

Transitional Dynamics in the Solow-Swan Growth Model with AK Technology and Logistic Population Change*

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Abstract

This paper offers an alternative way, based on the logistic population growth hypothesis, to yield transitional dynamics in the standard AK model with exogenous savings rate. Within this framework, we show that the dynamics of the capital stock per person and its growth rate can be non-monotonic over time. Moreover, even in the presence of negative growth, the capital stock per-capita can converge to a strictly positive level (different from the initial level) when time goes to infinity. In general, the analysis allows us to conclude that the dynamics of the Solow-Swan model with linear technology and logistic population growth is richer than the one with exponential population growth.

Key Words: Transitional Dynamics; AK model; Economic Growth; Population Dynamics; Physical Capital Investment

JEL Codes: C61; J10; O16; O41

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1 Introduction

One of the most relevant conclusions of the neoclassical growth model (Solow, 1956, and Swan, 1956, henceforth simply Solow-Swan) is that in the steady-state the growth rate of real per-capita output equals the rate of technological progress, which is taken as an exogenous variable. In the absence of exogenous technical change economic growth would be zero. Although neoclassical growth theory is unable to reveal the ultimate sources of long run growth, it provides a useful setting for the study of transitional dynamics and suggests that along the transition, starting from an initially low level of per-capita variables, their growth rates decline monotonically. This is the traditional (absolute and conditional) convergence result of the Solow-Swan model. As it is well known (Barro and Sala-i-Martin, 2004, Chap. 1) this result crucially hinges upon the assumption that the aggregate production function displays positive but diminishing returns to private inputs (including physical capital).

The so-called AK models (Romer, 1986; Rebelo, 1991, Jones and Manuelli, 1990) were developed between the end of the 1980s and the beginning of the 1990s as a response to the outcome of the neoclassical theory that, without technological progress, economic growth would eventually be doomed to be equal to zero. In particular, these models challenge the neoclassical assumption that returns to capital diminish as the capital stock increases and in their simplest version (Rebelo, 1991) postulate a world where the marginal and average products of capital are always constant. Under the latter hypothesis the result is reached that in the very long run per-capita GDP growth is both endogenous (in the sense that it stems from the structure of the model) and, under specific conditions on the parameter values, also persistently positive even without any technical change. However, unlike the neoclassical growth model, the standard AK formulation (Rebelo, 1991) also predicts that growth rates do not exhibit any tendency to (absolute or conditional) convergence. In other words, the model has no transitional dynamics (per-capita variables always grow at the same constant rate). This represents a substantial weakness of the model since conditional convergence seems to be an empirical regularity.

Recently, several attempts at saving the AK model from the criticism that it cannot explain convergence have been done. Acemoglu and Ventura (2002), Boucekkine et al. (2005), Carroll et al. (1997), Gomez (2008), and Kocherlacota and Yi (1995,1996) are comprehensive and rigorous examples.

Acemoglu and Ventura (2002) show how specialization and international trade may affect the process of world growth and convergence. The idea they propose is that in an open economy the parameter A will depend positively on a country's terms of trade. A country that accumulates more capital supplies more of the commodities in which it specializes to the rest of the world relative to the supply of foreign goods, and this drives its terms of trade down. The decline of the terms of trade, in turn, reduces A and discourages further accumulation of capital, so lowering the country's growth rate until it converges to the growth rate of the rest of the world (*terms-of-trade effect*).

Boucekkine et al. (2005) replace the standard hypothesis of exponential depreciation of physical capital that one may find in the basic AK model of Rebelo (1991) with the assumption that all machines have a fixed lifetime (a constant "scrapping time"). This assumption allows them "... to add to the AK model the minimum structure needed

to make the vintage capital technology economically relevant” (p. 40). They show that the use of this simple depreciation rule changes completely the dynamics of the standard AK model in the sense of making convergence to the balanced growth path no longer monotonic due to the existence of replacement echoes.

Through numerical simulations, Carroll et al. (1997) show that the introduction of habit formation¹ in the standard AK endogenous growth model may cause this model to exhibit transitional dynamics. Gomez (2008) proves formally that the convergence speed of the AK model with *external habits* is higher than that in the AK model with *internal habits*.²

Finally, Kocherlacota and Yi (1995,1996) show that the introduction of exogenous technological shocks may represent another useful approach in the direction of making the AK model consistent with the convergence result of the neoclassical growth theory.

