \prec A \cup \{a, b\}$ and $A \cup \{a, \underline{b}\} \sim B \cup \{b, \underline{b}\}$. By transitivity, $B \cup \{c, \underline{d}\} \prec A \cup \{a, b\}$. But we have previously shown that $A \cup \{a, b\} \prec B \cup \{c, \underline{d}\}$. A contradiction.

2. Assume by contradiction that $A \cup \{a, b\} \succ B \cup \{c, d\}$. This case is treated like the previous one.

Proof of lemma 5

Start with $\{b\} \succ \{a\} \succsim B \cup \{b\}$. By averaging, $\{b\} \succ B$. Write Bas $B = \{b_1, \ldots, b_r\}$ with $\{b_1\} \precsim \{b_2\} \precsim \ldots \precsim \{b_r\}$. Let b_j be such that $\{b_j\} \prec \{b\}$ and $\{b\} \precsim b_i$ for all $i > j$. The existence of such a b_j is guaranteed by averaging. By certainty equivalence, one can find a consequence b'_j in X such that $b'_j \sim \{b, b_j\}$. By Averaging, $b_j \prec b'_j \prec b$. Define A' by $A' = B \cup \{b'_j\} \setminus \{b_j\}$. By averaging and transitivity, one has $A' \succ B$. By restricted independence, $A' \cup \{b\} \succ B \cup \{b\}$. By construction, $A' \cup \{b_j\} = B \cup \{b'_j\}$. By restricted independence, $B \cup \{b\} \succ B \cup \{b'_j\}$. Hence $A' \cup \{b\} \succ B \cup \{b\} \succ B \cup \{b'_j\} = A' \cup \{b_j\}$. By richness, there exists some consequence a'such that $A' \cup \{a'\} \sim B \cup \{b\}$. By restricted independence, one has $b \succ a' \succ b'_j$, which, given the definition of A', establish that $a' \notin A'$.

Proof of lemma 6

We prove only part (i) of condition C, the proof of the other part being similar. Suppose that we have $\{a\} \succsim B \cup \{b\}$, $\{b\} \sim \{c, d\}$, $\{b\} \succ \{a\}$, $b \notin B$and $\{c, d\} \cap B = \emptyset$ for consequences a, b, c, d in X and some finite subset B of X. By Lemma 5, there exists a finite set $A'$and a consequence a'such that $A' \cup \{a'\} \sim B \cup \{b\}$, $a' \notin A'$and $\#A' = \#B$. By Lemma 4, we must have $A' \cup \{a', b\} \sim B \cup \{c, d\}$. By certainty equivalence, there exists a consequence a''such that $a'' \sim A' \cup \{a'\}$. By transitivity, $\{b\} \succ \{a\} \succsim A' \cup \{a'\} \sim \{a''\}$. By attenuation, $A' \cup \{a', b\} \prec \{a'', b\}$. By transitivity, $\{a'', b\} \succ B \cup \{c, d\}$. Restricted independence and $\{a\} \succsim \{a''\}$ imply $\{b, a\} \succsim \{b, a''\}$. Transitivity finally yields $\{a, b\} \succ B \cup \{c, d\}$.

References

Adams, E. W. (1965): "Elements of a Theory of Inexact Measurement," *Philosophy of Science*, 32, 205–228.

Ahn, D. S. (2008): "Ambiguity without a State Space," *Review of Economic Studies*, 75, 3–28.

Arrow, K. J., and L. Hurwicz (1972): "An Optimality Criterion for decision-making under ignorance," in *Uncertainty and Expectations in Economics: Essays in honour of G. L. S. Shackle*, ed. by C. F. Carter, and J. L. Ford, pp. 1–11. Basil Blackwell, Oxford.

Baigent, N., and Y. Xu (2004): "Decisions under complete ignorance," Mimeo,Institute of public economics, Graz university.

Barberà, S., W. Bossert, and P. K. Pattanaik (2004): "Ranking Sets of Objects," in *Handbook of Utility Theory, vol. 2: Extensions*, ed. by S. Barberà, P. Hammond, and C. Seidl, pp. 893–977. Kluwer, Dordrecht.

Barberà, S., B. Dutta, and A. Sen (2001): "Strategy-Proof Social Choice Correspondences," *Journal of Economic Theory*, 101, 374–394.

