Equilibrium Moment Restrictions on Asset Prices at Maturity

P Simmons &	N Tantisantiwong
Department of Economics	Department of Economics
University of York	University of York
York YO10 5DD	York YO10 5DD
UK	UK
email: ps1@york.ac.uk	email: nt522@york.ac.uk
tel: +44 1904 433763	tel:+44 1904 434670

December 8, 2009

Abstract

We model stochastic properties of equilibrium maturity prices which result from interaction between intertemporal traders closing out positions and noisy, price sensitive short term traders. The intertemporal traders can have arbitrary investment rules, preferences and information. We find a set of restrictions between second moments of equilibrium maturity prices. With two assets there is also a bound on the correlation between maturity prices.

JEL nos: D52; D82; G12

Keywords: risk averse investors, flow traders, asset prices, equilibrium moment restrictions

Equilibrium Moment Restrictions on Asset Prices At Maturity

A key question in finance is what determines the random properties of asset prices. There are different theories. Those based on CAPM or CCAPM with a one period horizon in which efficient investor behaviour leads to asset prices being linear functions of a single random variable (the market portfolio). The approach has been extended to allow for nonlinearities in the relation by including effects of skewness and kurtosis (Hung, 2008), multiperiod horizons in which the individual asset prices are not iid (Brennan and Cao, 1997; Adriaens and Melenberg, 2005), etc. But this type of literature takes the random properties of the underlying assets as an exogenous primitive and does not explain it in terms of fundamentals (Admati and Ross, 1985). Secondly there is the class of factor models in which there are common random variables driving asset prices. These random variables are either determined purely empirically (Fama and French, 1993; Chabi-Yo, 2008; Grauer, 1999) or via prior assumption. The factors can either be micro financial (Fama-French) or macroeconomic variables (Bekaert, Erb, Harvey and Viskanta, 1997). The factor models are more general than the CAPM/CCAPM - that is, there are no assumptions about the distribution of asset returns, investor preferences, and identification of market portfolio. A third class are speculative sunspot type models in which there is investor heterogeneity in information. Random information differences between investors are then translated into random processes for equilibrium asset prices. (Hillebrand and Wenzleburger, 2006)

Given the equilibrium asset price process implied by these models, properties of mean returns and also the relative volatility of asset returns or (in a multi-asset context) covariances between asset returns can be explored. In particular the volatility of exante and expost returns results in a variance bound. (Shiller, 1981; Engel, 2005; Bandi and Russell, 2006; Chabi-Yo, 2008). The basic approach is to use no arbitrage and rational expectations to derive representative investor equilibrium conditions. On the other hand, the market micro structure literature classifies traders into types (for example partially informed intertemporal traders, short term liquidity and noise traders) and uses properties of the market equilibrium conditions to explain how current prices aggregate information and sometimes also the time path of prices in the presence of processes for information flows over time (Lundholm, 2008; Kyle, 1989).

In this paper, we examine how risk averse investors with an intertemporal interest who are trading on the basis of their information interact with short term flow and noise traders to determine both means and second moments of equilibrium asset prices at maturity in a two-period multi-asset scenario. The intertemporal traders can follow any decision rules for investment and show heterogeneity in preferences or information. The flow traders trade according to current prices and we think of them as net consumers of assets. They may just be buying to improve liquidity, but in the case of physical assets such as commodities they may be marginal processors or consumers who enter/exit the market according to current price, or storage companies who are comparing current prices with storage costs. They may also just be noise traders acting randomly giving exogenous shocks to net asset demands at maturity; or equivalently in analytical effect, providing a channel through which fundamental net demand shocks impact on asset markets. With several assets, flow traders may be generalists who operate in several markets in response to the asset prices in all markets, or specialists who operate only in one market to an extent that depends on the price in that market. For example, a fruit packaging firm uses exactly the same equipment and labour to pack sultanas or apricots; depending on the current spot prices they may choose to package both or just one line of fruit. But a sugar beet processor has specialised plant and cannot diversify like this.

