Spatial dynamics and convergence: The spatial AK model

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Abstract

We study the optimal dynamics of an AK economy where population is uniformly distributed along the unit circle. Locations only differ in initial capital endowments. Despite constant returns to capital, we prove that transition dynamics will set in. In particular, we prove that the spatio-temporal dynamics, induced by the willingness of the planner to give the same (detrended) consumption over space and time, lead to convergence in the level of capital across locations in the long-run.

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1 Introduction

The issue of optimal and market allocation of economic activity across space has always been a central issue in economic theory from the seminal works of Hotelling (1929) and Salop (1979). Many authors have already investigated in depth the existence (and sometimes the non-existence) of optimal and/or market allocation in static models (among them, Starrett, 1974 and 1978). Recently, some authors have tried to tackle the issue of the spatial allocation of economic activity in dynamic frameworks, namely within economic growth frameworks (thus, with capital accumulation). This research line, initiated by Brito (2004), is nicely surveyed by Desmet and Rossi-Hansberg (2010). Two aspects turn out to be crucial: factor mobility and space modelling. On the first aspect, Brito and Boucekkine, Camacho and Zou (2009) consider frictionless capital mobility while Brock and Xepapadeas (2008) use the trick of a spatial externality to model the spatial component (without capital mobility). In the former papers, the production function at any place is neoclassical (decreasing returns in capital): capital flow from regions with low marginal productivity of capital to regions with high marginal productivity.

Concerning spatial modelling, both Brito and Boucekkine et al. consider an infinite spatial line, symmetrical to the infinite time horizon adopted in standard growth theory. But while Boucekkine et al. introduce spatial discounting (again mimicking time discounting) to ensure the convergence of the resulting utilitarian objective function, Brito postulates a non-utilitarian objective function and gets rid of spatial discounting. Brock and Xepapadeas (2008) take a compact interval to model space, therefore avoiding this technical problem. A striking finding of Boucekkine et al. is that the model with infinite time and space support, and utilitarian objective function with strictly concave preferences, which is admittedly the most natural candidate for spatial Ramsey model, suffers from a structural ill-posedness problem: in contrast to the non-spatial Ramsey model, the initial value of the co-state variable is no longer sufficient to determine the optimal paths. Rather than considering a non-standard objective function as in Brito (2004), we follow here the example of Brock and Xepapadeas (2008) and consider a more realistic compact modelling of space. However, in sharp contrast to these authors, we do not consider a compact interval on the real line but a circle, which is another traditional modelling of space in economics (Salop, 1979). A very important advantage of our modelling is that it does not require the specification of any space boundary conditions, to which the solution paths

are arguably extremely sensitive. We also keep capital mobility in line with Boucekkine et al. (2009).

In order to gather at least some partial clean analytical results, we will consider an AK technology. Using recent developments in dynamic programming in infinitely-dimensioned problems (Bensoussan et al., 2007), previously implemented in the literature of vintage capital theory (Fabbri and Gozzi, 2008), it is possible to identify an explicit value function under quite reasonable parametric conditions. As in the standard AK model, a benevolent social planner will choose a constant consumption level (that is detrended consumption): in our spatial set-up, this turns out to be true not only over time but also over space assuming homogeneous preferences and technology over space. Also, we will show that the aggregate capital stock, that's the aggregation of capital stocks across space, lies on a balanced growth path from $t = 0$, featuring the absence of transitional dynamics for the aggregate variables as in the standard AK theory. A major and striking departure from the standard theory will nonetheless emerge: while the aggregate capital stock shows no dynamics, an intense reallocation of capital will take place across space giving rise to transition dynamics at any point in space. Even more strikingly, these spatio-temporal dynamics will lead to the convergence of (time detrended) capital stocks across space to the same common value whatever the initial spatial distribution of capital under certain realistic assumptions.

Two crucial points should be made here. First of all, the spatial dynamics generated in the AK case are essentially different from those exhibited in Boucekkine et al.: in the former, there are no decreasing returns to capital, and the whole dynamics derive from the will of the benevolent planner to keep consumption level constant over time and space, and to move capital accordingly. Second, the similarity between the AK vintage model (as studied initially in Boucekkine et al., 2005, and revisited by Fabbri and Gozzi, 2008) and the AK spatial model should be noted here: in both, capital has a double heterogeneity (time and vintage in the vintage AK model, and time and space in the spatial AK), resulting in finitely-dimensioned optimal control problems; in both detrended consumption is constant, and capital transition dynamics emerge.

This said, the spatial modelling adopted in this paper allows to get much more specific and striking results concerning the property of economic convergence as outlined above. In our setting, returns to capital are constant, and one would not take for granted convergence of capital stocks over space,

that's transition dynamics equalizing capital across space. Typically, decreasing returns to capital are invoked as the engine of convergence in the neoclassical growth model. Things are much more involved if one keeps in mind that in our framework initial capital stock is heterogenous across space, which could be at first glance related to the literature of growth with heterogenous initial endowments (initiated by Stiglitz, 1969, and pushed forward by Chatterjee, 1994).[a] In this literature, decreasing returns to capital are not enough to ensure convergence of wealth across individuals: if they differ only in their initial endowments and share the same homothetic preferences, initial inequalities will be reproduced for ever! In a sense, our paper establishes a kind of symmetric property: convergence is obtained under constant return to capital. This said, one should not omit the specific nature of spatial dynamics. Among other specific results, we analytically show that convergence of capital stocks across space is guaranteed when the time discount rate is not to big, a requirement which does not appear in the growth literature with heterogenous endowments.

