Jumps in Asset Prices: A General Equilibrium Explanation

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1 Introduction

A fundamental question is how shocks transmit into the economy and the resulting allocation and prices. Every day the economy is submitted to shocks influencing the fundamental characteristics of the state of the economy. You slip in the bathroom during your morning shower hurting your back and thus your working capabilities are changed, the frost prevents the train from taking its usual route and thereby preventing a load full of passengers to arrive at their planned destination, etc. The question is then whether and how all these shocks affect the overall economy and the allocation of resources in the economy. Markets where changes in the economy are transmitted through are the financial kinds. Obviously, the financial markets are affected by the real economy in that they determine the inflow of funds and they affect the dividend stream of many securities. On the other hand, the financial markets affect the real economy by allowing consumers to reallocate income through time and uncertainty. A fundamental principle in finance is that prices are determined by their fundamental value, i.e., by their future dividend stream. Thus, if the dividends changes by a small amount, so will asset prices. Many asset pricing models assume that dividends follow continuous paths and hence prices should move continuous. However, these models have difficulties in generating the observed asset prices, in that among many things they fail to account for the "fat tails" in the distribution, i.e., the overrepresentation of large changes in prices, as expressed by Calvet & Fisher (2008)

> "In continuous-time setting, jumps in financial prices seem necessary to account for thick tails in asset returns, and the corresponding implied volatility smiles in near-maturity options."

We show how these jumps can be explained by incompleteness of the asset markets, i.e., by the inability of consumers to save and insure perfectly against shocks. The question is thus whether small shocks lead to small changes in the prices. Also, Shiller (1981) argues that asset prices fluctuate too much to be explained by the dividends and hence that the fundamental pricing theory is incorrect. However, we show how asset prices can exhibit excess volatility due to market incompleteness, while still maintaining the market efficiency hypothesis.

Why is it important that prices can jump even if the shocks to the economy are small? This is important since the presence of multiple equilibria is associated with a coordination problem among market participants. Moreover, our result shows that these jumps are connected to *real* jumps and hence have consequences for the welfare of the households. One can relate this to crashes on the stock exchanges experienced in Argentina and Russia, both examples where financial crashes and a real melt down of the economy happened simultaneously. Furthermore, our example shows that sudden drops in asset prices are *not* necessarily evidence that a bubble has burst, or that a bubble has been present. This makes it even more difficult for the monetary authorities to develop tools to remedy bubbles since prices can change rapidly in a short moment of time, and that they have to be compared with indicators of real activity such that e.g. GDP or others in order to find out whether there is a bubble. However, such indicators are only available after the authorities have to make a decision on whether they should intervene or not.

Our result furthermore shows the importance of working with a general equilibrium model, taking the interaction between the real economy and financial market into consideration.

An important question is how incompleteness of the financial markets affects the pricing of securities and the allocation of real commodities in economies. In the present paper we show that incompleteness of financial markets can induce discontinuities in commodity prices as the fundamental characteristics changes. We show that Walrasian equilibria, and thus perfect insurance opportunities, would eliminate such changes. Thus, our results provide a test with which incompleteness of markets can be verified. In order to give content to the phrase *"if the fundamental characteristics do not change significantly"*, we consider a non-atomic state space, more specifically, a continuum of states. We assume then that fundamentals, endowments, dividends and densities, are continuous in the states. Furthermore, we show that with differentiable utilities and real assets there is a continuum of equilibria, and hence the equilibrium exhibit real indeterminacy. Mas-Colell (1991) shows how indeterminacy is a generic phenomenon with real assets and incomplete markets when there is a finite number of securities and infinite states. More specifically, the result is as follows, for any real asset there exists an open set of endowments and utility functions where every element contains an equilibrium set with cardinality equal to the continuum. Our example confirms his result. Our proofs follow along the lines of Mas-Colell very tightly, in that the continuity property is proved using Pareto efficiency, while the example also closely relates to his example. However, we focus on the continuity results of the prices. Our result hinges on the fact that the spotmarket equilibrium set can have multiple elements, and thus, that the price expectations must be coordinated given the realization of a future state. Thus, the discontinuity is the result of changing expectations, changes that are not continuous. In the complete market case, these changes in expectations are forced to be eliminated since contracts can be signed which would leave the parties of an exchange better of and thus prices would change to equilibrate demand and supply.