Our intention is to derive moments of market equilibrium asset prices at maturity in a general and unrestricted framework. That is, we do not assume rational expectations, do not specify the context of the assets (e.g. are they a spot and futures price? or equities and bonds?), do not even need expected utility and we allow for heterogeneity of preferences and information. Other work has looked at some properties of moments of equilibrium asset prices. For example some studies look at links between second and higher monets of prices (Hung, 2008). The variance bound literature (Shiller, 1981; Kleidon,1986) and Hansen-Jagannathan bounds (Ferson and Siegel, 2003) derives the relations between ex-ante and ex-post variances, or the second moment as a function of other moments. However, we apply a somewhat different approach allowing derivation of moment restrictions in a multi-asset case, the relations between variances and covariances of different asset prices, and find a lower bound for correlation of asset prices. The moment restrictions we find in this framework can then be applied to a variety of special contexts.

The plan of the paper is to give the basic assumptions and derive the equilibrium prices in section 1 for the multi-asset case. In section 2, we compute the first and second moments. The moment restriction for a two-asset case is derived as an example and is utilised to find the lower bound of correlation. Finally, we conclude in section 3. Algebraic derivations are in the appendix.

1 The Framework

There are two periods, t and $t+1$. A representative risk averse investor i can choose net amounts of m real assets ($X_{it}^T = (x_{1t}, x_{2t}, ..., x_{mt})$) to buy at time t (short sales are allowed and there are no restrictions on saving/borrowing in a risk free asset). He can hold the assets for one period and sell all of his assets back to markets in the period $t+1$. The risk free interest rate is r_t. For exposition purposes only suppose that the i^{th} trader has quadratic preferences[1]. He maximises his mean-variance expected utility:

$$\max[E_i p_{t+1}^T - p_t^T/\rho]X_{it} - (\tau_i/2)X_{it}^T \Omega_i X_{it}$$

where p_t is a column vector of m asset prices ($p_t^T = (p_{1t}, p_{2t}, ..., p_{mt})$) and $E_i p_{t+1}$ is a column vector of expected prices. ρ is the discount rate $1/(1+r_t)$. τ_i is the risk-return trade-off and the first and second moments of prices at maturity as perceived by this trader reflect his/her information set.

Maximising expected utility with respect to X_{it} yields the i^{th} investor's net demands for X_{it}

$$X_{it} = \frac{1}{\tau_i}\Omega_i^{-1}[E_i p_{t+1} - p_t/\rho] \tag{1}$$

As usual these comprise an expected capital gain term and a risk diversification term which depends on the i^{th} investors view of the covariance between the two asset prices at $t+1$. Whatever the decision rule that investors use, there will be a system like (1) in which X_{it} is independent of the actual realisation of p_{t+1} since investments are made at t.This is the only crucial point we need to derive our moments of maturity prices.

When the markets come to clear at $t+1$, there are flow traders in each market and also exogenous net demand (noise) shocks ε_{jt+1} in both markets. In general, the flow traders net demand depends on the prices of m assets at $t+1$ according to the functions $f_j(p_{t+1})$ in market j.

To simplify the solution, the flow trader's net demand is assumed to be linear, (for example this could be given by quadratic costs of production for a small asset processor). The flow trader can trade in any asset markets. The market-clearing conditions in these two asset markets at time $t+1$ are that the aggregate net supply of risk averse investors be equal to the net demand from flow traders in each market plus the net demand from noise traders, ε_{t+1},

$$\begin{aligned} \sum_i X_{it} &= f(p_{t+1}) + \varepsilon_{t+1} \qquad (2) \\ &= A p_{t+1} + \varepsilon_{t+1} \end{aligned}$$

[1] We regard this as an approximation to an underlying utility function in which τ reflects the coefficient of absolute risk aversion.

where A is a general nonsymmetric, nonsingular square matrix with a dimension $m \times m$.

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mm} \end{bmatrix}$$

In general A has nonpositive diagonal elements (maturity traders react negatively to price in a market) but off diagonal elements may have an arbitrary sign. We assume that $E_t(\varepsilon_{t+1}) = \mu_{t+1}$ which allows for either an intercept in the flow trader demand or a baseline deterministic amount of noise demand and that Σ is the covariance matrix of ε_{t+1}.

