BANK OF ENGLAND

# A No-Arbitrage Structural Vector Autoregressive Model of the UK Yield Curve

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**Abstract**

This paper combines a Structural Vector Autoregression (SVAR) with a no-arbitrage approach to build a multifactor Affine Term Structure Model (ATSM). The resulting No-Arbitrage Structural Vector Autoregressive (NA-SVAR) model contains a pricing kernel which implies that expected excess returns are driven by the structural macroeconomic shocks. This is in contrast with a standard ATSM, in which agents are concerned with non-structural risks. As a simple application of a NA-SVAR model, we study the effects of supply, demand and monetary policy shocks on the UK yield curve. We show that all shocks affect the slope of the yield curve, with demand and supply shocks accounting for a large part of the time variation in bond yields. The short end of the yield curve is driven mainly by the expectations component, while the term premium matters for the dynamics of the long end of the yield curve.

**Key words:** Structural vector autoregression, interest rate risk, essentially affine term structure model

**JEL classification:** C32, E43, E44

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The views expressed in this paper are those of the authors, and not necessarily those of the Bank of England. The author wishes to thank Carlo Favero, Matthew Hurd, Mike Joyce, Peter Lildholdt, LinLin Niu, Monika Piazzesi, Ulf Soderstrom, Peter Westaway, participants at Royal Economic Society 2007 Conference, Bank of England and IGIER internal research seminars for their valuable suggestions and kind support.

**Contents**

**Summary**

Monetary policy makers control short-term interest rates. But, long term rates are no less important, since they influence borrowing costs and aggregate demand in the economy. Thus understanding the behavior of the whole yield curve is crucial to policy makers.

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The results can be summarized as follows. Demand and supply shocks have different effects on the yield curve. Both supply and demand shocks drive short-term interest rates, whereas demand shocks dominate long-term interest rates. Both, demand and supply shocks, affect the slope of the yield curve positively on impact. This result confirms the finding that the slope of the yield curve and economic activity are linked together. Finally, the monetary policy shock affects the whole yield curve, with the effect decreasing with maturity.

Our results are broadly consistent with the dynamics of yields implied by previous work in empirical macroeconomics. The advantage of our approach is that we are able to decompose long-term interest rates into expected short-term rates and term premia. The results show that the short end of the yield curve moves due to the changes in expectations, while the long end of the yield curve movements are due to the term premia dynamics.

Although the model performs well overall, it does not fit the long end of the yield curve well over the most recent sample, which suggests that including additional macroeconomic variables and shocks might improve the simple model.

## 1 Introduction

The main motivation of this paper is to try to understand how the term structure of nominal interest rates responds to fundamental economic shocks. The innovation of this paper is to assume that all risks affecting the pricing behavior of agents are related to structural macroeconomic shocks identified in a Structural Vector Autoregression (SVAR) framework.. This is in contrast with a standard empirical no-arbitrage Affine Term Structure Models (ATSM), in which agents are concerned with non-structural risks. Our approach allows us to enrich the partial equilibrium ATSM framework by macroeconomic theory, and thus to narrow the existing gap between no-arbitrage literature and macroeconomic models.

Our paper builds upon and extends two strands of research. The first is the finance no-arbitrage term structure literature, and the second is connected to empirical macroeconomic SVAR models. At the nexus of these two strands, we present a yield curve model that relates fundamental macroeconomic shocks to the bond pricing behavior of economic agents.

Theoretical and empirical research suggest that the dynamics of the term structure can be explained by a limited number of factors. But what is the nature of the factors driving the yield curve and how are they related to the economy? In canonical arbitrage-free term structure models, the factors driving the term structure are attributed to pure latent factors (see, for example, Duffie and Kan (1996), and Dai and Singleton (2000)). By contrast, a growing macro-finance literature links the dynamics of the term structure to macroeconomic variables, deviating from the pure latent structure, macroeconomists show that the shocks to macroeconomic factors can account for a large part of the time variation in bond yields. For example, in pioneering work introducing macroeconomic variables into a term structure model, Ang and Piazzesi (2003) claim that macro-factors explain up to 85% of the variation in US bond yields. More recently, for the UK, Lildholdt, Peacock and Panigirtzoglou (2007) confirm the importance of macroeconomic factors for the yield curve. They find that inflation and the output gap drive the short end of the yield curve, whereas long-run inflation dominates the long end. However, the shocks in these models are not fully structural. A simple structural model of the macroeconomy based on the Euler equation for consumption and pricing equation for firms is developed by Rudebusch and Wu (2004), who interpret the yield dynamics in terms of a structural macroeconomic model. Even so, they employ a reduced-form pricing kernel, which conflicts with the specification of the marginal rate of substitution in the Euler equation of their

model.

From the macroeconomic literature side, there are a few empirical studies using a SVAR framework to describe the joint dynamics of the macroeconomy and the yield curve. For instance, Evans and Marshall (1998) were among the first to study the effects of monetary policy shocks on the yield curve in a joint macroeconomic-term structure SVAR framework. By including single yields into standard macroeconomic SVARs, they describe the dynamics of the yield curve by studying the impulse responses of single yields, devoid of a dynamic term structure model of interest rates. Obviously, the inclusion of separate yields in a macroeconomic VAR is not an efficient approach, since it omits information contained in the whole yield curve. The impulse responses can be modelled only for the yields on observed bonds, while the method has no implications for the yields on bonds with non-traded or intermediate maturities. Moreover, the approach used by Evans and Marshall (1998) does not rule out arbitrage opportunities and hence cannot explain whether the changes in yields are due to a revision of expectations of short-term rates or due to changes in term premia.

We contribute to the macro-finance term structure literature by combining an arbitrage-free term structure model with a SVAR approach. The combination of the SVAR and the ATSM helps us to achieve several goals. Most importantly, we are able to model the yield curve across maturities and across time jointly with the macroeconomic dynamics. In particular, we relate the yield curve to fundamental structural macroeconomic shocks and impose cross-equation no-arbitrage restrictions on the parameters of yields. Additionally, the ability to decompose yields on expected risk-free rates and term premia helps us to better understand the channels through which fundamental shocks affect yields.

The model is estimated in two steps. First, we identify structural fundamental shocks from an SVAR based on macroeconomic variables and the policy rate. Second, following the nonlinear least squares approach, as in Ang, Piazzesi, and Wei (2006), we estimate the yield parameters (restricted due to no-arbitrage) by minimizing the sum of squared fitting errors of the model. The two-step procedure simplifies the estimation of highly parametrized ATSMs considerably.

As an application, we study fundamental shocks to the UK economy and establish their role in determining the term structure of UK nominal interest rates. A structural VAR is used to identify aggregate supply, aggregate demand, and monetary policy shocks. We then analyze the shocks'

effects on the nominal yield curve, with the help of the term structure model of interest rates. More specifically, we estimate a three-variable SVAR model based on the output gap, inflation and the short-term nominal interest rate. Following Blanchard and Quah (1989), supply shocks are identified through long-run restrictions, assuming that supply shocks alone have a long-run impact on the level of output. Therefore, we restrict aggregate demand and monetary policy shocks to have no long-run impact on output. To separate monetary policy shocks from demand shocks, we impose additional short-run restrictions on the variance-covariance matrix of residuals. Thus, in the context of the term structure model, we assume that agents are concerned with fundamental risks of the economy when they price bonds and, combining the SVAR and no-arbitrage approaches, we build a three-factor ATSM for the UK yield curve.

The results can be summarized as follows. We show that demand and supply shocks have different effects on the yield curve. Both supply and demand shocks drive the short end of the yield curve. However, the long end of the yield curve is determined almost entirely by demand shocks, affecting mostly the preference-related term premia rather than the expectations component. Both, demand and supply shocks, affect the slope of the yield curve positively on impact. This result confirms the finding that the slope of the yield curve and economic activity are linked together (see, for example, Estrella and Hardouvelis (1991), and Harvey (1988), who first documented the leading indicator properties of the yield curve slope for future economic activity). Finally, the monetary policy shock affects the whole yield curve, with the effect decreasing with maturity. Our results are broadly consistent with the dynamics of yields implied by unrestricted SVAR approach (as in Evans and Marshall (1998)). The advantage of our approach is the explicit model of the term premium, which allows us to show that the short end of the yield curve moves due to the changes in expectations, while the long end of the yield curve movements are due to the term premia dynamics.