Balasko (1988) tells a story to explain the content of the results of analyzing the equilibrium manifold. Here it is told that passing through irregular economies could, and must eventually, imply a "large" change in commodity prices. We formalize this story, and we show that in order for this story to be true, we need incomplete financial markets, so that traders cannot insure against these changes, since complete markets would prevent such jumps from occurring. Obviously, risk averse traders would like to insure against such risk.

Non-real Business Cycles can be argued to be large changes in prices and consumption, absent of large changes in fundamentals. We thus show that incomplete markets can be the cause of such phenomenon, which are not sunspot equilibria.

A related problem is the upper hemi continuity of the equilibrium correspondence, i.e., the equilibrium prices and allocations parameterized by the endowment. This property implies that any convergent sequence of endowments must have a converging sequence of equilibrium states. It is, however, easy to see that the equilibrium correspondence is upper hemi continuous, and, thus, our result shows that these questions are separated.

Let us briefly take an overview on how volatility in asset prices has been studied in a theoretical framework (We apologise for any unjustified omissions)

Calvet (2001) shows that with CARA utility functions interest rates can fluctuate over time but not over states since there are idiosyncratic shocks and aggregate certainty. Also, he shows that there is a unique equilibrium. Calvet & Fishers (2008) incorporate jumps in the drift and volatility components of the dividend process and they take a statistical approach to the asset pricing. They use a so-called multifractal model which diverges from the normal Gaussian model by allowing jumps in the drift and volatility. However, they do not explain these jumps, while our jumps are perfectly endogenous. Geanakoplos (1997) shows by means of an example that incompleteness of the markets can induce increased volatility on the prices on durable commodities. He considers a model of asset market which endogenize the default rate and the level of collateral. The use of durable commodities as collateral tends to increase the demand for these commodities and hence increase the price. Again, a general equilibrium model is required for this result, since commodities through collateralization is tied up together. Thus, the price of one commodity affects the other commodities as well. The paper Citanna and Schmedders (2002) studies volatility of security prices and financial innovation. Their results point in two directions, depending on the nature of risk, more specifically whether there is aggregate risk or not. When there is no aggregate risk, completing the asset market will generically reduce the asset price volatility. While in the case of aggregate risk, reducing the degree of incompleteness per se is not necessarily associated with a volatility reduction. We take the financial structure as exogenous. The paper Horsley and Wrobel (2005) shows that the equilibrium price has a continuous density. However, we provide a different prove of this result, as we exploit the relationship between Pareto efficiency and Walrasian equilibria given by the first fundamental theorem of welfare economics. Our proof moreover shows that the continuity is intimately related to the efficiency property, since any such allocation must be continuous.

Finally, let us just state some remarks on our indeterminacy result. In Debreu (1970) it is shown that in a GE generically with have a finite set of equilibria, while in Battinelli et al. (2002) it is shown that with *real* assets and a *finite set of states of nature* there is generically a finite set of equilibria. In the other extreme, with *nominal* securities Balasko & Cass (1989) shows that the indeterminacy is large and calculates the dimension of indeterminacy to be of the difference between the number of states and securities. Our example shows that the result of determinacy with real assets and finite states does not extend to the case of a continuum of states. We have not shown that our example is robust, however we claim without any proof that it is actually robust.

The paper is structured as follows: In section 2 we introduce notation and the equilibrium concepts, and further state our assumptions on the fundamental characteristics of the economy. Then in section 3 we state and prove our main results: that with complete markets the prices are continuous in states and an example illustrates that asset prices can have jumps when the asset market is incomplete with finitely many securities.

2 The model

Set-up

There is a finite number T of dates with $t \in \{0, \ldots, T\}$ and there is uncertainty. The set of states at date t is $S = [0, 1]$ with $s \in S$ and $\pi : S^{T+1} \to \mathbb{R}_+$ is the density on the set of states S^T. Let μ be the probability measure defined by π so $\mu(A) = \int_A \pi(s^T)\ ds^T$ for every Borel set $A \subset S^T$. The state of the first date $t = 0$ is $s_0 \in S$. There is a finite number of goods ℓ at every state with $j \in \{1, \ldots, \ell\}$. There is a finite number m of consumers with $i \in \{1, \ldots, m\}$. A collection of maps $p = (p_t)$, where $p_t : S^{t+1} \to \mathbb{R}^\ell_+$, is a price system for goods.