Solving for the equilibrium asset prices at $t+1$ gives

$$\begin{aligned} p_{t+1} &= A^{-1}[\sum_i X_{it} - \varepsilon_{t+1}] \\ &= Z - A^{-1}\varepsilon_{t+1} \end{aligned} \tag{3}$$

(3) also implies a relation between the maturity prices; For example partition A^{-1} as

$$A^{-1} = \begin{bmatrix} \alpha_{11} & \alpha_{1m} \\ \alpha_{m1}^T & \alpha_{mm} \end{bmatrix}$$

where α_{11} is $(m-1)$x$(m-1), \alpha_{mm}$ is a scalar and α_{1m}, α_{m1} are $(m-1)$x1 column vectors. Assume that α_{11} is nonsingular (basically that rank (A)=m) Similarly partition the column vectors

$$Z = A^{-1}\sum_i X_{it} = \begin{bmatrix} Z_{m-1} \\ z_m \end{bmatrix}$$

Solve from the first $m-1$ equations to get

$$\varepsilon_{m-1} = \alpha_{11}^{-1}Z_{m-1} - \alpha_{11}^{-1}\alpha_{1m}\varepsilon_m - \alpha_{11}^{-1}p_{m-1}$$

and replace this in the final equation of (3) to get asset prices at time $t+1$:

$$p_m = (z_m - \alpha_{m1}^T\alpha_{11}^{-1}Z_{m-1}) - (\alpha_{mm} - \alpha_{m1}^T\alpha_{11}^{-1}\alpha_{1m})\varepsilon_m + \alpha_{m1}^T\alpha_{11}^{-1}p_{m-1} \tag{4}$$

When the flow trader is a specialist in just one market and has net demand independent of the price in the other market, A is a diagonal matrix. In this case $\alpha_{m1}^T = 0, \alpha_{11}^{-1}$ is diagonal and (4) reduces to

$$p_m = z_m - a_{mm}^{-1}\varepsilon_m$$

and there is no systematic relation between the maturity prices in different markets.

2 The moments of maturity prices

In this section, we derive the expected values, variances and covariance of prices at t assuming that each investors beliefs at t about first and second moments at $t+1$ are exogenous and have no link to the equilibrium moments at $t+1$. First, we compute the mean prices from (3). Next we derive the second moments of prices. We show that there is no relationship between price volatilities when the shock in one market does not affect the price in another market. Finally we show that the sensitivity of the flow traders demand to both prices does lead to second moment restrictions.

2.1 First Moments

Taking the expected prices, variances and covariances of prices at $t+1$ as perceived by investor i at t to be exogenous, the mean equilibrium prices from (3) is

$$E(p_{t+1}) = A^{-1}[\sum_i X_{it} - \mu_{t+1}] \tag{5}$$

The mean equilibrium prices at maturity reflect the price expectations held in the previous period by the intertemporal traders and their risk preferences (via X_{it}) together with the sensitivity of flow traders to maturity prices.

2.2 Second Moments

It follows that the second moments of prices will be identical in the two models. Thus, the presence of risk free borrowing/saving possibilities has no impact on volatility at maturity. From (3), the covariance matrix of maturity prices is Λ

$$\Lambda = A^{-T}\Sigma A^{-1}$$

Partition the markets into the first $m-1$ and the last so that

$$\Lambda = \begin{bmatrix} \Lambda_{11} & \Lambda_{12} \\ \Lambda_{12}^T & \Lambda_{22} \end{bmatrix}$$

Λ_{11} is $(m-1)\times(m-1)$ and Λ_{12} is the column vector $(m-1)\times 1$ of the covariance between the price of the m^{th} asset and other asset prices. Λ_{22} is the variance of asset price p_m, so it is scalar. The covariance matrix of noise demand

$$\Sigma = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{12}^T & \Sigma_{22} \end{bmatrix}$$

where Σ_{11} is $(m-1)\times(m-1)$ and Σ_{12} is the column vector $(m-1)\times 1$ of the covariance between noise trading in the m^{th} asset market and that in other asset markets. Σ_{22} is the variance of noise trading in the m^{th} asset market, so it is scalar. The inverse of the matrix A is

$$A^{-1} = \begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21}^T & \alpha_{22} \end{bmatrix}_{mxm}$$

where α_{11} is $(m-1)\times(m-1)$; α_{12} and α_{21} are the column vectors $(m-1)\times 1$ of the flow trader's sensitivity to the price of asset m,and α_{22} is scalar. So

$$\begin{bmatrix} \Lambda_{11} & \Lambda_{12} \\ \Lambda_{12}^T & \Lambda_{22} \end{bmatrix} = \begin{bmatrix} \alpha_{11}^T & \alpha_{21} \\ \alpha_{12}^T & \alpha_{22} \end{bmatrix} \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{12}^T & \Sigma_{22} \end{bmatrix} \begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21}^T & \alpha_{22} \end{bmatrix}$$

