Parameter estimation for differential equations using fractal-based methods and applications to economics

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Abstract. Many problems from the area of economics and finance can be described using dynamical models. For them, in which time is the only independent variable and for which we work in a continuous framework, these models take the form of deterministic differential equations (DEs). We may study these models in two fundamental ways: the direct problem and the inverse problem. The *direct problem* is stated as follows: given all of the parameters in a system of DEs, find a solution or determine its properties either analytically or numerically. The *inverse problem* reads: given a system of DEs with unknown parameters and some observational data, determine the values of the parameters such that the system admits the data as an approximate solution. The inverse problem is crucial for the calibration of the model; starting from a series of data we wish to describe them using deterministic differential equations in which the parameters have to be estimated from data samples. The solutions of the inverse problems are the estimations of the unknown parameters and we use fractal-based methods to get them. We then show some applications to technological change and competition models.

Keywords: Differential equations, collage methods, inverse problems, parameter estimation, Lotka Volterra models, technological change, boat-fishery model.

1 Introduction: inverse problems for DEs with initial conditions

We now recall a method of solving inverse problems for differential equations using fixed point theory for contractive operators. Many inverse problems may be viewed in terms of the approximation of a target element x in a complete metric space (X, d) by the fixed point $\bar{x}$ of a contraction mapping $T : X \to X$. In practical applications, from a family of contraction mappings T_λ, $\lambda \in \Lambda \subset \mathbb{R}^n$, one wishes to find the parameter $\bar{\lambda}$ for which the approximation error $d(x, \bar{x}_\lambda)$ is as small as possible. Thanks to a simple consequence of Banach's fixed point

theorem known as the "Collage Theorem", most practical methods of solving the inverse problem for fixed point equations seek to find an operator T for which the *collage distance* $d(x, Tx)$ is as small as possible.

Theorem 1. *("Collage Theorem" [2]) Let (X, d) be a complete metric space and $T : X \to X$ a contraction mapping with contraction factor $c \in [0, 1)$. Then for any $x \in X$,*

$$d(x, \bar{x}) \leq \frac{1}{1-c} d(x, Tx), \tag{1}$$

where $\bar{x}$ is the fixed point of T.

One now seeks a contraction mapping T that minimizes the so-called *collage error* $d(x, Tx)$ – in other words, a mapping that sends the target x as close as possible to itself. This is the essence of the method of *collage coding* which has been the basis of most, if not all, fractal image coding and compression methods. Many problems in the parameter estimation literature for differential equations can be formulated in such a collage coding framework as showed in [9] and subsequent works [3,7,8,6]. We recall the basic steps of this approach. Starting from the differential equation:

$$\dot{x} = f(t, x), \quad x(0) = x_0, \tag{2}$$

let us consider the Picard integral operator associated with it,

$$(Tx)(t) = x_0 + \int_0^t f(s, x(s))\, ds. \tag{3}$$

We assume that f is Lipschitz in the variable x, that is $|f(s, x_1) - f(s, x_2)| \leq K|x_1 - x_2|$. Under these hypotheses T is Lipschitz on the space $C([-\delta, \delta] \times [-M, M])$ for some δ and $M > 0$. With respect to the L^2 distance, the following result holds.

Theorem 2. *[9] The operator T satisfies*

$$\|Tu - Tv\|_2 \leq c\|u - v\|_2 \tag{4}$$

for all $u, v \in C([-\delta, \delta] \times [-M, M])$ where $c = \delta K$.

Now let $\delta' > 0$ be such that $\delta' K < 1$. Let $\{\phi_i\}$ be a basis of functions in $L^2([-\delta', \delta'] \times [-M, M])$, then

$$f_{\mathbf{a}}(s, x) = \sum_{i=1}^{+\infty} a_i \phi_i(s, x). \tag{5}$$

Each sequence of coefficients $\mathbf{a} = \{a_i\}_{i=1}^{+\infty}$, then defines a Picard operator $T_{\mathbf{a}}$. Suppose further that each function $\phi_i(s, x)$ is Lipschitz in x with constants K_i.