The paper is organized as follows. Section 2 sketches the model. Section 3 displays the analytical results. Section 4 provides some numerical illustrations and exercises regarding the convergence and divergence questions outlined above. Section 5 concludes.[b]

2 The model

We first present the law of motion of capital in time and space. We shall follow here the very generic modelling adopted in the related literature (as recently re-used by Boucekkine et al., 2009). At a given point $(t, x) \in [0, \infty) \times \mathbb{R}$, physical capital $k(t, x)$ evolves according to

$$\frac{\partial k}{\partial t}(t, x) = Ak(t, x) - c(t, x) - \tau(t, x). \tag{1}$$

A represents the level of technology, which we assume to be constant over time and space. The production function is AK at any point in space. $c(t, x)$ and $\tau(t, x)$ stand respectively for consumption and trade balance at the trade balance at $(t, x) \in [0, \infty) \times \mathbb{R}$. Capital depreciation is zero everywhere. Finally, we assume that there is no adjustment or transportation cost when moving capital from a location to another. Such costs traditionally generate

[a] See also Caselli and Ventura (2000) for a more recent contribution.

[b] We report the heavy mathematical apparatus needed to solve the model in the appendix.

non-instantaneous adjustment (to the optimal outcomes), we switch off this dynamics engine in this model to only rely on spatial dynamics (if any).

Now we come to the space modelling. We assume that individuals are distributed homogeneously along the unit circle, which we denote by $\mathbb{T}$:

$$\mathbb{T} := \left\{ x \in \mathbb{R}^2 \ : \ |x| = 1 \right\}$$

Two remarks are in order here. First of all, our assumption of a non-growing and spatially homogeneous population distribution is made for convenience. Considering an heterogenous spatial population distribution would lead to introduce space weights in the objective social welfare function, which will break down the simple analytical solution derived in this paper. We prefer sticking to our analytical case for clarity. Second, as argued in the introduction section, the choice of the unit circle to represent space is not innocuous, it allows to avoid the specification of boundary conditions, which use to condition decisively the shape of optimal dynamics. It is worth mentioning here that the results obtained along this paper are robust to certain modifications of the space geometry. More precisely, we can study the problem on a more general n-dimensional compact and connected manifold without boundary but the mathematical apparatus has to be adapted accordingly.[c]

Let us focus on the trade balance in region B, an arc of the circle. We have that

$$\int_B \tau(t,x)\,\mathrm{d}x = \frac{\partial k}{\partial x}(t,b) - \frac{\partial k}{\partial x}(t,a),$$

the regional trade balance is equal to the capital entering from one or its borders less the capital flowing away from the other border. Using the Fundamental Theorem of Calculus we can rewrite the above expression as

$$\int_B \tau(t,x)\,\mathrm{d}x = \int_B \frac{\partial^2 k}{\partial x^2}(t,x)\,\mathrm{d}x. \tag{2}$$

Hence the following equality holds in region B:

$$\int_B \left(\frac{\partial k}{\partial t}(t,x) - \frac{\partial^2 k}{\partial x^2}(t,x) - Ak(t,x) + c(t,x) \right) \mathrm{d}x = 0. \tag{3}$$

The evolution of physical capital at any point $(t,x) \in \mathbb{R}^+ \times B$ is obtained using (3) when the length of B tends to zero, for any B arc of $\mathbb{T}$. Provided

[c] In this more general setting the Laplace operator on $\mathbb{T}$ considered here has to be replaced by the so-called Laplace-Beltrami operator on the manifold.

an initial distribution of physical capital $k_0(\cdot)$ on $\mathbb{T}$, the policy maker has to choose a control $c(\cdot,\cdot)$ to maximize the following functional

$$\max_{c(\cdot,\cdot)} J(k_0, c(\cdot,\cdot)) := \int_0^{+\infty} e^{-\rho t} \int_{\mathbb{T}} \frac{(c(t,x))^{1-\sigma}}{1-\sigma} \, \mathrm{d}x \, \mathrm{d}t. \tag{4}$$

subject to the state equation

$$\begin{cases} \dfrac{\partial k}{\partial t}(t,x) = \dfrac{\partial^2 k}{\partial x^2}(t,x) + Ak(t,x) - c(t,x) \\ k(t,0) = k(t,2\pi) \\ k(0,x) = k_0(x). \end{cases} \tag{5}$$

We define $\mathcal{U}_{k_0}$ as the set of admissible controls for the problem (4)-(5), which depends on the initial distribution k_0:

$$\mathcal{U}_{k_0} := \left\{ c(\cdot,\cdot) \in L^2(\mathbb{R}^+ \times \mathbb{T}; \mathbb{R}^+) \ : \ k(t,x) \geq 0 \ \text{ for all } (t,x) \in \mathbb{R}^+ \times \mathbb{T} \right\} \tag{6}$$

The value function of our problem starting from k_0 is defined as

$$V(k_0) := \sup_{c(\cdot,\cdot) \in \mathcal{U}_{k_0}} J(k_0, c(\cdot,\cdot))$$

A few comments are in order here. In first place, it should be noted that the social welfare objective function considered "treats" equally all the individuals independently of their location, that is independently of their initial capital endowment. One would invoke ethical criteria to justify a spatially-weighted objective function, for example to account for initial inequalities. Here we show that without this kind of weighting, capital stocks will end up equalized across space. It should be also noted that we concentrate here on planner's optimal solutions, we do not address the issue of decentralization, which would probably bring us close to consider spatial weighting in the spirit of Negishi (1960). The next section is devoted to uncover the main analytical properties of our central planner problem.