Thus, the covariance matrix of maturity prices (see appendix) satisfies

$$\begin{aligned} \Lambda_{12} &= (I - \frac{\alpha_{21}\alpha_{12}^T\alpha_{11}^{-T}}{\alpha_{22}+\alpha_{21}^T\alpha_{11}^{-1}\alpha_{12}})\Lambda_{11}\alpha_{11}^{-1}\alpha_{12} + \frac{\alpha_{21}}{\alpha_{22}+\alpha_{21}^T\alpha_{11}^{-1}\alpha_{12}}\Lambda_{22} \\ &\quad +(\alpha_{11}^T - \frac{2\alpha_{21}\alpha_{12}^T}{\alpha_{22}+\alpha_{21}^T\alpha_{11}^{-1}\alpha_{12}})\Sigma_{12}(\alpha_{22}-\alpha_{21}^T\alpha_{11}^{-1}\alpha_{12}) \end{aligned} \tag{6}$$

(6) is a key result of this paper. It shows that there are $m-1$ linear restrictions between elements of the covariance matrix of maturity prices that subsequently we can specialise to suit a variety of contexts depending on the parameters in the matrix A, which are set behaviourally by the context in which the flow traders operate. Summarising

Proposition 1 *There are $m-1$ linear restrictions between the variances and covariances of equilibrium risky asset prices at maturity. The intercept involves the covariance between markets of the fundamental shocks.*

In the special case in which the trading shocks in the last market are uncorrelated with those of other markets $\Sigma_{12}=0$ and the set of linear restrictions is homogeneous.

To see the detail of these restrictions in terms of the impact of flow traders on markets take an example with only two risky assets; $m=2$. So

$$\Sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix}$$

is 2x2.(6) becomes

$$\lambda_{12} = (1-\frac{\alpha_{21}\alpha_{12}\alpha_{11}^{-1}}{\alpha_{22}+\alpha_{21}\alpha_{11}^{-1}\alpha_{12}})\lambda_{11}\alpha_{11}^{-1}\alpha_{12}+\frac{1}{\alpha_{22}+\alpha_{21}\alpha_{11}^{-1}\alpha_{12}}\alpha_{21}\lambda_{22} \tag{7}$$
$$+(\alpha_{22}-\alpha_{21}\alpha_{11}^{-1}\alpha_{12})(\alpha_{11}-\frac{2\alpha_{21}\alpha_{12}}{\alpha_{22}+\alpha_{21}\alpha_{11}^{-1}\alpha_{12}})\sigma_{12}$$

which can be written as

$$\lambda_{12} = \frac{\alpha_{22}\alpha_{12}}{\alpha_{11}\alpha_{22}+\alpha_{21}\alpha_{12}}\lambda_{11}+\frac{\alpha_{11}\alpha_{21}}{\alpha_{11}\alpha_{22}+\alpha_{21}\alpha_{12}}\lambda_{22} \tag{8}$$
$$+\frac{(\alpha_{11}\alpha_{22}-\alpha_{21}\alpha_{12})^2}{\alpha_{11}\alpha_{22}+\alpha_{21}\alpha_{12}}\sigma_{12}$$

Recall that the α_{ij} are elements of the inverse of A :

$$\alpha_{11} = \frac{a_{22}}{a_{11}a_{22}-a_{12}a_{21}}, \alpha_{22}=\frac{a_{11}}{a_{11}a_{22}-a_{12}a_{21}}, \alpha_{12}=\frac{-a_{12}}{a_{11}a_{22}-a_{12}a_{21}}$$
$$\alpha_{21} = \frac{-a_{21}}{a_{11}a_{22}-a_{12}a_{21}}$$

so (8) becomes

$$\lambda_{12} = -\frac{a_{11}a_{12}}{a_{22}a_{11}+a_{12}a_{21}}\lambda_{11}-\frac{a_{22}a_{21}}{a_{22}a_{11}+a_{12}a_{21}}\lambda_{22}+\frac{\sigma_{12}}{(a_{22}a_{11}+a_{12}a_{21})}$$