Although the model performs well overall, it does not fit the long end of the yield curve properly, which suggests that including more factors might improve the simple model.

The rest of the paper is structured as follows. In Section 2 we outline the basic assumptions and implications of the canonical ATSM. We also show how to modify the model and impose a macroeconomic structure. Section 3 shows how to combine the SVAR and ATSM approaches. The details of the estimation method follow in Section 4. Our results are then presented in Section 5. Finally, some conclusions and possible extensions are provided in Section 6.

## 2 Related Literature

In this section, we outline the basic concepts of two related strands of literature, around which this paper is developed, and briefly highlight their main features.

### *2.1 No-arbitrage term structure models*

The fundamental assumption of no-arbitrage term structure models is that there are only a few variables, $X_t$, relevant for bond pricing (Duffie and Kan (1996), Dai and Singleton (2000)). These models price all bonds in economy by specifying the dynamics of the state vector $X_t$, setting initial conditions for the bond pricing rule, and specifying the market prices of risk, $\Lambda_t$. In a discrete-time setup the three basic assumptions of ATSM take the following form:

1) The transition equation for the state vector relevant for pricing bonds follows the Gaussian VAR:

$$X_t = \mu + \Phi X_{t-1} + \Sigma \varepsilon_t, \qquad \textbf{(1)}$$

where $X_t$ is an $(n \times 1)$-vector of state variables, $\varepsilon_t$ is an $(n \times 1)$ -vector of i.i.d. shocks with zero mean and identity covariance matrix; $\mu$ is $(n \times 1)$ , $\Phi$ is $(n \times n)$ , $\Sigma$ and is $(n \times n)$.

2) The one-period interest rate is a linear function of the state variables:

$$r_t = \delta_0 + \delta_1 X_t, \qquad \textbf{(2)}$$

where $\delta_0$ is a scalar, and $\delta_1$ is an $(1 \times n)$ −vector.

3) The prices of risk associated with shocks $\varepsilon_t$, denoted by $\Lambda_t$, are an affine function of the state of the economy (see Duffee (2002)):

$$\Lambda_t = \lambda_0 + \lambda_1 X_t \qquad \textbf{(3)}$$

for the $(n \times 1)$ vector $\lambda_0$, and the $(n \times n)$ matrix $\lambda_1$. We use market prices of risk to specify a stochastic discount factor that transforms the physical distribution of bond prices into its risk-neutral equivalent.[1] In the case of the ATSM, the stochastic discount factor takes the form:

$$M_{t+1} \equiv \exp(-r_t) \exp(-\frac{1}{2}\Lambda_t'\Lambda_t - \Lambda_t'\varepsilon_{t+1}) \qquad \textbf{(4)}$$

[1] If investors are risk neutral, the assumption of no-arbitrage implies that $P_{t,n} = E_t(e^{-r_t} P_{t+1,n-1})$. However, if investors are risk averse, then their pricing behavior differs from that of risk neutral agents, since the former take into account the amount of risk affecting the future prices. In this case, a stochastic discount factor, $M_t$, transforms the observed distribution of prices into a risk-neutral equivalent distribution: $P_{t,n} = E_t(M_{t+1} P_{t+1,n-1})$.

Under these assumptions, the price and yield of any maturity are affine functions of the state variables:

$$p_{t,n} = A_n + B_n' X_t, \quad \textbf{(5)}$$

$$y_{t,n} = -\frac{1}{n}\left(A_n + B_n' X_t\right), \quad \textbf{(6)}$$

where $y_{t,n}$ is a continuously compounded yield on the bond of maturity $n$ at time $t$; scalar $A_n$ and the $(n \times 1)$ vector $B_n$ depend on the parameters $\{\lambda_0, \lambda_1, \delta_0, \delta_1, \mu, \Phi, \Sigma\}$ and, as shown in the Appendix, they are the solutions of the system of difference equations:

$$\begin{aligned} A_{n+1} &= A_n + B_n'\mu - B_n'\Sigma\lambda_o + \tfrac{1}{2}B_n'\Sigma\Sigma' B_n - \delta_0 \\ B_{n+1} &= (\Phi - \Sigma\lambda_1)' B_n - \delta_1' \end{aligned} \quad \textbf{(7)}$$

The no-arbitrage ATSM (NA-ATSM) has gained huge popularity in the finance literature due to the fact that the implied affine functions of a few unobservable (latent) factors explain almost all movements of the yield curve (see Duffie and Kan (1996), or Dai and Singleton (2000)). Nevertheless, the pure ATSM has not gained the same popularity among economists since it is not suitable for many macroeconomic policy applications. There is no theory behind the NA-ATSM apart from the no-arbitrage assumption and the economic nature of the latent factors is unclear. Observing that the short-term rate is an instrument for monetary policy, macroeconomists have proposed a possible solution: to combine No-Arbitrage ATSM models with macroeconomic models.

Attempts to incorporate macroeconomic theory into the no-arbitrage models can be divided into three groups. First, one can assume that the state vector is completely unobservable and after that can search for its macroeconomic interpretation by Taylor rules, or by other typical macroeconomic relations (see Rudebusch and Wu (2004) who interpret latent factors as the perceived inflation target and output gap, and therefore enrich the state dynamics with variables related to inflation and output). Second, certain authors argue that all factors related to bond pricing are observable. One example of this method is the paper by Ang, Piazzesi and Wei (2006), who combine an empirical unrestricted VAR macro-model with a No-Arbitrage representation of yields. The paper by Ang, Bekaert and Wei (2007) belongs to the third group, which is a combination of the first two approaches. The authors suppose that the state vector

relevant for the bond pricing consists of both, latent and observable, factors. This approach is very popular and there are a number of papers which adopt it: Ang and Piazzesi (2003), Dai and Philippon (2004), Hördhal, Tristani, and Vestin (2006), Lildholdt, Peacock and Panigirtzoglou (2007) among others. Our model also belongs to the third, "mixed", group of ATSMs, as we assume that the state vector consists of both, latent and observable factors.

### *2.2 Structural VAR models*

Returning to the second strand of the literature, our work belongs to a class of macroeconomic Structural VAR models that allow a researcher to transform the reduced-form VAR model into a system of structural equations of the economy. The identification of structural shocks is an extremely controversial venture since, by imposing different identifying assumptions, it is possible to derive different conclusions about important economic questions. Restrictions depend on the variables included and on the shocks to be identified. Standard restrictions employed in the literature impose constraints on the short run or long-run impact of particular shocks on variables or informational gaps. For example, the assumption that output is not observed by central banks when making decisions on interest rates results in a *short-run zero restriction*. Instead, a *short-run sign restriction* is necessary to make sure that the impact of contractionary monetary policy shock is non-negative on the interest rate, and non-positive on real GDP growth and inflation. If only technology shocks had a permanent impact on output and no other shock affected real activity in the long-run, then an econometrician would employ a *long-run zero restriction.*

Different empirical studies have used a SVAR framework to describe the joint dynamics of the macroeconomy and the yield curve. For instance, Evans and Marshall (1998) were among the first to study the effects of monetary policy shocks on the yield curve in a joint macroeconomic-term structure SVAR framework.[2] They use the following vector autoregression:

$$\begin{bmatrix} Z_t \\ y_{t,n} \end{bmatrix} = \begin{bmatrix} A(L) & 0 \\ C(L) & D(L) \end{bmatrix} \begin{bmatrix} Z_{t-1} \\ y_{t-1,n} \end{bmatrix} + \begin{bmatrix} a & 0 \\ c & b \end{bmatrix} \begin{bmatrix} \varepsilon_t^Z \\ \varepsilon_t^n \end{bmatrix} \qquad \textbf{(8)}$$

However, such SVARs describe the dynamics of the yield curve by studying the impulse responses of single yields, devoid of a dynamic term structure model of interest rates.