3 Spatial dynamics in the AK model: analytical results

We shall start by characterizing a crucial property of the optimal solution: As in the pre-exiting AK frameworks (with or without vintage capital), the planner will choose a constant consumption level over time and space, and

all aggregate variables will grow at a constant growth rate from $t = 0$. The next proposition establishes the existence of a unique explicit value function and gives a first characterization of optimal consumption.

Theorem 3.1. *Suppose that*

$$A(1-\sigma) < \rho \tag{7}$$

and consider $k_0 \in L^2(\mathbb{T})$ *a positive initial distribution of physical capital. Define*

$$\eta := \frac{\rho - A(1-\sigma)}{2\pi\sigma}. \tag{8}$$

Provided that the trajectory $k^*(t,x)$ *driven by the feedback control (constant in* x*)*

$$c^*(t,x) = \eta \int_{\mathbb{T}} k^*(t,y)\,\mathrm{d}y \tag{9}$$

remains positive,[d] $c^*(t,x)$ *is the unique optimal control of the problem. Moreover the value function of the problem is finite and can be written explicitly as*

$$V(k_0) = \alpha \left(\int_{\mathbb{T}} k_0(x)\,\mathrm{d}x \right)^{1-\sigma} \tag{11}$$

where

$$\alpha = \frac{1}{1-\sigma} \left(\frac{\rho - A(1-\sigma)}{2\pi\sigma} \right)^{-\sigma}. \tag{12}$$

Proof. See Appendix B. □

The theorem deserves some comments. First of all, the parametric condition (7) is exactly the one needed in the counterpart standard AK theory to assure that the objective function is finite (or equivalently, that the transversality condition is met). It is also needed here to guarantee that the value function of the problem is finite. The value function $V(k_0)$ is given explicitly by (11): notice that it depends only on the aggregate initial capital stock. Two different capital distributions will give the same value function, and

[d] The trajectory driven by the feedback control is the unique solution of

$$\begin{cases} \dfrac{\partial k(t,x)}{\partial t} = \partial_x^2 k(t,x) + Ak(t,x) - \eta \displaystyle\int_{\mathbb{T}} k(t,y)\,\mathrm{d}y \\ k(t,0) = k(t,2\pi) \\ k(0,x) = k_0(x). \end{cases} \tag{10}$$

therefore the same optimal consumption rule (here given by equation (9)): the optimal solution features therefore a kind of pooling of resources, which is hardly surprising given the social optimum framework considered here. Finally, one has to notice that both the value function and the feedback control are similar to those of the standard AK model. However, the proportionality parameters η and α are specific to the space modelling adopted in this paper. We now extract the optimal solutions for the aggregate capital stock and the induced optimal consumption rule at any point (t, x).

Proposition 3.2. *Under the same assumptions of Theorem 3.1 the aggregate capital* $K(t) := \int_{\mathbb{T}} k(t, y)\,\mathrm{d}y$*, along the optimal trajectories, is*

$$K(t) = K(0)e^{\beta t} \tag{13}$$

where

$$\beta := \left[\frac{A-\rho}{\sigma}\right]. \tag{14}$$

and $K(0) := \int_{\mathbb{T}} k_0(y)\,\mathrm{d}y$.

Proof. See Appendix B. □

Notice that the optimal aggregate capital stock evolves exactly as in the standard AK model: it grows at the same standard growth rate, $\beta = \frac{A-\rho}{\sigma}$, from $t = 0$. From (7), one can then use (13) to obtain that the optimal control is $c^*(t, x) = \eta K(0)e^{\beta t}$. As announced repeatedly above, we obtain the crucial property that the planner will choose the same (detrended) consumption level for all individuals whatever their location and generation. Thus, the aggregate variables have no transition dynamics just like in the standard AK theory. Nonetheless, if the planner is willing to keep (detrended) consumption constant over time and space, he has to move capital over time and space accordingly. What could be the induced optimal spatio-temporal dynamics? The next theorem gives the answer to this appealing question. Indeed, once we obtain an explicit solution for the optimal choice of the policy maker, we can replace it into (5) to study the evolution of physical capital. This can be done explicitly in our setting under an additional parametric condition.

Theorem 3.3. *Assume that the hypotheses of Theorem 3.1 are satisfied. Suppose that*

$$\rho < A(1-\sigma) + \sigma. \tag{15}$$

Then, along the optimal trajectory, the detrended capital

$$k_D(t,x) := \frac{k(t,x)}{e^{\beta t}} \tag{16}$$

converges uniformly (and a fortiori *pointwise), as a function of x, to the constant function $\frac{K(0)}{2\pi}$ when t tends to infinity. In other words*

$$\lim_{t\to\infty}\left(\sup_{x\in\mathbb{T}}\left|k_D(t,x)-\frac{K(0)}{2\pi}\right|\right)=0.$$

Proof. See Appendix B. □

Theorem 3.3 is the main result of this paper: it shows that the spatio-temporal dynamics induced by the willingness of the planner to give the same (detrended) consumption over space and time lead to equalize the capital level across locations in the long-run. Hence, spatio-temporal dynamics do eliminate the initial inequalities in capital endowments in our spatial AK model. As mentioned in the introduction, this result is striking in many respects, notably with respect to the traditional theory of convergence, which typically builds on representative agents settings. In the latter, decreasing returns to capital drive convergence while AK technologies are associated with divergence (due to the absence of transition dynamics). If one has in mind the existing (rather thin) literature on the link between growth and inequalities where individuals have heterogenous initial endowments, then our result could be interpreted at first glance as symmetrical to the traditional property highlighted by Chatterjee (1994): under decreasing returns to capital, initial inequalities will be reproduced for ever. This said, the spatial dynamics entailed in our model are specific (though they derive from a generic law of motion of capital in space and time). In particular, our convergence result is guaranteed when $\rho < A(1-\sigma)+\sigma$. Otherwise, everything can happen. Notice however, that this condition is very largely satisfied in realistic parameterizations of the model. The next section provides a complementary numerical investigation.

