# Optimality and Diversifiability of Mean Variance and Arbitrage Pricing Portfolios

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# Agenda

- Focus of this paper: large $N$ characterization of mean-variance and arbitrage-pricing portfolios.
- Set-up: dynamic factor model.
- Results for portfolios weights and return (Theorems), abstracts from estimation uncertainty.
- Discussion of implications for:
  - Diversification and granularity.
  - Role of factors' conditional distribution.
  - Limit approximations.
  - Short-selling.
  - Sub-optimal trading strategies.

# This paper's contributions

- The paper shows 'new' results for MV and AP trading strategies:

  (i) It extends known results on diversification (common versus idiosyncratic risk) of MV and AP portfolios to non-exact pricing cases.

  (ii) It characterizes asymptotic behaviour of portfolio weights: in non-exact pricing cases MV and AP portfolio weights asymptotically equivalent and, moreover, functionally independent of factors conditional moments

  (iii) (technical) provides primitive conditions on asset returns distribution that extends typical high-level assumptions used in asset pricing literature.

- Remark 1. Our analysis here abstracts from estimation: results are pointwise in $t$ and we let $N \longrightarrow \infty$ (no double asymptotics but cross-sectional asymptotics).
- Remark 2. Generally speaking, our analysis sheds light on the issue of 'how to construct a market-beta neutral portfolio?' One needs to identify and estimate accurately factor loadings corresponding to strong factors! Nothing else matters.

# A factor model of asset returns

- Set-up: "dynamic" factor model: $N$-dimensional vector of asset returns $\mathbf{r}_t$ driven by a $k \times 1$ vector of common factors and the $N \times 1$ vector of idiosyncratic components: $k$ is fixed as $N \longrightarrow \infty$.

$$\mathbf{r}_t - r_{0,t-1}\mathbf{e} = \boldsymbol{\mu}_{t-1} + \mathbf{B}\mathbf{z}_t + \boldsymbol{\varepsilon}_t$$

  where $A_{(N),t} \equiv \left(\cup_{i=1}^{N} \mathcal{A}_{i.}\right) \cup \left(\cup_{i=1}^{N} \mathcal{A}_{it}\right) \cup A_t$, so that $r_t \in A_{(N),t}$ for any $N$

- Both factors and idiosyncratic shocks conditionally heteroskedastic with zero conditional means

$$\mathbf{z}_t \mid \mathcal{A}_{(N),t-1} \sim (0, \Omega_{t-1}), \quad \boldsymbol{\varepsilon}_t \mid \mathcal{A}_{(N),t-1} \sim (0, \mathbf{G}_{t-1}), \quad (1)$$

  $k \times k$ matrix $\Omega_t > 0$, $N \times N$ matrix $\mathbf{G}_t > 0$ with $\mathbf{H}_t = \mathbf{G}_t^{-1}$.

- It can be alternatively assumed $\mathbf{z}_t \mid \mathcal{A}_{(N),t-1} \sim (0, \mathbf{I}_k)$ with $\mathbf{B}_t = \mathbf{B}\Omega_t^{\frac{1}{2}}$.

# Covariance matrix of asset returns

- $\mathbf{G}_t$ need not be diagonal. Even bounded max eigenvalue condition assumed by Chamberlain and Rothschild (1983) not required. We only need $\varepsilon_t$ to be cross sectionally weakly dependent in the sense discussed in Chudik, Pesaran, and Tosetti (2010). that allows $\rho(\mathbf{G}_t) = O(N^{\alpha})$, for $\alpha < 1$.
- Asset return conditional variance-covariance matrix follows as:

$$\begin{aligned} & E\left[(\mathbf{r}_t - r_{0,t-1}\mathbf{e} - \boldsymbol{\mu}_{t-1})(\mathbf{r}_t - r_{0,t-1}\mathbf{e} - \boldsymbol{\mu}_{t-1})' \mid \mathcal{A}_{(N),t-1}\right] \\ = \quad & \Sigma_{t-1} = \mathbf{B}\Omega_{t-1}\mathbf{B}' + \mathbf{G}_{t-1}. \end{aligned}$$

- No need to specify any parametric form for $\mathbf{H}_t^{-1}$ and $\Omega_t$.
- Our set up nests various dynamic factor models with time-varying conditional second moment: Diebold and Nerlove (1989), Harvey et al (1994), King et al (1994), Chib et al (2002), Fiorentini et al (2004), Connor et al (2006) among others.