Consumers are described by their identical consumption sets $X = \mathbb{R}^\ell$, endowments $\omega_i = (\omega_i^t)_t$, where endowments at date t is described by a map $\omega_i^t : S^{t+1} \to X$, and state utility function $u_i : X \to \mathbb{R}$. A collection of maps $x_i = (x_i^t)_t$, where $x_i^t : S^{t+1} \to \mathbb{R}^\ell$, is a consumption bundle. An allocation of goods $x = (x_i)_i$ is a list of individual consumption bundles.

Walras equilibrium

Let $s^t = (s_0, \ldots, s_t)$ denote the history of states to date t, then the problem of consumer i is:

$$\max_{x_i} \quad \int_{S^T} \pi(s^T) u(x_i^0(s_0), \ldots, x_i^T(s^T))\ ds^T$$

$$\text{s.t.} \quad \int_{S^T} \sum_t p_t(s^t) \cdot x_i^t(s^t)\ ds^t \ \leq \ \int_{S^T} \sum_t p_t(s^t) \cdot \omega_i^t(s^t)\ ds^t$$

Please note that the problem of consumer i may not have a solution because none of the integrals may be defined. For now it is hoped that informally the problem makes sense.

Definition 1 *A Walras equilibrium (p, x) is a price system for goods and an allocation of goods such that:*

- *x_i is a solution to the problem of consumer i for all i, and;*
- *markets clear $\sum_i x_i^t(s^t) = \sum_i \omega_i^t(s^t)$ for all t and s^t.*

Radner equilibrium

There is a finite number n of assets with $k \in \{1, \ldots, n\}$ where the dividend of asset k at date t is described by a map $a_k^t : S^{t+1} \to \mathbb{R}^\ell$. A collection of maps $q = (q_t)$, where $q_t : S^{t+1} \to \mathbb{R}^n$, is a price system for assets. A collection of maps $z_i = (z_i^t)_t$, where $z_i^t : S^{t+1} \to \mathbb{R}^k$, is portfolio plan. An allocation of assets $z = (z_i)$ is a list of portfolio plans. Portfolios are restricted to be in Z where $Z \subset \mathbb{R}^n$ is bounded from below, convex and closed and the no-trade portfolio is in the interior of Z.

A price system (p, q) is a price system for goods and a price system for assets. An allocation (x, z) is an allocation of goods and an allocation of assets.

Let $a_t(s^t)$ be the $\ell \times n$-matrix of dividends $(a_t^1(s^t) \ldots a_t^n(s^T))$ at date t in state s^t and let s^t be the history $(s_0, \ldots, s_t)$ of states up to date t, then the problem of consumer i is:

$$\max_{(x_i, z_i)} \int_{S^T} \pi(s^T) u(x_i^0(s_0), \ldots, x_i^T(s^T)) \, ds^T$$

$$\text{s.t.} \quad \begin{cases} p_1(s_0) \cdot x_i^0(s_0) + z_i^0(s_0) \cdot q_0(s_0) \ \leq \ p_1(s_0) \cdot \omega_i^0(s_0) \\ p_t(s^t) \cdot x_i^t(s^t) + z_i^t(s^t) \cdot q_t(s^t) \\ \qquad \leq \ p_t(s^t) \cdot \omega_i^t(s^t) + z_i^{t-1}(s^{t-1}) \cdot (q_t(s^t) + p_t(s^t) a^t(s^t)) \\ \qquad \text{for } t \in \{1, \ldots, T-1\} \\ p_T(s^T) \cdot x_i^T(s^T) \ \leq \ p_T(s^T) \cdot \omega_i^T(s^T) + z_i^{t-1}(s^T) \cdot (p_T(s^T) a^T(s^T)) \\ z_i^t(s^t) \ \in \ Z \text{ for } t \in \{0, \ldots, T-1\} \end{cases}$$

Definition 2 *A Radner equilibrium $((p, q), (x, z))$ is a price system and an allocation such that:*

- *(x_i, z_i) is a solution to the problem of consumer i for all i, and;*
- *market clears $\sum_i x_i^t(s^t) = \sum_i \omega_i^t(s^t)$ for all t and s^t.*

Assumptions

The consumers are supposed to satisfy the following assumptions:

(A.1) $\omega_i^t \in \mathrm{C}^1([0,1]^{t+1}, X)$.