4 Spatial dynamics in the AK model: computational results

We perform a numerical exercise to shed more light on the transitional dynamics of the spatial AK-model[e]. We use the isomorphism between the circumference $\mathbb{T}$ and the closed interval $[0, 2\pi]$ once we have identified the two boundary points 0 and 2π. There are three parameters key to our modelling, ρ, A and σ. In the Ak-model, $A = Y/k$ and consequently we choose $A = 1/3$ as a reasonable ratio output to physical capital. Then we fix $\rho = 0.07$ and $\sigma = 0.8$, the set $\{\rho, A, \sigma\}$ satisfies both (7) and (15).

Table 1: Parameters values

Total Factor Productivity	A	1/3
Time discount rate	ρ	0.07
Intertemporal elasticity of substitution	σ	0.8

Provided that conditions (7) and (15) hold[f], Theorem 3.3 proves that detrended capital at any point (t, x) converges to the constant function $\frac{K(0)}{2\pi}$ when t tends to infinity. To illustrate this result, we study the case of an economy made of two regions, the first region, $[0, \pi]$ is initially endowed with twice the capital of the second region, $[\pi, 2\pi]$, namely

$$k_0(x) = \begin{cases} 20, x \in [0, \pi], \\ 10, x \in [\pi, 2\pi]. \end{cases}$$

Theorem 3.3 shows that the policy maker allocates the same amount of consumption to all locations x at any time t. In the standard Ak-model the optimal trajectory instantaneously adjusts to the optimal consumption and production plan. This behavior does no longer hold when physical capital is allowed to move across space as figures 1 and 2 show. Capital moves from

[e]Provided that conditions (7) and (15) hold, the optimal trajectory of physical capital can be simulated using the Fourier series of k_D:

$$k_D(t, x) = \frac{K(0)}{2\pi} + \sum_{\substack{n \in \mathbb{N} \\ n \neq 0}} \frac{e^{(A - n^2 - \beta)t}}{\sqrt{\pi}} \left(\cos(nx) \int_{\mathbb{T}} k_0(x) \cos(nx) dx + \sin(nx) \int_{\mathbb{T}} k_0(x) \sin(nx) dx \right)$$

For our exercise $n = 20$.

[f]For $A = 1/3$ and $\sigma = 0.8$, the range of values for ρ which satisfies conditions (7) and (15) is $]0.0667, 0.8667[$.

rich locations towards poor ones, but adjustment is not instantaneous since it takes time for capital to achieve its final location. As capital moves a wave appears and it brings about the emergence of a temporary production agglomeration in the rich region. Simultaneously, a depressed area is formed in the center of the poor region. When capital moves from left to right, all locations in the rich region send capital to the poor region to reach the optimal path. The depressed area appears since capital moves from $t = 0$ and from left to right, also within the poor region, without waiting for the locations to achieve a level close to optimal to pass capital. With time, both the agglomeration and depression lose force and we can observe a complete convergence to the spatially homogenous steady state value for detrended capital from $t = 10$. The graphs show physical capital across space at different moments: on the top panel of figure 1 at times $t = 0.02$, $t = 0.2$, $t = 1$ and $t = 10$. The bottom panel shows the optimal trajectory of detrended physical capital for $(x, t) \in [0, 2\pi] \times [0, 8]$[g].

Figure 2 illustrates the importance of condition (15). For this example $\rho = A(1-) + \sigma = 0.8667$ and we can see that detrended capital does not converge to an spatially homogenous distribution. A noteworthy aspect is that detrended physical capital does converge to a steady state, and it does so very fast.

[g]Convergence towards the steady state would have been slower if transportation costs, adjustment costs, capital depreciation or trade barriers had been present.