# Conditional means of asset returns

- Conditional mean of the asset returns:

$$E(\mathbf{r}_t - r_{0,t-1}\mathbf{e} \mid \mathcal{A}_{(N),t-1}) \equiv \boldsymbol{\mu}_{t-1} = \mathbf{v}_{t-1} + \mathbf{B}\,\boldsymbol{\lambda}_{t-1},$$

- Here:

$\mathbf{v}_{t-1}$ pricing error,
$\boldsymbol{\lambda}_{t-1}$ factor risk premia,
$\mathbf{B}$ factor loadings.

- Assume $\mathbf{v}_t$ row-wise independent from $\mathbf{B}$, and $\mathbf{H}_t$.
- Given the focus of the analysis, we take specifications of $\boldsymbol{\mu}_{t-1}$ and $\mathbf{v}_{t-1}$ as given.
- Dynamics can be allowed for through serial correlations in $\boldsymbol{\lambda}_{t-1}$.

# APT Restrictions

- APT starts from

$$\boldsymbol{\mu}_{t-1} = \mathbf{B}\hat{\boldsymbol{\lambda}}_{t-1} + \hat{\mathbf{v}}_{t-1}, \tag{2}$$

where

$$\hat{\boldsymbol{\lambda}}_{t-1} = \left(\mathbf{B}'\mathbf{H}_{t-1}\mathbf{B}\right)^{-1}\mathbf{B}'\mathbf{H}_{t-1}\boldsymbol{\mu}_{t-1} \tag{3}$$

is the GLS estimator and the regression residuals $\hat{\mathbf{v}}_{t-1}$ satisfies $\mathbf{B}'\mathbf{H}_t\hat{\mathbf{v}}_{t-1} = 0$.

- We make assumptions on population quantities $\boldsymbol{\lambda}_{t-1}, \mathbf{v}_{t-1}$ rather than on sample quantities $\hat{\boldsymbol{\lambda}}_{t-1}, \hat{\mathbf{v}}_{t-1}$ given our aim of establishing limit of portfolio weights and portfolios returns.

- $\mathbf{B} = (\ \boldsymbol{\beta}_1, ..., \ \boldsymbol{\beta}_N)'$ $N \times k$ matrix of factor loadings such that as $N \to \infty$:

$$N^{-1}\mathbf{B}'\mathbf{e} \to_p \boldsymbol{\mu}_{\beta}, \ \ N^{-1}\mathbf{B}'\mathbf{H}_t\mathbf{B} \to_p \mathbf{D}_t > 0. \tag{4}$$

  $\mathbf{e} = (1, 1, ..., 1)'$. This result require row-wise independence of $\mathbf{B}$ from $\mathbf{H}_t, \varepsilon_t$
- Factors are strong in the sense that $\mathbf{B}'\mathbf{B} = O_p(N)$
- Great deal of cross-sectional dependence is permitted. Primitive conditions are provided.
- Can generalize to heterogeneous non-random $\mathbf{B}$.

# Three assumptions about pricing errors

- Assumption on pricing consequences of different no-arbitrage restrictions (recall $\boldsymbol{\mu}_t = \mathbf{B}\boldsymbol{\lambda}_t + \mathbf{v}_t$).
  (i) exact pricing: for any $N$

$$\mathbf{v}_t = 0.$$

  (ii) asymptotic arbitrage pricing: as $N \to \infty$

$$\mathbf{e}_i'\mathbf{H}_t\mathbf{v}_t = O_p(N^{-\frac{1}{2}}),\ N^{\frac{1}{2}}\mathbf{e}'\mathbf{H}_t\mathbf{v}_t \to_p c_t,\ \ \mathbf{v}_t'\mathbf{H}_t\mathbf{v}_t \to_p d_t.$$