The main objective of our paper is to offer a different way to recover transitional dynamics in standard AK-type models. The alternative framework we present here is based on the notion of population change. More concretely we extend Guerrini (2006), who has already extensively analyzed the role of a variable population growth rate within the neoclassical growth model, by assuming a linear (AK) aggregate production function without diminishing returns to capital and a logistic-type population growth law. We continue to follow Solow-Swan in assuming that the savings rate is exogenous. The reason is that we want to stress here the importance of the logistic population growth model as a mechanism able *per-se* to restoring transitional dynamics in the simplest version of the AK model. Therefore, concerning the nature of the savings rate, we make the easiest possible hypothesis (that of exogeneity).

Within this framework we show that, contrary to the standard AK model, the capital stock per-worker is not necessarily always growing. More generally, we conclude that the dynamics of the Solow-Swan model with AK production function and logistic population growth is richer than that with exponential population growth.

The paper is structured as follows. In Section 2 we present a brief reminder of the basic Solow-Swan model with AK technology and exponential population growth. In Section 3 we replace the assumption of exponential population growth with that of logistic population growth. In section 4 we characterize the steady-state equilibria of the Solow-Swan model with linear technology and logistic population growth and analyze the transitional dynamics of it. Section 5 concludes.

2 The Solow-Swan model with AK technology and constant labor growth rate: a brief reminder

There is a single good (Y_t) produced by means of only one factor of production, physical capital (K_t), according to an aggregate production function exhibiting constant returns to scale (CRS), i.e. $Y_t = F(K_t) = AK_t$, where A is an exogenous positive constant that reflects the level of the technology. Because of the presence of CRS to

¹In habit-formation models individuals’ utility depends on their current consumption and on how their current consumption compares to a reference consumption level (the habits stock).

²In models with *internal habits* the habits stock is formed from own past consumption levels. In models with *external habits*, instead, the habits stock is formed from economy-wide average past consumption levels.

the only reproducible factor-input the economy is capable of endogenous growth. The supply of savings is assumed to be proportional to aggregate income, i.e. $S_t = sY_t$, where s denotes the exogenous saving rate. Since the economy is closed and there is no public sector, the change in the capital stock equals gross investment less depreciation, i.e. $\dot{K}_t = sY_t - \delta K_t$, where δ is the constant obsolescence rate of physical capital and a dot over a variable represents differentiation with respect to time. If we define a new variable, the capital-labor ratio $k_t = K_t/L_t$, then the growth rate in the capital stock per-worker³ is given by $\dot{k}_t/k_t = \dot{K}_t/K_t - \dot{L}_t/L_t$. Therefore, by substituting the production function into the law of motion of capital, and by multiplying through by k_t , we get

$$\dot{k}_t = (N - n)k_t, \quad N \equiv sA - \delta, \quad (1)$$

where $n = \dot{L}_t/L_t$ denotes the constant labor (or population) growth rate. This first order differential equation in k_t is the fundamental equation of the Solow-Swan model with a production function which is linear in physical capital. Together with the initial condition $k_0 > 0$, Eq. (1) completely determines the entire time path of the per-capita capital stock. In addition, given this path, we can compute the paths for $y_t = Ak_t$, per-capita output, and $c_t = (1 - s)y_t$, per-capita consumption. From Eq. (1) it is also possible to conclude:

Lemma 1. *For all $t > 0$, the time path of per-worker capital stock is*

$$k_t = \frac{k_0 L_0 e^{Nt}}{L_t}. \quad (1')$$

Proof. Immediate from Eq. (1) with $N \neq n$. □

In Eq. (1'), L_t and L_0 represent the size of the labor-force at time t and at time 0, respectively. We assume $L_0 > 0$. Putting together Eqs. (1) and (1') we get the following Proposition.

Proposition 1. *If $N = n$, then any level of initial capital will be a steady-state with zero growth in the per-worker capital stock. If $N < n$, there will always be negative growth and eventually the economy will converge in the long-run to a capital stock per-worker equal to zero. If $N > n$, the economy will always grow at a positive and constant rate, irrespective of the level of the capital-labor ratio that it starts from, and in the very long-run will approach an infinitely large level of capital per-worker.*

In words, Proposition 1 says that depending on the value of N relative to n in the Solow-Swan growth model with AK technology and constant population change, economic growth can be either equal to zero (with the per-capita capital stock remaining constant over time at its initial level), or negative (with the per-capita capital stock monotonically converging over time toward zero), or else positive (with the per-capita capital stock monotonically converging over time toward infinity).

