Jevons, Jenkin, and Walras on demand-and-supply analysis in the theory of exchange

Franco Donzelli
Department of Economics, Business and Statistics
University of Milano

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Abstract

In his economic writings Jevons insists on the allegedly fundamental role played by the so-called "laws of supply and demand" in his theory of exchange; yet no demand-and-supply analysis is actually employed in deriving such theory, as developed in Chapter 4 of *The Theory of Political Economy* (*TPE*). This is all the more puzzling in the light of the following two facts: 1) in his 1868 correspondence with Jevons, Fleeming Jenkin provides a complete geometrical solution of the exchange equilibrium problem based on the use of demand and supply curves, but his suggestion is wholly neglected by Jevons in the first edition of *TPE* (1871); 2) in his 1874 open letter to Jevons, Walras explicitly criticizes his correspondent for his defective treatment of the "laws of supply and demand", suggesting an alternative analytical solution of the exchange problem based on the use of demand and supply functions; yet Jevons entirely disregards Walras's remarks in preparing the second edition of *TPE* (1879). This paper compares Jevons's, Jenkin's and Walras's approaches to the exchange equilibrium problem, explaining the analytical and epistemological reasons that underlie Jevons's neglect of his correspondents' criticism and advice.

Keywords: Jevons, Jenkin, Walras, equilibrium, disequilibrium, demand and supply, theory of exchange

JEL codes: B13, B16, B31, C62, D41, D50

1 Introduction

After sketching his main ideas in a few pages of a short paper publicly delivered in 1862 and eventually published in full in 1866, W. Stanley Jevons extensively develops his theory of exchange in Chapter 4 of his path-breaking book, *The Theory of Political Economy* (henceforth *TPE*), of which the first edition appears in 1871 and the second one, the last during the author's lifetime, in 1879.

One of the most surprising features of Chapter 4 of *TPE* is that, in spite of the reiterated emphasis laid by Jevons on the allegedly fundamental role played by the so-called "laws of supply and demand" in his theory of exchange, no formal demand-and-supply analysis is actually employed by the author in deriving such theory nor, in spite of what Jevons himself occasionally claims[1], can be deduced from the formal statement of the theory, as can be found in *TPE*. This peculiar aspect of Jevons's approach, repeatedly noticed in the literature, has prompted a number of ingenious explanations, among which the one put forward by White (1989), resting on epistemological grounds, is especially notable[2]. It does not seem, however, that the deepest analytical reasons underlying Jevons's modelling choices have yet been spelled out in the literature. In the present paper we intend to fill this gap by exploring the hypothesis that the startling lack of any demand- and supply analysis in Jevons's theory of exchange may have to do with some basic difficulties surrounding the interpretation and formalization of the equilibrium and disequilibrium concepts that the so-called "marginalistic" approach, started precisely by Jevons with his writings of the mid-1860s and early 1870s, is bound to employ in trying to explain the traders' interactions, choices and behavior.

With a view to unraveling this entangled issue, it is convenient to compare Jevons's theory of exchange, the only part of his overall theoretical system on which we intend to focus attention in this paper, with the almost contemporary theory of exchange developed by Léon Walras in his first two

[1] See, e.g., what Jevons writes in the Section of Chapter IV entitled "Illustrations of the Theory of Exchange":

> The ordinary laws of supply and demand, when properly stated, are the practical manifestation of the theory. (Jevons, 1970, p. 148))

[2] See also White (1994), (2001), and (2004).

mémoires, published in 1874 and 1876, respectively, as well as in Section II of the *Eléments d'économie politique pure* (henceforth *Eléments*), published in two installments in 1874 and 1877, where the solution of the exchange problem crucially rests on a fully-fledged demand-and-supply analysis of the traders' choices and behavior. As is well-known, since 1874 Walras repeatedly underlines, both in private correspondence with Jevons and in public contributions to journals, books and collections of papers, that there exist a few important differences between his own approach and Jevons's, differences that have somehow to do with the absence of a true and proper demand-and-supply analysis in the latter's approach. In view of this, it may be interesting to explain why Jevons neither reacts to Walras's critical remarks, nor tries to revise his own account of the theory of exchange in the second edition of *TPE* in such a way as to meet at least some of Walras's criticisms and implicit or explicit suggestions. But, perhaps, it is even more interesting to explain why Jevons does not apparently make any effort to incorporate into the first edition of *TPE* at least some traits of the almost complete demand-and-supply analysis that the engineer-economist Fleeming Jenkin is able to put forward, in an impressive anticipation of a few distinctive features of Walras's later approach, in a private exchange of correspondence with Jevons taking place in 1868, i.e., well in advance of the date (1870) in which Jevons will start to put in writing his first draft of *TPE*.

The paper will be structured as follows. In Section 2 we shall discuss the conceptual apparatus underlying Jevons's theory of exchange, as put forward in Chapter IV of *TPE*. Section 3 will be devoted to a formal statement of such theory. The chief conceptual and analytical questions left unsettled by Jevons's theory of exchange will be summarized in Section 4. In Section 5 we shall state and discuss Walras's theory of exchange, as put forward in his first two *mémoires* and in the *Eléments*, especially focusing on the role played by demand-and-supply analysis in the development of the theory. Section 6 will examine Walras's critique of Jevons's theory of exchange. Section 7 will contrast Jenkin's stance on the theory of exchange with Jevons's. In Section 8 we shall compare Jenkin's, Jevons's, and Walras's respective positions on the role of demand-and-supply analysis in the theory of exchange. Section 9 concludes.

2 Jevons's theory of exchange: the conceptual apparatus

In order to facilitate both the exposition and the discussion of Jevons's theory of exchange, our immediate objective in this and the following twoSections, and especially its comparison with Walras's and Jenkin's approaches, a task that will be undertaken in Sections 4 to 8 below, it is convenient to adopt a common symbolic notation, conforming to contemporary standards. Hence, both here and in the following, we shall stick to the formalism to be introduced now. When necessary, and without further notice, this formalism will replace, even in textual quotations, the one originally employed by the authors concerned, often unacceptable nowadays; on no occasion, however, such replacement will entail any substantive alteration of the original assumptions or propositions.

Let us then consider a pure-exchange economy with a finite number $L \geq 2$ of commodities, denoted by $l = 1, 2, ..., L$, and a finite number $I \geq 2$ of consumers-traders (henceforth indifferently referred to as either consumers or traders), denoted by $i = 1, 2, ..., I$. Each consumer i is characterized by a consumption set $X_i = \{x_i \equiv (x_{1i}, x_{2i}, ..., x_{Li})\} = \mathbb{R}^L_+$, a utility function $u_i : X_i \to \mathbb{R}$, and endowments $\omega_i \equiv (\omega_{1i}, \omega_{2i}, ..., \omega_{Li}) \in \mathbb{R}^L_+ \backslash \{0\}$. Let $x = (x_1, ..., x_I) \in X = \times_{i=1}^I X_i \subset \mathbb{R}^{LI}_+$ be an allocation; $\bar{\omega} = (\bar{\omega}_1, \bar{\omega}_2, ..., \bar{\omega}_L) = \sum_{i=1}^I \omega_i \in \mathbb{R}^L_{++}$ the aggregate endowments; $A^{L\times I} = \left\{ x \in X \mid \sum_{i=1}^I x_i = \bar{\omega} \right\}$ the set of feasible, non-wasteful allocations. A pure-exchange economy with these characteristics will be denoted by $\mathcal{E}^{L\times I} = \left\{ (X_i, u_i(\cdot), \omega_i)_{i=1}^I \right\}$ in the following. When $L = 2$ and $I = 2$, the pure-exchange, two-commodity, two-consumer economy $\mathcal{E}^{2\times 2} = \left\{ \left(\mathbb{R}^2_+, u_i(\cdot), \omega_i\right)_{i=1}^2 \right\}$ will be called an Edgeworth Box economy, after Edgeworth (1881).

In the first thirteen Sections of Chapter IV of *TPE*, and especially in the two central analytical Sections, respectively entitled "The Theory of Exchange" and "Symbolic Statement of the Theory", Jevons discusses what is to all analytical purposes an Edgeworth Box Economy, $\mathcal{E}_J^{2\times 2} = \left\{ \left(\mathbb{R}^2_+, u_i(\cdot), \omega_i\right)_{i=1}^2 \right\}$, satisfying a few further specific assumptions concerning the traders' endowments and utility functions. As to the endowments, both traders are assumed to be "cornered", that is, to hold a positive quantity of one commodity only; specifically, in the following we shall assume: $\omega_1 = (\bar{\omega}_1, 0)$ and $\omega_2 = (0, \bar{\omega}_2)$. As to the utility functions, each consumer i is supposed to be characterized by a cardinal, additively separable utility function, that is:

$$u_i(x_i) = v_{1i}(x_{1i}) + v_{2i}(x_{2i}), \forall x_i \in X_i, \quad i = 1, 2.$$

Moreover, the utility functions are (tacitly) assumed in this part of Chapter IV to be twice continuously differentiable and to satisfy the following restrictions on the signs of the first- and second-order pure partial derivatives:

$$\nabla u_i(x_i) = \left(\frac{\partial u_i\,(x_i)}{\partial x_{1i}}, \frac{\partial u_i\,(x_i)}{\partial x_{2i}} \right) = (v'_{1i}(x_{1i}), v'_{2i}(x_{2i})) >> 0, \forall x_i \in X_i, \quad i = 1, 2$$

and

$$\left(\frac{\partial^2 u_i\,(x_i)}{\partial x_{1i}^2}, \frac{\partial^2 u_i\,(x_i)}{\partial x_{2i}^2} \right) = (v''_{1i}(x_{1i}), v''_{2i}(x_{2i})) << 0, \forall x_i \in X_i, \quad i = 1, 2.$$

Given the additive separability assumption, the first-order partial derivative of i's utility function with respect to commodity l, representing the marginal utility function (Jevons's "degree of utility" function) of commodity l for consumer i, turns out to be a function of the quantity of commodity l only. The above inequalities therefore imply that consumer i's marginal utility functions are positive and monotonically decreasing throughout, for $i = 1, 2$ and $l = 1, 2$.

Let us now consider an "act of exchange" taking place between the two traders. Such an "act" involves the exchange of either "infinitely small" or "finite" quantities of the two commodities (Jevons, 1970, pp. 138-9): it will be called 'differential' in the former case and 'finite' in the latter. In either case the quantity of the commodity given in exchange will be taken to be negative (that is, $dx_{li} < 0$ or $\Delta x_{li} < 0$, if commodity l is given by trader i, for $i, l = 1, 2$), while the quantity of the commodity received in exchange will be taken to be positive (that is, $dx_{li} > 0$ or $\Delta x_{li} > 0$, if commodity l is received by trader i, for $i, l = 1, 2$). Since an "act of exchange" is necessarily bilateral, the vectors of the quantities traded satisfy the following conditions: $(dx_{11}, dx_{21}) = -(dx_{12}, dx_{22})$, if the "act" is 'differential'; $(\Delta x_{11}, \Delta x_{21}) = -(\Delta x_{12}, \Delta x_{22})$, if the "act" is 'finite'[3]. Finally, let $dx_l =$

[3] Referring to J. S. Mill's analysis of demand and supply, Jevons (1970, p. 143) writes:

> Mill's equation [...] states that the quantity of commodity given by A is equal to the quantity received by B. This seems at first sight to be a mere

$|dx_{li}|$ (resp., $\Delta x_l = |\Delta x_{li}|$) be the absolute value of the quantity exchanged of commodity l in a 'differential' (resp., 'finite') "act of exchange"; then, following Jevons (1970, pp. 138), a ratio of the type $\frac{dx_2}{dx_1} = \left|\frac{dx_{21}}{dx_{11}}\right| = \left|\frac{dx_{22}}{dx_{12}}\right| > 0$ (resp., $\frac{\Delta x_2}{\Delta x_1} = \left|\frac{\Delta x_{21}}{\Delta x_{11}}\right| = \left|\frac{\Delta x_{22}}{\Delta x_{12}}\right| > 0$) will be called a 'differential' (resp., 'finite') "ratio of exchange". As we shall see, a special kind of 'finite' "ratio of exchange" plays a fundamental role in Jevons's theory of exchange: it is the 'finite' "ratio" $\frac{x_2}{x_1}$, where $x_1 = -\Delta x_{11} = -(x_{11} - \omega_{11}) = \bar{\omega}_1 - x_{11} = \Delta x_{12} = x_{12} - \omega_{11} = x_{12}$ and $x_2 = -\Delta x_{22} = -(x_{22} - \omega_{22}) = \bar{\omega}_2 - x_{22} = \Delta x_{21} = x_{21} - \omega_{21} = x_{21}$.