Theorem 3. *[9] Let* $\|\mathbf{K}\|_2 = \left(\sum_{i=1}^{+\infty} K_i^2\right)^{\frac{1}{2}}$ *and* $\|\mathbf{a}\|_2 = \left(\sum_{i=1}^{+\infty} a_i^2\right)^{\frac{1}{2}}$. *Then*

$$|f_{\mathbf{a}}(s,x_1) - f_{\mathbf{a}}(s,x_2)| \leq \|\mathbf{a}\|_2 \|\mathbf{K}\|_2 |x_1 - x_2| \tag{6}$$

for all $s \in [-\delta', \delta']$ *and* $x_1, x_2 \in [-M, M]$.

Given a target solution $x(t)$, we now seek to minimize the collage distance $\| x - T_{\mathbf{a}} x \|_2$. The square of the collage distance becomes

$$\begin{aligned} \Delta(\mathbf{a})^2 &= \|x - T_{\mathbf{a}} x\|_2^2 \\ &= \int_{-\delta}^{\delta} \left| x(t) - \int_0^t \sum_{i=1}^{+\infty} a_i \phi_i(s, x(s)) ds \right|^2 dt \end{aligned} \tag{7}$$

and the inverse problem can be formulated as

$$\min_{\mathbf{a} \in \Lambda} \Delta(\mathbf{a}), \tag{8}$$

where $\Lambda = \{\mathbf{a} \in \mathbb{R}^{+\infty} : \|\mathbf{K}\|_2 \|\mathbf{a}\|_2 < 1\}$. The minimization may be performed by means of classical minimization methods on a subspace of finite dimension. Of course, the approximation error goes to zero when the dimension goes to infinity.

2 The Technological Competition Model

The technological competition model is given by

$$\begin{aligned} \frac{dx_1}{dt}(t) &= f_1(x_1, x_2) = \frac{a_1}{K_1} x_1 \left(K_1 - x_1 - \alpha_2 x_2\right) \\ \frac{dx_2}{dt}(t) &= f_2(x_1, x_2) = \frac{a_2}{K_2} x_2 \left(K_2 - x_2 - \alpha_1 x_1\right), \end{aligned}$$

where all of the parameters are positive and a_1, a_2, α_1 and α_2 are less than one. Observe that the nonnegative quadrant is invariant. This means that if we start with $(x_1(0), x_2(0)) \geq (0,0)$ then we have $(x_1(t), x_2(t)) \geq (0,0)$ for all time. That is, in the applied meaningful cases, x_1 and x_2, as determined by our model, are always nonnegative. The linearization of the vector field (f_1, f_2) is

$$Df(x_1,x_2) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{pmatrix} = \begin{pmatrix} a_1 - \frac{a_1}{K_1}\left(2x_1 + \alpha_2 x_2\right) & -\frac{a_1 \alpha_2 x_1}{K_1} \\ -\frac{a_2 \alpha_1 x_2}{K_2} & a_2 - \frac{a_2}{K_2}\left(2x_2 + \alpha_1 x_1\right). \end{pmatrix}$$

Solving for equilibria, we obtain

$$(0,0), (0,K_2), (K_1,0), \text{ and } (x_1^\star, x_2^\star) = \left(\frac{K_1 - \alpha_2 K_2}{1 - \alpha_1 \alpha_2}, \frac{K_2 - \alpha_1 K_1}{1 - \alpha_1 \alpha_2}\right).$$

And so, evaluating the linearization at each of the equilibrium points, we calculate that

$$Df(0,0) = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}.$$

Since $Df(0,0)$ is positive definite, we know that $(0,0)$ is an unstable equilibrium (a source). On the other hand, at the next two points we obtain

$$Df(0,K_2) = \begin{pmatrix} \frac{a_1}{K_1}(K_1 - \alpha_2 K_2) & 0 \\ -a_2\alpha_1 & -a_2 \end{pmatrix}$$

and

$$Df(K_1,0) = \begin{pmatrix} -a_1 & -a_1\alpha_2 \\ 0 & \frac{a_2}{K_2}(K_2 - \alpha_1 K_1) \end{pmatrix}.$$