In the special case of uncorrelated fundamental shocks $\sigma_{12}=0$ and the restriction becomes

$$\lambda_{12} = -\frac{a_{11}a_{12}}{a_{11}a_{22}+a_{12}a_{21}}\lambda_{11}-\frac{a_{21}a_{22}}{a_{11}a_{22}+a_{12}a_{21}}\lambda_{22} \tag{9}$$

Recall that $a_{jj}<0$ but a_{jk} is of ambiguous sign. If $a_{jk}<0$ which could occur with strong complementarities for flow traders then $\lambda_{12}>0$ but in the more likely case $a_{jk}>0$ and then the covariance of maturity prices could be negative.

There is also an implicit lower bound on the absolute value of the correlation $|\rho|$ between the maturity prices when $\sigma_{12}=0$. From (9)

$$\lambda_{12}^2 = \left(-\frac{a_{11}a_{12}}{a_{11}a_{22}+a_{12}a_{21}}\lambda_{11}^2-\frac{a_{21}a_{22}}{a_{11}a_{22}+a_{12}a_{21}}\lambda_{22}^2\right)^2 \tag{10}$$

$$\lambda_{12}^2 = \frac{a_{11}^2a_{12}^2}{(a_{11}a_{22}+a_{12}a_{21})^2}\lambda_{11}^4+\frac{a_{22}^2a_{21}^2}{(a_{11}a_{22}+a_{12}a_{21})^2}\lambda_{22}^4+2\frac{a_{11}a_{22}a_{12}a_{21}}{(a_{11}a_{22}+a_{12}a_{21})^2}\lambda_{11}^2\lambda_{22}^2$$

$$\rho^2 = \lambda_{12}^2/\lambda_{11}^2\lambda_{22}^2$$
$$= \frac{a_{11}^2a_{12}^2}{(a_{11}a_{22}+a_{12}a_{21})^2}\frac{\lambda_{11}^2}{\lambda_{22}^2}+\frac{a_{22}^2a_{21}^2}{(a_{11}a_{22}+a_{12}a_{21})^2}\frac{\lambda_{22}^2}{\lambda_{11}^2}+2\frac{a_{11}a_{22}a_{12}a_{21}}{(a_{11}a_{22}+a_{12}a_{21})^2}$$

But since

$$2\frac{a_{11}a_{22}a_{12}a_{21}}{\left(a_{11}a_{22}+a_{12}a_{21}\right)^2}>0$$

no matter what the sign of a_{jk} is,

$$\rho^2>\frac{a_{11}^2a_{12}^2}{\left(a_{11}a_{22}+a_{12}a_{21}\right)^2}\frac{\lambda_{11}^2}{\lambda_{22}^2}+\frac{a_{22}^2a_{21}^2}{\left(a_{11}a_{22}+a_{12}a_{21}\right)^2}\frac{\lambda_{22}^2}{\lambda_{11}^2}$$

The squared correlation between maturity prices is bounded below by the right hand side.

Another special case arises when the flow traders are specialists responding to the maturity price in only one market ($a_{jk}=0$) in which case

$$\lambda_{11}^2=\frac{\sigma_{11}}{a_{11}^2},\lambda_{22}^2=\frac{\sigma_{22}}{a_{22}^2},\lambda_{12}=\frac{\sigma_{12}}{a_{11}a_{22}}$$

and there is then no second moment restriction. The magnitude of liquidity shocks totally determines the volatility of maturity prices. Summarising

Proposition 2 *An implicit lower bound on the absolute value of the correlation $|\rho|$ between the maturity prices is found when $\sigma_{12}=0$ and flow traders are generalists. The bound of correlation depends on the variance of asset prices and flow traders' sensitivities to both asset prices.*

3 Conclusions

Existing asset pricing theories focus on representative investor behaviour to derive the first moment or bounds of asset prices by appealing to no arbitrage and efficient pricing considerations. At the same time these restrictions are often empirically rejected. Here, we have taken a framework in which equilibrium asset prices at maturity reflect the interaction between intertemporal traders who are closing out positions and short term flow and noise traders.