Given an Edgeworth Box economy $\mathcal{E}_J^{2\times 2} = \left\{ \left(\mathbb{R}_+^2, u_i(\cdot), \omega_i\right)_{i=1}^2 \right\}$, satisfying the above assumptions on endowments and utilities, for each consumer one can define a concept that will prove of some use in the following. Precisely, let consumer i's marginal rate of substitution of commodity 2 for commodity 1 when i's consumption is x_i, $MRS_{21}^i(x_i)$, be defined as the quantity of commodity 2 that consumer i would be willing to exchange for one unit of commodity 1 at the margin, in order to keep his utility unchanged at the original level $u_i(x_i)$. From this definition it follows that:

$$MRS_{21}^i(x_i) \equiv \left|\frac{dx_{2i}}{dx_{1i}}\right|_{u_i(x_i+dx_i)=u_i(x_i)} = \frac{\frac{\partial u_i(x_i)}{\partial x_{1i}}}{\frac{\partial u_i(x_i)}{\partial x_{2i}}} = \frac{v_{1i}'(x_{1i})}{v_{2i}'(x_{2i})},\ i = 1, 2.$$

In his writings Jevons ignores the notion of the marginal rate of substitution. Yet, he does know and systematically employ the notion of the marginal utility of commodity l for consumer i, which, under the stated assumptions on the properties of the utility functions, is well-defined and bounded away from 0 everywhere in the consumption set. Moreover, though not explicitly discussing the concept of the marginal rate of substitution as such, he does implicitly make use of it in his analysis, since he computes the ratios of the values of the marginal utility functions of either consumer corresponding to specific consumption bundles and examines the role of such ratios in solving the exchange equilibrium problem.

In the following seven Sections of Chapter IV, starting from that entitled "Problems in the Theory of Exchange" (Jevons, 1970, p. 150), Jevons tries to extend the scope of his analysis in various directions, by relaxing

truism, for this equality must necessarily exist if any exchange takes place at all.

both the assumptions defining the Edgeworth Box economy, to which his attention is exclusively confined in the initial part of Chapter IV, and some of the special assumptions concerning the traders' characteristics (consumption sets and utility functions), which further constrain his discussion in the same part of that Chapter: in particular, in the Sections entitled "Complex Cases in the Theory", "Competition in Exchange" and "Acquired Utility of Commodities" (Jevons, 1970, pp. 152-5 and 168-70), he examines exchange economies with more than two traders and/or more than two commodities; in the Sections entitled "Problems in the Theory of Exchange", "Failures in the Equations of Exchange" and "Negative and Zero Value" (Jevons, 1970, pp. 150-2 and 155-66), he drops or relaxes either the assumption that commodities be perfectly divisible, or the assumptions that the utility functions be continuous or differentiable, or finally the assumptions on the signs of the derivatives of the utility functions. While at the end of the next Section of this paper we shall briefly discuss some of Jevons's attempts at generalizing his approach, up until then we shall confine our attention to the core of Jevons's theory of exchange, as it emerges from the first thirteen Sections of Chapter IV of *TPE*.

As already explained, the formal model employed in the initial part of Chapter IV unambiguously refers to an Edgeworth Box economy, exhibiting the special features mentioned above. Yet, the verbal interpretation of the model, as put forward by Jevons in the three Sections that immediately precede the analytical statement of the theory, i.e., in the Sections entitled "Definition of Market", "Definition of Trading Body", and "The Law of Indifference" (Jevons, 1970, pp. 132-9), is not free of major ambiguities. At least three of them, significantly affecting Jevons's modeling choices, must be discussed before embarking upon an examination of the model proper: the first ambiguity concerns the very notion of a trader or, to use Jevons's expression, of a "trading body"; the second has to do with time structure of the analysis, or the distinction between "statics" and "dynamics"; the third concerns what Jevons calls, in the second edition of *TPE*, the "law of indifference" (in the first edition the same "law" had been labeled as the "principle of uniformity").

As to the "trading body" concept, Jevons provides the following elusive definition:

> By a *trading body* I mean, in the most general manner, any body either of buyers or sellers. The trading body may be a single

> individual in one case; it may be the whole inhabitants of a continent in another; it may be the individuals of a trade diffused in a country in a third. (Jevons, 1970, p. 135)

So Jevons's "trading body" can be either an individual decision maker, in conformity with the standard contemporary interpretation of the concept of a trader, or an aggregate of individuals. But, at least at first, also the aggregative interpretation of the concept is apparently justified by Jevons on individualistic grounds. For, a few lines after the previous sentence, he writes:

> We must use the expression with this wide meaning, because the principles of exchange are the same in nature, however wide or narrow may be the market considered. Every trading body is either an individual or an aggregate of individuals, and the law, in the case of the aggregate, must depend on the fulfilment of law in the individuals. (Jevons, 1970, p. 135)

Yet, not even this sort of individualistic justification is entirely satisfactory, after all, for

> the economic laws representing the conduct of large aggregates of individuals will never represent exactly the conduct of any one individual. If we could imagine that there were a thousand individuals all exactly alike [...], then the average laws of supply and demand deduced from the conduct of such individuals would agree with the conduct of any one individual. But a community is composed of persons differing widely in their powers, wants, habits and possessions. In such circumstances the average laws applying to them will come under what I have elsewhere called the 'Fictitious Mean', that is to say, they are numerical results which do not pretend to represent the character of any existing thing. (Jevons, 1970, p. 136; see also p. 86)

As can be seen, when the "trading body" concept is interpreted as an "aggregate of individuals", Jevons wavers between two alternative positions: on the one hand, he would like to endorse the idea that the law ruling the behavior of such an aggregate can be exactly deduced from the laws ruling the behavior of the individuals composing it; on the other hand, however, he is apparently prepared to recognize that such an exact deduction can only be hoped for in the special case in which all the individuals are alike, so that in all practical cases one has to put up with the "Method of Fictitious Means", according to which the laws governing the conduct of the aggregate are average laws that cannot be exactly traced back to the laws governing the conduct of the individuals.

This ambiguity is never dispelled by Jevons. As a matter of fact, in dealing with the formalized part of his theory, he invariably interprets the two "trading bodies" appearing in it as if they were two individual decision units, a stance that we shall adopt in the following. Yet, he never gives up the alternative interpretation of a "trading body" as an aggregate, without specifying, however, whether the laws of the aggregate can or cannot be formally deduced from the laws of the individuals. The fact is that Jevons needs both interpretations, for reasons that will become clear in the following.

As to the time structure of the analysis, Jevons may appear, at first sight, to hold a well-defined methodological stance, which directly inspires his modeling choices in the theory of exchange:

> We must carefully distinguish [...] between the statics and the dynamics of this subject. The real condition of industry is one of perpetual motion and change. [...] If we wished to have a complete solution of the problem in all its natural complexity, we should have to treat it as a problem of motion - a problem of dynamics. But [...] it is only as a purely statical problem that I can venture to treat the action of exchange. Holders of commodities will be regarded not as continuously passing on these commodities in streams of trade, but as possessing certain fixed amounts which they exchange until they come to equilibrium. (Jevons, 1970, 138)

From this passage the following prescriptions seem to emerge with reasonable clarity: first, Jevons's theory of exchange is deliberately restricted to the "statics of the subject"; secondly, to take a statical view of the exchange problem means, for Jevons, to regard the "holders of commodities [...] as possessing certain fixed amounts" of the same commodities, and presumably also as being characterized by fixed preferences, and consequently to "determine the results of exchange" under the assumption of fixed data (Jevons, 1970, p. 143); thirdly, since the traders are explicitly said to "exchange until they come to equilibrium", it would appear that, according to Jevons, the analysis of the equilibration process is not inconsistent, at least in principle, with the "statics of the subject", provided that the data of the economy are not allowed to change during the process.

Yet this liberal interpretation of the statical method is immediately disavowed by Jevons himself. The need to give up the analysis of the equilibration process is initially justified on practical grounds, as the following sentence shows:

> It is much more easy to determine the point at which a pendulum will come to rest than to calculate the velocity at which it will move when displaced from that point of rest. Just so, it is a far more easy task to lay down the conditions under which trade is completed and interchange ceases, than to attempt to ascertain at what rate trade will go on when equilibrium is not attained. (Jevons, 1970, p. 138)

This, however, is not yet the end of the story. For the contraposition between the statics and the dynamics of the exchange problem is after all much more dramatic than it would appear from the mechanical analogy of the pendulum recalled in the above passage: for in mechanics there is no theoretical, but only a practical, distinction between the statical and the dynamic perspective; on the contrary, in economics the contrast between the two methods is also theoretical, implying that some dynamic tools are precluded to statics and, for this reason, the equilibration process cannot fall under the jurisdiction of statical analysis. All this, according to Jevons, has to do with the fundamental distinction, already introduced above, between two notions of "ratio of exchange", the 'differential' and the 'finite':

> Strictly speaking, the ratio of exchange at any moment is that of dx_2 to dx_1, of an infinitely small quantity of one commodity to the infinitely small quantity of another which is given for it. The ratio of exchange is really a differential coefficient.

But, this being said, the difference between the statics and the dynamics of the exchange problem

> will present itself in this form: dynamically we could not treat the ratio of exchange otherwise than as the ratio of dx_2 and dx_1,

> infinitesimal quantities of commodity. Our equations would then be regarded as differential equations, which would have to be integrated. But in the statical view of the question we can substitute the ratio of the finite quantities x_2 and x_1. (Jevons, 1970, p. 138)

A clarification of Jevons's confused sentences on the respective jurisdictions of statics and dynamics in the theory of exchange has to be postponed to Section 8 below, where we shall also explain why Jevons takes such a wavering position on whether or not the analysis of the equilibration process can be part of the statics of the subject. Here we want to draw the reader's attention to Jevons's cryptic statement that, "in the statical view of the question", the 'finite' "ratio of exchange", $\frac{x_2}{x_1}$, can be substituted for the 'differential' one, $\frac{dx_2}{dx_1}$. Such substitution, as we shall see in a moment, plays a crucial role in Jevons's solution of the exchange problem. This is probably the reason why Jevons tries to justify it not only, as we have just seen, as a peculiar trait of statics, but also as a sort of by-product of "a general law of the utmost importance in economics", which Jevons proposes to call "the *law of indifference*" (Jevons, 1970, p. 137). Now, as already mentioned above, such "law" is the third highly ambiguous concept on which Jevons erects his theoretical system. Hence, before coming back to the issue of the justification that the "law" is supposed to provide to the proposed substitution of $\frac{x_2}{x_1}$ for $\frac{dx_2}{dx_1}$, it is necessary to elucidate the meaning of the "law" itself.