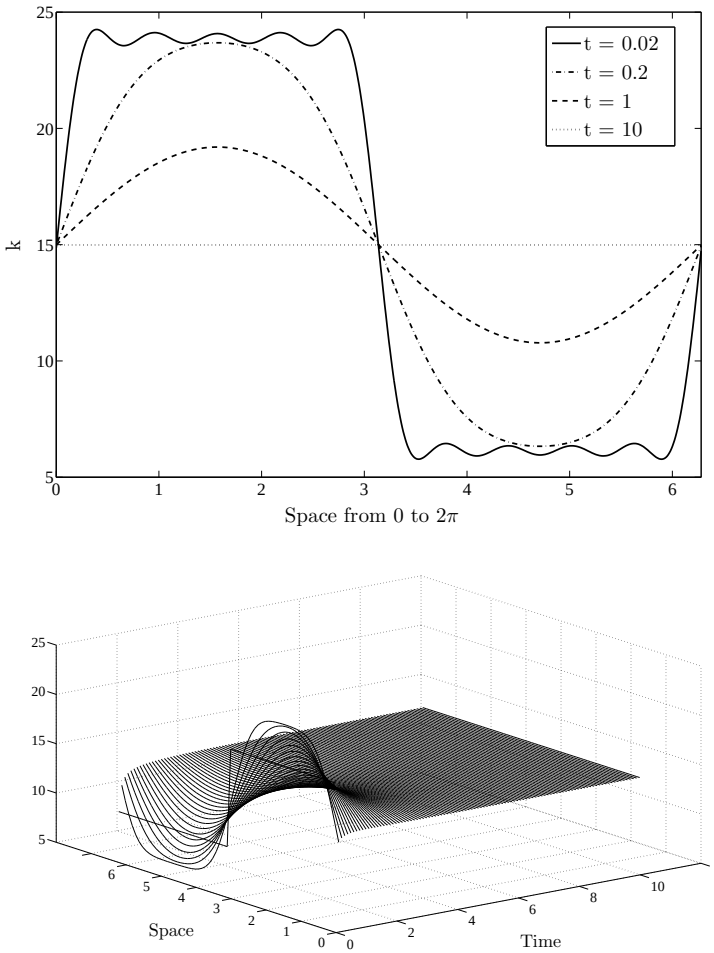

Figure 1: On the top panel: physical capital distribution at $t = 0.02, 0.2, 1, 10$. The bottom panel shows $k(t, x)$ where $(t, x) \in [0, 8] \times [0, 2\pi]$.

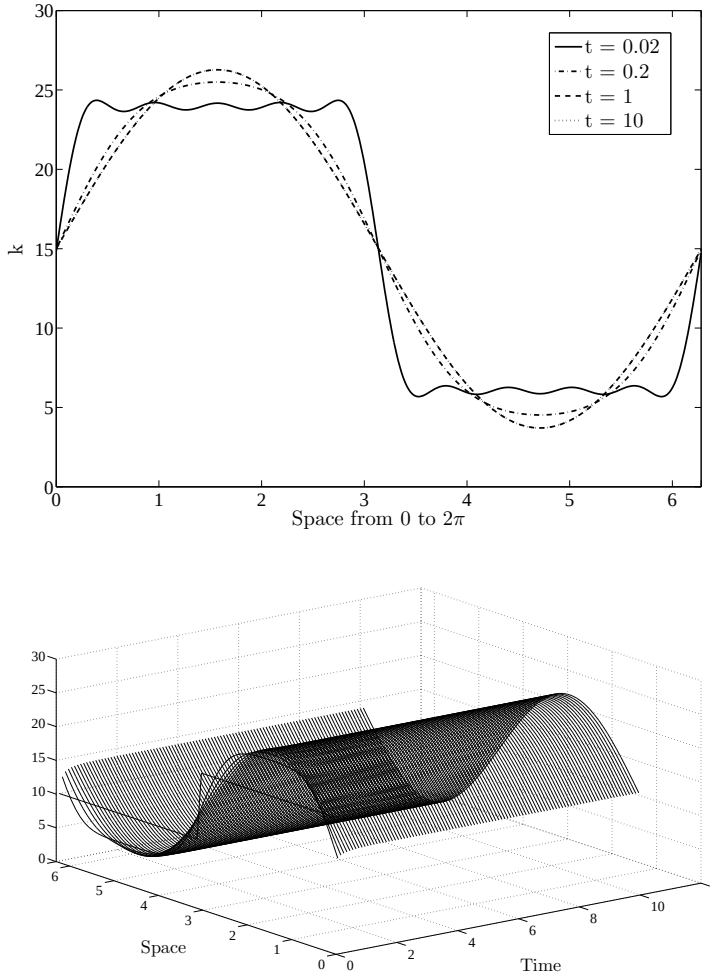

Figure 2: On the top panel: physical capital distribution at $t = 0.02, 0.2, 1, 10$. The bottom panel shows $k(t, x)$ where $(t, x) \in [0, 8] \times [0, 2\pi]$.

5 Concluding remarks

In this paper, we have shown how spatial dynamics, introduced through a generic law of motion of capital, interact with the mechanics inherent to the AK model. The main result of our work is that the spatio-temporal dynamics, induced by the willingness of the planner to give the same (detrended) consumption over space and time, lead to convergence in the level of capital across locations in the long-run. An important opened line of research is to study to which extent our results are robust to the introduction of spatial weights in the objective function to cope with specific ethical criteria. Another one, more ambitious, would address decentralization in an attempt to extend Negishi's work to this type of spatial models.

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Appendices

A The Hilbert space setting

Consider the Hilbert space $L^2(\mathbb{T})$ and define the operator

$$\begin{cases} D(G) := H^2(\mathbb{T}) \\ G(f) = \Delta f. \end{cases}$$

G is the generator of the C_0-semigroup e^{tG} (the heat semigroup) on $L^2(\mathbb{T})$. The operator G is self-adjoint (so $D(G^*) = D(G)$ and $G^*\phi = G\phi = \Delta\phi$). The state equation (5) can be rewritten as an evolution equation in $L^2(\mathbb{T})$

$$\begin{cases} \dot{k}(t) = Gk(t) + Ak(t) - c(t) \\ k(0) = k_0. \end{cases} \tag{17}$$