  (iii) unconstrained (no arbitrage): as $N \to \infty$

$$\mathbf{e}_i'\mathbf{H}_t\ \mathbf{v}_t = O_p(1),\ \frac{\mathbf{e}'\mathbf{H}_t\ \mathbf{v}_t}{N} \to_p c_t,\ \frac{\mathbf{v}_t'\mathbf{H}_t\ \mathbf{v}_t}{N} \to_p d_t.$$

- When exact pricing condition holds, our factor model is CAPM/intertemporal CAPM.

- Turning to arbitrage pricing, note that under APT we should not be able to find a portfolio $\mathbf{w}_t$ such that as $N \to \infty$

$$var\left[\mathbf{w}'_{t-1}(\mathbf{r}_t - r_{0t}\mathbf{e}) \mid \mathcal{A}_{(N),t-1}\right] \to 0,\ \mathbf{w}'_{t-1}\boldsymbol{\mu}_{t-1} \geq v > 0 \ \ a.s.$$

otherwise there will be possibility of making unabounded risk free returns. Under this setting we must have

$$\hat{\mathbf{v}}'_t\mathbf{H}_t\hat{\mathbf{v}}_t = \mathbf{v}'_t\left[\mathbf{H}_t - \mathbf{H}_t\mathbf{B}(\mathbf{B}'\mathbf{H}_t\mathbf{B})^{-1}\mathbf{B}'\mathbf{H}_t\right]\mathbf{v}_t = O_p(1). \quad (5)$$

Our conditions on $\mathbf{v}_t, \mathbf{B}, \mathbf{H}_t$ is slightly stronger than required by APT (we require the existence of limits rather than upper bounds).

Further limit conditions are required such that as $N \longrightarrow \infty$,

$$\frac{\mathbf{e}'\mathbf{H}_t\mathbf{e}}{N} \longrightarrow_p a_t > 0, \tag{6}$$

$$\frac{\mathbf{B}'\mathbf{H}_t\mathbf{B}}{N} \longrightarrow_p \mathbf{D}_t > 0, \text{(see above)} \tag{7}$$

$$\frac{\mathbf{B}'\mathbf{H}_t\mathbf{H}_t\mathbf{B}}{N} \longrightarrow_p \mathbf{E}_t \geq 0, \tag{8}$$

$$\mathbf{B}'\mathbf{H}_t\mathbf{e}_i = O_p(1), \;\; \mathbf{e}'\mathbf{H}_t\mathbf{e}_i = O_p(1) \tag{9}$$

where $\mathbf{e}_i$ is the $i^{th}$ column of the identity matrix $\mathbf{I}_N$.

# MV and AP Portfolios

- Results on portfolio weights and returns
- Mean-Variance portfolios:

$$\mathbf{w}_{t-1}^{MV} = \frac{1}{\kappa_{t-1}} \Sigma_{t-1}^{-1} \boldsymbol{\mu}_{t-1}. \tag{10}$$

with portfolio return

$$\rho_t^{MV} = (\mathbf{r}_t - r_{0t-1}\mathbf{e})' \mathbf{w}_{t-1}^{MV} + r_{0t-1}. \tag{11}$$

- Arbitrage-Pricing portfolios: used to establish APT (see Ingersoll, 1984, Theorem 1)

$$\mathbf{w}_{t-1}^{AP} = \frac{1}{\kappa_{t-1}} \mathbf{H}_{t-1} \hat{\mathbf{v}}_{t-1} \tag{12}$$

with associated portfolio return

$$\rho_t^{AP} = (\mathbf{r}_t - r_{0t-1}\mathbf{e})' \mathbf{w}_{t-1}^{AP} + r_{0t-1}. \tag{13}$$

Recall that $\mathbf{B}'\mathbf{H}_t \hat{\mathbf{v}}_{t-1} = 0$.