When the "law" is first introduced (without mentioning its name), its meaning is explained as follows:

> A market [...] is theoretically perfect only when all traders have perfect knowledge of the conditions of supply and demand, and the consequent ratio of exchange; and in such a market, as we shall now see, there can only be one ratio of exchange of one uniform commodity at any moment. (Jevons, 1970, pp. 133)

On the contrary, when, a few pages later, the "law" is named and explicitly defined, its meaning is explained as follows:

> When a commodity is perfectly uniform or homogeneous in quality, any portion may be indifferently used in place of an equal portion: hence, in the same market, and at the same moment, all portions must be exchanged at the same ratio. [...] Hence

> follows what is undoubtedly true, with proper explanations, that *in the same open market, at any one moment, there cannot be two prices for the same kind of article.* (Jevons, 1970, pp. 136-7)

These two definitions prompt the following three remarks. In the first place, Jevons is uncertain as to how the fundamental implication of his "law of indifference" ought to be specified: while on many occasions he asserts that the "law" implies the sameness of the "ratio of exchange of one uniform commodity", on a few other occasions, instead, he speaks of the sameness of the "price" of such a "uniform commodity"[4]. Yet, except for the Sections discussing the "law", no mention of the "price" concept is made in the remainder of Chapter IV, where the concept of a "ratio of exchange" is employed in the entire discussion of the formal model: the relationship existing between "prices" (always understood as "money prices") and "ratios of exchange" is spelled out only in a Section of Chapter V of *TPE* entitled "Relations of Economic Quantities" (1970, pp. 203-5), where Jevons discusses the relations between his own terminology and the expressions used in either preexisting theories or every-day language. Jevons's occasional use of the term "price" in discussing the "law" is probably due to his perceiving how odd it may appear to speak of the sameness of the "ratio of exchange of one uniform commodity", without even mentioning the other commodity with which the first is supposed to be exchanged. As we shall see, the absence of the "price" concept in the statement of the theory of exchange is one of the fundamental weaknesses of Jevons's approach.

In the second place, Jevons is careful in specifying that the "law" holds "at any one moment", but typically does not hold over time:

> Though the price of the same commodity must be uniform at any one moment, it may vary from moment to moment, and must be conceived in a state of continual change. Theoretically speaking, it would not usually be possible to buy two portions of

[4]It might be noted that Cournot (1838, pp. 51-2, fn.), in a much earlier statement of a principle somehow anticipating Jevons's "law", had defined a "market", in the sense of economic theory, as a "territory" where "prices [...] take the same level throughout, with ease and rapidity". The relevant passage from Cournot's *Recherches* is quoted by Jevons in the second edition of *TEP* (1970, pp. 132-3, fn.). It should be noted, however, that Jevons becomes acquainted with Cournot's writings only after the publication of the first edition of *TEP*.

> the same commodity *successively* at the same ratio of exchange [...]. (Jevons, 1970, p. 137)

In the third place, the "law" is explicitly said to hold in a "perfect" or "open" market. But a "market", in turn, is defined in such a demanding way, as far as the traders' knowledge and motivations are concerned, as to turn the "law" into a statement which is "undoubtedly true, with proper explanations", that is, into an almost tautological statement:

> By a market I shall mean two or more persons dealing in two or more commodities, whose stocks of those commodities and intentions of exchanging are known to all. It is also essential that the ratio of exchange between any two persons should be known to all the others [...] and there must be perfectly free competition, so that anyone will exchange with anyone else for the slightest apparent advantage. (Jevons, 1970, p. 133)

It is clear that Jevons would have liked to be able to prove the "law" as a theorem, starting from more primitive assumptions on traders' knowledge, motivations and kinds of market interaction. But he is unable to do so: in particular, there is no attempt in Chapter IV of *TPE* to employ those "arbitrage" conditions that had already been successfully exploited by Cournot for similar purposes in Chapter III of his *Recherches*[5]. So, in the end, Jevons piles up assumptions on the characteristics of the traders and the functioning of the market which are never really used to prove anything, but have the only purpose of making the "law" plausible.

3 Jevons's theory of exchange: a formal statement

This is particularly evident when one looks at the chief analytical use to which Jevons puts the "law". Such use, as already mentioned, has to do with the relation between 'differential' and 'finite' "ratios of exchange". In fact, after

[5] See Cournot (1838, pp. 29-43). It is ironic that, in the Preface to the second edition of *TPE*, Jevons (1970, p. 58) should qualify Chapter III of Cournot's *Recherches*, where the theory of arbitrage is fully developed in the context of a discussion of foreign exchanges, as "not particularly useful".

asserting that, "in the statical view of the question", the 'differential' "ratio", $\frac{dx_2}{dx_1}$, can be replaced by the 'finite' one, $\frac{x_2}{x_1}$, Jevons (1970, p. 139) continues as follows:

> Thus, from the self-evident principle, stated on p. 137, that there cannot, in the same market, at the same moment, be two different prices for the same uniform commodity, it follows that *the last increments in an act of exchange must be exchanged in the same ratio as the whole quantities exchanged*. [...] This result we may express by stating that the increments concerned in the process of exchange must obey the equation

$$\frac{dx_2}{dx_1} = \frac{x_2}{x_1} \tag{1}$$

Now, in this application of the "law", which is by far the most important one from Jevons's own viewpoint, the "law of indifference" really boils down to a trivial truism. First of all, the "law" is supposed to apply to an "act of exchange", which it is indeed difficult to regard as an event involving more than two individual traders, where "two commodities are bartered in the ratio of x_1 for x_2", x_1 and x_2 being the "finite" traded quantities of the commodities concerned. Then the result is simply obtained by observing that "every mth part of x_1 is given for the mth part of x_2, [...] so that, at the limit, even an infinitely small part of x_1 must be exchanged for an infinitely small part of x_2, in the same ratio of the whole quantities", which is indeed a platitude. Yet, it should be noted that this application of the "law", however trivial, does all the same set some constraint on the interpretation of equation (1), constraint that unfortunately Jevons tends to forget: for, since the "law" holds "at one moment" and the "act of exchange" is instantaneous as well, Jevons should not feel authorized to speak of "the increments concerned in the process of exchange", unless he were thinking of an "instantaneous process", which is a contradiction in terms.

After such a toilsome introduction of the main conceptual tools, Jevons eventually sets out to derive the competitive equilibrium conditions for his model, called the "equations of exchange". He starts by asking "at what point the exchange will cease to be beneficial" for the traders. After pointing out that such "question must involve both the ratio of exchange and the degrees of utility", he provides a tentative answer by developing a specific numerical example. Assuming, "for a moment", that the "['finite'] ratio of exchange" between two commodities be "established" at a given level, he verbally arrives at the conclusion that the benefits from exchange for either trader cease when the ratio of each trader's "degrees of utility" is equal to the inverse of the 'differential' "ratio of exchange" of the two commodities, which in turn is assumed to be equal to "the established ['finite'] ratio of exchange". The "point" so determined is called the "point of equilibrium" by Jevons (1970, pp. 139-40).

Now, there is a surprising omission in Jevons's introductory discussion of the determination of the so-called "point of equilibrium": for, since the provisionally "established ratio of exchange" is generally not what would currently be called a competitive equilibrium ratio, the quantities of commodities that the traders want to exchange at Jevons's "point of equilibrium" cannot generally be exchanged; but not a single word is uttered by Jevons about this eventuality and its possible consequences.

Similarly surprising is the diagram (Fig. 5 of *TPE*, reproduced as Fig. 1 below) used to illustrate the verbal argument. Even if Jevons (1970, p. 140) makes it clear from the beginning that "it is hardly possible to represent this theory completely by means of a diagram", all the same the only diagram actually employed for this purpose is much more disappointing than one might reasonably have expected: first of all, this diagram represents the marginal utility curves of one single trader, while the surrounding verbal discussion concerns both; in the second place, the two marginal utility curves, drawn with respect to rotated horizontal axes and superposed on one another, are said to identify the "point of equilibrium" of the trader concerned by means of their "point of intersection", under the assumption that "the ratio of exchange [...] be that of unit for unit, or 1 to 1". But, once again, nothing is said about the reason for selecting precisely that "ratio of exchange" or about the consequences of that selection, in the likely case the "1 to 1 ratio" were not what would currently be referred to as the competitive equilibrium "ratio".

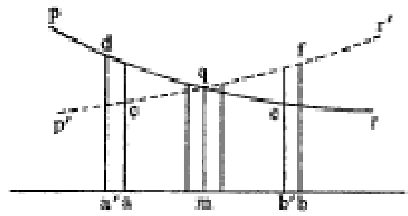

Fig. 1

While postponing a full explanation of these bewildering omissions to Section 8 below, let us now examine how Jevons, taking for granted that equation (1) holds, eventually proceeds to determine the equilibrium conditions for his Edgeworth Box economy, $\mathcal{E}_J^{2\times 2} = \left\{ \left(\mathbb{R}_+^2, u_i\left(\cdot\right), \omega_i \right)_{i=1}^2 \right\}$, in the Section of Chapter IV of *TPE* entitled "Symbolic Statement of the Theory". Right at the beginning of this Section, Jevons points out that the subsequent discussion will concern what is supposed to hold "in a state of equilibrium" (1970, p. 141). Now, as we have just seen, the expression "point of equilibrium" had been loosely used by Jevons in the preceding page (p. 140) to denote the traders' utility maximizing choices at an arbitrarily given "ratio of exchange". Yet, from the nature of the argument developed in the Section starting on p. 141, it is clear that Jevons is now employing the "equilibrium" concept in a much stricter sense, corresponding to what would be termed a 'competitive equilibrium' nowadays. Thus, in order to make our account of Jevons's theory more understandable, we shall employ the expression 'competitive equilibrium' when the context unmistakably calls for it, even if Jevons nowhere makes use of that expression. Hence, letting x_1^* and x_2^* be the quantities of the two commodities traded in such a 'competitive equilibrium' state, $\frac{x_2^*}{x_1^*}$ turns out to be the competitive equilibrium 'finite' "ratio of exchange". Given the assumptions on the traders' characteristics, one obviously has: $x_1^* = s_{11}^* = d_{12}^*$ and $x_2^* = d_{21}^* = s_{22}^*$, where $s_{11}^* = \bar{\omega}_1 - x_{11}^*$ and $d_{12}^* = x_{12}^*$ are the 'competitive equilibrium' quantities of commodity 1 respectively supplied by 1 and demanded by 2, while $d_{21}^* = x_{21}^*$ and $s_{22}^* = \bar{\omega}_2 - x_{22}^*$ are the 'competitive equilibrium' quantities of commodity 2 respectively demanded by 1 and supplied by 2.

Then, for any pair of differentials dx_1^* and dx_2^* such that $\frac{dx_2^*}{dx_1^*} = \frac{x_2^*}{x_1^*}$, for trader 1 one has:

$$v'_{11}(\bar{\omega}_1 - x_1^*)dx_1^* = v'_{21}(x_2^*)dx_2^*,$$

or

$$\frac{v'_{11}(\bar{\omega}_1 - x_1^*)}{v'_{21}(x_2^*)} = \frac{dx_2^*}{dx_1^*}. \tag{2}$$

A similar condition holds for trader 2:

$$\frac{v'_{12}(x_1^*)}{v'_{22}(\bar{\omega}_2 - x_2^*)} = \frac{dx_2^*}{dx_1^*}. \tag{3}$$

Hence, by substituting (1) into both (2) and (3), one gets:

$$\frac{v'_{11}(\bar{\omega}_1 - x_1^*)}{v'_{21}(x_2^*)} = \frac{x_2^*}{x_1^*} = \frac{v'_{12}(x_1^*)}{v'_{22}(\bar{\omega}_2 - x_2^*)}, \tag{4}$$

which are Jevons's "equations of exchange", representing his fundamental result in the "theory of exchange". The way in which such "equations" are first obtained and then interpreted by Jevons prompts the following remarks.