With an abuse of the notation we denote by $k(\cdot)$ the solution of (17). This is justified by the fact that $k(t)(x) = k(t,x)$ where the latter is the solution of (5). The set of the admissible controls in the infinite dimensional setting is $\left\{c \in L^2\left(\mathbb{R}^+; L^2(\mathbb{T};\mathbb{R}^+)\right) \,:\, k(t)(x) \geq 0 \text{ for all } (t,x)\right\}$. It is exactly equivalent to the set $\mathcal{U}_{k_0}$ defined in (6) so the abuse of the notation "c" is justified too. The mild form of (17) is

$$k(t) = e^{Gt}k_0 + \int_0^t e^{(t-s)G}\left(Ak(s) - c(s)\right) \mathrm{d}s.$$

Let us denote by $\langle \cdot, \cdot \rangle$ the scalar product on $L^2(\mathbb{T})$ and by $\mathbb{1}$ the function

$$\begin{cases} \mathbb{1} \colon \mathbb{T} \to \mathbb{R} \\ \mathbb{1}(x) \equiv 1. \end{cases}$$

Note that $\mathbb{1} \in D(G)$. Using such a notation the functional (4) can be rewritten as

$$\int_0^{+\infty} e^{-\rho t} \langle \mathbb{1}, U(c(t)) \rangle \, \mathrm{d}t$$

where, for a given $\eta \in L^2(\mathbb{T})$, $U(\eta) \colon \mathbb{T} \to \mathbb{R}$ is the function $U(\eta) \colon x \mapsto \frac{(\eta(x))^{1-\sigma}}{1-\sigma}$.

The Hamilton-Jacobi-Bellman equation (HJB equation hereafter) of the system is defined as

$$\rho v(k) = \langle k, G\nabla v(k) \rangle + A \langle k, \nabla v(k) \rangle + \sup_{c \in L^2(\mathbb{T};\mathbb{R}^+)} \left\{ - \langle c, \nabla v(k) \rangle + \langle \mathbb{1}, U(c) \rangle \right\}. \tag{18}$$

B Proofs

This sections contains the proofs of the theorems and of the propositions presented in the text. We use the setting and the notation introduced in Appendix A. The

statement of the theorems in Section 3 are formulated, for the reader convenience, without using the infinite dimensional language (we used there the PDE formalism), but, of course, they are completely equivalent to the related infinite-dimensional ones we present here.

Proof of Theorem 3.1. We prove the theorem using the dynamic programming. The proof is presented in some steps:

Step 1: *We find an explicit solution*[h] *of the HJB equation (18) on the open set* $\Omega := \left\{k \in L^2(\mathbb{T}) \ : \ \langle k, \mathbb{1}\rangle > 0\right\}$.

We look for a solution of (18) of the following form: $v(k) = \alpha \langle k, \mathbb{1}\rangle^{1-\sigma}$ for some positive real number α, so that $\nabla v(k) = \alpha(1-\sigma)\langle k, \mathbb{1}\rangle^{-\sigma}\mathbb{1}$. Note that, for all $k \in \Omega$, $\nabla v(k) \in D(G)$ and that $\nabla v \colon \Omega \to D(G)$ is continuous (when $D(G)$ is endowed with the graph norm). Substituting in (18) we obtain:

$$\rho\alpha \langle k, \mathbb{1}\rangle^{1-\sigma} = \alpha(1-\sigma)\langle k, \mathbb{1}\rangle^{-\sigma}\langle k, G\mathbb{1}\rangle + A\alpha(1-\sigma)\langle k, \mathbb{1}\rangle^{-\sigma}\langle k, \mathbb{1}\rangle + \sup_{c \in L^2(\mathbb{T};\mathbb{R}^+)} \left\{-\alpha(1-\sigma)\langle k, \mathbb{1}\rangle^{-\sigma}\langle c, \mathbb{1}\rangle + \langle \mathbb{1}, U(c)\rangle\right\}.$$

Observing that $G\mathbb{1} = 0$ and that the supremum is attained when $c = (\alpha(1-\sigma))^{-1/\sigma}\langle k, \mathbb{1}\rangle\mathbb{1}$ the expression above becomes:

$$\rho\alpha \langle k, \mathbb{1}\rangle^{1-\sigma} = A\alpha(1-\sigma)\langle k, \mathbb{1}\rangle^{1-\sigma} - 2\pi\alpha(1-\sigma)(\alpha(1-\sigma))^{-1/\sigma}\langle k, \mathbb{1}\rangle^{1-\sigma} + 2\pi\frac{\left[(\alpha(1-\sigma))^{-1/\sigma}\langle k, \mathbb{1}\rangle\right]^{1-\sigma}}{1-\sigma}. \quad (19)$$

From (19) we have

$$\rho = A(1-\sigma) - 2\pi(1-\sigma)(\alpha(1-\sigma))^{-1/\sigma} + 2\pi(\alpha(1-\sigma))^{-1/\sigma}$$

so there exists a solution of the requested form when

$$\alpha = \frac{1}{1-\sigma}\left(\frac{\rho - A(1-\sigma)}{2\pi\sigma}\right)^{-\sigma}.$$

Before passing to step 2 we make an observation that will be useful later: given an admissible (and hence positive) control $c(\cdot)$, the related trajectory $k(\cdot)$ is given by the solution of (17). Therefore at every time and at every point x of the space $k(\cdot)$ remains below the solution of $\dot{\bar{k}}(t) = G\bar{k}(t) + A\bar{k}(t)$ with $\bar{k}(0) = k_0$. In particular,