Thus, the AK model with exogenous savings rate and constant population growth predicts that physical capital accumulation can generate sustained and positive growth

³We simplify the analysis by assuming that the labor-force is equal to the total population. With this assumption per-capita and per-worker variables do coincide.

even in the absence of any disembodied technological progress: an economy that starts from a stock of capital per-worker equal to k_0 will perpetually accumulate physical capital and its capital stock per-capita will rise at a constant rate toward infinity if $N > n$. This condition states that, for this event to occur, the savings rate (adjusted by the level of technology, sA), net of the depreciation rate of physical capital (δ), should be greater than the constant population growth rate (n).

3 The Solow-Swan model with AK technology and logistic labor growth rate

In this section we modify the basic Solow-Swan model with AK technology by considering a different population growth law. In the model of the previous section population was assumed to grow according to $\dot{L}_t = nL_t$, where n is the given population growth rate. The main problem of this assumption is that population grows exponentially, i.e. $L_t = L_0 e^{nt}$, with $L_0 > 0$, and so, if $n > 0$, L_t tends to infinity as time goes to infinity, which is clearly unrealistic. To remove the prediction of unbounded population size in the very long-run Verhulst (1838) considered the hypothesis that any stable population would show a saturation level. Therefore, he proposed to augment the exponential population growth model by the multiplicative factor $-bL_t$, where b is a positive constant such that $n - bL_0 > 0$:

$$\frac{\dot{L}_t}{L_t} = n - bL_t, \quad b > 0. \quad (2)$$

This equation is known as the Verhulst equation and the underlying population growth model is known as logistic model. Sometimes Eq. (2) is written as $\dot{L}_t/L_t = n(1 - L_t/\Omega)$, where $\Omega = n/b$ is called by demographers the carrying capacity. As a function of time, Eq. (2) is a Bernoulli differential equation whose solution is

$$L_t = \frac{nL_0 e^{nt}}{n - bL_0 + bL_0 e^{nt}}. \quad (3)$$

Hence, the L_t -dependent population growth rate ($n - bL_t$) in Eq. (2) allows to have a finite limiting population, n/b . From Eq. (3), we derive that L_t is monotone increasing from L_0 to $L_\infty = n/b$. With the inclusion of a logistic-type population growth law the economy of the modified AK model is, thus, described by

$$\begin{cases} \dot{k}_t &= [N - (n - bL_t)] k_t, \\ \dot{L}_t &= L_t(n - bL_t). \end{cases} \quad (4)$$

Given $k_0 > 0$ and $L_0 > 0$, Eq. (4) becomes a Cauchy problem, which has a unique solution (k_t, L_t) defined on $[0, \infty)$.

4 Steady-state analysis and transitional dynamics

In this section we start by defining and characterizing the steady-state equilibria of the model. A steady-state in this economy is defined as a situation in which the growth

rate of the per-capita physical capital stock and the growth rate of labor are both equal to zero. We denote the steady-state equilibrium values of k_t and L_t by k_* and L_* , respectively. In studying the steady-states of our model, we confine our analysis only to interior solutions, i.e. we exclude economically meaningless solutions such as $k_* = 0$ or $L_* = 0$.

Lemma 2. *The economy has infinite steady-state equilibria if $N = 0$, and no steady-state if $N \neq 0$.*

Proof. To solve for the steady-state equilibrium we impose the growth rates in system (4) to be zero. This leads to: $Nk_t = 0$ and $n - bL_t = 0$. From these, we can conclude that, if $N \neq 0$, the economy has no (non-trivial) steady-state, whereas, if $N = 0$, for any value $k_* > 0$, the point (k_*, L_*) , where $L_* = n/b$, is a (non-trivial) steady-state equilibrium. \square

The next Lemma characterizes the evolution of the capital-labor ratio over time.

Lemma 3. *For all $t > 0$, the time path of per-worker capital is given by*

$$k_t = \frac{k_0 L_0 e^{Nt}}{L_t}. \quad (5)$$

Proof. The first equation of the dynamical system (4) is an homogenous first-order differential equation, whose solution, from elementary differential calculus, is known to be given by $k_t = k_0 e^{\int_0^t (N - \dot{L}_t/L_t) dt} = k_0 e^{Nt - \ln(L_t/L_0)} = k_0 L_0 e^{Nt}/L_t$. \square

Thus, even with logistic population growth, the time path of per-worker capital (k_t) coincides with the one we would get with exponential population growth. In particular, for given $k_0 > 0$ and $L_0 > 0$, k_t depends negatively on current population size L_t (dilution effect). The next Proposition analyzes the relationship between economic growth and the net adjusted savings rate (N) in the Solow-Swan model with AK technology and logistic population growth.