Each matrix has one negative eigenvalue, with the sign of other one determined by a relationship between K_1, K_2, and one of the α_is. All three of these equilibrium points have at least one component equal to zero, corresponding to one of the competing technologies begin eliminated from the market. The origin is a special equilibrium point, in that we only arrive at it if we start at it: if neither of the two technologies is present at the start, both of them will never appear. The equilibrium $(0,K_2)$ corresponds to technology x_2 triumphing over technology x_1. The single negative eigenvalue corresponds to the case that the market only has technology x_2 at the start. x_1 never appears, so we arrive at an equilibrium state where technology x_2 is the only one in the market. In the case that $K_1 - \alpha_2 K_2 < 0$, it is possible for a market with both technologies present to approach a state where x_2 has eliminated x_1. If $K_1 - \alpha_2 K_2 > 0$, then we can never reach $(0,K_2)$ if we start with both technologies present. Similar remarks can be made about $(K_1,0)$. However, notice that if we try to make both of these boundary equilibria stable, we require both $K_1 - \alpha_2 K_2 < 0$ and $K_2 - \alpha_1 K_1 < 0$, which means that

$$K_1 < \alpha_2 K_2 < \alpha_2(\alpha_1 K_1) \Rightarrow 1 < \alpha_1\alpha_2.$$

But this is a contradiction if both α_is are less than one. As a result, we can make at most one of the nontrivial boundary equilibria stable. The final and most interesting equilibrium point can correspond to coexistence of the two technologies in the case that both $x_1^\star$ and $x_2^\star$ are positive. This situation occurs when

$$K_1 - \alpha_2 K_2 > 0 \text{ and } K_2 - \alpha_1 K_1 > 0. \tag{9}$$

These conditions are familiar, corresponding to the case when both boundary equilibria cannot be reached by interior solutions. In this case, we calculate that

$$Df(x_1^\star, x_2^\star) = \begin{pmatrix} -\frac{a_1}{K_1}\frac{K_1-\alpha_2 K_2}{1-\alpha_1\alpha_2} & -\frac{a_1\alpha_2}{K_1}\frac{K_1-\alpha_2 K_2}{1-\alpha_1\alpha_2} \\ -\frac{a_2\alpha_2}{K_2}\frac{K_2-\alpha_1 K_1}{1-\alpha_1\alpha_2} & -\frac{a_2}{K_2}\frac{K_2-\alpha_1 K_1}{1-\alpha_1\alpha_2} \end{pmatrix},$$

with determinant

$$a_1 a_2 \frac{(K_1 - \alpha_2 K_2)(K_2 - \alpha_1 K_1)}{K_2 K_1 (1 - \alpha_1\alpha_2)} > 0.$$

Since the determinant is positive and $(Df(x_1^\star, x_2^\star))_{11} < 0$, we conclude that if we are in the case where our system exhibits a positive equilibrium then it is

asymptotically stable—in fact with basis of attraction the positive quadrant! Notice that if either inequality in (9) is replaced by the corresponding equation then our equilibrium point coalesces with one of the boundary equilibria. If either inequality is in fact negative, then the equilibrium point we are discussing is not physically realizable.

Figure 1 presents a solution trajectory in the case that $K_1 = 320$, $K_2 = 100$, $\alpha_1 = 0.2$, $\alpha_2 = 0.5$, $a_1 = 0.3$, and $a_2 = 0.6$. Note that the inequalities in (9) are satisfied.

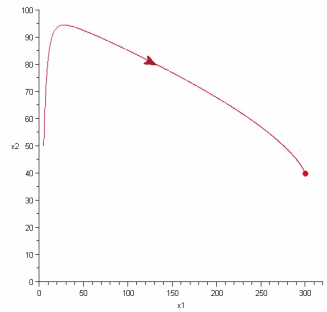

Fig. 1. All solution trajectories starting in the positive quadrant approach the positive equilibrium because (9) holds.

Figure 2 presents a solution trajectory in the case that $K_1 = 125$, $K_2 = 320$, $\alpha_1 = 0.2$, $\alpha_2 = 0.5$, $a_1 = 0.3$, and $a_2 = 0.6$. In this case, we have no positive equilibrium, but the equilibrium point $(0, 320)$ is asymptotically stable.