The term structure literature, instead, has been operating with rather arbitrary identification

[2]See also Bagliano and Favero (1998), who estimate VARs with macroeconomic variables, as well as short- and long-term yields.

schemes. For example, in the work by Ang and Piazzesi (2003), there are 5 factors (3 latent and 2 principal components for real activity and inflation) and the variance-covariance matrix of residuals is assumed to be block-diagonal: macro-factors are recursively identified, latent factors are orthogonal. Unfortunately, it is unclear from the model how these shocks should be interpreted. Rudebusch and Wu (2004) work with two latent factors and VAR dynamics enriched by inflation and the output gap. Again, the interpretation of their shocks is vague. Lildholdt, Panigirtzoglou and Peacock (2007) introduce 3 factors (two unobserved factors and inflation as the observed factor) and assume a diagonal variance-covariance matrix of residuals. Therefore, their shocks are orthogonal, and hence not fully structural.

There are two recent papers that try to identify macroeconomic shocks in a no-arbitrage ATSM framework. These are the papers by Ang, Dong and Piazzesi (2005) and Dai and Philippon (2004). Dai and Philippon (2004) were the first to estimate an ATSM based on Structural VAR dynamics. They present a macro-finance model where the pricing kernel is driven by several shocks, one of which is a structural fiscal policy shock identified using the long-run identification strategy by Blanchard and Perotti (2002). The approach of Ang, Dong and Piazzesi (2005) is very different: they are able to identify monetary policy shocks without imposing structural restrictions on the variance-covariance matrix of residuals. The authors show that the same ATSM accommodates several types of Taylor Rules: a benchmark Taylor Rule, a backward-looking Taylor Rule, and a forward-looking Taylor Rule.

## 3 The Model

In contrast to previous no-arbitrage studies, our paper identifies all fundamental macroeconomic shocks as in the SVAR literature, and then uses ATSM to price bonds with respect to the identified shocks.

In our model, the state vector, $X_t$, of the yield curve is represented by $n$ observable macroeconomic factors and one latent factor, $f_t$. We denote $Z_{0t}$ as a $(n \times 1)-$ vector of observed macroeconomic variables and describe the short term interest rate by the relationship:

$$r_t = \delta_0 + \delta_1 X_t \equiv \delta_0 + \boldsymbol{\delta}_z Z_{0t} + \delta_f \, f_t, \qquad \textbf{(9)}$$

where $X_t \equiv \begin{bmatrix} Z_{0t} \\ f_t \end{bmatrix}$.

While the number of observable macroeconomic factors could be arbitrary, it is important to have only one latent factor, since in this case we can map the state vector into the vector of observable variables, $Z_t = (Z_{0,t},\ r_t)'$[3], and therefore to greatly simplify the standard estimation method. Given **(9)**, the state vector $X_t$ is mapped to $Z_t$ by the relation:

$$Z_t = \begin{bmatrix} \mathbf{0}_{(n\times 1)} \\ \delta_0 \end{bmatrix} + \begin{bmatrix} I_{(n\times n)} & \mathbf{0}_{(n\times 1)} \\ \boldsymbol{\delta}_{z(1\times n)} & \delta_f \end{bmatrix} X_t \quad \textbf{(10)}$$

$$\equiv M_{0((n+1)\times 1)} + M_{((n+1)\times(n+1))} \cdot X_t \quad \textbf{(11)}$$

Observed macroeconomic variables are assumed to follow the SVAR process:

$$AZ_t = \alpha + B(L)Z_{t-1} + \varepsilon_t, \quad \textbf{(12)}$$

with $A = A_{(n+1)\times(n+1)},\ \alpha = \alpha_{(n+1)\times 1},\ B = B_{(n+1)\times(n+1)}$, or, in a reduced form representation:

$$Z_t = \tilde{\mu} + \tilde{\Phi}(L)Z_{t-1} + \tilde{\Sigma}\varepsilon_t \quad \textbf{(13)}$$

where $\tilde{\Phi}(L)$ is a polynomial in the lag operator, and the macroeconomic shocks to be identified are given by $\varepsilon_t,\ \varepsilon_t \sim N(0, I),\ \tilde{\Sigma}\tilde{\Sigma}' = V$.

Interestingly, the dynamics of the vector $Z_t$ are not affected by the no-arbitrage restrictions, since the short-term rate, $r_t$, is a risk-free rate and hence has no need to be adjusted for risk. Once the shocks are identified in the macroeconomic SVAR **(13)**, equations **(9)** and **(12)** imply that the state vector follows the VAR process:

$$X_t = \mu + \Phi X_{t-1} + \Sigma\varepsilon_t, \quad \textbf{(14)}$$

where

$$\mu = M^{-1}\left(\tilde{\mu} - (I - \tilde{\Phi})\, M_0\right) \quad \textbf{(15)}$$

$$\Phi = M^{-1}\tilde{\Phi}M, \quad \textbf{(16)}$$

$$\Sigma = M^{-1}\tilde{\Sigma} \quad \textbf{(17)}$$

[3] According to Dai and Singleton (2000), affine term structure models require three latent factors to match yield curve dynamics.

This structure is a standard assumption about the dynamics of the state vector in affine term structure models.

Equations **(14)**, **(3)**, and **(9)** constitute a standard affine term structure model with one latent and $n$ observable factors. The main feature of the model, however, is that the pricing kernel is a function of *structural macroeconomic shocks*, $\varepsilon_{t+1}$, identified by the restrictions on SVAR **(12)**. In general, our approach allows us to explain the yield curve with any kind of SVAR model based on the macroeconomic variables and the short-term interest rate. We call the approach a No-Arbitrage Structural Vector Autoregressive (NA-SVAR) Model of the Yield Curve and, in what follows, we propose a simple three-variable SVAR model as an application of our approach.

### *3.1 A Simple Example*

We want to explain the dynamics of the state vector by three standard fundamental shocks. These are supply, demand and monetary policy shocks, $\varepsilon_t = \left(\varepsilon_t^S, \varepsilon_t^D, \varepsilon_t^{MP}\right)'$. Since the model defines the short-term rate as a linear combination of the state variables in $X_t$, a natural assumption is that the observable state variables are the rate of inflation, $\pi_t$, and a real activity variable. Due to the unobservability of potential output and the output gap, we use annual GDP growth, $g_t$, as our proxy for real activity. Thus, $X_t = (g_t,\ \pi_t,\ f_t)'$ , similar to Ang, Dong, Piazzesi (2005), and we describe a short-term interest rate by Taylor Rule:

$$r_t = \delta_0 + \delta_1 X_t \equiv \delta_0 + \delta_g g_t + \delta_\pi \pi_t + \delta_f\ f_t, \qquad \textbf{(18)}$$

However, we restrain ourselves from interpreting the latent factor as a monetary policy shock, since the simple benchmark Taylor Rule **(18)** is likely to be misspecified.[4] Instead, we identify monetary policy shocks from our Structural VAR framework.

The identification scheme is based on the approach by Blanchard and Quah (1989), who use a long-run restriction to identify aggregate supply and aggregate demand shocks in a bivariate model of GNP and unemployment. Blanchard and Quah (1989) attain identification by limiting aggregate demand not to have a permanent effect on the level of GNP. Here we also use the Blanchard-Quah identifying strategy to separate supply shocks from other economic surprises. We interpret the fluctuations in the state vector as due to two types of shocks: shocks that have permanent effect on output and shocks that do not. The first type of shock is interpreted as a

[4]For instance, empirical applications find that the lagged interest rate is an omitted variable from the benchmark Taylor rule, a result that is interpreted as interest rate smoothing by the monetary policy authorities (see, for example, Clarida, Galì and Gertler (2000)).

supply shock. In addition, given that the one-period interest rate is among the SVAR variables, one of the non-permanent fundamental shocks could be interpreted as a monetary policy shock.

As it is shown in the Appendix, imposing the long-run exclusion restriction identifying supply shocks is equivalent to imposing the following constraint on the dynamics of the state vector:

$$D \equiv (I_{(3\times 3)} - \tilde{\Phi})^{-1}\tilde{\Sigma} = \begin{bmatrix} \cdot & 0 & 0 \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix}, \tag{19}$$

where $\cdot$ denotes a free parameter.