In the first place, it should be noted that, in deriving the "equations of exchange", Jevons makes explicit use of equation (1), expressing the formal version of the "law of indifference". This has some important consequences on the interpretation of Jevons's equilibrium concept: for, as we have seen, the "law of indifference" only holds at a specified time instant; but then Jevons's equilibrium concept must be given an "instantaneous" interpretation as well. This means that the equilibrium allocation should be imagined as instantaneously reached by the two traders, by means of one single "act of exchange", taking place at one and the same 'competitive equilibrium' "ratio of exchange", and leading them directly from the initial endowment to the final equilibrium allocation.

In the second place, it should be stressed that Jevons's argument is entirely couched in terms of the 'competitive equilibrium' values of the traded quantities of the two commodities, x_1^* and x_2^*, from which Jevons directly derives the 'finite', 'competitive equilibrium' "ratio of exchange", $\frac{x_2^*}{x_1^*}$, and indirectly also the 'differential' one, $\frac{dx_2^*}{dx_1^*}$, since the two "ratios" are assumed equal by virtue of the "law of indifference". This means, however, that quantities or "ratios" different from the 'competitive equilibrium' ones are nowhere mentioned or even alluded to by Jevons in the development of his formal argument. But this is also the reason why Jevons nowhere introduces, let alone discusses, an equation like

$$\frac{v'_{11}(\bar{\omega}_1 - x_1)}{v'_{21}(x_2)} = \frac{dx_2}{dx_1} = \frac{v'_{12}(x_1)}{v'_{22}(\bar{\omega}_2 - x_2)}, \tag{5}$$

defining Edgeworth's "contract curve" (1881, p. 21), where x_1 and x_2 are the traded quantities of the two commodities that satisfy the functional relationship implicitly defined by equation (5), while $\frac{dx_2}{dx_1}$ is the 'differential' "ratio of exchange" that, at every point along the "contract curve", equals the common value of the traders' marginal rates of substitution between the two commodities prevailing at that point. Given that Edgeworth's analysis in *Mathematical Psychics* starts precisely from Jevons's "theory of exchange" in *TPE* (1881, pp. 20, 39, and App. V, pp. 104-16), the lack of any hint in Jevons's "theory" to such equation or the associated "curve" may appear surprising. But this is just another instance of the intrinsic limits of Jevons's approach, to which we shall come back in Section 8 below.

Finally, it remains to discuss what relationship, if any, can be established between Jevons's "theory of exchange", as expressed by his "equations of exchange" (equations (4) above), and the so-called "laws of supply and demand". According to Jevons (1970, p. 143), his "theory"

> is perfectly consistent with the laws of supply and demand; and if we had the functions of utility determined, it would be possible to throw them into a form clearly expressing the equivalence of supply and demand. (Jevons, 1970, p. 143)

The second part of this statement is relatively cryptic: it probably means that, if the marginal utility functions could be analytically specified, what typically cannot be done, at least according to Jevons, then for each commodity one could numerically compute the equilibrium values of the quantities demanded and supplied, thereby confirming the "equivalence of supply and demand". But, apart from this, should one ask in which sense the alleged "perfect consistency" between "theory" and "laws" ultimately reveals itself, Jevons's answer couldn't but appear quite disappointing:

> We may regard x_1 as the quantity demanded on one side and supplied on the other; similarly, x_2 is the quantity supplied on the one side and demanded on the other. Now, when we hold

> the two equations to be simultaneously true, we assume that the x_1 and x_2 of one equation equal those of the other. The laws of supply and demand are thus a result of what seems to me the true theory of value or exchange. (Jevons, 1970, pp. 143-4)

As a matter of fact, this sentence simply asserts that, at a 'competitive equilibrium' of a two-trader, two-commodity model, the 'equilibrium' quantity of either commodity demanded by either trader equals the 'equilibrium' quantity of the same commodity supplied by the other trader. Since in this case the 'equilibrium' condition is at least explicitly mentioned, Jevons's statement does not in effect boil down to J. S. Mill's truism mentioned in footnote 3 above. Yet, the real content of Jevons's statement is far too limited to justify his conclusion that "the laws of supply and demand are [...] a result of [his] theory of value or exchange": indeed, at the end of Jevons's discussion of his Edgeworth Box model, the true meaning of the "laws of supply and demand" still remains in the dark.

As mentioned at the beginning of the previous Section, once the formal statement of the two-trader, two-commodity model is completed, Jevons strives to extend his analysis in various directions: in particular, he tries to generalize his "equations of exchange" to economies with more than two traders and/or more than two commodities. All his attempts rest on the assumption that

> [t]he exchanges in the most complicated case may [...] always be decomposed into simple exchanges, and every exchange will give rise to two equations sufficient to determine the quantities involved. (Jevons, 1970, p. 154)

Unfortunately, however, this assumption is unfounded: for Jevons's "theory of exchange", which is molded on the requirements of the simple two-trader, two-commodity economy, is insufficient to cope with an economy with a larger number of traders and/or commodities, unless it is reinforced by the insertion of new concepts and assumptions. Hence, Jevons's reductionist strategy ends up in a failure: in spite of what he seems to suggest (1970, p. 154), Jevons is really unable to deal with traders who are not "cornered"; moreover, when he tries to put forward a theory of multilateral commodity exchanges, involving $L > 2$ commodities, he ends up with a set of $\frac{L(L-1)}{2}$ bilateral exchanges, each involving a pair of commodities and each viewed as independent of the others, so that the $\frac{L(L-1)}{2}$ "ratios of exchange", obtained by applying the simple "equations of exchange" of Jevons's simple two-commodity model, do not satisfy the Cournot-Walras arbitrage conditions, to which we shall come back in Section 5 below.

4 Jevons's theory of exchange: six unsettled questions

By examining Jevons's "theory of exchange" in the previous two Sections, we have been able to identify a number of shortcomings characterizing it: precisely, we have singled out three major ambiguities marring its conceptual apparatus and three major gaps weakening its analytical structure. Conceptual ambiguities and analytical gaps are obviously related. All the same, it is now convenient to list them separately, briefly summarizing their features.

The first ambiguity has to do with Jevons's peculiar concept of "trading bodies", that are alternatively viewed as either individual traders, or representative agents, or "fictitious means". The second ambiguity arises from Jevons's interpretation and use of the concepts of statics, dynamics and equilibrium: after sharply distinguishing between statics and dynamics, and contending that the exchange problem ought be tackled from a purely statical point of view, Jevons is apparently willing to concede that the study of the equilibration process is not inconsistent with statics; yet, after recognizing that a truly dynamic analysis would require integrating suitable differential equations describing the trading process over time, he ends up with endorsing a strictly "instantaneous" interpretation of the equilibrium concept. The third ambiguity is that surrounding the so-called "law of indifference", which is initially viewed as the outcome of a market equilibration process taking place under stringent assumptions about the traders' knowledge and information, but then reduced to an almost trivial truism in the only relevant application.

Let us turn now to the gaps. The first one is revealed by two related facts: on the one hand, the price concept is almost entirely neglected by Jevons; on the other, no serious demand-and-supply analysis is associated to his "theory of exchange". The second gap has to do with the missing equation of Edgeworth's "contract curve": in effect, no such equation is put forward by Jevons, even if he apparently comes very close to identifying its properties.

Finally, the third gap is due to Jevons's inability to extend his "theory of exchange" beyond the narrow boundaries of the two-trader, two-commodity model.

All these shortcomings call for an explanation, a critical assessment, and possible suggestions for improvement. Two economists contemporary with Jevons, Fleeming Jenkin and Léon Walras, undertake this task during the most productive period of Jevons's lifetime, as far as economic theory is concerned, that is, during the period ranging from the publication of the "Brief Account" in 1866 and the appearance of the second edition of *TPE* in 1879; their contributions, moreover, are directly brought to Jevons's attention by Jenkin and Walras, respectively, in private correspondence or public debates. In the following we are interested in explaining, on the one hand, why and how the two economists develop their critical views of Jevons's "theory of exchange", and, on the other, why and how Jevons reacts (or, better, does not react) to their critical remarks and propositions[6]. Even if Jenkin is the first to enter the stage, it is preferable to start from Walras's contribution: Walras's approach, in fact, is incomparably more systematic and theoretically self-conscious than Jenkin's; hence, even if it is slightly posterior, it will prove instrumental in shedding light on the latter's earlier contribution as well.

5 Walras's pure-exchange, two commodity model

Walras's critical remarks concerning Jevons's "theory of exchange" are explicitly put forward on a few occasions that will be recalled in the next Section. Yet, such scant explicit observations simply represent the distilled essence of a huge mass of underlying reflections, which only occasionally come to the surface. As a matter of fact, the bulk of Walras's critique of Jevons's approach is implicitly contained in the former's own theory of exchange, made

[6]Edgeworth's *Mathematical Psychics* is published in 1881, that is, just before Jevons's premature death in 1882: hence, even if Edgeworth's essay is explicitly conceived as a critical defence and generalization of Jevons's approach, the latter's lack of reactions to Edgeworth's fundamental contribution is easily justified, in this case, by Jevons's unfortunate lack of time. Since this paper is focused on Jevons's interaction (or lack of interaction) with his critics, we have chosen not to discuss Edgeworth's work, apart from a few occasional remarks, for in this case Jevons's lack of reaction requires no special explanation. The theoretical relation between Edgeworth's approach and Jevons's, however, will be the subject of a companion paper.

up of two related models (the two- and the multi-commodity model), to which one has to turn in order to understand the real differences between the two economists' positions.

Walras develops his pure-exchange, two-commodity model in the first of his published theoretical writings, the *mémoire* "Principe d'une théorie mathématique de l'échange" which appears in 1874, as well as in the first part of Section II of the first edition of the *Eléments*, whose first installment is published in the same year[7]. Walras's two-commodity model is obviously propedeutical to his exchange model with an arbitrary finite number of commodities, whose discussion immediately follows, in the *Eléments*, that of the simpler two-commodity model[8]. Yet, in order to facilitate the intended comparison between Walras's theory of exchange and Jevons's, which, as we have seen, essentially boils down to a two-commodity model, we shall focus attention on the two-commodity model in Walras's case as well. In the next Section, however, in facing the issue of the generalizability of the theory, we shall also devote a few remarks to Walras's multi-commodity model.

In his pure-exchange, two-commodity model Walras studies an exchange economy $\mathcal{E}_W^{2\times I} = \left\{ \left(\mathbb{R}_+^2, u_i\left(\cdot\right), \omega_i \right)_{i=1}^{I} \right\}$, with $I \geq 2$, satisfying a few specific assumptions concerning the traders' endowments and utility functions. As to the endowments, all the traders are assumed to be "cornered", that is, to hold a positive quantity of one commodity only; specifically, in the following we shall assume $\omega_i = (\omega_{1i}, 0)$, with $\omega_{1i} > 0$, for $i = 1, ..., I'$, with $I' \geq 1$, and $\omega_i =$

[7] The first edition of the *Eléments* is published in two installments, which respectively appear in 1874 and 1877. Three further editions are published during Walras's lifetime: the second edition appears in 1889, the third in 1996, and the fourth in 1900. A fifth posthumous edition, containing a few changes arranged by Walras himself before his death, will be published in 1926. In this paper all references will be made to the comparative *variorum* edition of the *Eléments*, published in 1988, collating the texts of all the previous editions. When the text varies across editions and it is necessary or convenient to refer to one or more specific editions, the number(s) of the edition(s) referred to will be specified in bold, after the page number(s).