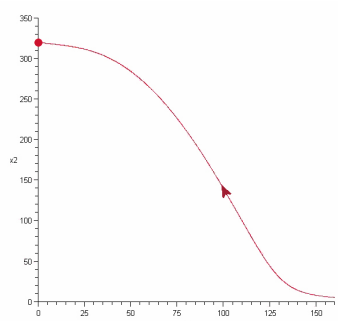

Fig. 2. All solution trajectories starting in the positive quadrant approach $(0, K_2)$ in this case.

2.1 Inverse Problem

The model equations are

$$\frac{dx_1}{dt}(t) = f_1(x_1, x_2) = a_1 x_1 - \frac{a_1}{K_1} x_1^2 - \alpha_2 \frac{a_1}{K_1} x_1 x_2$$
$$\frac{dx_2}{dt}(t) = f_2(x_1, x_2) = a_2 x_2 - \frac{a_2}{K_2} x_2^2 - \alpha_1 \frac{a_2}{K_2} x_1 x_2$$

The inverse problem of interests is: given observed values $x_1(t_i)$ and $x_2(t_i)$ for $1 \leq i \leq N$, say, approximate the values of the parameters a_1, a_2, α_1, α_2, K_1, and K_2. We use collage coding, finding the system of the form

$$\frac{dx_1}{dt}(t) = b_1 x_1 + c_1 x_1^2 + d_1 x_1 x_2 \tag{10}$$
$$\frac{dx_2}{dt}(t) = b_2 x_1 + c_2 x_2^2 + d_2 x_1 x_2 \tag{11}$$

for which the corresponding L^2 collage distance is minimized. Having found the coefficients b_i, c_i, and d_i, $i = 1, 2$, we obtain the approximation of the physical parameters via

$$a_i = b_i, \ K_i = -\frac{b_i}{c_i}, \text{ and } \alpha_2 = \frac{d_1}{c_1}, \ \alpha_1 = \frac{d_2}{c_2}.$$

Example: We set $K_1 = 125$, $K_2 = 320$, $\alpha_1 = 0.2$, $\alpha_2 = 0.5$, $a_1 = 0.3$, and $a_2 = 0.6$, and solve numerically the system of differential equations. We gather observed data by adding low amplitude Gaussian noise to sampled values of the numerical solution. For $x_1(t)$, we gather 100 sample values at the times $t = \frac{i}{100}$, $0 \leq i \leq 99$; we add normally distributed noise with distribution $\mathcal{N}(0, 0.4)$. We fit a piecewise tenth-degree polynomial to each consecutive set of ten data points to produce our target function for $x_1(t)$. We follow the same procedure to produce a target function for $x_2(t)$, this time with from a $\mathcal{N}(0, 1.6)$ distribution. Upon minimizing the collage distance for (10)-(11), we obtain to five decimal places

$$b_1 = 0.25243, \ c_1 = -0.00208, \ d_1 = -0.00107, \ x_{10} = 160.14811,$$

and

$$b_2 = 0.57433, \ c_2 = -0.00180, \ d_2 = -0.00027, \ x_{20} = 7.21813.$$

From these values, we get the approximations

$$a_1 = 0.25243, \ K_1 = 121.23341, \ \alpha_2 = 0.51641,$$

and

$$a_2 = 0.57433, \ K_2 = 319.72885, \ \alpha_1 = 0.15181.$$

As we decrease the noise or take other measures to increase the accuracy of our target functions, the approximations we obtain by minimizing the collage distance become increasingly accurate.

3 The Economic Resource Model

We consider a common access fishery model,

$$\begin{aligned}\dot{a}(t) &= \gamma(\bar{p}Hb(t) - \bar{c})a(t)\\ \dot{b}(t) &= B(\bar{b} - b(t))b(t) - Ha(t)b(t),\end{aligned}$$

where the first equation models fishing effort by the fishermen, quantified by the number of boats on the water, and the second equation models the fish population. (With some tweaking, this model is the equivalent to the self-regulating predator-prey model found in biomathematics. Here, the fish are analogous to the prey and fishermen (or boats) to the predators.)