(A.2) $u_i \in \mathrm{C}^2(X, \mathbb{R})$ with $Du_i(x_i) \in \mathbb{R}_{++}^\ell$ for all x_i and $v^T D^2 u_i(x_i) v < 0$ for all x_i and v such that $Du_i(x_i) \cdot v = 0$ and $v \neq 0$.

(A.3) Suppose $\lim_{n\to\infty} \min\{x_{in}^1, \ldots, x_{in}^\ell\} = -\infty$, then $\lim_{n\to\infty} u_i(x_{in}) = -\infty$.

The economy is supposed to satisfy the following assumptions:

(A.4) $\pi \in \mathrm{C}^1([0,1]^{T+1}, \mathbb{R}_{++})$.

(A.5) $a_k^t \in \mathrm{C}^1([0,1]^{t+1}, \mathbb{R}_+^\ell)$ for all k and t.

Note that Assumption (A.3) implies that the set $\{x \in X \,|\, u_i(x) \geq \bar{u}\}$ is bounded from below for all $\bar{u}$.

3 Properties of equilibria

Theorem 1 *Suppose that (p, x) is a Walras equilibrium. Then $p_t(s^t) \in C^1(S^{t+1}, \mathbb{R}_{++}^\ell)$ for all t and s^t a.e.*

Proof: We prove this by showing firstly that

Lemma 1 *Suppose that (p, x) is a Walras equilibrium. Then we have that $x_i^t(s^t) \in C^1(S^{t+1}, \mathbb{R}_{++}^\ell)$ for all t and i, and s^t a.e.*

Proof: We show this by using the equivalence between the Walrasian equilibria and Pareto efficient allocations, and that we can characterize the pareto set using solutions to the problem

$$\max_{x=(x_i)_i} \int_{S^T} \pi(s^T) u(x_1^0(s_0), \ldots, x_i^T(s^T))\, ds^T$$

$$\text{s.t.} \begin{cases} \int_{S^T} \pi(s^t) u(x_i^t)\, ds^T \;\geq\; \int_{S^T} \pi(s^t) u(y_i^t)\, ds^T \\ \qquad \text{for } i \in \{2, \ldots, m\} \\ \sum_i x_i^t(s^t) - \omega_i^t(s_t) \;\leq\; 0 \\ \text{for } t \in \{0, \ldots, T-1\}, s^t \in S^t \end{cases}$$

for some feasible allocation y. We refer to this problem as (*). We then show that for every allocation x that solves a problem like this, must necessarily have that $x\colon S^{T+1} \to \mathbb{R}^{\ell m}$ is a continuous map. We show it for the case of $T = 2$, but it easily extended to the more general case for an arbitrary $T < \infty$. We denote by $\mathcal{A}$ the set of feasible allocations, and $\mathcal{A}(s)$ as the set of state s feasbile allocations. Furthermore, denote by $U^i(x)$ the expected utility of consumer i of a consumption plan x.

Lemma 2 *Let x be a solution to (*), then $x_1^1(s) = \arg\max_{\xi \in \mathcal{A}(s)} u^1(x_0^1, \xi^1)$ such that $u^j(x_0^j, \xi^j) \geq u^j(x_0^j, x_1^j(s))$ for $j = 2, \ldots, m$, for all $s \in S$*