Our main result is that so long as the short term traders react to asset prices in more than one market there is a set of linear restrictions on the covariance matrix of the equilibrium asset prices. These restrictions are testable just from asset price data. The restrictions are robust in that they are independent of the investment decision rules used by the intertemporal traders and also hold with or without short selling. We also find the lower bound of correlation between two asset prices.

One interesting extension would be to apply rational expectations to the intertemporal traders. That is these traders would base their initial positions on expectations of maturity prices, the earlier information and risk preferences, short term traders partly reflect noise.

The setting can be applied to derivative markets, especially futures markets in which intertemporal traders are closing out positions close to maturity. It can also be applied to prices of any related assets, commodities and industries such as bond and equity prices, bio-fuel commodity and oil prices, prices of house and construction materials.

A Appendix

A.1 The second moments of prices

The variance-covariance matrix of m asset prices:

$$\begin{bmatrix} \Lambda_{11} & \Lambda_{12} \\ \Lambda_{12}^T & \Lambda_{22} \end{bmatrix} = \begin{bmatrix} \alpha_{11}^T & \alpha_{21} \\ \alpha_{12}^T & \alpha_{22} \end{bmatrix} \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{12}^T & \Sigma_{22} \end{bmatrix} \begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21}^T & \alpha_{22} \end{bmatrix}$$

$$= \begin{bmatrix} \left(\alpha_{11}^T\Sigma_{11} + \alpha_{21}\Sigma_{12}^T\right)\alpha_{11} + \left(\alpha_{11}^T\Sigma_{12} + \alpha_{21}\Sigma_{22}\right)\alpha_{21}^T & \left(\alpha_{11}^T\Sigma_{11} + \alpha_{21}\Sigma_{12}^T\right)\alpha_{12} + \left(\alpha_{11}^T\Sigma_{12} + \alpha_{21}\Sigma_{22}\right)\alpha_{22} \\ \left(\alpha_{12}^T\Sigma_{11} + \alpha_{22}\Sigma_{12}^T\right)\alpha_{11} + \left(\alpha_{12}^T\Sigma_{12} + \alpha_{22}^T\Sigma_{22}\right)\alpha_{21}^T & \left(\alpha_{12}^T\Sigma_{11} + \alpha_{22}^T\Sigma_{12}^T\right)\alpha_{12} + \left(\alpha_{12}^T\Sigma_{12} + \alpha_{22}^T\Sigma_{22}\right)\alpha_{22} \end{bmatrix}$$

from which

$$\Lambda_{11} = \alpha_{11}^T\Sigma_{11}\alpha_{11} + \alpha_{21}\Sigma_{12}^T\alpha_{11} + \alpha_{11}^T\Sigma_{12}\alpha_{21}^T + \alpha_{21}\Sigma_{22}\alpha_{21}^T \tag{11}$$

$$\Lambda_{22} = \alpha_{12}^T\Sigma_{11}\alpha_{12} + \alpha_{22}\Sigma_{12}^T\alpha_{12} + \alpha_{12}^T\Sigma_{12}\alpha_{22} + \alpha_{22}\Sigma_{22}\alpha_{22} \tag{12}$$

$$\Lambda_{12} = \alpha_{11}^T\Sigma_{11}\alpha_{12} + \alpha_{21}\Sigma_{12}^T\alpha_{12} + \alpha_{11}^T\Sigma_{12}\alpha_{22} + \alpha_{21}\Sigma_{22}\alpha_{22} \tag{13}$$

From first equation (11),

$$\begin{aligned} \Sigma_{11} &= \alpha_{11}^{-T}\Lambda_{11}\alpha_{11}^{-1} - \alpha_{11}^{-T}\alpha_{21}\Sigma_{12}^T\alpha_{11}\alpha_{11}^{-1} - \alpha_{11}^{-T}\alpha_{11}^T\Sigma_{12}\alpha_{21}^T\alpha_{11}^{-1} - \alpha_{11}^{-T}\alpha_{21}\Sigma_{22}\alpha_{21}^T\alpha_{11}^{-1} \\ &= \alpha_{11}^{-T}\Lambda_{11}\alpha_{11}^{-1} - \alpha_{11}^{-T}\alpha_{21}\Sigma_{12}^T - \Sigma_{12}\alpha_{21}^T\alpha_{11}^{-1} - \alpha_{11}^{-T}\alpha_{21}\Sigma_{22}\alpha_{21}^T\alpha_{11}^{-1} \end{aligned}$$