# Relationships of MV and AP portfolios

To clarify analogies between MV and AP portfolios, set

$$\breve{\Sigma}_t^{-1} = \mathbf{H}_t - \mathbf{H}_t\mathbf{B}(\mathbf{B}'\mathbf{H}_t\mathbf{B})^{-1}\mathbf{B}'\mathbf{H}_t,$$

as compared to

$$\Sigma_t^{-1} = \mathbf{H}_t - \mathbf{H}_t\mathbf{B}(N^{-1}\Omega_t^{-1} + N^{-1}\ \mathbf{B}'\mathbf{H}_t\mathbf{B})^{-1}N^{-1}\mathbf{B}'\mathbf{H}_t.$$

Then for any finite $N > k$, the AP portfolio weights satisfy

$$\mathbf{w}_{t-1}^{ap} = \breve{\Sigma}_{t-1}^{-1}\hat{\mathbf{v}}_{t-1} = \Sigma_{t-1}^{-1}\hat{\mathbf{v}}_{t-1} = \breve{\Sigma}_{t-1}^{-1}\mathbf{v}_{t-1} = \breve{\Sigma}_{t-1}^{-1}\boldsymbol{\mu}_{t-1}.$$

The AP portfolio return satisfies

$$\rho_t^{ap} = r_{0,t-1} + (\boldsymbol{\varepsilon}_t + \hat{\mathbf{v}}_{t-1})'\breve{\Sigma}_{t-1}^{-1}\hat{\mathbf{v}}_{t-1},$$

and the difference between the MV and the AP portfolio returns satisfies

$$\rho_t^{mv} - \rho_t^{ap} = \boldsymbol{\varepsilon}_t'\Sigma_{t-1}^{-1}\mathbf{B}\hat{\boldsymbol{\lambda}}_{t-1} + \hat{\boldsymbol{\lambda}}_{t-1}'\mathbf{B}'\Sigma_{t-1}^{-1}\mathbf{B}(\hat{\boldsymbol{\lambda}}_{t-1} + \mathbf{z}_t).$$

Remarks:

- AP expressed as function of true pricing error $\mathbf{v}_t$.
- Estimation of APT portfolio weights does not require factor risk premia $\boldsymbol{\lambda}_t$.
- Difference between MV and AP portfolio returns involves an $O_p(N^{-\frac{1}{2}})$ term $\boldsymbol{\varepsilon}_t'\Sigma_{t-1}^{-1}\mathbf{B}\hat{\boldsymbol{\lambda}}_{t-1}$, and an $O_p(1)$ term $(\mathbf{z}_t' + \hat{\boldsymbol{\lambda}}_{t-1}')\mathbf{B}'\Sigma_{t-1}^{-1}\mathbf{B}\hat{\boldsymbol{\lambda}}_{t-1}$, irrespective of the assumed form of no-arbitrage.
- AP portfolio return independent of the factors and their risk premia for any $N$.
- Under exact pricing $\rho_t^{AP} = r_{0t-1}$.

# A well diversified portfolio

- We consider a slightly different definition from Chamberlain (1983) of well-diversification.

Definition 1 (well-diversification) The portfolio $\mathbf{w}$ is well diversified if

$$\| \mathbf{w} \| \rightarrow_p 0 \text{ as } N \rightarrow \infty.$$

Remark (a) Since

$$\mathbf{w}' \mathbf{G}_{t-1} \mathbf{w} \leq \| \mathbf{w} \| \, \rho(\mathbf{G}_{t-1}) \;\; a.s.$$

well-diversification of $\mathbf{w}$ implies that idiosyncratic risk vanishes in mean square if the maximum eigenvalue of $\mathbf{G}_{t-1}$ does not grow too quickly.