Proof: Assume that there exists some s where the statement is not true, i.e., there exists s and $\xi \in \mathcal{A}(s)$ such that $u^1(x_0^1, \xi^1) > u^j(x_0^1, x^1(s))$ and $u^j(x_0^j, \xi^j) \geq u^j(x_0^j, x_1^j(s))$ for $j = 2, \ldots, n$. Since u^1 is continuous, there exists an open neighborhood $U^1 \subset \mathbb{R}^\ell$ of $x_1^1(s)$ such that $u^1(x_0, \xi^1) > u^1(x_0, x_1^1(s))$ for all $\xi^1 \in U$. By an appropriate redistribution we can wlog. assume that $u^j(x_0^j, \xi^j) > u^j(x_0^j, x_1^j(s))$ for all $j = 1, \ldots, n$. Hence, there exists $U = \prod_i U^i$ open neighborhood of ξ such that $u^j(x_0^j, \bar{\xi}^j) > u^j(x_0^j, x_1^j(s))$ for all $\bar{\xi} \in U$. Since r_1 is continuous there exists an open neighborhood $S' \subset S$ of s, such that $S' \subset r_1^{-1}(V)$ for every V open with $r_1(s) \in V$. Hence there exists a function $\xi\colon S' \to \mathbb{R}^{m\ell}$ such that $u^j(x_0^j, \xi^j(s')) > u^j(x_0^j, x_1^j(s'))$ for all $s' \in S'$, and the allocation

$$y(s') = \begin{cases} \xi(s') & s' \in S' \\ x(s') & s' \in S \setminus S' \end{cases}$$

will satisfy that $U^j(y^j) > U^j(x^j)$ for all $j = 1, \ldots, n$. A contradiction.

□

Lemma 3 *If $x, y \in \mathcal{A}$, $U^j(x^j) \geq U^j(y^j)$ for all j and at least one with strict inequality, then there exists $z \in \mathcal{A}$ such that $U^j(z^j) > U^j(y^j)$ for all j.*

Proof:

Assume that $U^1(x^1) > U^1(y^1)$, let $z^1 = tx^1$ where $t < 1$ satisfies that $U^1(tx^1) > U^1(y^1)$, then consider $z^j = x^j + \frac{t}{n-1}x^1 > x^j$. The result follows by continuity and strict monotinicity of preferences.

□

Assume that x_1 is discontinuous, i.e. that there exists a s_0, a neighborhood V of $x_1(s_0)$, such that for every U open with $s_0 \in U$ there exists some $s \in U$ such that $x_1(s) \notin V$.

Let $V(s_0) \subset \mathbb{R}^{m\ell}$ be a neighborhood of $x_1(s_0)$ such that for all $U \ni s_0$

$$x_1^{-1}(V(s_0)) \cap (S \setminus U) \neq \emptyset.$$

Consider then (U_n) a monoton decreasing sequence of neighborhoods, such that $\lim_{m\to\infty} \bigcap_{n\geq 1}^m U_n = \{x\}$, so that we can find a sequence $(s_n)_n$, $s_n \to s_0$ such that $s_n \in x_1^{-1}(V(s_0)) \cap (S \setminus U_n)$ or: $x_1(s_n) \notin V(s_0)$ for all n. Moreover, it holds that $u^j(x_0^j, x_1^j(s_n)) \geq u^j(y_0^j, y_1^j(s_n))$ for all n, and by continuity of u^j it holds that $u^j(x_0^j, \lim_n x_1^j(s_n)) \geq u^j(y_0^j, \lim_n y_1^j(s_n))$. Thus, there exists some N such that for every $n \geq N$ will we have that $u^1(x_0, x_1^1(s_0)) > u^1(x_0, x_1^1(s_n))$. By Lemma 3 there exists an allocation $\xi(s_n) \in \mathcal{A}_1(s_n)$ such that $u^j(x_0^j, \xi_1^j(s_n)) > u^j(y_0^j, y_1^j(s_n))$ with n sufficiently large, and thus x cannot be the solution to (*) according to Lemma 2.

□

Then we use that $\pi(s)\partial u_i(x^t) = \lambda(s^t)p_t(s^t)$ for some continuous $\lambda\colon S^{T+1} \to \mathbb{R}_+$, and then we conclude that p_t must be continuous in S^{t+1}.

$\square$

As can be seen in the proof, if we define prices $p^t = \pi\tilde{p}^t$ and refer to $\tilde{p}$ as the internal price system then one can show that $\tilde{p}$ is continuous without the assumption of continuity of π.