[h] We say that v from some open set O in $L^2(\mathbb{T})$ to $\mathbb{R}$ is a solution (on O) of (18) if $v \in C^1(O; D(G))$ (where $D(G)$ is endowed with the graph norm) and v solves (18) at every point of O.

for all $t \geq 0$, $\left\langle \bar{k}(t), \mathbb{1} \right\rangle \geq \langle k(t), \mathbb{1} \rangle$. $\bar{k}(t)$ can be expressed as $\bar{k}(t) = e^{tA} e^{tG} k_0$ so that $\left\langle \bar{k}(t), \mathbb{1} \right\rangle = e^{tA} \left\langle \bar{k}_0, \mathbb{1} \right\rangle$. This means that for every choice of $c(\cdot)$ we have

$$\begin{aligned} \left| e^{-\rho t} v(k(t)) \right| &= e^{-\rho t} \alpha \left\langle k(t), \mathbb{1} \right\rangle^{1-\sigma} \leq e^{-\rho t} \alpha \left\langle \bar{k}(t), \mathbb{1} \right\rangle^{1-\sigma} \\ &= e^{-\rho t} e^{tA(1-\sigma)} \left\langle \bar{k}_0, \mathbb{1} \right\rangle = e^{-(\rho - A(1-\sigma)t)} \left\langle \bar{k}_0, \mathbb{1} \right\rangle \xrightarrow{t \to \infty} 0 \end{aligned} \tag{20}$$

where we obtain the last limit thanks to Hypothesis (7).

Step 2: *We prove that the feedback control provided by the solution is admissible.*

The feedback control provided by the solution is

$$\begin{cases} \phi \colon L^2(\mathbb{T}) \to L^2(\mathbb{T}) \\ \phi(k) := \arg\max_{c \in L^2(\mathbb{T})} \left\{ -\alpha(1-\sigma) \left\langle k, \mathbb{1} \right\rangle^{-\sigma} \left\langle c, \mathbb{1} \right\rangle + \left\langle \mathbb{1}, U(c) \right\rangle \right\} = \\ \qquad\qquad = (\alpha(1-\sigma))^{-1/\sigma} \left\langle k, \mathbb{1} \right\rangle \mathbb{1} = \eta \left\langle k, \mathbb{1} \right\rangle \mathbb{1} \end{cases} \tag{21}$$

where $\eta = \frac{\rho - A(1-\sigma)}{2\pi\sigma}$, the related trajectory is the solution of the mild equation

$$k(t) = e^{Gt} k_0 + \int_0^t e^{(t-s)G} \left(Ak(s) - \eta \left\langle k, \mathbb{1} \right\rangle \mathbb{1} \right) \mathrm{d}s. \tag{22}$$

Proving that such an equation has a unique solution $k^*(t)$ and that $k^*(t)(x) = k^*(t,x)$ where $k^*(t,x)$ is the solution of (10) are standard facts (see for example Bensoussan et al. (2007)). The control we want to prove to be admissible (and that we prove to be optimal in step 3) is that defined, for all $t \geq 0$, as $c^*(t) := \phi(k^*(t))$. Since by hypothesis $k^*(t)(x)$ remains positive, then $c^*(t)$ remains positive too and then it is admissible.

Step 3: *We prove that the feedback control provided by the solution is optimal (proving at the same time that the solution of the HJB equation we found is indeed the value function).*

To prove that $c^*(\cdot)$ is an optimal control we have to prove that for every other admissible control $\tilde{c}(\cdot)$ we have $J(k_0, c^*(\cdot)) \geq J(k_0, \tilde{c}(\cdot))$.

Let us call $\tilde{k}(\cdot)$ the trajectory related to the admissible control $\tilde{c}(\cdot)$ and let us denote by $w(t,k) \colon \mathbb{R} \times L^2(\mathbb{T}) \to \mathbb{R}$ the function $w(t,k) := e^{-\rho t} v(k)$. We have:

$$\begin{aligned} v(k_0) - w(T, \tilde{k}(T)) = w(t, \tilde{k}(0)) - w(T, \tilde{k}(T)) &= -\int_0^T \frac{\mathrm{d}}{\mathrm{d}t} w(t, \tilde{k}(t)) \, \mathrm{d}t \\ &= \int_0^T e^{-\rho t} \left[\rho v(\tilde{k}(t)) - \left\langle G\tilde{k}(t) + A\tilde{k}(t) - \tilde{c}(t), \nabla v(\tilde{k}(t)) \right\rangle \right] \mathrm{d}t \end{aligned}$$

The last expression makes sense thanks to the regularizing properties of the heat semigroup (in particular for all $t > 0$, $\tilde{k}(t) \in D(G)$). Passing to the limit in the last expression, as $t \to \infty$, (using (20)) we have

$$v(k_0) = \int_0^{+\infty} e^{-\rho t} \left[\rho v(\tilde{k}(t)) - \left\langle A\tilde{k}(t) - \tilde{c}(t), \nabla v(\tilde{k}(t)) \right\rangle - \left\langle \tilde{k}(t), G \nabla v(\tilde{k}(t)) \right\rangle \right] \mathrm{d}t$$