# Market neutrality

Definition 2 (asymptotic market neutrality) The portfolio $\mathbf{w}$ is said to be asymptotically market (or beta) neutral if

$$\| \mathbf{B}'\mathbf{w} \| \rightarrow_p 0 \text{ as } N \rightarrow \infty.$$

Remark (a) For the AP portfolio, Definition 2 applies for all finite $N > k$ as well, since

$$\mathbf{B}'\mathbf{w}_t^{ap} = \mathbf{0}, \text{ for any } N > k, \text{ by construction.}$$

Remark (b) When $\mathbf{w}$ satisfies Definition 2, contribution to portfolio return of both the common risk, $\mathbf{z}_t$, and the risk premia, $\boldsymbol{\lambda}_t$, vanish.

Remark (c) A portfolio that satisfies Definition 2 need not be well diversified, as acknowledged for instance by Hubermann (1982, p. 187).

Remark (d) Definition 2 relevant if the portfolio weights do not decay to zero too quickly.

**(exact no-arbitrage pricing)** Under the exact pricing condition
(i) For any $i$

$$Nw_{it}^{mv} - \mathbf{e}_i'\mathbf{H}_t\mathbf{B}\mathbf{D}_t^{-1}\Omega_t^{-1}\boldsymbol{\lambda}_t \rightarrow_p 0 \tag{14}$$

and

$$w_{it}^{ap} = 0.$$

(ii) If it is further assumed that

$$\begin{aligned} N^{-1/2}\mathbf{e}'\mathbf{H}_{t-1}\varepsilon_t &\quad | \quad \mathcal{A}_{t-1} \rightarrow_d N(0, a_{t-1}), \\ N^{-1/2}\mathbf{B}'\mathbf{H}_{t-1}\varepsilon_t &\quad | \quad \mathcal{A}_{t-1} \rightarrow_d N(\mathbf{0}, \mathbf{D}_{t-1}), \end{aligned}$$

then

$$\begin{aligned} \rho_t^{mv} - r_{0,t-1} &\rightarrow_p \boldsymbol{\lambda}'_{t-1}\Omega_{t-1}^{-1}(\boldsymbol{\lambda}_{t-1} + \mathbf{z}_t), \\ \left(\frac{\mu_{\rho,t-1}^{mv} - r_{0,t-1}}{\sigma_{\rho,t-1}^{mv}}\right) &\rightarrow_p (\boldsymbol{\lambda}'_{t-1}\Omega_{t-1}^{-1}\boldsymbol{\lambda}_{t-1})^{\frac{1}{2}}, \end{aligned}$$

and

$$\rho_t^{ap} = r_{0,t-1}.$$

To summarize, **under exact no-arbitrage pricing**:

- $\mathbf{w}^{mv}$ is well-diversified (Definition 1), but it is not market neutral (Definition 2). $\mathbf{w}^{ap}$ satisfies Definitions 1 for any $N > k$.
- The limit mv portfolio excess return, and its (ex ante) Sharpe ratio, are only a function of the factors characteristics.
- The ap portfolio excess return is identically zero.
- The mv portfolio weights are $O_p(N^{-1})$, and a function of the factors' characteristics.

(**asymptotic no-arbitrage pricing**) In this case we have
(i) For any $i$

$$N^{\frac{1}{2}} w_{it}^{mv} - \mathring{w}_{it} \longrightarrow_p 0$$

and

$$N^{\frac{1}{2}} w_{it}^{ap} - \mathring{w}_{it} \longrightarrow_p 0,$$

where

$$\mathring{w}_{it} = N^{\frac{1}{2}} \mathbf{e}_i' \mathbf{H}_t \mathbf{v}_t - (c_t / b_t) \mathbf{e}_i' \mathbf{H}_t \mathbf{B} \mathbf{A}_t^{-1} \boldsymbol{\mu}_\beta.$$