This model is referred to as a "common access" model because there are no barriers to entry. That is, fishermen are free to enter and exit the industry as they wish without penalty, cost, legal restriction, or any other stipulations which make entry difficult. In practice, we have entry so long as profits are positive. In the case of zero profits, we will neither have entry nor exit until other factors influence the dynamics of interaction between our players, the fish and fishermen. For instance, if there is suddenly a large number of fish, more fishermen will enter the industry in hopes of realizing potential gains from profit. The meaning of each term in our model is given in the following list:

$$\begin{aligned}
a(t) &= \text{number of boats at time } t\\
b(t) &= \text{number of fish at time } t\\
\bar{b} &= \text{sustainable fish population, } \bar{b} > 0\\
B &= \text{scaling term} = \frac{\text{growth rate of fish}}{\bar{b}},\ 0 < B < 1\\
H &= \text{technological constant, converts effort into catch}, 0 < H < 1\\
\bar{c} &= \text{marginal constant cost per boat}, \bar{c} > 0\\
\bar{p} &= \text{market price per fish}, \bar{p} > 0\\
R &= pHb(t)a(t) = \text{total industry revenue at time } t, R > 0\\
E &= R - ca(t) = \text{industry profits at time } t, E > 0\\
\gamma &= \text{scaling term}, \gamma > 0\\
\gamma c &= \text{rate at which fishermen leave the water}, c > 0.
\end{aligned}$$

In the first equation, $\bar{c}a(t)$ is the total cost of the boats per unit time, at time t. The product $\bar{p}Hb(t)a(t)$ is the total revenue per unit time. The difference $\bar{p}Hb(t)a(t) - \bar{c}a(t)$ is the profit at time t. The first equation says that the rate of change of the fishing effort is proportional to the profit. The first term of the left hand side of the second equation, $B(\bar{b} - b(t))b(t)$, is in the usual logistic form, describing the natural dynamics of the fish population. The bracketed term $(\bar{b} - b(t))$ is the element which makes this model self-regulating. The second term, $Ha(t)b(t)$, represents the total harvest.

We find that the model has three equilibria

$$(0,0),\ (0,\bar{b}),\ \text{and}\,(a^\star,b^\star)=\left(\frac{B}{H}\left[\bar{b}-\frac{\bar{c}}{H\bar{p}}\right],\frac{\bar{c}}{H\bar{p}}\right).$$

The final equilibrium corresponds to coexistence of the fish and fisherman populations in the case that $\bar{b}-\frac{\bar{c}}{H\bar{p}}$ is positive. The linearization of the vector field is

$$Df(a,b)=\begin{pmatrix}\gamma\bar{c} & \gamma\bar{p}Ha\\ -Hb & B\bar{b}-2Bb-Ha\end{pmatrix}$$

Evaluating at the origin, we have

$$Df(0,0)=\begin{pmatrix}\gamma\bar{c} & 0\\ 0 & B\bar{b}\end{pmatrix}.$$

We conclude that $(0,0)$ is unstable. At the equilibrium point $(0,\bar{b})$, we find

$$Df(0,\bar{b})=\begin{pmatrix}\gamma\bar{c} & 0\\ -H\bar{b} & -B\bar{b}\end{pmatrix},$$

with eigenvalues of each sign. The equilibrium point is an unstable saddle point. The stable ray of contraction on the b-axis corresponds to the fact that in the absence of fishermen the fish population approaches the sustainable fish population value, $\bar{b}$. Finally, in the case $\bar{b}-\frac{\bar{c}}{H\bar{p}}>0$, at the positive equilibrium point $(a^\star,b^\star)$, we determine that

$$Df\left(a^\star,b^\star\right)=\begin{pmatrix}0 & \gamma\bar{p}B\left(\bar{b}-\frac{\bar{c}}{H\bar{p}}\right)\\ -\frac{\bar{c}}{\bar{p}} & -\frac{B\bar{c}}{H\bar{p}}\end{pmatrix}.$$

We calculate that

$$\det\left(Df\left(a^\star,b^\star\right)\right)=\gamma\bar{c}B\left(\bar{b}-\frac{\bar{c}}{H\bar{p}}\right)>0$$
$$\operatorname{trace}\left(Df\left(a^\star,b^\star\right)\right)=-\frac{B\bar{c}}{H\bar{p}}<0.$$

We conclude that the coexistence equilibrium is a stable sink.

To illustrate the results, we set $\bar{b}=1000000$, $B=\frac{7}{480000}$, $H=0.5$, $\bar{c}=100000$, $\bar{p}=2$, $\gamma=\frac{1}{20000}$, and use the the initial values $b_0=100000$ and $a_0=22$ to generate the phase portrait in Figure 3.