To calculate the dynamic effects of the supply shock, we require estimates of $\tilde{\Phi}$, which could be obtained by ordinary least squares estimation, and the first column of $\tilde{\Sigma}$, which in turn can be obtained once we know the matrix $D$. Indeed, note that

$$\begin{aligned} DD' &= (I_{(3\times 3)} - \tilde{\Phi})^{-1}\tilde{\Sigma}\,\tilde{\Sigma}'(I_{(3\times 3)} - \tilde{\Phi}')^{-1} \\ &= (I_{(3\times 3)} - \tilde{\Phi})^{-1}V_0(I_{(3\times 3)} - \tilde{\Phi}')^{-1} \\ &= \begin{bmatrix} d_{11} & 0 & 0 \\ d_{21} & d_{22} & d_{23} \\ d_{31} & d_{32} & d_{33} \end{bmatrix}\begin{bmatrix} d_{11} & d_{21} & d_{31} \\ 0 & d_{22} & d_{32} \\ 0 & d_{23} & d_{33} \end{bmatrix}' \\ &= \begin{bmatrix} d_{11}^2 & d_{21}d_{11} & d_{31}d_{11} \\ d_{21}d_{11} & d_1 & d_2 \\ d_{31}d_{11} & d_2 & d_3 \end{bmatrix} \end{aligned} \tag{20}$$

Thus, the first element of $D$, $d_{11}$, can be obtained given $\tilde{\Phi}$ and the variance-covariance matrix of residuals $V_0$, assuming that $d_{11} > 0$. Given $d_{11}$, the elements $d_{21}, d_{31}$ can be easily estimated together with the first column of $\tilde{\Sigma}$.

Additionally, we impose short-run restrictions to distinguish between the two remaining shocks. We impose standard zero restrictions to identify monetary policy shocks, under which the

monetary policy actions have no immediate effect on inflation and real output[5]:

$$\tilde{\Sigma} = \begin{bmatrix} \cdot & \cdot & 0 \\ \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot \end{bmatrix}. \tag{21}$$

Demand shocks are thereby identified by residual.

## 4 Estimation

### *4.1 Data*

We use monthly data on continuously compounded nominal spot yields from the Bank of England's dataset, assuming them to be default-risk-free. These yield curve data are estimated by fitting a cubic spline through general collateral (GC) repo rates and conventional government bonds[6]. To estimate the model we use a wide range of maturities: 1 month, 9 months, 12 months, 3 years, 5 years, 7 years, and 10 years.

While the relationship between yields might be stable over time, the relationship between interest rates and macroeconomic variables is likely to have changed over time. Thus we limit our analysis to the recent monetary policy regime, i.e. the period of inflation targeting. Thus we have restricted our sample period to be from October 1992 to December 2006.

As a proxy for inflation, we use annual CPI inflation.[7] Annual monthly GDP growth real time estimates are taken from the NIESR dataset. The series are displayed in Chart 1.

### *4.2 Model Identification*

For our particular sample and variables, we specify the dynamics of the state vector by a VAR(1):

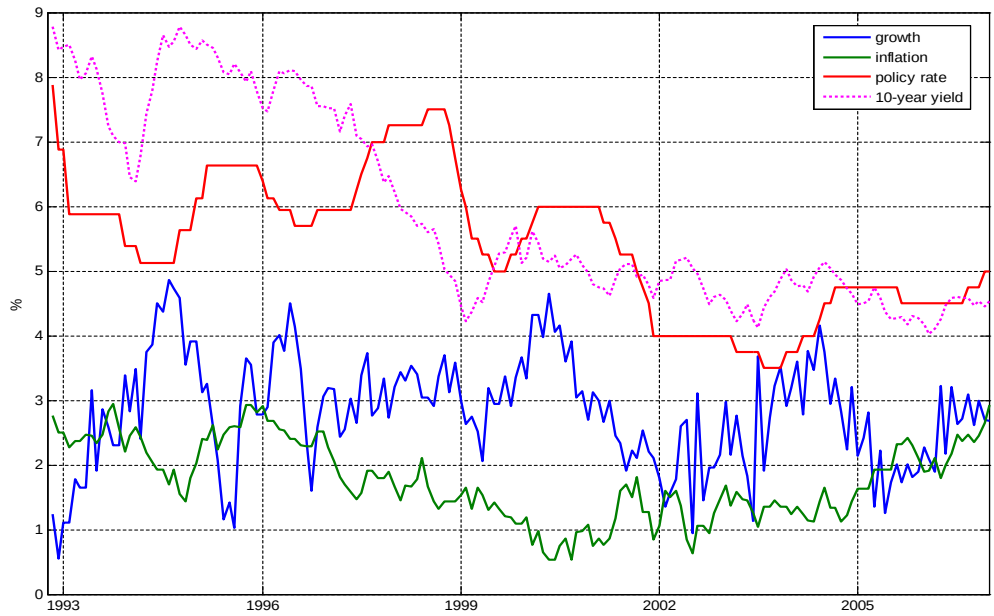
$$Z_t = \tilde{\mu} + \tilde{\Phi}_1 Z_{t-1} + \tilde{\Sigma}\varepsilon_t. \tag{22}$$

[5]Since in this case the model would be overidentified, we alternatively relax the assumption that monetary policy shock has no effect on output in the short run and thus let $\tilde{\Sigma}(1,3)$ be a free parameter. The results do not change significantly.

[6]See Anderson and Sleath (1999). The data are available from the Bank of England website, www.bankofengland.co.uk.

[7]The results are similar for RPIX inflation, which was the UK's target measure during 1992-2003.

**Chart 1: Observed macroeconomic time series**




It is easy to verify that in our case the exclusion restriction **(19)** for identification of supply shocks takes the form:

$$\left(I_{(3\times 3)} - \tilde{\Phi}_1\right)^{-1} \tilde{\Sigma} = \begin{bmatrix} \cdot & 0 & 0 \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix}. \qquad \textbf{(23)}$$

The impulse response $\gamma_h$ of $Z_{t+h}$ to a unit shock in $\varepsilon_t$ can then be computed as

$$\gamma_h = \tilde{\gamma}_h \tilde{\Sigma} \qquad \textbf{(24)}$$

$$\tilde{\gamma}_h = \tilde{\Phi}_1 \tilde{\gamma}_{h-1} \qquad \textbf{(25)}$$

with initial conditions $\tilde{\gamma}_{-1} = 0,\ \tilde{\gamma}_0 = I_{3\times 3}$. Here the $(j, l)$ element of $\gamma_h$ represents the response of $jth$ element in $Z_{t+h}$ to a unit shock in the $lth$ element of $\varepsilon_t$. (See, for example, Christiano, Eichenbaum and Evans (1999)).

In constructing confidence intervals for impulse responses, we use bootstrap simulated distribution percentiles, i.e extract the relevant bands directly from the ordered replications at

each horizon. Since the simple bootstrap method suffers from small sample biasedness and lack of scale invariance, we use a bootstrap-after-the-bootstrap procedure suggested by Kilian (1998). The approach is summarized in Canova (2005), Chapter 4.

Let $\alpha = \{\lambda_0, \lambda_1, \delta_0, \delta_1, \mu, \Phi, \Sigma\}$ be the set of free parameters in the model. To estimate yield curve parameters and extract the latent factor, we use the approach attributed to Chen and Scott (1993). In this setting, $N$ unobservable factors are computed by assuming that $N$ bond yields are measured without error. Other interest rates are assumed to be measured with error and provide over-identifying restrictions. In our joint macro-finance model, there is only one latent factor, and thus we assume that only one yield, a 1-month rate, $r_t$, is observed exactly. Although in the approach of Chen and Scott (1993), the choice of the yields measured without errors is arbitrary, in our case it is natural to assume that the one-month rate is observed without error, as it is represented by the perfectly observed policy rate. The vector $Y_t^{Er}$ of the remaining bond yields is instead observed with independently distributed zero-mean errors. This assumption has the additional important advantage that we can implement a two-step estimation procedure, which drastically decreases the number of parameters to be estimated compared to a one-step maximum likelihood approach.