(ii) Further if

$$\mathbf{v}'_{t-1}\mathbf{H}_{t-1}\varepsilon_t \mid \mathcal{A}_{t-1} \to_d N(0, d_{t-1}),$$

then

$$\rho_t^{mv} - r_{0,t-1} \mid \mathcal{A}_{t-1} \to_d \frac{e_{t-1}}{b_{t-1}} + x_t + \boldsymbol{\lambda}'_{t-1}\Omega_{t-1}^{-1}\boldsymbol{\lambda}_{t-1} + \boldsymbol{\lambda}'_{t-1}\Omega_{t-1}^{-1}\mathbf{z}_t,$$

$$\left(\frac{\mu_{\rho,t-1}^{mv} - r_{0,t-1}}{\sigma_{\rho,t-1}^{mv}}\right) \to_p \left(\frac{e_{t-1}}{b_{t-1}} + \boldsymbol{\lambda}'_{t-1}\Omega_{t-1}^{-1}\boldsymbol{\lambda}_{t-1}\right)^{\frac{1}{2}},$$

and

$$\rho_t^{ap} - r_{0,t-1} \mid \mathcal{A}_{t-1} \to_d \left(\frac{e_{t-1}}{b_{t-1}}\right) + x_t,$$

$$\left(\frac{\mu_{\rho,t-1}^{ap} - r_{0,t-1}}{\sigma_{\rho,t-1}^{ap}}\right) \to_p \left(\frac{e_{t-1}}{b_{t-1}}\right)^{\frac{1}{2}},$$

where $x_t \sim N(0, e_{t-1}/b_{t-1})$.

To summarize, **under asymptotic no-arbitrage pricing**:

- $w_t^{mv}$ is neither well-diversified nor market neutral.
- The limit mv portfolio excess return, and the associated ex ante Sharpe ratio, are functions of both factors and asset-specific characteristics.
- The ex ante Sharpe ratio is positive and bounded.
- The same features apply to the limit ap portfolio excess return with the notable difference of being functionally independent of the common factors.
- In general, the limit Sharpe ratio for the ap portfolio is smaller than that of the mv portfolio.
- The mv and ap portfolio weights are both $O_p(N^{-\frac{1}{2}})$. Their limit approximation, $\mathring{w}_{it}$, is the same and does not depend on the distribution of the common factors, $\mathbf{z}_t$.

# Portfolio diversification

- Under the exact pricing case, the MV portfolio weights can be written as

$$\mathbf{w}_t^{mv} = N^{-1}\mathbf{H}_t\mathbf{B}\mathbf{D}_t^{-1}\Omega_t^{-1}\boldsymbol{\lambda}_t + O_p(N^{-3/2}),$$

and

$$\mathbf{w}_t^{mv\prime}\mathbf{w}_t^{mv} = N^{-1}\boldsymbol{\lambda}_t'\Omega_t^{-1}\mathbf{D}_t^{-1}\left(\frac{\mathbf{B}'\mathbf{H}_t\mathbf{H}_t\mathbf{B}}{N}\right)\mathbf{D}_t^{-1}\Omega_t^{-1}\boldsymbol{\lambda}_t + O_p(N^{-5/2})$$

yielding $\mathbf{w}_t^{mv\prime}\mathbf{w}_t^{mv} \longrightarrow_p 0$ as $N \longrightarrow \infty$. Hence, in this case the MV portfolio is well-diversified in the sense of Definition 1.

- The above result does not carry over to other two cases: in the asymptotic no-arbitrage case we have

$$\begin{aligned}\mathbf{w}_t^{mv\prime}\mathbf{w}_t^{mv} &= \mathbf{v}_t'\mathbf{H}_t\mathbf{H}_t\mathbf{v}_t - 2(c_t/b_t)\left(\frac{\mathbf{v}_t'\mathbf{H}_t\mathbf{H}_t\mathbf{B}}{N}\right)\mathbf{A}_t^{-1}\mu_\beta \\ &\quad +(c_t/b_t)^2\mu_\beta'\mathbf{A}_t^{-1}\left(\frac{\mathbf{B}'\mathbf{H}_t\mathbf{H}_t\mathbf{B}}{N}\right)\mathbf{A}_t^{-1}\mu_\beta \\ &\quad +O_p(N^{-1/2}),\end{aligned}$$

which is $O_p(1)$. In this case the weights are not granular and the possibility that one or more assets in the portfolio will be given sizeable weights can not be ruled out even if $N \to \infty$.