[8] In the first three editions of the *Eléments* the two-commodity model is put forward in the first part (approximately the first half) of Section II, the second part of that Section being devoted to the discussion of the exchange model with more than two commodities. In the fourth and fifth editions, instead, the old Section II is split into two Sections, the new Sections II and III, respectively devoted to the exchange model with only two and more than two commodities. The exchange model with an arbitrary finite number of commodities is also separately discussed in Walras's second *mémoire*, called "Equations de l'échange", which is published in 1876.

$(0, \omega_{2i})$, with $\omega_{2i} > 0$, for $i = I'+1, ..., I$, with $I > I'$, so that $\bar{\omega} = (\bar{\omega}_1, \bar{\omega}_2) = (\Sigma_{i=1}^{I'} \omega_{1i}, \Sigma_{i=I'+1}^{I} \omega_{2i}) \in \mathbb{R}^2_{++}$. As to the utility functions, each consumer $i = 1, ..., I$ is supposed to be characterized by a cardinal, additively separable utility function, satisfying the same restrictions on the signs of the first- and second-order pure partial derivatives as in Jevons's model[9]. Under these assumptions, if the consumers' number were $I = 2$, then Walras's exchange economy $\mathcal{E}_W^{2\times I}$ would become indistinguishable from Jevons's Edgeworth Box economy, $\mathcal{E}_J^{2\times 2}$. In general, however, one must suppose that, in Walras's model, I be greater than 2, while I' and $I - I'$ are both greater than 1.

Now let $p = (p_{12}, 1) \in \mathbb{R}^2_{++}$ be the price system, expressed in terms of commodity 2 taken as the numeraire. Since in the economy there are only two commodities, one can only have one independent relative price, $p_{12} = \frac{1}{p_{21}}$. Walras assumes the traders to behave competitively: precisely, Walras's 'Perfect Competition Assumption' implies that the traders take prices as given and choose their optimal trade plans, called "dispositions à l'enchère" by Walras (1988, p. 83), in such a way as to maximize their respective utility functions, under their respective Walrasian budget constraints. For all $p = (p_{12}, 1)$, the optimization problem to be solved by trader i can be formally written as:

$$MRS_{21}^i(x_{1i}, x_{2i}) = \frac{v'_{1i}(x_{1i})}{v'_{2i}(x_{2i})} = p_{12} \tag{6}$$

$$px_i = p\omega_i, \tag{7}$$

where equation (6) is Walras's "condition de satisfaction maxima" of trader i and equation (7) is the Walrasian budget constraint of the same trader, for $i = 1, ..., I$ (Walras, 1988, pp. 111-7).

By solving this system, one obtains the individual demand and supply functions for the two commodities for each trader, that is:

[9] As a matter of fact, Walras typically assumes the marginal utility function of commodity l for consumer i to go to zero for $x_{li} < \infty$ (Walras, 1988, pp. 107-11). The assumption that the marginal utility of each commodity be strictly positive and monotonically decreasing over each consumer's entire consumption set is made here with a view to simplifying our discussion: in fact, it allows us to dodge all boundary problems and to obtain demand and supply functions that are well-defined for all positive prices. This assumption can anyhow be dispensed with, at the cost of complicating somewhat the analysis.

$$
\begin{aligned}
s_{1i}(p_{12}) &= \bar{\omega}_{1i} - x_{1i}(p_{12}) \\
d_{2i}(p_{12}) &= x_{2i}(p_{12})
\end{aligned}
$$

for $i = 1, ..., I'$, and

$$
\begin{aligned}
d_{1i}(p_{12}) &= x_{1i}(p_{12}) \\
s_{2i}(p_{12}) &= \bar{\omega}_{2i} - x_{2i}(p_{12})
\end{aligned}
$$

for $i = I' + 1, ..., I$.

Then, by aggregating over the traders, we obtain the aggregate demand and supply functions for both commodities, that is:

$$
\begin{aligned}
d_1(p_{12}) &= \Sigma_{i=I'+1}^{I} d_{1i}(p_{12}) = \Sigma_{i=I'+1}^{I} x_{1i}(p_{12}), \\
s_1(p_{12}) &= \Sigma_{i=1}^{I'} s_{1i}(p_{12}) = \Sigma_{i=1}^{I'} (\bar{\omega}_{1i} - x_{1i}(p_{12})) = \bar{\omega}_1 - \Sigma_{i=1}^{I'} x_{1i}(p_{12}), \text{ and}
\end{aligned}
$$

$$
\begin{aligned}
d_2(p_{12}) &= \Sigma_{i=1}^{I'} d_{2i}(p_{12}) = \Sigma_{i=1}^{I'} x_{2i}(p_{12}), \\
s_2(p_{12}) &= \Sigma_{i=I'+1}^{I} s_{2i}(p_{12}) = \Sigma_{i=I'+1}^{I} (\bar{\omega}_{2i} - x_{2i}(p_{12})) = \bar{\omega}_2 - \Sigma_{i=I'+1}^{I} x_{2i}(p_{12}),
\end{aligned}
$$

from which we can get the excess demand functions:

$$
\begin{aligned}
z_1(p_{12}) &= d_1(p_{12}) - s_1(p_{12}) = \Sigma_{i=I'+1}^{I} x_{1i}(p_{12}) - \bar{\omega}_1 + \Sigma_{i=1}^{I'} x_{1i}(p_{12}) = \\
&= \Sigma_{i=1}^{I} x_{1i}(p_{12}) - \bar{\omega}_1 = x_1(p_{12}) - \bar{\omega}_1,
\end{aligned}
$$

$$
\begin{aligned}
z_2(p_{12}) &= d_2(p_{12}) - s_2(p_{12}) = \Sigma_{i=1}^{I'} x_{2i}(p_{12}) - \bar{\omega}_2 + \Sigma_{i=I'+1}^{I} x_{2i}(p_{12}) = \\
&= \Sigma_{i=1}^{I} x_{2i}(p_{12}) - \bar{\omega}_2 = x_2(p_{12}) - \bar{\omega}_2.
\end{aligned}
$$

Finally, by equating to zero the excess demand functions, we obtain the market clearing equations, one for each commodity:

$$
z_l(p_{12}^W) = x_l(p_{12}^W) - \bar{\omega}_1 = 0, \qquad l = 1, 2. \tag{8}
$$

Yet, given Walras' Law, that is, $pz = p_{12}z_1(p_{12}) + z_2(p_{12}) \equiv 0$, $\forall p = (p_{12}, 1) \in \mathbb{R}^2_{++}$, only one equation provides an independent equilibrium condition. By solving either equation, therefore, we obtain the Walrasian competitive equilibrium relative price, p_{12}^W (Walras, 1988, pp. 136-7).

This "solution" is "analytique". But one can also give it the "forme géométrique", by drawing for each commodity the demand and supply curves and finding their intersection. Anyhow, the "solution" thus determined, by means that can indifferently be "analytique" or "géométrique" , still remains the "solution mathématique" for Walras. Under the stated conditions, a "solution" exists, even if it is not necessarily unique (Walras, 1988, p. 97). Let us assume, for simplicity, that it is unique. Then one can ask how such "solution" is concretely determined "sur le marché". According to Walras, this empirical determination occurs as follows. One relative price p_{12} being "crié", the corresponding excess demand for commodity 1, $z_1(p_{12})$, will be determined "sans calcul, mais néanmoins conformément à la condition de satisfaction maxima": then p_{12} increases or decreases, according to whether $z_1(p_{12})$ is greater or less than zero, and the process goes on until the equilibrium price, p_{12}^W, is eventually reached (Walras, 1988, pp. 137-8). This is the first instance of Walras's celebrated *tâtonnement* construct, that Walras subsequently applies to all his equilibrium models with a view to explaining how, in each case, the "solution mathématique" is empirically arrived at "sur le marché".

6 Walras's critique of Jevons's theory of exchange

With his model of a pure-exchange two-commodity economy, Walras arrives at results partially similar to those arrived at by Jevons a few years before with his "theory of exchange". This partial similarity, on the other hand, is explicitly recognized by both of them - with many a qualification on Walras's side, as will be seen - in the well-known exchange of correspondence taking place in May 1874, shortly after the publication of Walras's first *mémoire*[10]: in particular, assuming the number of traders in Walras's model to be $I = 2$,

[10] This exchange, reproduced in the June 1874 issue of the *Journal des Economistes*, and later in the various editions of Walras's *Théorie Mathématique de la richesse sociale* (1877, 1883), can now be found in (Walras 1993, pp. 47-52).

not only the economy analyzed by Walras turns out to be formally identical with the Edgeworth Box economy examined by Jevons, but also Walras's equations (6), (7) and (8), though reached through a different route, boil down to Jevons's "equations of exchange" (4).

Yet, in spite of the similarities in some of the results eventually obtained, not only does Walras's overall approach significantly differ from Jevons's, but the former's exchange models also prove to be a potentially much richer theory than the latter's "theory of exchange": in particular, starting from his different methodological and analytical premises, Walras is able to confront and answer some, though not all, of the questions left open by Jevons's "theory of exchange", questions that have been summarized in Section 4 above. The roots of such divergence are partly identified by Walras himself both in his 1874 open letter to Jevons (Walras, 1993, pp. 49-50) and, more extensively, in the paragraph devoted to Jevons's "theory of exchange", especially prepared for the second edition of the *Eléments* (1889), which however appears when Jevons has been dead for seven years (Walras, 1988, pp. 251-3, **2-5**).

The fundamental difference between the two approaches lies in the different interpretation of the perfect competition hypothesis: as we have seen, in fact, the 'Perfect Competition Assumption' means for Walras, but not for Jevons, that the traders make their optimizing choices by taking prices as given parameters. But, as Walras himself does not fail to point out (1988, p. 253, **2-5**), this has momentous implications for the structure of the two authors' theories: for, while in Walras's exchange models the "prix" become the fundamental "inconnues du problème", in Jevons's "theory of exchange" that role is played by the "quantitités éxchangées". By explicitly introducing the commodity prices into the picture, however, Walras is able to extensively generalize the version of the "law of indifference" actually employed by Jevons in constructing his "theory of exchange", a version that, as explained above, is really nothing but a truism. What Walras actually resorts to in building his exchange models is a much more general "law", which might be called the 'Law of One Price', even if Walras never uses the latter expression in his writings. Such 'Law' is already implicit in the 'Perfect Competition Assumption', as it has been formulated above. Yet, it might be useful to state it separately, in view of the crucial role it plays in explaining the contrast between Walras's and Jevons's approaches: as employed by Walras, the 'Law of One Price' consists in the assumption that, at any given instant, one and the same price system is simultaneously announced to all traders, both at equilibrium and out of equilibrium.

When combined with the 'Perfect Competition Assumption', the 'Law of One Price' implies that in Walras's exchange models the traders are supposed to choose their trade plans, at the given prices, in all possible states of the economy, that is, not only at equilibrium, but also out of equilibrium. However, out of equilibrium not all the chosen trade plans can be actually carried out. But this implies that some mentalistic concepts for which no observable counterpart can be found, such as the trade plans chosen by at least some traders when the economy is out of equilibrium, necessarily enter Walras's exchange models. Nothing similar can instead be found in Jevons's "theory of exchange": for, in this case, prices are never announced to traders, nor disequilibrium trade plans are ever explicitly taken into account. Jevons comes very close to allowing for this possible occurrence, when he discusses the traders' utility maximizing choices at an arbitrarily given "ratio of exchange"; but, as explained in Section 3 above, he never really crosses the line, for he stubbornly refuses to analyze the possible consequences of such disequilibrium choices, leaving the whole discussion apparently unfinished. From this point of view, therefore, Walras (1993, pp. 49-50) is completely right in asserting that Jevons does not derive his "*équation d'échange*" from an explicit "considération de *satisfaction maximum*", as embodied, in particular, in Walras's equation (6): to use that equation, in fact, would imply taking prices as the fundamental "inconnues du problème", as well as accepting that mentalistic unobservable concepts, such as disequilibrium trade plans, may enter the theoretical picture; but these are two steps that Jevons is evidently unwilling to take. In Jevons's "equations of exchange", indeed, the traders' utilities are maximized, under the constraint that the traders' exchanges take place at the 'competitive equilibrium' "ratio of exchange"; but no price and no trades other than the 'competitive equilibrium' ones, which are all observable magnitudes, appear in Jevons's "equations", which, as we have seen, are derived under the assumption that the economy already is "in an equilibrium state".