Proposition 2. *Let $\gamma_{k_t} \equiv \dot{k}_t/k_t$ and $\gamma_{L_t} \equiv \dot{L}_t/L_t$ denote the per-worker capital stock growth rate and the labor growth rate, respectively.*

- 1) *Let $N = 0$. Then $\gamma_{k_t} = -\gamma_{L_t} < 0$ for each $t \in [0, \infty)$. The growth rate of the per-capita capital stock (γ_{k_t}) tends to zero when t goes to infinity.*
- 2) *Let $N < 0$. Then $\gamma_{k_t} < 0$ for each $t \in [0, \infty)$. The growth rate of the per-capita capital stock (γ_{k_t}) is negative even when t goes to infinity.*
- 3) *Let $N > 0$. If $N \geq n - bL_0$, then $\gamma_{k_t} \geq 0$ for each $t \geq 0$. If $N < n - bL_0$, then there exists a unique $T > 0$ such that $\gamma_{k_t} < 0$ for each $t \in [0, T)$ and $\gamma_{k_t} \geq 0$ for each $t \in [T, \infty)$. The growth rate of the per-capita capital stock (γ_{k_t}) is positive when t goes to infinity. The value of T is given by $(1/n) \ln[(n - bL_0)(n - N)/bL_0N]$.*

Proof. The statement follows immediately once we rewrite $\dot{k}_t = (N - \gamma_{L_t})k_t$ as $\gamma_{k_t} = N - \gamma_{L_t}$, and recall that $0 \leq \gamma_{L_t} \leq n - bL_0$. Let $N < n - bL_0$. What we know is that both γ_{L_t} and N are less than $n - bL_0$, and this is clearly not enough to make any conclusion about the sign of γ_{k_t} . Consequently, the behavior of γ_{k_t} seems to be undetermined in this case. However, replacing Eq. (3) into Eq. (2) yields

$$\gamma_{L_t} = \frac{n(n - bL_0)}{n - bL_0 + bL_0 e^{nt}}. \quad (6)$$

Since $\dot{\gamma}_{L_t} < 0$, we derive that γ_{L_t} is a monotone decreasing function from $\gamma_{L_0} = n - bL_0$ to $\gamma_{L_\infty} = 0$. This fact allows us to conclude that there is a unique value of t , say T , where the curves γ_{L_t} and N meet each other. Moreover, $\gamma_{L_t} > N$ if $t \in [0, T)$, and $\gamma_{L_t} \leq N$ if $t \in [T, \infty)$. Finally, setting γ_{L_t} equal to N , and solving the corresponding equation we get the value T . \square

Corollary 1.

- 1) Let $N = 0$. Starting from k_0 , the capital stock k_t decreases monotonically to $k_0 L_0 b / n < k_0$.
- 2) Let $N < 0$. Then k_t decreases monotonically from k_0 to 0.
- 3) Let $N > 0$. If $N \geq n - bL_0$, then k_t increases monotonically from k_0 to ∞ . If $N < n - bL_0$, then there exists a unique $T > 0$ such that k_t decreases monotonically to k_T in $[0, T)$, and it increases monotonically to ∞ in $[T, \infty)$.

Proof. Let $N = 0$. Then Eq. (5) becomes $k_t = k_0 L_0 / L_t$. Therefore, k_t converges to the positive number $k_0 L_0 b / n$ as t grows to infinity. In addition, from Proposition 2 we know that $\gamma_{k_t} < 0$, which implies $\dot{k}_t < 0$, i.e. k_t decreases monotonically. Let $N < 0$. Again from Proposition 2 and Eq. (5), rewritten as $k_t = k_0 L_0 / (e^{-Nt} L_t)$, we can conclude that k_t decreases monotonically to 0. Let $N > 0$. If $N \geq n - bL_0$, then Proposition 2 yields that k_t is a monotone increasing function. Moreover, by taking t to infinity in Eq. (5) we get $k_\infty = \infty$. Similarly, the statement when $N < n - bL_0$. \square