# Comparisons with some sub-optimal portfolios

- We consider the global-minimum-variance (GMV) and the equal weighted (EW) portfolios.
- GMV portfolio weights, $\mathbf{w}_t^{gmv} = (w_{1t}^{gmv}, w_{2t}^{gmv}, ..., w_{Nt}^{gmv})'$ solution to the problem:

$$\mathbf{w}_t^{gmv} = \text{argmin}_{\mathbf{w}} \left(\mathbf{w}'\Sigma_t\mathbf{w}\right), \text{ such that } \mathbf{w}'\mathbf{e} = 1,$$

yielding

$$\mathbf{w}_t^{gmv} = \frac{\Sigma_t^{-1}\mathbf{e}}{\mathbf{e}'\Sigma_t^{-1}\mathbf{e}}.$$

- Well known that this portfolio does not belong to the efficient frontier, unless $\mu_{i,t-1} = \mu_{t-1}$ for all $i$.
- This portfolio is still of interest since it does not require the estimation of expected returns. Jagannathan and Ma (2003) report comparable performance to MV portfolio.

- Let

$$\mathring{w}_{it}^{gmv} = N^{-1}\left[\left(\frac{b_t}{a_t}\right)\mathbf{e}_i'\mathbf{H}_t\mathbf{e} - \mathbf{e}_i'\mathbf{H}_t\mathbf{B}\mathbf{A}_t^{-1}\boldsymbol{\mu}_\beta\right]. \tag{15}$$

- Then, under any of the no-arbitrage conditions listed in Assumption 5 we have:

$$N(w_{it}^{gmv} - \mathring{w}_{it}^{gmv}) \rightarrow_p \ 0. \tag{16}$$

For portfolio returns and the *ex ante* Sharpe ratio we have under exact no-arbitrage, for a random variable $g_t \sim N(0, a_{t-1}/b_{t-1})$,

$$N^{\frac{1}{2}}(\rho_t^{gmv} - r_{0,t-1}) \mid \mathcal{A}_{t-1} \to_d g_t,$$

$$N^{\frac{1}{2}}\left(\frac{\mu_{\rho,t-1}^{gmv} - r_{0,t-1}}{\sigma_{\rho,t-1}^{gmv}}\right) \to_p \sqrt{a}_{t-1}\boldsymbol{\mu}'_{\beta}\mathbf{A}_{t-1}^{-1}\Omega_{t-1}^{-1}\boldsymbol{\lambda}_{t-1}.$$

- Under asymptotic no-arbitrage

$$N^{\frac{1}{2}}(\rho_t^{gmv} - r_{0,t-1}) \mid \mathcal{A}_{t-1} \to_d g_t + \frac{c_{t-1}}{a_{t-1}},$$

$$\left(\frac{\mu_{\rho,t-1}^{gmv} - r_{0,t-1}}{\sigma_{\rho,t-1}^{gmv}}\right) \to_p \frac{c_{t-1}}{\sqrt{a_{t-1}b_{t-1}}}.$$

- For the EW portfolio, defined by $\mathbf{w}_t^{ew} = N^{-1}\mathbf{e}$, we have

$$\rho_t^{ew} - r_{0,t-1} = N^{-1}(\mathbf{r}_t - r_{0,t-1}\mathbf{e})'\mathbf{e} \to_p \boldsymbol{\mu}_{v,t-1} + \boldsymbol{\mu}_\beta'\boldsymbol{\lambda}_{t-1},$$

where $N^{-1}\mathbf{e}'\mathbf{v}_t \to_p \mu_{vt}$, and

$$\left(\frac{\mu_{\rho,t-1}^{ew} - r_{0,t-1}}{\sigma_{\rho,t-1}^{ew}}\right) \to_p \frac{\boldsymbol{\mu}_{v,t-1} + \boldsymbol{\mu}_\beta'\,\boldsymbol{\lambda}_{t-1}}{\boldsymbol{\mu}_\beta'\Omega_{t-1}\boldsymbol{\mu}_\beta}.$$

- The EW portfolio is well-diversified, but is not market neutral. Well-diversification occurs when $\rho\left(\mathbf{G}_{t-1}\right) = o_p(N)$.
- ex ante Sharpe ratio of the equal weighted portfolio is bounded in $N$, but *need not be positive*.
- Relatively favourable evidence provided in the empirical literature for the EW portfolio (see De Miguel et al (2009)) most likely is due to negative impact of estimation uncertainty on the performance of MV and AP portfolios.