Moreover, by assuming the 'Law of One Price' and making the 'Perfect Competition Assumption', Walras can free himself from the restrictions on the number of traders that Jevons is instead forced to respect in developing his "theory of exchange": as a matter of fact, from the very beginning Walras's pure-exchange, two-commodity model, unlike Jevons's, is supposed to describe the functioning of an economy with any finite number of traders. This is possible in Walras's case for, in view of the 'Law of One Price' and the 'Perfect Competition Assumption', Walras can immediately aggregate the traders' optimal trade plans, irrespective of their number, and can consequently define the aggregate demand and supply functions for each commodity without a hitch. As Walras rightly points out (1993, p. 50), such functions cannot instead be obtained in Jevons's case, due to the latter's inability to take prices as the "inconnues du problème" and his related refusal to deal with unobservables. Moreover, and for the same reasons, there is no need, in Walras's case, to have recourse to Jevons's ambiguous concept of a "trading body", nor to use any other "fictitious mean", in order to find an indirect way to describe the aggregate behavior of a group of traders: Jevons is compelled to resort to such indirect devices since, unlike Walras, he starts from a theory that, from a rigorous and formal point of view, only applies to two individual traders, as Walras does not fail to emphasize (1988, p. 253, **2**-**5**).

Another implication of Walras's modeling choices, especially of his adoption of the two assumptions on which we have dwelled above, is the ease with which he can extend his two-commodity model with "cornered" traders to a multi-commodity model with traders characterized by arbitrary endowments: to this end, Walras needs only supplement his original assumptions with a "theory of arbitrage", as already developed in a different context by Cournot (1838, Chapter III), from whom Walras borrows the idea and the associated solution (Walras, 1988, pp. 161-73, **1** and **2**-**5**). But Walras can exploit Cournot's intuition, building upon it a truly general equilibrium model of a multi-commodity exchange economy with traders characterized by arbitrary endowments, only because he, unlike Jevons, chooses to take prices as the fundamental "inconnues du problème": failing this, no generalized budget constraint for "non-cornered" traders can be defined and no consistency requirement among triples of prices, based on indirect exchanges among traders acting in more than two markets, can really be satisfied.

Finally, with his theory of the *tâtonnement*, Walras is apparently able to provide an answer to the issue of equilibrium attainment, an issue that, as explained in Section 2 above, had been left by Jevons in a state of ambiguity and confusion. Yet Walras's approach, in spite of his firm belief in the validity of the suggested solution, is far from unobjectionable. As a matter of fact, in 1874, at the time when Walras first formalizes his analysis of the process of adjustment towards equilibrium in his exchange models, he is probably convinced that the dynamics of the *tâtonnement* process is compatible with observable disequilibrium behavior, even if he is unable to put forward any formal theory of such behavior[11]. Yet, after the appearance of Bertrand's critical review of Walras's *Théorie mathématique de la richesse sociale* (1883), Walras is forced to recognize that any sort of observable out-of-equilibrium trading would necessarily entail some change in the data of the theory, hence in the equilibrium eventually reached by the equilibration process; therefore, in order to avoid this otcome, that he had been anxious to rule out from the beginning (1988, p. 146), Walras is led to explicitly introduce a 'No-Trade-Out-Of-Equilibrium Assumption', turning the *tâtonnement* process in the exchange models into a purely virtual process in 'logical' time, over which nothing observable is allowed to take place[12]. So, in the end, not differently from Jevons's "state of equilibrium", also Walras's equilibrium state must be supposed to be instantaneously arrived at, even if in the latter case the equilibrium attainment is apparently explained by means of a 'durationless process', i.e., a process which consumes no amount of 'real' time to carry its effects through.

7 Jenkin *vs.* Jevons on the theory of exchange

It is somewhat surprising that Jevons should neither emphasize the patent differences between his approach and Walras's, nor apparently try to contrast

[11] As far as the exchange models are concerned, Walras's belief that the *tâtonnement* construct is not inconsistent with the traders carrying out actual observable trades when markets are out of equilibrium can only be indirectly inferred from the introductory examples of the out-of-equilibrium working of specific markets he develops in both his first *mémoire* (Walras, 1993, pp. 31-2) and the first edition of the *Eléments* (Walras, 1988, pp. 71-2, **1**) for illustration purposes. As far as the production models are concerned, however, it is absolutely certain that, at the time of writing the first edition of the *Eléments* and well beyond it, Walras is convinced that the dynamics of the *tâtonnement* process entails observable disequilibrium behavior. Donzelli (2007) discusses in detail the evolution over the various editions of the *Eléments* and related writings of Walras's ideas on the nature of the *tâtonnement* construct and its relationship with equilibrium and disequilibrium analysis.

[12] See Walras (1885, p. 312, fn. 1), (1988, pp. 71-2, **2-5**); see also the *mémoire* attached to a letter sent by Walras to Pareto, as reproduced in Jaffé (1965, Vol. II, p. 630). In the fourth edition of the *Eléments*, appearing in 1900, Walras eventually generalizes the 'No-Trade-Out-Of-Equilibrium Assumption' to all sorts of models and *tâtonnement* processes, including the models and processes involving production activities, by adopting the so-called "hypothèse des *bons*" (Walras, 1988, pp. 309, 377, 447, **4-5**).

Walras's critical remarks. As a matter of fact, immediately after becoming acquainted with Walras's pure-exchange two-commodity model, as put forward in Walras's first *mémoire*, Jevons politely reacts by sending a letter to Walras where he stresses the similarities between the two theories (Walras 1993, pp. 47-9). Then, soon after being informed by Walras himself that, in the latter's opinion, barring one specific "point de contact", their respective theories might prove to be "quelque peu divergentes" (Walras 1993, pp. 47-9), Jevons, in writing a paper on "The Progress of the Mathematical Theory of Political Economy", restates his own version of the "theory of exchange" as if it were fully representative of Walras's own version as well (Jevons, 1874, pp. 75-85).

Even more startling, however, are Jevons's remarks about the so-called "laws of supply and demand", remarks that can be found in the same 1874 paper that has just been mentioned. In this regard, Jevons asserts that "the laws of supply and demand [...] are easily deduced from the theory of exchange", where by "theory of exchange" he means precisely the theory developed "in the work of M. Walras, or in my 'Theory of Political Economy'"; in Jevons's opinion, such deduction is particularly easy, since the "laws of supply and demand" are "the principal inferences" that can be derived from the "theory of exchange" (Jevons, 1874, pp. 83-4). But, as we have seen in the previous Sections, while in Walras's works the "laws of supply and demand" are extensively discussed in a way that is quite close to contemporary standards, nothing similar can instead be found in Jevons's writings. On the other hand, this missing piece in Jevons's "theory of exchange" had already been pointed out by Walras himself when, in commenting upon Jevons's first paper in theoretical economics (the "Brief Account", published in 1866), he had written:

> Je ne vois pas non plus que vous [...] fondiez [l'équation d'échange] sur la considération de *satisfaction maximum*, qui est pourtant si simple et si claire. Je ne vois pas non plus que vous en tiriez l'équation de la *demand effective* en fonction du *prix*, qui s'en déduit si aisément, et qui est si essentielle à la solution du problème de la détermination des prix d'équilibre. (Walras, 1993, p. 50)

Finally, it is also surprising that Jevons should not take into account Walras's objections and implicit suggestions in preparing the second edition of *TPE* (1879). In the second edition, in effect, one can find a number of changes and additions with respect to the first one, but none of them is apparently related to Walras's observations: Jevons's revisions only concern those parts of the book with which he had been dissatisfied from the beginning or against which some critical remarks had been made by some reviewers of the first edition[13].

Of course, one can provide an easy explanation for Jevons's lack of reactions to the stimuli proceeding from Walras. In fact, it is only in 1874 that Walras eventually enters the theoretical debate with the publication of his first *mémoire* on the theory of exchange; hence, it is only since that date that Walras's novel approach can possibly start producing some effects on Jevons's view of the exchange problem. By that date, however, Jevons has been working for more than a decade at the construction of his "theory of exchange": hence, one may plausibly conjecture that it is too late for Jevons to positively respond to a sudden external impulse, let alone to accept a major change in his entrenched habits of thought and direction of research. Yet, even if this sort of explanation has some appeal, it is not entirely convincing: for, six years before Walras's fatal appearance on the scene, Jevons had already been exposed to some of the new ideas that Walras would have forcefully brought to his attention in 1874; in particular, Jevons had already been informed not only on how individual demand and supply functions can be obtained from the solution of traders' utility maximization problems, but also on how such functions can be concretely employed to solve the equilibrium determination problem in Jevons's Edgeworth Box model. As we know since the publication of Jevons's 1863-1872 correspondence (Black, 1977), all the events just summarized had occurred in 1868, i.e., well before the date (1870) to which the first draft of *TPE* can be traced, thanks to Fleeming Jenkin, an engineer-economist particularly active in the economic debates raging in Great Britain in the late 1860s and early 1870s[14], who had an exchange of correspondence with Jevons precisely in that year.

[13] In the second edition, in particular, the Section "Analogy to the Theory of the Lever" (Jevons, 1970, pp. 144-7) is appended to the central theoretical Section "Symbolic Statement of the Theory", already present in the first, in order to rebut the objections concerning the so-called "problem of integration" raised, among others, by both the author of an anonymous review of the first edition, published on the *Saturday Review* in November 1871 and probably due to the mathematician G.W. Hemming of St John's College, Cambridge, and by Alfred Marshall in a review published on *The Academy* in April 1872 (see Black, 1981, pp. 141-6 and 152-7).

[14] See, in particular, Jenkin (1868), (1870), and (1871), later reprinted in Jenkin (1931).

As will be seen, Jenkin's remarkable contribution essentially consists in an almost complete theory of both the derivation of the traders' individual demand and supply curves from their marginal utility curves, and the graphical determination of a competitive equilibrium along quasi-Walrasian lines. Jenkin's contribution is contained in a short series of two or three letters he sends to Jevons at the beginning of March 1868, in reply to letters by the latter that have unfortunately not been retraced: Jenkin's first letter is dated March 4, 1868, while the other one is (or the other two are) dated March 11, 1868[15]. Even if Jevons's letters are lost, it is apparent from Jenkin's discussion that his correspondent's arguments must have been closely similar to, if not identical with, those which Jevons would have later developed in *TPE*: in particular, it is clear that the only kind of diagram used by Jevons in his letters must have been the same as that drawn in Fig. 5 of *TPE* (Jevons, 1970, p. 140), upon which we dwelled in Section 3 above[16]. At the same time, it is also apparent that not a single piece of the notable demand-and-supply apparatus put forward by Jenkin in his correspondence finds its way into Jevons's later writings, including the two editions of *TPE* published during Jevons's lifetime.