Since the SVAR of the observed variables **(13)** is not affected by no-arbitrage restrictions, we can estimate the VAR parameters $\tilde{\mu}$, $\tilde{\Phi}$, $\tilde{\Sigma}$ by ordinary least squares (OLS) in the first step. Then, in the second step, given the VAR estimates, we map the observed variables into the state vector by equation **(26)** and get the estimates for the state dynamics and the risk price parameters by the GMM method proposed by Ang, Piazzesi and Wei (2006). Namely, we minimize the sum of squared fitting errors of the model

$$\min_{\{\lambda_0,\lambda_1,\delta_0,\delta_1\}} \sum_{n=1}^{N} \sum_{t=1}^{T} \left(y_{t,n} - \hat{y}_{t,n}\right)^2,$$

where we compute model-implied yields, $\hat{y}_{t,n} = -\frac{1}{n}\left(A_n + B_n' X_t\right)$, with the factors, $X_t$, extracted from the inverted state relation **(10)** of the model:

$$X_t = M^{-1}[Z_t - M_0], \qquad \textbf{(26)}$$

with

$$M_0 = \begin{bmatrix} 0 & 0 & \delta_0 \end{bmatrix}', \; M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \delta_g & \delta_\pi & \delta_f \end{bmatrix}$$

The standard errors are calculated using two-step GMM, with moments from each stage of the

procedure as in Ang, Piazzesi, and Wei (2006).

Due to the presence of the latent factor, the model is not identified: the latent factor could be arbitrarily scaled and shifted, producing observationally equivalent systems. We therefore normalize the model by imposing the loading of the short rate on the latent factor $\delta_f = 1$, and by constraining the mean of the latent factor to be zero. This normalization relates the short rate equation **(9)** to a Taylor rule, where the latent factor plays a role of a residual. We further assume that the market prices of risk affecting the time variation of the risk premia are represented by matrix $\lambda$. Since several elements were insignificant when we initially estimated the model with full matrix $\lambda_1$, we restrict them to be zeros:

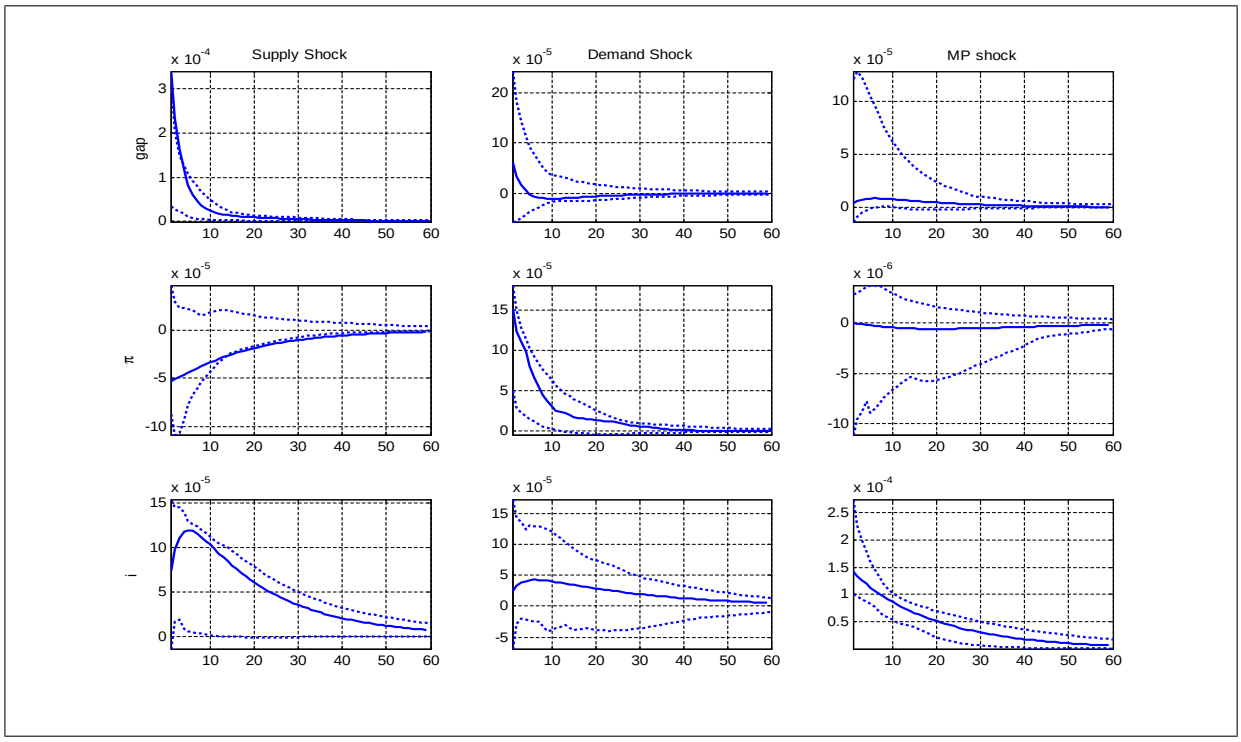
$$\lambda_1 = \begin{pmatrix} \lambda_{11} & 0 & \lambda_{13} \\ 0 & 0 & \lambda_{23} \\ 0 & 0 & \lambda_{33} \end{pmatrix}.$$

## 5 Results

To explain the dynamics of the yield curve, we first apply OLS to estimate the reduced form VAR for GDP growth, CPI inflation and the short-term interest rate. The structural residuals are then retrieved by the help of long-run and short-run restrictions. The responses of the observed variables to the fundamental shocks are shown in Chart 2. The impulse response functions of the VAR confirm the findings of Blanchard and Quah (1989): supply shocks permanently affect the level of output due to their strictly positive effect on output growth and negatively affects inflation on impact. After a negative response to the positive supply shock, inflation increases and comes back to zero after approximately four years. A positive supply shock has a significant positive and persistent effect on the short-term interest rate. A positive demand shock increases both prices and output growth, but the effect of demand on the output seems to be statistically insignificant.

The estimated parameters determining market prices of risk are reported in Table A. The parameters in $\lambda_0$, affecting only the constant yield coefficient $\left(-\frac{1}{n}A_n\right)$ are significantly different from zero (see (6)), which means that the fundamental risks affect the average term spreads and expected returns. The parameters in $\lambda_1$, affecting both the time variation in the yields and indirectly the constant yield coefficient, are significantly different from zero as well. Thus the

**Chart 2: Impulse response functions of observable variables.**




hypothesis that $\lambda_{0i} = \lambda_{1i} = 0$ for all $i$, under which the risk premium is zero and investors are risk neutral, is rejected by the model.

| Table A: Market prices of risk estimates. | | | | | |
|---|---|---|---|---|---|
| Prices of risk, $\lambda_0$ | | | Prices of risk, $\lambda_1$ | | |
| | | | $\lambda_{11}$ | -144.4 | (32.9) |
| | | | $\lambda_{21}$ | 0 | – |
| | | | $\lambda_{31}$ | 0 | – |
| $\lambda_{0g}$ | 0 | – | $\lambda_{12}$ | 0 | – |
| $\lambda_{0\pi}$ | 0.08 | (0.03) | $\lambda_{22}$ | 0 | – |
| $\lambda_{0f}$ | 0.26 | (0.12) | $\lambda_{32}$ | 0 | – |
| | | | $\lambda_{13}$ | -14.20 | (10.60) |
| | | | $\lambda_{23}$ | -13.43 | (8.83) |
| | | | $\lambda_{33}$ | 9.89 | (10.02) |
| Note: Estimated standard errors in parenthesis | | | | | |

Table B reports first and second moments of the observed yields and those predicted by the estimated model. We see that the model fits the short end of the yield curve reasonably well, but the long end is not fully captured: long-term yields implied by the model are less volatile than in the data. This unexplained variance is attributed to the measurement errors. The issue of a large "excess volatility" in long-term interest rates is not new in the literature. It was raised more than twenty-five years ago by Shiller (1979) and it is still relevant today (see, for instance, Ellingsen and Söderstrom (2005), who confirm that observed long-term interest rates seem to be excessively sensitive to fundamental innovations.).