# Limit portfolios

- Conditional distribution of factors $z_t$ can be irrelevant, as far as form of MV limiting portfolios is concerned.
- Contribution of factors risk premia $\lambda_t$ and $\Omega_t$ could vanish at a fast rate.
- Empirical implication: can we avoid specifying, let alone estimating, $\lambda_t$ and $\Omega_t$ (and set them equal to zero matrices)?

- Obviously, for finite $N$, this implies approximation error (except for AP portfolios).
- But using the limit portfolio formulae permits avoiding model and estimation risk (namely consequences of incorrectly specifying and poorly estimating $\Omega_t$ and $\lambda_t$).
- This suggests (radical) shrinkage-type estimator for $\Sigma_t^{-1}$.
- (Performance will be illustrated with Monte-Carlo exercises in a follow up estimation paper)

# Short-selling

- When $\mathbf{H}_t$ diagonal

$$w_{it}^{mv} \longrightarrow_p h_{ii,t} \left[ v_{it} - \frac{\mu_{vt}^h \left( \boldsymbol{\mu}_\beta' \Sigma_\beta^{-1} \boldsymbol{\beta}_i \right)}{1 + \boldsymbol{\mu}_\beta' \Sigma_\beta^{-1} \boldsymbol{\mu}_\beta} \right],$$

- and if the factor loadings are also mutually orthogonal,

$$w_{it}^{mv} \longrightarrow_p \frac{h_{ii,t}}{1 + \boldsymbol{\mu}_\beta' \Sigma_\beta^{-1} \boldsymbol{\mu}_\beta} \left[ \begin{array}{c} v_{it} + \left( \frac{\mu_{\beta 1}}{\sigma_{\beta 1}} \right)^2 \left( v_{it} - \tilde{v}_{it}^{(1)} \right) \\ + \ldots + \left( \frac{\mu_{\beta k}}{\sigma_{\beta k}} \right)^2 \left( v_{it} - \tilde{v}_{it}^{(k)} \right) \end{array} \right].$$

- where $\tilde{v}_{it}^{(j)} = \mu_{vt}^h \frac{\beta_{ij}}{\mu_{\beta j}}$, $\mu_{vt}^h$ is a weighted average of the pricing errors, $v_{jt}$.

- The term $\tilde{v}_{it}^{(j)}$ measures the extent to which the pricing error of the $i$th asset deviates from the mean pricing errors across all assets, corrected by $\beta_{ij}/\mu_{\beta j}$.
- Negative portfolio weight can arise whenever $\mid \beta_{ik} \mid > \mid \mu_{\beta k} \mid$.
- The effect gets magnified for large factor loading ratios, $\mu_{\beta k}/\sigma_{\beta^{(k)}}$.
- Large dispersion $(\sigma_{\beta^{(k)}})$ implies *ceteris paribus* smaller chances of finding negative weights.

# Final remarks

Conclusion: new results for MV and AP trading strategies:

(i) establish well diversification and market neutrality for MV and AP portfolios under various form of no-arbitrage (understand how to build market neutral strategies).
(ii) Under certain conditions, (first-order) limit approximation of AP and MV portfolio weights are functionally independent from conditional distribution of factors.
(iii) Primitive conditions on asset returns distribution are derived.

Implications for

- Diversification.
- Relevance of factors conditional distribution.
- Short-selling.
- Practical implementation of MV/AP portfolios (no need to estimate conditional mean and covariance of factors, although we do need estimation of factors!)

# Agenda of future research:

This paper presents characterization of MV/AP trading strategies without model and estimation risk. We are now looking at extent to which these properties are preserved when we allow for both (double asymptotics in $N$, $T$).