Thirty years after the publication of the Jenkin-Jevons correspondence, the reasons why Jevons should not have taken advantage, in writing *TPE*, of the suggestions freely put at his disposal by Jenkin are still somewhat mysterious. In the case of Jenkin, the reasons for Jevons's neglect of his correspondent's enlightening contribution cannot be the same as in the case of Walras: for, even if Jevons had already arrived at a reasonably well-defined general conception of his "theory of exchange" in the early 1860s, in 1868 he was certainly much more open-minded than he would have been six years later, particularly because in 1868 *TPE* had yet to be written, while in 1874 it was a published work with a reasonably well-established

[15] From the materials published in Black (1977, pp. 166-78), it is unclear whether the two documents dated March 11 are really distinct letters or, what is more likely, just two parts of one and the same letter, written at different times of the same day and mailed together in the same envelope.

[16] As will be recalled, that diagram is drawn by Jevons (1970, pp. 140-1) under the assumption that "the ratio of exchange [between the two commodities] be that of unit for unit, or 1 to 1". As we shall see, Jenkin develops in his letters a much more sophisticated graphical apparatus, under the assumption of a variable "ratio of exchange". Yet he starts precisely from Jevons's simplified diagram, referring to the latter's assumption of a "1 to 1 exchange ratio" as "a device [...] only required to facilitate the graphical expression" (Black, 1977, p. 171).

reputation. Moreover, the attention paid by Jevons to Jenkin's contributions was certainly very high, as is confirmed by Jevons himself when he reveals that it was "partly in consequence" of the publication of Jenkin's 1870 paper on "The Graphic Representation of the Laws of Supply and Demand" that he "was led to write and publish the Theory in 1871" (Black, 1977, p. 166). Why then not a single one of the revolutionary ideas put forward by Jenkin in 1868 percolates through Jevons's mind, eventually surfacing in the first edition of *TPE*? Before answering this question, it is necessary to examine in greater detail Jenkin's ideas, as sketched in his correspondence with Jevons.

In the first letter, in discussing a standard two-trader, two-commodity model, Jenkin develops a graphical apparatus allowing him to determine the quantities of both commodities demanded or supplied by either trader, under the assumption of a given "ratio of exchange", which, in the apparent wake of Jevons, is taken equal to 1 "only [...] to facilitate the graphical expression". The two traders, respectively labelled "Jones" and "Brown", are cornered in the two commodities: Jones (resp., Brown) holds the overall endowment of "silk" (resp., "cotton"). Jenkin then proceeds as follows: first, for each trader, the marginal utility curve of one commodity is superposed on the reversed marginal utility curve of the other, the quantities of both commodities being therefore measured along the same horizontal axis, but in opposite directions; secondly, the diagram representing the two marginal utility curves of one trader is rotated by 180° on the commodity axis and then placed side by side with the other trader's diagram, in such a way that the commodity axes coincide. In the resulting diagram, reproduced as Fig. 2 below, "Jones" is the top trader, while "Brown" is the bottom one; "silk" is measured from left to right, while "cotton" is measured from right to left; "x" and "y" (resp., "x_1" and "y_1") are the quantities of "cotton" and "silk" that "Jones" (resp., "Brown") are willing to exchange: in the diagram $x = y$ and $x_1 = y_1$, since the "ratio of exchange" is provisionally taken equal to 1, in the diagram one necessarily has $x = y$ and $x_1 = y_1$ (Black, 1977, pp. 170-1. Jenkin's relevant diagram is reproduced as Fig. 2 below). By means of this device, eight variables (namely, the quantities and the marginal utilities of the two commodities for either trader) can be simultaneously represented on the same plane. Under the assumption of cardinal utility, Jenkin's apparatus provides the same information, though in a more cumbersome way, as would be provided by an Edgeworth Box apparatus describing the same economy.

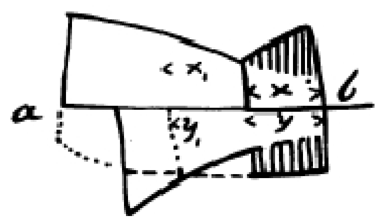

Fig. 2

Now, for Jenkin it is an easy task to show that, given an arbitrary "ratio of exchange", the two traders' demands and supplies are generally not compatible. In case they are not, Jenkin assumes, as a matter of course, that an exchange will take place all the same: the quantities actually traded are determined in conformity with the rule currently known as "the short-side rule". Specifically, in the above diagram it turns out that $x = y < x_1 = y_1$; hence, the quantities traded would be those corresponding to Jones's desires, i.e., x and y, while Brown, who happens to find himself on the long side, would be rationed. The shaded areas in the diagram represent the increases in the traders' total utilities ensuing from their trading the specified amounts of the two commodities at the given "ratio of exchange": while Jones's utility would be maximized by such a trade (under the constraint of a fixed "ratio of exchange" equal to 1), Brown's wouldn't. Now, for Jenkin, the real problem is to determine the "ratio" at which the traders are "both perfectly satisfied & would exchange neither more nor less"; but, he concludes, "I see in this theory no means of determining [that] ratio [...]" (Black, 1977, pp. 172).

Jenkin's first letter ends more or less with this negative conclusion. Yet, in the following letter(s), Jenkin makes considerable progress towards solving the problem he himself has raised, even if, as we shall see, he is the first not to be satisfied with the suggested solution. As a matter of fact, in the second letter Jenkin is able to extend his graphical apparatus in such a way as to take explicitly into account the effects of changes in the "ratio of exchange" between the two commodities, initially taken as fixed. It should be noted that, while the notation and the diagrams employed in the first letter are similar or identical to the notation and diagram (Fig. 5) used by Jevons in *TPE*, none of the developments we are now discussing can be found in those parts of *TPE* where Jevons deals with the same subject.

To get his result, Jenkin proceeds as follows. First, one of the two commodities, the same for both traders, is tacitly taken as the numeraire of the economy: the 'price' of this commodity is set equal to 1, while the "ratio of exchange" becomes the 'relative price' of the other (it should be noted that Jenkin, like Jevons, never uses the terms 'price' or 'relative price' in his discussion). Secondly, for each trader Jenkin tacitly divides the marginal utility functions of the two commodities by their respective 'prices', thereby obtaining a pair of 'weighted marginal utility' functions, whose graphs are then plotted (Black, 1977, p. 173). Finally, by progressively changing the "ratio of exchange" between the two commodities, i.e., the 'relative price' of one commodity in terms of the other taken as the numeraire, each trader's 'weighted marginal utility' curves are made to move, so that their intersection changes, describing for each trader a sequence of quantities demanded and supplied of the two commodities. The process goes on until a pair of consistent intersection points, one for each trader, is graphically discovered and a competitive equilibrium determined (Black, 1977, pp. 176, 178. Jenkin's relevant diagrams are reproduced below as Fig. 3 and Fig. 4, respectively[17]). Under the assumption of cardinal utility functions, Jenkin's graphical method of solution is analogous to, though more awkward than, the method that can be used in an Edgeworth Box in order to graphically determine the competitive equilibrium allocation: such method, as is well-known, consists in rotating the budget line around the endowment point until a common tangency point with a pair of indifference curves, one for each trader, is arrived at.

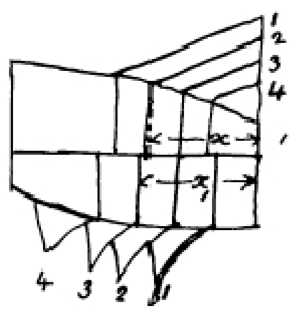

Fig. 3

[17]It should be noted that Jenkin makes a slip in drawing the diagram reproduced as Fig. 3: for, while asserting that the equilibrium rate of exchange is the "rate numbered 2" in his diagram, he keeps to this statement only as far as the top trader ("Jones") is concerned; for the bottom trader ("Brown"), instead, the equilibrium rate is mistakenly numbered '3'. The equilibrium "rate" is identified by the consistency of the traders' trade plans, i.e., by the fact that $x = x_1$.

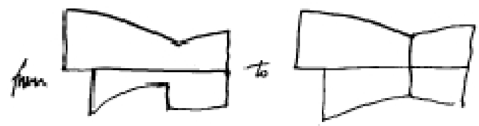

Fig. 4

On top of this, however, another method of solution is put forward by Jenkin. Instead of plotting on a plane a sequence of 'weighted marginal utility' curves, Jenkin adopts a three-dimensional representation, which can be obtained by the previous one by adding a third orthogonal axis on which to represent the changing values of the 'relative price'. Then, for each trader, two solids can be obtained, each bounded above by the surface described by the moving 'weighted marginal utility' curves. Now, as can be seen from the diagram reproduced below as Fig. 5, "the variation in the quantity [of cotton] Jones might like to exchange might be represented by the horizontal projection of the curve caused by the intersection of two solids" (Black,1977, p. 174).

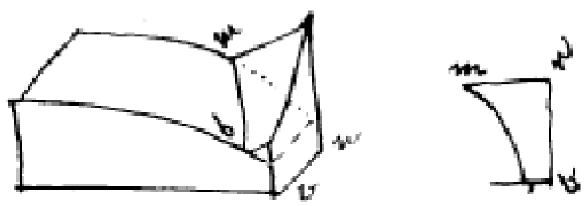

Fig. 5

Proceeding in this way, Jenkin is able to derive the demand and supply curves for either trader, which he calls the "exchange curves". Then he goes on as follows:

> Calling *mb* Jones [sic] exchange curve [for one commodity] determined as above, & *op* Browns [sic] exchange curve [for the same commodity], putting these two curves together as sketched we shall find the rate fixed by your [i.e., Jevons's] equations. (Black, 1977, p. 175. The relevant diagrams are reproduced below as Fig. 6 and Fig. 7)

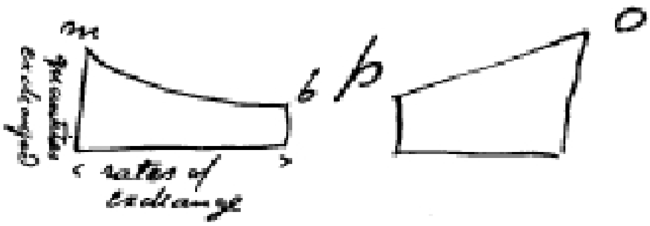

Fig. 6

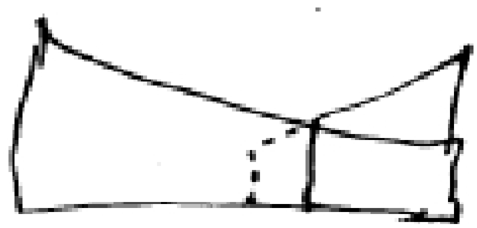

Fig. 7

8 Jenkin, Jevons and Walras on demand-and-supply analysis

Since it is practically certain that such apparatus was unknown to Jevons before his correspondence with Jenkin in 1868, Jenkin might have played with respect to Jevons a role similar to that played by Paul Piccard with respect to Walras. As is well-known, in a note written at Walras's request, presumably in October 1872, Paul Piccard, then professor of mechanics and a colleague of Walras's at the Académie de Lausanne, solves the problem of analytically deriving demand and supply functions from marginal utility functions in a two-commodity context, a problem that Walras had been unable to solve up to that time[18]. From the point of view of the formalism employed, Jenkin's contribution is less advanced and less precise than

[18] Piccard's note is reproduced both in Jaffé (1965, Vol. I, pp. 308-311) and in Walras (1993, pp. 693-5). On Piccard's role in the development of Walras's *Eléments*, see Jaffé (1977).

Piccard's: Jenkin's solution, in fact, is purely geometrical, and moreover relatively clumsy, while Piccard's is analytical and neat; furthermore, Jenkin's letters, evidently written in a hurry, are full of slips and mistakes, which however are not missing even in the much more pondered note by Piccard. On the other hand, the scope of Jenkin's discussion is wider than Piccard's, since the former provides a full graphical solution of the equilibrium problem in the two-trader case, while Piccard does not provide any analytical solution of the equilibrium problem in the exchange model with an arbitrary finite number of traders, probably because not required to do so by Walras, who had already been able to solve the problem by himself (starting, however, from empirically given aggregate demand and supply functions without any microfoundation). Somewhat paradoxically, however, the wider scope of Jenkin's note does not turn to Jevons's advantage.