In words, Proposition 2 and Corollary 1 suggest that in the Solow-Swan model with AK technology and logistic-type population growth the dynamics of γ_{k_t} and k_t is crucially related to the value of $N \equiv sA - \delta$ (the adjusted savings rate, net of physical capital depreciation). When $N = 0$, then the growth rate of the economy is always (namely, for each finite t) negative and the per-capita capital stock converges monotonically from k_0 to a positive constant, lower than k_0 . When $N < 0$, instead, the growth rate of the economy is still always negative, but now the per-capita capital stock converges monotonically from k_0 to zero. This case parallels the case where $N < n$ in the Solow-Swan model with AK technology but constant exponential population growth. Finally, when $N > 0$ the dynamics of γ_{k_t} and k_t can be monotonic or not over time depending on whether $N \geq n - bL_0$ or $0 < N < n - bL_0$. When $N \geq n - bL_0 > 0$, the dynamics of γ_{k_t} and k_t is still monotonic over time with the growth rate of the economy being always positive (or, at most, equal to zero, $\gamma_{k_t} \geq 0$) and the per-capita capital stock increasing monotonically from k_0 toward infinity. The most interesting case, however,

is when $0 < N < n - bL_0$. In this interval of parameter values the evolution over time of γ_{k_t} and k_t becomes non-monotonic. In more detail, when $t \in [0, T)$, the growth rate of the economy is negative and k_t decreases monotonically toward a finite k_T . When $t \in [T, \infty)$, the growth rate of the economy becomes positive (or, at most, equal to zero) and k_t increases monotonically from k_0 to infinity. Therefore, in the Solow-Swan model with AK technology and non-constant population growth, economic growth can be positive or equal to zero either when $n - bL_0 \leq N$ (in this case $\gamma_{k_t} \geq 0$ for each $t \geq 0$), or when $N \in (0, n - bL_0)$. In the latter case $\gamma_{k_t} \geq 0$ only for a sufficiently large t , namely $t \in [T, \infty)$. This is an important difference with the Solow-Swan growth model with AK technology and constant exponential population growth where a non-negative growth rate of variables in per-capita terms can be achieved (and for each time t) only by postulating a net adjusted saving rate ($sA - \delta$) not lower than the exogenous and constant population growth rate ($N \geq n$). Another relevant difference with the Solow-Swan growth model with AK technology and constant exponential population growth is that, even in the presence of a negative growth rate, the per-capita stock of capital can converge to a strictly positive constant when t goes to infinity (this happens whenever $N = 0$).

The next Corollary summarizes the long-run solution of the model in the variables, (k_t, L_t) depending on the value N .

Corollary 2. *The long-run behavior of the model's solution is as follows:*

$\lim_{t \rightarrow \infty} (k_t, L_t) = (k_0 L_0 b / n, n/b)$ if $N = 0$, $\lim_{t \rightarrow \infty} (k_t, L_t) = (0, n/b)$ if $N < 0$, and $\lim_{t \rightarrow \infty} (k_t, L_t) = (\infty, n/b)$ if $N > 0$.

5 Conclusion

Due to its relative simplicity, the AK model (Rebelo, 1991) has gained an important place within the new growth theory. By assuming that the marginal and average products of capital are always constant, this theory is able to predict endogenous and persistently positive growth rates even in the absence of any technological progress. The main drawback of the AK approach to economic growth, however, resides in the fact that it cannot explain convergence: per-capita variables always grow at the same constant rate (the model does not exhibit any transitional dynamics). In this paper we have proposed a different way (alternative to other possible solutions proposed till now by economic literature) to get back transitional dynamics in the AK model with exogenous savings rate. Our idea consists in introducing a logistic-type population growth law in an otherwise standard Solow-Swan model with linear aggregate technology. As it is well known, the main problem behind the assumption of constant population growth is that as time goes to infinity population size goes to infinity as well, which is clearly unrealistic. Using the logistic, as opposed to the exponential, population growth hypothesis (Verhulst, 1838) has the advantage that population size tends to a finite saturation level in the very long-run. We have shown that under this hypothesis the standard AK model is able to generate transitional dynamics. More generally, we can conclude that the dynamics of the Solow-Swan model with AK production function and logistic population growth is richer than the one with exponential population

growth. For future research it would be interesting to analyze whether and, eventually, how the logistic population growth hypothesis might affect the dynamics of other and more sophisticated endogenous growth models (such as those with endogenous technological progress).

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