Corollary 1 *Suppose that (p, x) is a Walras equilibrium and that $a = (a_k)_k$, where $a_k = (a_k^t)_t$ and $a_k^t : S^{t+1} \to \mathbb{R}^\ell$, is an asset structure such that (p, q, x, z) is a Radner equilibrium. Then $q_k^t(s^t) \in C^1(S^{t+1}, \mathbb{R})$ for all k, t, and s^t a.e.*

Proof:

This follows easily using the fact that

$$\begin{aligned} q_k^t(s^t) &= \exp[\lambda^{t+1} p_{t+1} \cdot a_k^{t+1} \mid s^t] \\ &= \frac{1}{\mu(S^{T+1}(s^t))} \int_{S^{T+1}(s^t)} \pi(s^T)\lambda^{t+1}(s^{t+1})p_{t+1}(s^{t+1}) \cdot a_k^{t+1}(s^{t+1})\ ds^T \end{aligned}$$

for every asset k according to the no-arbitrage property and that all functions λ^{t+1}, p_{t+1} and a_k^t are continuous in S, and where

$$S^{T+1}(s_0^t) = \{s^T \in S^T \mid s^\tau = s_0^\tau \forall \tau \le t\}.$$

Indeed, if $s_n^t \to s^t$, then obviously $\mu(S^{T+1}(s_n^t)) \to \mu(S^{T+1}(s^t))$ and

$$\int_{S^{T+1}(s_n^t)} \pi(s^T)\lambda^{t+1}(s^{t+1})p_{t+1}(s^{t+1}) \cdot a_k^{t+1}(s^{t+1})\ ds^T \to \int_{S^{T+1}(s^t)} \pi(s^T)\lambda^{t+1}(s^{t+1})p_{t+1}(s^{t+1}) \cdot a_k^{t+1}(s^{t+1})\ ds^T,$$

since we have that

$$\int_{S^{T+1}(s^t)} \pi(s^T)\lambda^{t+1}(s^{t+1})p_{t+1}(s^{t+1}) \cdot a_k^{t+1}(s^{t+1})\ ds^T = \int_{S^{T+1}} \chi_{S^{T+1}(s^t)}\pi(s^T)\lambda^{t+1}(s^{t+1})p_{t+1}(s^{t+1}) \cdot a_k^{t+1}(s^{t+1})\ ds^T,$$

and obviously $\chi_{S^{T+1}(s_n^t)} \longrightarrow \chi_{S^{T+1}(s^t)}$

□

Theorem 2 *There exists an economy, a Radner equilibrium for the economy (p, q, x, z), and $A \subset S$ with $\mu(A) > 0$ such that $q_k^t(s^t) \notin C(S^{t+1}, \mathbb{R})$ for some k and t and every $s_t \in A$.*

Proof: Consider an economy with three dates $T = 2$, one good per state $\ell = 1$, two consumers $m = 2$ and one asset $n = 1$. For the density $\pi : S \to \mathbb{R}_{++}$ suppose that $\pi(s_1) = 1$ for all $s_1 \in S = [0, 1]$, S being the state space and the information σ-algebras are $\mathcal{F}_0 = \mathcal{B}$, $\mathcal{F}_1 = \{s\}_{s \in S} = \mathcal{F}_2$. Thus, there is perfect information at date $t = 1$ and we can identify any state $s \in S^3$ with a state in S. The asset is a riskless bond with zero rate of interest.

Endowments at the first date are supposed to be identical $\omega_2^0 = \omega_1^0 = \omega^0$ and endowments at the last two dates are supposed to be reverse, i.e., $\omega_2^1(s_1) = \omega_1^2(1 - s_1)$ and $\omega_2^2(s_1) = \omega_1^1(1 - s_1)$. Similarly, von-Neuman-Morgenstern utility functions are supposed to be identical for the first date and reverse for the last two dates, i.e., $u_2(x^0, x^1, x^2) = u_1(x^0, x^2, x^1)$ for every $x^0, x^1, x^2 \in \mathbb{R}$. Note that this implies that $\frac{\partial u_1(x^0, x^1, x^2)}{\partial x^t} = \frac{\partial u_2(x^0, x^2, x^1)}{\partial x^t}$ for every $t = 0, 1, 2$ and all $x^0, x^1, x^2 \in \mathbb{R}$.