As a matter of fact, while Walras is eager to grasp the manna falling from Piccard's heaven, as well as to exploit the new conceptual and analytical tools made available by his colleague in order to erect upon them his whole theoretical system, Jevons, instead, is not prepared to accept Jenkin's gift, also because that gift, unlike Piccard's, is partly poisonous. The fact is that neither Jenkin, nor, to an even greater degree, Jevons himself are willing to bear the logical consequences of adopting an approach to the solution of the equilibrium determination problem in a pure-exchange economy that will later come to be known as the Walrasian demand-and-supply, or excess-demand, approach. As for Jenkin, his reluctance to follow through the logic of the Walrasian approach can be inferred both from what he does explicitly write in his 1868 correspondence with Jevons, and from his shunning, in his later writings on demand-and-supply analysis[19], the individual demand-and-supply apparatus, based on the maximizing behavior of competitive traders, that he had himself tentatively developed in 1868. As for Jevons, his unwillingness to pay the costs implicit in adopting a demand-and-supply analysis of the Walrasian type is chiefly, but not exclusively, revealed by his bewildering omissions in *TPE*.

Let us consider Jenkin's position first. Immediately after drawing a demand-and-supply equilibrium diagram relative to the non-numeraire commodity, similar to the one reproduced in Fig. 7 above, Jenkin makes the following remark:

> If *ml* be Jones [sic] exchange curve & *np* Browns [sic] exchange

[19] Especially in Jenkin (1870).

> curve their intersection determines the rate of exchange at which each would be satisfied to exchange exactly the same quantity, but there is no motive operating on their minds to induce them to agree on this rate. (Black, 1977, p. 177)

Now, this passage may appear to raise the very same indeterminacy issue as will be raised a few years later by Edgeworth in his *Mathematical Psychics* (1881), with reference to the so-called "simple contract" problem faced by an "isolated couple" of traders. But this is really not so, for Jenkin's curves are obtained under the apparent assumption that the two traders make their trade choices as if they were competitive utility maximizers, who adjust quantities to given "rates of exchanges", while Edgeworth's "two isolated bargainers" are definitely not supposed to take prices as given nor to trade at a constant "rate of exchange" until an equilibrium is eventually reached (Edgeworth, 1981, pp. 31, 40, 42, 47-8, 109).

Now, when in a pure-exchange, two-commodity model the traders are assumed to behave as competitive utility maximizers, two consequences necessarily follow: 1) at a given 'relative price' (or "ratio of exchange", or "rate of exchange"), the traders choose unobservable trade plans, typically unexecutable, which can be turned into observable, executable trades only at equilibrium; 2) while the adjustment process towards equilibrium is brought about by progressively changing the 'relative price' in conformity to the usual price adjustment rule, according to which the change in the 'relative price' is a sign-preserving function of the aggregate excess-demand for the corresponding commodity, this change cannot itself be the product of the individual traders' choices, but can only be effected by an objective mechanism, pursuing a superindividual aim. The rules of the competitive game, therefore, seem to demand a twofold disconnection: the first rule requires that, out of equilibrium, the individuals' unobservable plans of action be distinguished from the individuals' observable actions; the second rule implies that the mechanism in charge of the equilibration process be driven by a law of motion which is independent of the individuals' rationality and preferences.

Walras, after some uncertainties and oscillations, eventually accepts the rules of the competitive game. As to the second one, it is the essence of the Walrasian *tâtonnement* construct that the adjustment towards equilibrium be carried out by the "mécanisme de la concurrence sur le marché", which is conceived as an objective device, independent of the individuals' will and desires (Walras, 1988, p. 93). As to the first rule, however, Walras yields to the requirements of the competitive discipline only towards the end of his active scientific life, after cultivating for a long time the dream that some sort of observable disequilibrium behavior can be not only allowed for, but also accounted for by the theory.

Now, if Walras, the very founder of the general equilibrium approach, wavers for such a long time about the first rule, it is certainly not surprising that Jenkin, whose theoretical awareness is incomparably inferior to Walras's, should never accept it. As a matter of fact, Jenkin's stance is peculiar: for he accepts the competitive idea that the traders make their choices by taking the 'relative price' as a fixed parameter, but he does not accept the consequent requirement that the traders' trade plans be distinguished from their actual trades and, in particular, that no trades be carried out at disequilibrium 'prices'. On the contrary, according to Jenkin, plans must always be carried out, even out of equilibrium. But then he is forced to concoct an explanation for disequilibrium behavior; and this, in turn, makes his adoption of the "short-side rule" intelligible.

As to the adjustment process, Jenkin never accepts the idea that there exists an objective mechanism, moreover of a virtual type, which is at work in the economy: for him, the driving force of the adjustment process must lie in the subjective motivations of the individual members of the economy. But then he is caught in the following dilemma: on the one hand, given his quasi-competitive premises, the 'relative price' becomes the most natural candidate for playing the role of the state variable in any plausible equilibration process; on the other, since he wants the adjustment process to be the outcome of the individuals' motivated choices, he needs to find a set of concurrent motivations leading the traders to jointly change the 'relative price' in a definite direction. But, of course, no such set of concurrent motivations can possibly exist: for, in an Edgeworth Box economy with cornered traders, making their choices as if they were competitive utility maximizers, no change in the 'relative price' can be to the advantage of both traders[20]. This negative conclusion is forcefully stressed by Jenkin in the following sentence, which refers to the same demand-and-supply equilibrium diagram as above:

> You appear to me to assume that the ratio of exchange would be that fixed by the intersection of the curves ml and np, but in order that this should be true it would be necessary that the

[20] This point is lucidly emphasized by Edgeworth (1881, p. 116).

aggregate utilities to *each* party should increase up to that point which is not true. (Black, 1977, p. 177; see also p. 178)

According to Jenkin, therefore, such a 'competitive equilibrium' "rate of exchange" can well be identified, but "it is not the true rate which will obtain in any market except accidentally", since there is "no motive power tending to change" the "rate" in the appropriate direction. (Black, 1977, pp. 177-8).

Given Jenkin's negative conclusions about the possibility of buttressing Jevons's 'competitive equilibrium' theory, i.e., his "exchange equations", with a demand-and-supply analysis of the Walrasian type, it is after all not surprising that Jevons should not feel encouraged to exploit to his advantage the tools provided by his correspondent. But the reasons underlying Jevons's refusal to adopt a Walrasian or quasi-Walrasian approach to equilibrium determination in the theory of exchange are probably even more basic than those leading Jenkin to doubt of the usefulness of that approach. For Jevons, in fact, the very distinction between a trade plan and an "act of trade", a distinction that is required by the Walrasian type of demand-and-supply analysis, is unconceivable: all trades of which one can legitimately speak are the observable outcomes of bilateral bargains involving two traders at a time; there is no such thing as a trade plan, disconnected from an "act of trade", which can in principle be carried out; in short, no counterfactuals are allowed for in the "theory of exchange".

Trading is conceived by Jevons as a physical activity, comparable to the actions, motions, and processes discussed by mechanics. Any trade is viewed as bringing about a joint change in the commodity holdings of the traders involved; such change is comparable to the joint displacement of two material points constrained to move together in ordinary space[21]. This mechanical analogy so strongly affects Jevons's reasoning that he is mistakenly led to think that, as the motion of a material point in mechanics can be obtained by integrating its velocity function with respect to time, so in economic dynamics the joint 'trajectory' described in the space of allocations by the commodity holdings of two traders involved in a bilateral trading process can (or at least should) be obtained by integrating with respect to time the 'differential' "ratio of exchange" between the two commodities determined, at each instant, by the trading activity of the two traders (Jevons, 1970, p. 138). This analogy is grossly misleading, however: for, unlike the motion of

[21] Such a simile is worked out in detail by Edgeworth (1981, p. 24), in an attempt to rationalize Jevons's approach.

a material point in space, the 'motions' of the commodity holdings of one or more "trading bodies" in the space of allocations need not take place continuously, by differential displacements, nor be viewed as a function of time. In effect, as we have seen, Jevons himself is eventually forced to conclude that, in his own "theory of exchange", the "equilibrium state" determined by the "exchange equations" must be regarded as instantaneously reached through a 'finite' trade, carried out at a constant "ratio of exchange" in a single step (Jevons, 1970, pp. 138-9, 143).

Jevons's repeated recourse to mechanical analogies, far from providing any solution to the substantive issues at stake, reveals what is perhaps the main cause of his inability to overcome the limitations of his theory: under the spell of his mechanical bias, Jevons is unable to recognize the ultimate problem of the individualistic social sciences, namely, the problem of the interaction among individually optimizing agents, a problem for which no reliable analogy can be found in the physical sciences[22].

9 Conclusions

With specific reference to the "theory of exchange", Jevons is able to identify the 'competitive equilibrium' conditions for a two-trader, two-commodity model of an Edgeworth Box economy with 'cornered' traders. But he is unable either to convincingly explain how such an equilibrium is attained, or to extend his restricted model to more general exchange economies, allowing for 'non-cornered' traders, on the one hand, and an arbitrary finite number of traders and/or commodities, on the other.

In order to correct the many shortcomings of his "theory of exchange" and to overcome its limitations, Jevons might have embraced either one of two alternative approaches: either the game-theoretic, coalitional approach, leading to the solution concept of the 'core', that Edgeworth will fully develop

[22] Contrary to what has frequently been maintained (see, e.g., Mirowski (1989)), 'marginalistic' or 'neoclassical' economics ought not to be regarded as an offspring, albeit illegitimate, of the physical sciences. Of course it is true that, at the dawning of the approach, the attempts to slavishly imitate such sciences and to find spurious analogies therein abound. But, as Jevons's, Walras's, Edgeworth's and Pareto's unsuccessful efforts in that direction abundantly prove, such endeavors have only slackened the pace of 'neoclassical' economics and hindered the understanding of its true foundations. On Pareto's 'mechanical dream', as can be found expressed especially in Pareto (1896-97), see Donzelli (1997).

in 1881, taking precisely Jevons's "theory" as his starting point; or the general competitive equilibrium approach, fully developed by Walras in 1874-7, but partly anticipated by Jenkin in his correspondence with Jevons in 1868.

Jevons is completely blind to the first way-out, even if it probably represents the most natural extension of his own view of the trading process as a sequence of bilateral bargains: Jevons's blindness to this solution is indirectly proven by his inability to write down and characterize the equation of the "contract curve", even if he comes very close to perceiving its existence. But Jevons cannot endorse the Walrasian solution either, for it conflicts with his aversion to the use of counterfactual reasoning and non-observational concepts in the "theory of exchange".

Due to these reasons, Jevons's "theory" cannot be generalized and has to remain formally restricted to the two-trader, two-commodity, 'competitive equilibrium' model, with reference to which it had originally been conceived. The surreptitious attempts made by Jevons to provide a more extensive and liberal interpretation of the "theory" explain the ambiguities surrounding some of its crucial concepts and propositions, such as the concept of a "trading body" and the so-called "law of indifference". Jevons's refusal to take prices as unknown parameters in explaining individual choices, a refusal probably linked to his physicalist or mechanical bias, prevents him not only from exploiting Jenkin's theoretical suggestions, as advanced in the latter's 1868 correspondence with Jevons, but also from employing any sort of demand-and-supply analysis of the Walrasian type in the development of his "theory of exchange", and, finally, from discussing anything like an equilibration process based on the adjustment of prices to the virtual occurrence of aggregate excess-demands on the markets for commodities.

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