Barberà, S., and P. K. Pattanaik (1984): "Extending an Order on a Set to the Power Set: Some Remarks on Kanai and Peleg's Approach," *Journal of Economic Theory*, 32, 185–191.

Benoît, J. P. (2002): "Strategic Manipulation in Voting Games When Lotteries and Ties are Permitted," *Journal of Economic Theory*, 102, 421–436.

Bossert, W. (1989): "On the extension of preferences over a set to the power set: An axiomatic characterization of a quasi-ordering," *Journal of Economic Theory*, 49, 84–92.

——— (1997): "Uncertainty aversion in non-probabilistic decision models," *Mathematical Social Sciences*, 34, 191–203.

Bossert, W., P. Pattanaik, and Y. Xu (2000): "Choice under Complete Uncertainty: Axiomatic Characterization of some Decision Rules," *Economic Theory*, 16, 295–312.

Casadesus-Masanell, R., P. Klibanoff, and E. Ozdenoren (2000): "Maxmin expected utility over Savage acts with a set of priors," *Journal of Economic Theory*, 92, 35–65.

Fishburn, P. (1970): *Utility Theory for Decision Making*. John Wiley, New York.

Fishburn, P. C. (1972): "Even-Chance Lotteries in Social Choice Theory," *Theory and Decision*, 3, 18–40.

——— (1984): "Comment on the Kannai-Peleg impossibility theorem for extending orders," *Journal of Economic Theory*, 32, 176–179.

——— (1992): "Signed orders and power set extensions," *Journal of Economic Theory*, 56, 1–19.

Gärdenfors, P. A. (1976): "On Definitions of Manipulation of Social Choice Function," in *Aggregation and Revelation of Preferences*, ed. by J. Laffont. North Holland, Amsterdam.

Gilboa, I., and D. Schmeidler (1989): "Maxmin Expected Utility with non-unique prior," *Journal of Mathematical Economics*, 18, 99–110.

Heiner, R. A., and D. J. Packard (1984): "A Uniqueness Result for Extending Orders; with Applications to Collective Choice as Inconsistency Resolution," *Journal of Economic Theory*, 32, 180–184.

Jaynes, E. T. (2003): *Probability Theory: the Logic of Science*. Cambridge University Press, Cambridge UK.

Kannai, Y., and B. Peleg (1984): "A Note on the Extension of an Order on a Set to the Power Set," *Journal of Economic Theory*, 32, 172–175.

Keynes, J. M. (1921): *A treatise on probabilities*. Macmillan, New York.

Krantz, D., R. D. Luce, P. Suppes, and A. Tversky (1971): *Foundations of Measurement, vol. 1*. Academic Press, New York London.

Luce, R. D., and H. Raiffa (1957): *Games and Decisions*. John Wiley, New York.

Maskin, E. (1979): "Decision Making under Ignorance with Implications for Social Choice," *Theory and Decision*, 11, 319–337.

Nehring, K., and C. Puppe (1996): "Continous extension of an order on a set to the power set," *Journal of Economic Theory*, 68, 456–479.

Neumann, J. V., and O. Morgenstern (1947): *Theory of Game and Economic Behavior*. Princeton University Press, Princeton, NJ, USA.

Nitzan, S., and P. K. Pattanaik (1984): "Median-Based extensions of an Ordering over a Set to the Power Set: An Axiomatic Characterization," *Journal of Economic Theory*, 34, 252–261.

Ozyürt, S., and R. Sanver (2006): "A General Impossibility Result on Strategy-Proof Social Choice Hyperfunctions," Mimeo,Department of Economics,New York University.

Pattanaik, P. K., and B. Peleg (1984): "An Axiomatic Characterization of the Lexicographic Maximin Extension of an Ordering over a Set to the Power Set," *Social Choice and Welfare*, 1, 113–122.

Peleg, B., and H. Peters (2005): "Nash Consistent Effectivity Functions through Lottery Models," The Hebrew University of Jerusalem, Center for the study of Rationality working paper no 404.

Puppe, C. (1995): "Freedom of Choice and Rational Decisions," *Social Choice and Welfare*, 12, 137–154.

Savage, J. L. (1954): *The Foundation of Statistics*. Wiley, New York.

Schmeidler, D. (1989): "Subjective probability and expected utility without additivity," *Econometrica*, 57, 571–587.

Scott, D. (1964): "Measurement Structure and Linear Inequalities," *Journal of Mathematical Psychology*, 1, 233–247.