3.1 Inverse Problem

We are interested in solving the inverse problems: given data values $b(t_i)$, $i=1,\ldots,M$ and $a(t_j)$, $j=1,\ldots,N$, find values of the physical variables b, B, H, c, p, and γ so that the solution to the system agrees approximately with the data.

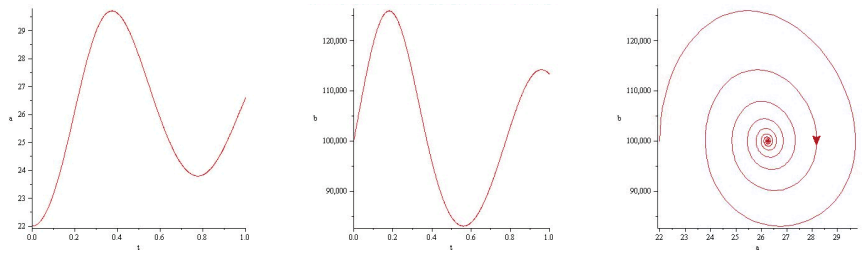

Fig. 3. Left to right: graphs of $a(t)$ versus t, $b(t)$ versus t, and the phase portrait $b(t)$ versus $a(t)$.

Example: To generate solution data, we set $\bar{b} = 1000000$, $B = \frac{7}{480000}$, $H = 0.5$, $\bar{c} = 100000$, $\bar{p} = 2$, and $\gamma = \frac{1}{20000}$, solve numerically for $b(t)$ and $a(t)$, and sample the solutions at uniformly-spaced times in $[0, 1]$, adding low-amplitude Gaussian noise with amplitude ε_b and ε_a, respectively. We fit piecewise polynomial target functions to these noisy data values and minimize the L^2 collage distance corresponding to the differential equations

$$\dot{b}(t) = c_1 b(t) + c_2 b^2(t) + c_3 a(t) b(t) \tag{12}$$

$$\dot{a}(t) = c_4 a(t) b(t) + c_5 a(t). \tag{13}$$

The results for different noise amplitudes are summarized in Table 1.

ε_b	ε_a	b_0	a_0	c_1	c_2	c_3	c_4	c_5
0	0	100000	22.0000	14.5833	-0.00001	-0.5000	0.00005	-5.00000
0.05	0.02	100680	21.9916	14.5735	-0.00001	-0.4998	0.00005	-4.9535
0.08	0.04	101134	22.0046	14.5343	-0.00001	-0.4998	0.00005	-4.9128
0.10	0.06	101476	22.0279	14.4795	-0.00001	-0.4998	0.00005	-4.8755
0.15	0.05	101964	21.9608	14.5714	-0.00001	-0.4990	0.00005	-4.8661
0.20	0.10	102815	22.0237	14.4100	-0.00001	-0.4983	0.00005	-4.7674

Table 1. Minimal Collage Distance Coefficients for the Resource Model Inverse Problem.

We observe that $c_1 = B\bar{b}$, $c_2 = -B$, $c_3 = -H$, $c_4 = \gamma\bar{p}H$, and $c_5 = -\gamma\bar{c}$. If we assume that $\bar{p} = 2$ is known, since it is the price determined by the market, we can calculate the remaining parameters from the minimal collage distance coefficient values. We obtain the results in Table 2.

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ε_b	ε_a	b_0	a_0	$\bar{b}$	B	H	$\bar{c}$	$\bar{p}$	γ
0	0	100000	22.00	1000000	0.0000100	0.500	100000	2	0.0000500
0.05	0.02	100680	21.99	995241	0.0000146	0.500	99878	2	0.0000496
0.08	0.04	101134	22.01	1014389	0.0000143	0.500	99848	2	0.0000492
0.10	0.06	101476	22.03	1046756	0.0000138	0.500	99852	2	0.0000488
0.15	0.05	101964	21.96	971935	0.0000150	0.499	99543	2	0.0000489
0.20	0.10	102815	22.02	1056825	0.0000136	0.498	99455	2	0.0000479

Table 2. Minimal Collage Distance Parameter Values for the Resource Model Inverse Problem.

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