**Table B: Model implied and observed moments.**

| | 1st moments | | 2nd moments | |
|---|---|---|---|---|
| | Data | Model | Data | Model |
| 1-month | 5.42 | 5.42 | 1.09 | 1.09 |
| 9-month | 5.34 | 5.35 | 1.03 | 1.01 |
| 1-year | 5.39 | 5.37 | 1.05 | 0.99 |
| 3-year | 5.69 | 5.66 | 1.26 | 1.00 |
| 5 - year | 5.87 | 5.83 | 1.35 | 1.00 |
| 7- year | 5.92 | 5.91 | 1.42 | 1.04 |
| 10 - year | 5.95 | 5.87 | 1.51 | 1.09 |

### *5.1 Model Dynamics and Fundamental Shocks*

Charts 3-5 show how our identified fundamental shocks affect the yield curve. More precisely, each plot shows the reaction of all maturities to a one unit standard deviation impulse to each fundamental shock. The monetary policy shock shifts the level of the yield curve upward with the effect decreasing with maturity, as reported by Ang and Piazzesi (2003). Supply shocks by contrast have a smaller effect on short-term interest rates, but a larger effect on long rates, thus increasing the slope of the yield curve on the impact, which is the same pattern Evans and Marshall (2001) present for an unrestricted SVAR of the US yield curve. In our model, demand shocks increase long term interest rates on average, with the effect increasing with maturity. These results confirm the well-known empirical finding that the slope of the yield curve and economic activity are interrelated: an increase in the slope tends to indicate higher GDP growth in the future (see Estrella and Hardouvelis (1991), and Harvey (1988)). Monetary policy in our model has a negligible impact on output and prices, which suggest that the example at hand is a simplistic representation of the monetary policy reaction function. Including additional macroeconomic variables, to which monetary policy could react (for example, fiscal variables), might improve the identification of shocks.

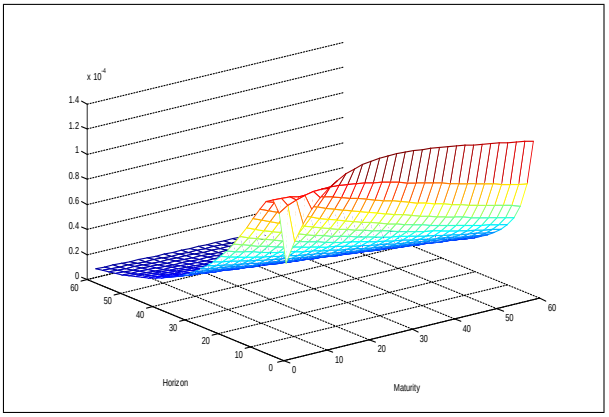



**Chart 3: The impulse response function of the yield curve with respect to a one standard deviation supply shock. X-axis: maturity in months. Y-axis: projection horizon, months.**

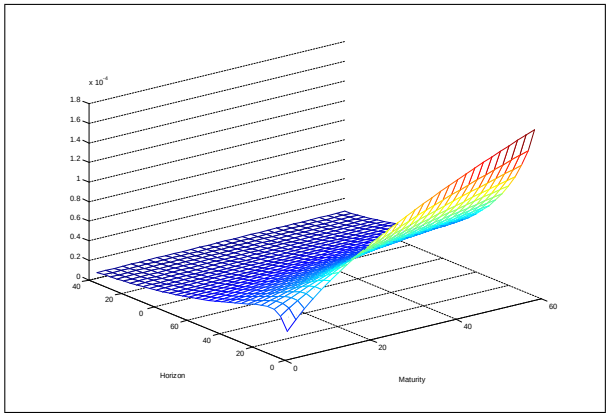



**Chart 4: The impulse response function of the yield curve with respect to a one standard deviation demand shock. X-axis: maturity in months. Y-axis: projection horizon, months.**

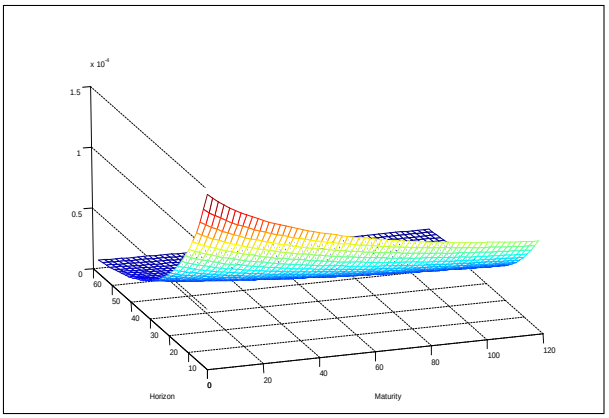



**Chart 5: The impulse response function of the yield curve with respect to a one standard deviation monetary policy shock. X-axis: maturity in months. Y-axis: projection horizon, months.**

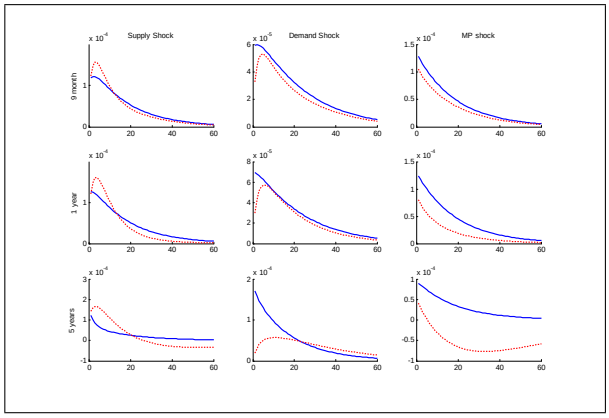



**Chart 6: Solid line: model implied responses of the yield curve to the fundamental shocks. Dotted line: impulse responses of the yield curve according to the model by Evans and Marshall (1998) (see equation (27)).**

In addition, we would like to compare the impulse responses of yields produced by our model to that by Evans and Marshall (1998), that is without imposing No-Arbitrage restrictions. However, we met several difficulties in implementing a direct comparison. For example, the lag length optimal for the VAR of the state vector produces very imprecise estimates of the yield equation in **(8)**, or, for certain cases, the companion matrix in **(8)** could have eigenvalues outside the unit circle. Thus, in order to compare the most appropriate impulse responses, we impose additional zero restrictions on the yield parameters in VAR **(8)** and estimate the following form of VAR:

$$\begin{bmatrix} Z_t \\ y_{t,n} \end{bmatrix} = \begin{bmatrix} \tilde{\mu} \\ c_{0n} \end{bmatrix} + \begin{bmatrix} \tilde{\Phi}(L) & 0 \\ C^n(L) & 0 \end{bmatrix} \begin{bmatrix} Z_{t-1} \\ y^n_{t-1} \end{bmatrix} + \begin{bmatrix} \tilde{\Sigma} & 0 \\ c_n & 1 \end{bmatrix} \begin{bmatrix} \varepsilon^Z_t \\ \varepsilon^n_t \end{bmatrix} \qquad \textbf{(27)}$$

These restrictions assure that the yields implied by both models are functions of the same state variables, since our model explains the dynamic of yields by the following state-space model:

$$\begin{cases} Z_t = \tilde{\mu} + \tilde{\Phi}(L)Z_{t-1} + \tilde{\Sigma}\varepsilon_t \\ y_{t,n} = \alpha_n + \beta_n(L)Z_t + \zeta^n_t \ , \end{cases} \qquad \textbf{(28)}$$

where $\zeta^n_t$ is a measurement error of the $n-$maturity yield. The difference between **(27)** and **(28)** is the No-Arbitrage restrictions imposed on the coefficients $\alpha_n, \beta_n$. Chart 6 shows the impulse responses of the yields implied by our model and by Evans and Marshall (1998)'s approach. The only inconsistency is seen when comparing the impact of monetary policy shocks on the long end of the yield curve. While both models imply that the long-term yields should increase on impact, Evans and Marshall (1998)'s model suggests that the effect of the monetary policy shock does not vanish even after ten years, which is rather counterintuitive. In general, both models produce similar short-term yield impulse responses to the whole range of the shocks.

While the approach by Evans and Marshall (1998) describes the dynamics of observed yields consistently with the NA-SVAR approach, it cannot explain whether changes are due to the revision of expectations or due to changes in risk premia. The advantage of our approach is that we have a complete model of the yield curve with its decomposition into expectations of risk free rate and risk premia. Chart 7 explains which of the yield components, expectations or risk premia, has a major impact on the dynamics of the yield curve. We find that the expectations component, $\sum_{i=0}^{n-1} E_t(r_{t+i})$, explains almost all movements of the short end of the model implied yield curve, while it has little explanatory power for the long end of the yield curve. The results are consistent with traditional term-structure models, which regard long-term interest rates as tightly linked to an average of expected future short-term interest rates. Thanks to this averaging, the traditional models imply that news about the cyclical dynamics of the economy should have a larger effect on short-term interest rates than on long-term interest rates. Therefore, our results

complement standard term structure models by explaining the movements of the long end of the curve by changes in preferences (reflected in term premia), rather than by news about the cyclical dynamics of economics (reflected in the expectations component).