Substitute this in second equation (12) and remember Σ_{22} is a scalar

$$\begin{aligned} \Lambda_{22} &= \alpha_{12}^T\alpha_{11}^{-T}\Lambda_{11}\alpha_{11}^{-1}\alpha_{12} - \alpha_{12}^T\alpha_{11}^{-T}\alpha_{21}\Sigma_{12}^T\alpha_{12} - \alpha_{12}^T\Sigma_{12}\alpha_{21}^T\alpha_{11}^{-1}\alpha_{12} - \alpha_{12}^T\alpha_{11}^{-T}\alpha_{21}\Sigma_{22}\alpha_{21}^T\alpha_{11}^{-1}\alpha_{12} \\ &\quad + \alpha_{22}\Sigma_{12}^T\alpha_{12} + \alpha_{12}^T\Sigma_{12}\alpha_{22} + \alpha_{22}\Sigma_{22}\alpha_{22} \\ &= \alpha_{12}^T\alpha_{11}^{-T}\Lambda_{11}\alpha_{11}^{-1}\alpha_{12} - \alpha_{12}^T\alpha_{11}^{-T}\alpha_{21}\Sigma_{12}^T\alpha_{12} - \alpha_{12}^T\Sigma_{12}\alpha_{21}^T\alpha_{11}^{-1}\alpha_{12} + \alpha_{22}\Sigma_{12}^T\alpha_{12} + \alpha_{12}^T\Sigma_{12}\alpha_{22} \\ &\quad + (\alpha_{22}\alpha_{22} - \alpha_{12}^T\alpha_{11}^{-T}\alpha_{21}\alpha_{21}^T\alpha_{11}^{-1}\alpha_{12})\Sigma_{22} \\ \Sigma_{22} &= \left[\Lambda_{22} - \alpha_{12}^T\alpha_{11}^{-T}\Lambda_{11}\alpha_{11}^{-1}\alpha_{12} + \alpha_{12}^T\alpha_{11}^{-T}\alpha_{21}\Sigma_{12}^T\alpha_{12} + \alpha_{12}^T\Sigma_{12}\alpha_{21}^T\alpha_{11}^{-1}\alpha_{12} - \alpha_{22}\Sigma_{12}^T\alpha_{12} - \alpha_{12}^T\Sigma_{12}\alpha_{22}\right] / \\ &\quad (\alpha_{22}\alpha_{22} - \alpha_{12}^T\alpha_{11}^{-T}\alpha_{21}\alpha_{21}^T\alpha_{11}^{-1}\alpha_{12}) \end{aligned}$$

where $\alpha_{22}\alpha_{22} - \alpha_{12}^T\alpha_{11}^{-T}\alpha_{21}\alpha_{21}^T\alpha_{11}^{-1}\alpha_{12}$ is a scalar. Substitute Σ_{11} and Σ_{22} into the third equation, (13)