Given $c_i^0 \in \mathbb{R}$, let $f(\cdot; c_i^0) : \mathbb{R}_{++}^2 \times \mathbb{R}_{++} \to \mathbb{R}_{++}^2$ denote the demand function for the consumer with endowments $e_i(s_1) \equiv (\omega_i^1(s_1), \omega_i^2(s_1))$ and utility function $v_i(\cdot; c_i^0) : \mathbb{R}_{++}^2 \to \mathbb{R}$ defined by $v_i(x_i^1, x_i^2; c_i^0) \equiv u_i(c_i^0, x_i^1, x_i^2)$. Then $(p, s_1) \in \mathbb{R}_{++}^2 \times S$ is an equilibrium for the Edgeworth box economy $\mathcal{E}(s_1; (c_i^0)_i) = (e_i(s_1), v_i(\cdot, ; c_i^0))_i$ if and only if

$$f_1(p, p \cdot e_1(s_1); c_1^0) + f_2(p, p \cdot e_2(s_1); c_2^0) \;=\; e_1(s_1) + e_2(s_1).$$

Obviously, given our symmetry assumptions we have that

Lemma 4 *(p_1, p_2, s_1) is an equilibrium for $\mathcal{E}(s_1; (c_i^0)_i)$ if and only if $(p_2, p_1, 1 - s_1)$ is an equilibrium for $\mathcal{E}(1 - s_1; (d_i^0)_i)$, where $(d_1^0, d_2^0) = (c_2^0, c_1^0)$.*

Suppose that equilibrium prices are normalized such that the sum of the prices is one and let $E \subset \mathbb{R}^2_{++} \times S$ be the equilibrium set for the collection of Edgeworth economies $(\mathcal{E}(s_1; (c_i^0)_i)_{s_1}$, where $c_i^0 = \omega_i^0(s_0)$, so

$$\begin{aligned} E &= \{ (p, s_1) \,|\, (p, s_1; (c_i^0)_i) \text{ is an equilibrium for } \mathcal{E}(s_1; (c_i^0)) \} \\ &= \{ (p, s_1) \,|\, (p, s_1; \omega^0) \text{ is an equilibrium for } \mathcal{E}(s_1; \omega^0) \}. \end{aligned}$$

and denote $E(s_1) = \{p \mid (p, s_1) \in E\}$ the equilibrium "slice" at s_1.

Suppose that E is S-shaped as shown in Figure 1 with $(\frac{1}{2}, \frac{1}{2}) \in E(\frac{1}{2})$, and let $r : S \to \mathbb{R}^2_{++}$ be a function such that $r_1(s_1)$ is the lowest equilibrium price for $s_1 < 1/2$, $r_1(s_1) = (1/2, 1/2)$ for $s_1 = 1/2$ and $r_1(s_1)$ is the highest equilibrium price for $s_1 > 1/2$, i.e.,

$$r(s_1) = \begin{cases} \arg\min_{(p_1,p_2) \in E(s_1)} p_1 & s_1 < \frac{1}{2} \\ (\frac{1}{2}, \frac{1}{2}) & s_1 = \frac{1}{2} \\ \arg\max_{(p_1,p_2) \in E(s_1)} p_1 & s_1 > \frac{1}{2} \end{cases} .$$

Note that r is a measurable, but *dis*continuous, selection from E. Further, we have that $r(s_1) = (r_1(s_1), r_2(s_1)) = (r_2(1 - s_1), r_1(1 - s_1))$ for every s_1.

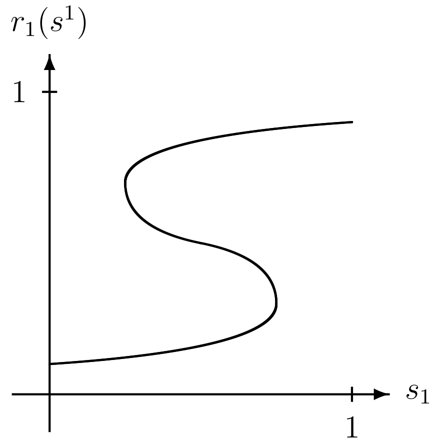

Figure 1: *The equilibrium set E.*

In order to construct a Radner equilibrium $((p, q), (x, z))$:

- let the allocation x be defined by

$$\begin{aligned} x_i^0 &= \omega_i^0 \\ x_i^t(s_1) &= f_i^t(r(s_1), r(s_1) \cdot e_i(s_1); \omega^0) \qquad t = 1, 2 \end{aligned}$$

  for $i = 1, 2$;

- let the portfolio plan z be defined by $z_i^0 = 0$ and

$$\begin{aligned} z_i^1(s_1) &= \frac{r_1(s_1)}{r_2(s_1)}(\omega_i^1(s_1) - f_i^1(r(s_1), e_i(s_1); \omega_i^0)) \\ &= f_i^2(r(s_1), e_i(s_1); \omega_i^0) - \omega_i^2(s_1); \end{aligned}$$

- let the price system p be defined by $p^2(s^2) = p^1(s^1) = p^0(s_0) = 1$, and;

- let the asset prices q be defined by $q^1(s_1) = \frac{r_2(s_1)}{r_1(s_1)} > 0$ for every s_1, and let $q^0 > 0$ be such that

$$\int \left(-q^0 \frac{\partial u_1(x_1(s_1))}{\partial x^0} + (1 + q^1(s_1)) \frac{\partial u_1(x_1(s_1))}{\partial x^1} \right) ds_1 = 0, \qquad (1)$$

  or

$$q^0 = \frac{\displaystyle\int (1 + q^1(s_1)) \frac{\partial u_1(x_1(s_1))}{\partial x^1} \, ds_1}{\displaystyle\int \frac{\partial u_1(x_1(s_1))}{\partial x^0} \, ds_1}.$$

Then (p, q, x, z) is a Radner equilibrium and the asset price at date 1 is discontinuous at $s_1 = 1/2$.

$\square$

Take alternatively any symmetric neighborhood $U \subset S$ of $\frac{1}{2}$, and let $\tilde{r}: S \to \mathbb{R}^2_{++}$ be $\tilde{r}(s_1) = r(s_1)$ whenever $s_1 \notin U$ and

$$\tilde{r}(s_1) = \begin{cases} \arg\min_{(p_1, p_2) \in E(s_1)} p_1 & s_1 \in U \cap \mathbb{Q} \\ \arg\max_{(p_1, p_2) \in E(s_1)} p_1 & s_1 \in U \cap (\mathbb{R} \setminus \mathbb{Q}) \end{cases}.$$

Then $\tilde{r}$ is Lebesgue-measurable but discontinuous on the set U of non-zero measure, and hence the resulting asset prices q are too.

It is somewhat surprising to see that there is a "continuum" of equilibria in this economy, in the sense that for every selection r of E which is "symmetric", in the sense that $r(s_1) = (r_1(s_1), r_2(s_1)) = (r_2(1-s_1), r_1(1-s_1))$ for every s_1, induces an equilibrium, since the allocation $x(r)$ is "symmetric" and hence equation (1) is satisfied. Also the allocation will generically differ. Contrary to the case of finite states and real assets, we here have indeterminacy.

Also, the collection $\mathcal{U}$ of open subsets of $[0,1]$ has atleast a dimensionality of 2, since the map $([0,1]^2 \setminus \triangle) \to P([0,1]) := (a,b) \mapsto]a,b[$ is a one-to-one map and obviously continuous endowing the collection $\mathcal{U}$ with the Hausdorff metric. Also, for every $(a,b),(c,d) \in \mathbb{R}^2 \setminus \triangle$ we have that every $(a,b) \leq (x,y) \leq (c,d)$ implies that $]c,d[\in \mathcal{U}$, i.e., the image of our injection contains a connected set.

These remarks leads us to the following corollary

Corollary 2 *There exists an economy, a open, connected set $R \subset Y$ of a linear space, and for every $r \in R$ a Radner equilibrium $(p|_r, q|_r, x|_r, z|_r)$, and $A \subset S$ with $\mu(A) > 0$ such that $q_k^t|_r(s^t) \notin C(S^{t+1}, \mathbb{R})$ for some k and t and every $s_t \in A$.*

The parameterization is obviously by the set of measurable, symmetric selections on E.

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