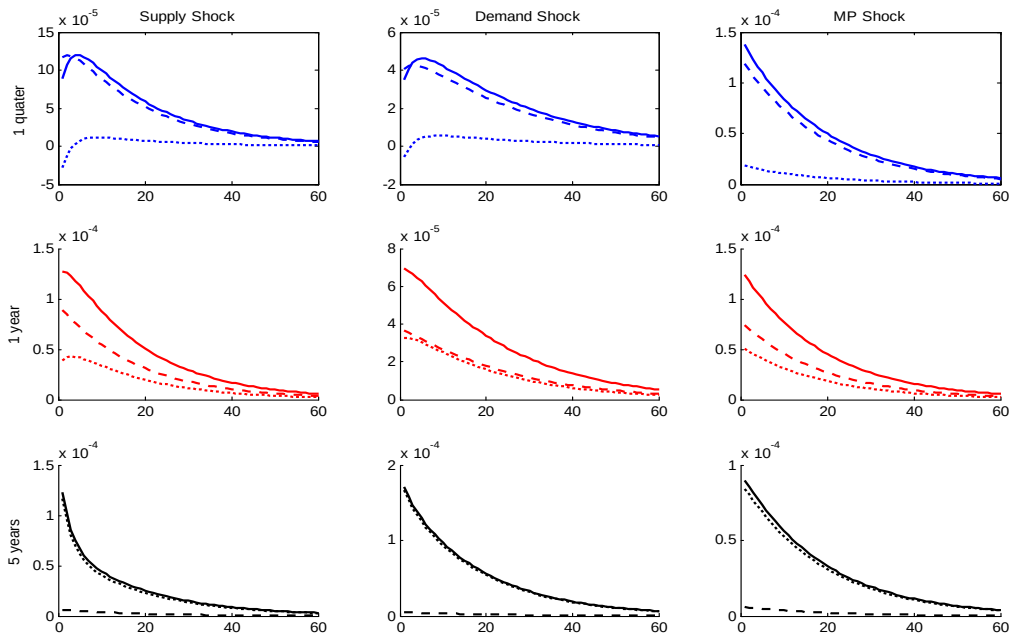



**Chart 7: The impulse response functions of the yield curve decomposition with respect to a one standard deviation structural shock. Solid line: model-implied yields. Dotted line: risk premia. Dashed line: risk free expectations.**

Finally, a useful supplementary description of yield curve dynamics can be obtained from the variance decomposition shown in Table C. The 1-month yield is driven by all three shocks, but predominantly by monetary policy. The 12-month yield is driven by supply and monetary policy shocks and, to a lesser extent, by demand shocks. Movements in the 10-year yield can be attributed to changes in demand: shocks to demand determine the long-end of the term structure by more than 80%. This result becomes intuitive once we remember that, at the long end of the yield curve, demand shocks affect the preferences related term premium rather than the expectations component. Interestingly, monetary policy shocks have very little explanatory power for the variance of the long end of the yield curve (less than 7%). This implication is consistent with recently observed data: UK long-term interest rates have not responded to the tighter monetary policy in 2006 and have remained at low levels.

**Table C: Model variance decomposition.**

| Forecast Horizon | Supply | Demand | MP |
|---|---|---|---|
| 1-month yield | | | |
| 1 month | 20.7 | 2.1 | 76.9 |
| 12 months | 46.3 | 6.2 | 47.5 |
| 60 month | 48 | 8.2 | 43.8 |
| 12-month yield | | | |
| 1 month | 44.6 | 13.1 | 42.3 |
| 12 months | 47.1 | 14.7 | 38.2 |
| 60 month | 46.3 | 16.3 | 37.4 |
| 10-year yield | | | |
| 1 month | 10.8 | 82.6 | 6.5 |
| 12 months | 4.2 | 87.6 | 8.2 |
| 60 month | 3.4 | 87.9 | 8.7 |

### *5.2 Model Decomposition*

In this subsection, we examine the model's implied decomposition of fitted nominal rates into expected future risk-free rates and term premia. Chart 8 shows this decomposition for short-, medium- and long-term maturities. As might have been expected, for shorter maturities, movements in expected future rates are more volatile than for longer maturities. The volatility of long maturity yields, instead, is explained by term premia. In particular, with almost constant average expected short-term yields over 15-year, the decomposition attributes all movements in 15-year model-implied yields to term premia.

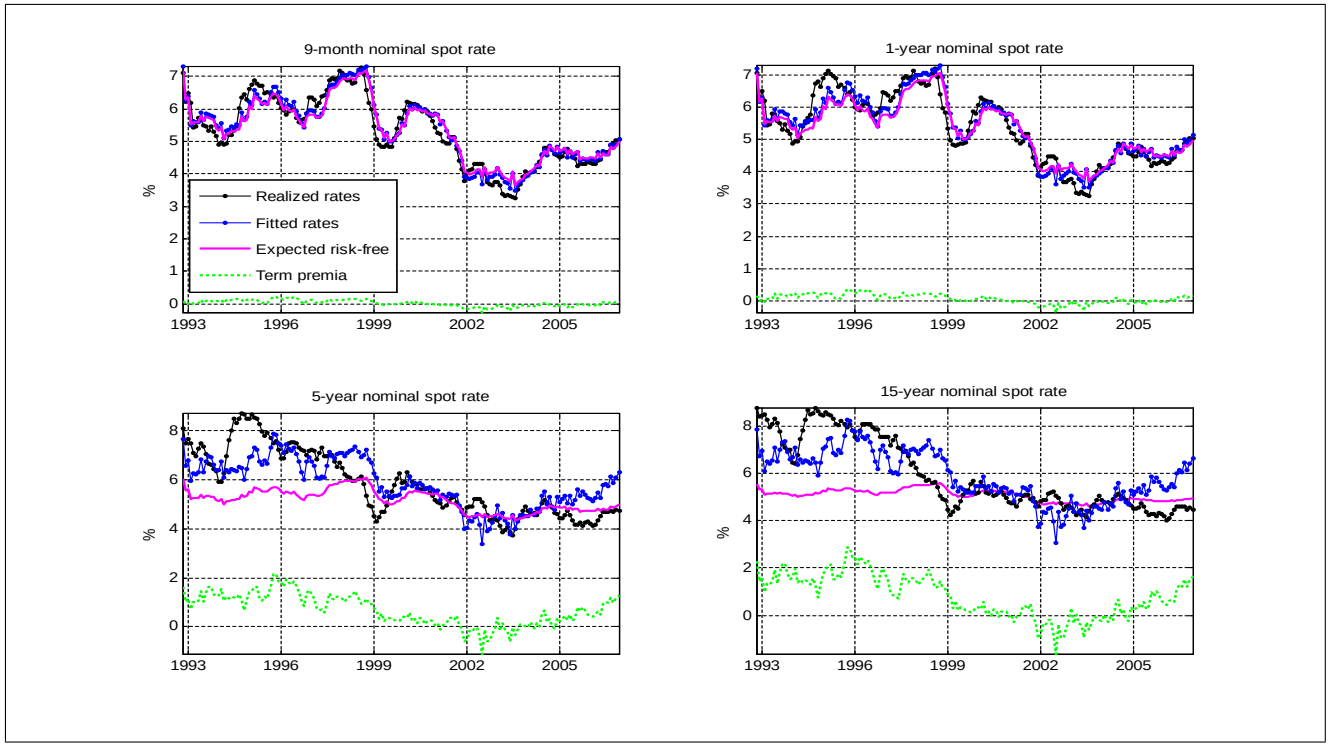



**Chart 8: Decompositions of model implied short-, medium- and long-term yields on risk-free expectations and term premia.**

Interestingly, the model suggests that, as a function of maturity, the average term premium has a hump shape: it increases up to 11-year maturity and then gradually decreases. For example, on average, investors required a positive premium of 0.03% on 1-year maturity bonds, 0.96% on 10-year maturity bonds and 0.86% on 15-year maturity bonds. Over the sample period, our short term premium estimates appear to be very small and stable, while the model suggest that the term premia for long term maturities decreased significantly in the mid 1990s and then was negative during 2001-2004.