$$\begin{aligned} \Lambda_{12} &= \alpha_{11}^T\left(\alpha_{11}^{-T}\Lambda_{11}\alpha_{11}^{-1} - \alpha_{11}^{-T}\alpha_{21}\Sigma_{12}^T - \Sigma_{12}\alpha_{21}^T\alpha_{11}^{-1} - \alpha_{11}^{-T}\alpha_{21}\Sigma_{22}\alpha_{21}^T\alpha_{11}^{-1}\right)\alpha_{12} \\ &\quad + \alpha_{21}\Sigma_{12}^T\alpha_{12} + \alpha_{11}^T\Sigma_{12}\alpha_{22} + \alpha_{21}\Sigma_{22}\alpha_{22} \\ &= \Lambda_{11}\alpha_{11}^{-1}\alpha_{12} - \alpha_{21}\Sigma_{12}^T\alpha_{12} - \alpha_{11}^T\Sigma_{12}\alpha_{21}^T\alpha_{11}^{-1}\alpha_{12} + \alpha_{21}\Sigma_{12}^T\alpha_{12} + \alpha_{11}^T\Sigma_{12}\alpha_{22} \\ &\quad + \frac{\alpha_{21}(\alpha_{22} - \alpha_{21}^T\alpha_{11}^{-1}\alpha_{12})}{(\alpha_{22}^2 - \alpha_{12}^T\alpha_{11}^{-T}\alpha_{21}\alpha_{21}^T\alpha_{11}^{-1}\alpha_{12})} \\ &\quad \left[\Lambda_{22} - \alpha_{12}^T\alpha_{11}^{-T}\Lambda_{11}\alpha_{11}^{-1}\alpha_{12} + \alpha_{12}^T\alpha_{11}^{-T}\alpha_{21}\Sigma_{12}^T\alpha_{12} + \alpha_{12}^T\Sigma_{12}\alpha_{21}^T\alpha_{11}^{-1}\alpha_{12} - \alpha_{22}(\Sigma_{12}^T\alpha_{12} + \alpha_{12}^T\Sigma_{12})\right] \\ &= \Lambda_{11}\alpha_{11}^{-1}\alpha_{12} - \alpha_{11}^T\Sigma_{12}\alpha_{21}^T\alpha_{11}^{-1}\alpha_{12} + \alpha_{11}^T\Sigma_{12}\alpha_{22} + \frac{\alpha_{21}(\alpha_{22} - \alpha_{21}^T\alpha_{11}^{-1}\alpha_{12})}{(\alpha_{22}^2 - \alpha_{12}^T\alpha_{11}^{-T}\alpha_{21}\alpha_{21}^T\alpha_{11}^{-1}\alpha_{12})} \\ &\quad \left[\Lambda_{22} - \alpha_{12}^T\alpha_{11}^{-T}\Lambda_{11}\alpha_{11}^{-1}\alpha_{12} + \alpha_{12}^T\Sigma_{12}(\alpha_{12}^T\alpha_{11}^{-T}\alpha_{21} + \alpha_{21}^T\alpha_{11}^{-1}\alpha_{12}) - 2\alpha_{22}\alpha_{12}^T\Sigma_{12}\right] \end{aligned}$$

Note $\Sigma_{12}^T\alpha_{12} = \alpha_{12}^T\Sigma_{12}$, $\alpha_{12}^T\alpha_{11}^{-T}\alpha_{21} = \alpha_{21}^T\alpha_{11}^{-1}\alpha_{12}$ and these terms are scalars.

$$\begin{aligned}
\Lambda_{12} &= \Lambda_{11}\alpha_{11}^{-1}\alpha_{12} + \frac{\alpha_{21}}{(\alpha_{22}+\alpha_{21}^T\alpha_{11}^{-1}\alpha_{12})}(\Lambda_{22} - \alpha_{12}^T\alpha_{11}^{-T}\Lambda_{11}\alpha_{11}^{-1}\alpha_{12}) \\
&\quad +(\alpha_{22} - \alpha_{21}^T\alpha_{11}^{-1}\alpha_{12})(\alpha_{11}^T\Sigma_{12} - \frac{\alpha_{21}}{(\alpha_{22}+\alpha_{21}^T\alpha_{11}^{-1}\alpha_{12})}(\Sigma_{12}^T\alpha_{12} + \alpha_{12}^T\Sigma_{12})) \\
&= (I - \frac{a_{21}\alpha_{12}^T\alpha_{11}^{-T}}{(\alpha_{22}+\alpha_{21}^T\alpha_{11}^{-1}\alpha_{12})})\Lambda_{11}\alpha_{11}^{-1}\alpha_{12} + \frac{\alpha_{21}}{(\alpha_{22}+\alpha_{21}^T\alpha_{11}^{-1}\alpha_{12})}\Lambda_{22}) \\
&\quad +(\alpha_{11}^T - \frac{2\alpha_{21}\alpha_{12}^T}{(\alpha_{22}+\alpha_{21}^T\alpha_{11}^{-1}\alpha_{12})})\Sigma_{12}(\alpha_{22} - \alpha_{21}^T\alpha_{11}^{-1}\alpha_{12})
\end{aligned}$$

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