Although our simple NA-SVAR model provides a good fit for short and medium maturities, it cannot explain the recent conundrum, i.e. the fall in UK long-term yields that started in 2004. The model implied long-term yields have been persistently higher those observed in the data since 2005, producing very large residuals. This inconsistency can be easily explained by the properties of the model, which implies that long-term yields are positive linear functions of macroeconomic variables that have been increasing in the UK economy since 2005. This result suggests that, while CPI inflation, GDP growth and policy rates are important for bond pricing

from a theoretical and empirical perspective, they are not sufficient to explain long maturity prices in the UK. This is very much in line with similar conclusions in Rudebusch, Swanson and Wu (2006), who applied the macro-finance model by Rudebusch and Wu (2004) to a more recent sample: "*Low level of long-term bond yields in the U.S. during the 2004–2005 period does appear to be a conundrum when viewed through a macro-finance lens. Specifically, neither of the two macro-finance empirical models we consider is able to explain the recent low level of, or the fall in, long-term bond yields. This finding is remarkable given that both models fit the earlier long-term yield data quite well.*"

Indeed, a simple macro-finance model, like ours or that of Rudebusch and Wu (2004), does not consider any recent sector-specific features, such as increased demand for long maturity bonds from UK pension funds. Moreover, the model's assumption of no structural change represents another caveat with the framework. However, these two caveats would probably apply to most of the other available term structure models.

## 6 Conclusions

This paper attempts to identify fundamental economic shocks in an ATSM framework using SVAR. Further, we show how bond prices can respond to the fundamental shocks of the economy. As a simple example of the NA-SVAR approach, we have chosen a three macroeconomic factor model of the UK yield curve. We interpret fluctuations in the state vector as due to two types of shocks: shocks that have permanent effects on output and shocks that do not. The first type of shock is interpreted as a supply shock, whereas the other shocks are related to demand and monetary policy.

Under this interpretation we show that supply shocks affect the whole yield curve, while positive demand shocks increase mostly the long end of the yield curve on impact and thus increase the slope of the yield curve. Demand and supply shocks account for a large part of the time variation in bond yields. In particular, demand shocks explain more than 80% of the variation in the long end of the UK yield curve. Moreover, we find that the short end of the yield curve is driven mostly by expectations, while the long end of the curve is mostly driven by changes in risk premia.

As a next step, additional structural identified shocks could be employed to study the behavior of

the yield curve. This would be a promising direction for NA-SVAR models. In addition, since the NA-SVAR model incorporates nominal yields and inflation expectations, it has implications for the real yield curve. That could also be a fruitful avenue for future research.

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## Appendix 1. Bond Prices under No-Arbitrage

In this section we specify the recursive structure of the coefficients in the bond pricing equation. Putting together all assumptions made in Section 2.1, we get

$$
\begin{aligned}
p_t(n+1) &= E_t\{m_{t+1} + p_{t+1}(n)\} + \frac{1}{2}Var_t\{m_{t+1} + p_{t+1}(n)\} \\
&= E_t\left\{-R_{1,t} - \frac{\lambda_t'\lambda_t}{2} - \lambda_t'\varepsilon_{t+1} + A_n + B_n'X_{t+1}\right\} + \frac{1}{2}Var_t\{m_{t+1} + p_{t+1}(n)\} \\
&= -R_{1,t} - \frac{\lambda_t'\lambda_t}{2} + A_n + E_t[B_n'(\mu + \Phi X_t + \Sigma\varepsilon_{t+1})] + \frac{1}{2}Var_t\left\{-\lambda_t'\varepsilon_{t+1} + B_n'\Sigma\varepsilon_{t+1}\right\} \\
&= -\delta_0 - \delta_1'X_t - \frac{\lambda_t'\lambda_t}{2} + A_n + B_n'(\mu + \Phi X_t) + \frac{1}{2}Var_t\left\{\left(B_n'\Sigma - \lambda_t'\right)\varepsilon_{t+1}\right\} \\
&= -\delta_0 - \delta_1'X_t - \frac{\lambda_t'\lambda_t}{2} + A_n + B_n'(\mu + \Phi X_t) + \frac{\left(-\lambda_t' + B_n'\Sigma\right)\left(-\lambda_t' + B_n'\Sigma\right)'}{2} \\
&= \left(-\delta_0 + A_n + B_n'\mu + \frac{1}{2}B_n'\Sigma\Sigma'B_n - B_n'\Sigma\lambda_0\right) + \left(-\delta_1' - B_n'\Sigma\lambda_1 + B_n'\Phi\right)X_t
\end{aligned}
$$

We get $A_{n+1}$, $B_{n+1}$ as a solution of the system of difference equations with initial condition $A_1 = \delta_0,\ B_1 = -\delta_1'$ :

$$
\left\{
\begin{array}{l}
A_{n+1} = A_n + B_n'\mu - B_n'\Sigma\lambda_o + \frac{1}{2}B_n'\Sigma\Sigma'B_n - \delta_0 \\
B_{n+1} = (\Phi - \Sigma\lambda_1)'B_n - \delta_1'
\end{array}
\right\} \qquad \textbf{A1}
$$

Thus, the continuously compounded yield on a zero-coupon bond of maturity $n$, is an affine structure of the state:

$$
y_t(n) = -\frac{1}{n}(A_n + B\prime_n X_t) \equiv a_n + b_n'X_t \qquad \textbf{A2}
$$

## Appendix 2. Supply shock identification

In our empirical example, we identify supply shocks by constraining other disturbances to have a zero long-run effect on output. Technically, we impose long-run restrictions, assuming that supply shock $\varepsilon_t^S$ is the only shock that has a permanent effect on output. Formally, this exclusion constraint is specified by:

$$\lim_{j \longrightarrow \infty} \left[ E_t g_{t+j} - E_{t-1} g_{t+j} \right] = f(\varepsilon_t^S). \qquad \textbf{(B-1)}$$

Below I show under which conditions the LHS of **(B-1)** will depend only on supply shocks.

$$\lim_{j \longrightarrow \infty} \left[ E_t g_{t+j} - E_{t-1} g_{t+j} \right] = e1_{(1\times 3)} \cdot \left( (I_{(3\times 3)} - \tilde{\Phi})^{-1} \tilde{\Sigma} \varepsilon_t \right),$$

where $e1_{(1\times 3)} = [1\ 0\ 0]$ is a selection row vector. In terms of the state vector dynamics,

$$\begin{aligned}
E_t X_{t+j} &= \tilde{\Phi}\ X_{t+j-1} + \tilde{\Sigma}\varepsilon_t = \tilde{\Phi}(\tilde{\Phi} X_{t+j-2} + \tilde{\Sigma}\varepsilon_t) + \tilde{\Sigma}\varepsilon_t \\
&= \tilde{\Phi}^2 X_{t+j-2} + \tilde{\Phi}\tilde{\Sigma}\varepsilon_t + \tilde{\Sigma}\varepsilon_t = ... \\
&= \tilde{\Phi}^{j+1} X_{t+j-1} + \tilde{\Phi}^{j}\tilde{\Sigma}\varepsilon_t + \tilde{\Phi}^{j-1}\tilde{\Sigma}\varepsilon_t + ... + \tilde{\Sigma}\varepsilon_t; \\
E_{t-1} X_{t+j} &= \tilde{\Phi}^{j+1} X_{t+j-1},
\end{aligned}$$

implying that

$$\begin{aligned}
\lim_{j \longrightarrow \infty} \left[ E_t X_{t+j} - E_{t-1} X_{t+j} \right] &= \lim_{j \longrightarrow \infty} (\tilde{\Phi}^{j} + \tilde{\Phi}^{j-1} + ... + I_{(3\times 3)}) \tilde{\Sigma}\varepsilon_t \\
&= (I_{(3\times 3)} - \tilde{\Phi})^{-1} \tilde{\Sigma}\varepsilon_t,
\end{aligned}$$

Adding the selection vector into the last formula completes the proof.