and then

$$
\begin{aligned}
&v(k_0) - J(k_0, \tilde{c}(\cdot)) \\
&\quad = \int_0^{+\infty} e^{-\rho t} \Bigg[\Bigg(\rho v(\tilde{k}(t)) - \left\langle A\tilde{k}(t), \nabla v(\tilde{k}(t)) \right\rangle - \left\langle \tilde{k}(t), G\nabla v(\tilde{k}(t)) \right\rangle \Bigg) \\
&\qquad\qquad + \Bigg(\left\langle \tilde{c}(t), \nabla v(\tilde{k}(t)) \right\rangle - \langle \mathbb{1}, U(\tilde{c}(t)) \rangle \Bigg) \Bigg] \, \mathrm{d}t \\
&\quad = \int_0^{+\infty} e^{-\rho t} \Bigg[\Bigg(\sup_{c \in L^2(\mathbb{T};\mathbb{R}^+)} \left\{ - \left\langle c, \nabla v(\tilde{k}(t)) \right\rangle + \langle \mathbb{1}, U(c) \rangle \right\} \Bigg) \\
&\qquad\qquad - \Bigg(- \left\langle \tilde{c}(t), \nabla v(\tilde{k}(t)) \right\rangle + \langle \mathbb{1}, U(\tilde{c}(t)) \rangle \Bigg) \Bigg] \, \mathrm{d}t \geq 0.
\end{aligned}
$$

We used the fact that v is a solution of (18). Last expression gives $v(k_0) - J(k_0, \tilde{c}(\cdot)) \geq 0$ and from the same expression we can also easily see that $v(k_0) - J(k_0, c^*(\cdot)) = 0$ (indeed $c^*(\cdot)$ is defined using the feedback defined in (21)). So, for all admissible $\tilde{c}$, $v(k_0) - J(k_0, \tilde{c}(\cdot)) \geq 0 = v(k_0) - J(k_0, c^*(\cdot))$ and then $J(k_0, \tilde{c}(\cdot)) \leq J(k_0, c^*(\cdot))$ and then c^* is optimal. In particular, since $v(k_0) = J(k_0, c^*(\cdot)) = 0$ and c^* is an optimal control, $v(k_0)$ is the value function at k_0. The uniqueness of the optimal control follows from standard convexity considerations. □

Proof of Proposition 3.2. Along the optimal trajectories we have

$$
\begin{aligned}
k(t) &= e^{Gt} k_0 + \int_0^t e^{(t-s)G} \left(Ak(s) - c(s) \right) \, \mathrm{d}s \\
&= e^{Gt} k_0 + \int_0^t e^{(t-s)G} \left(Ak(s) - \eta \left\langle k(s), \mathbb{1} \right\rangle \mathbb{1} \right) \, \mathrm{d}s. \qquad (23)
\end{aligned}
$$

so

$$
K(t) = \langle k(t), \mathbb{1} \rangle = \left\langle k_0, e^{Gt}\mathbb{1} \right\rangle + \int_0^t \left\langle \left(Ak(s) - \eta \left\langle k(s), \mathbb{1} \right\rangle \mathbb{1} \right), e^{(t-s)G}\mathbb{1} \right\rangle \, \mathrm{d}s
$$

where we used that e^{tG} is self-adjoint. Moreover using that $e^{Gt}\mathbb{1} = \mathbb{1}$ the expression above becomes:

$$
= K(0) + \int_0^t K(s) \left[A - 2\pi \frac{\rho - A(1-\sigma)}{2\pi\sigma} \right] \mathrm{d}s = K(0) + \int_0^t K(s) \left[\frac{A - \rho}{\sigma} \right] \mathrm{d}s.
$$

So $K(t)$ has to be equal to $K(0)e^{\beta t}$ where $\beta = (A-\rho)/\sigma$ and the claim is proved. □

Proof of Theorem 3.3. We begin showing that $k_D(t)$ converges to $\frac{K(0)\mathbb{1}}{2\pi}$ in the $L^2(\mathbb{T})$ norm when $t \to +\infty$. We write $k_D(t,x)$ using Fourier series. For $n \in \mathbb{Z}$, we

call e_n the function $e_n\colon \mathbb{T} \to \mathbb{R}$ given by

$$e_n(x) := \begin{cases} \frac{\cos(nx)}{\sqrt{\pi}} & \text{if } n \geq 1 \\ \frac{\sin(-nx)}{\sqrt{\pi}} = \frac{\sin(nx)}{\sqrt{\pi}} & \text{if } n \leq -1 \\ \frac{1}{\sqrt{2\pi}} & \text{if } n = 0. \end{cases}$$

They are a complete orthonormal Hilbert basis of $L^2(\mathbb{T})$. Consider the selfadjoint operator $\tilde{G}$ on $L^2(\mathbb{T})$

$$\tilde{G} := G + A$$

having the same domain of G. Equation (23) can be written in the following equivalent form:

$$k(t) = e^{\tilde{G}t}k_0 - \int_0^t e^{(t-s)\tilde{G}}\eta \langle k(s), \mathbb{1}\rangle \mathbb{1}\, \mathrm{d}s.$$

The Fourier coefficients can be found using such an expression:

$$\begin{aligned} \langle k_D(t), e_n\rangle &= e^{-\beta t} \langle k(t), e_n\rangle \\ &= e^{-\beta t} \left\langle e^{\tilde{G}t}k_0, e_n \right\rangle - e^{-\beta t} \int_0^t \left\langle e^{(t-s)\tilde{G}}\eta \langle k(s), \mathbb{1}\rangle \mathbb{1}, e_n \right\rangle \mathrm{d}s \\ &= e^{-\beta t} \left\langle k_0, e^{\tilde{G}t}e_n \right\rangle - e^{-\beta t} \int_0^t \eta \langle k(s), \mathbb{1}\rangle \left\langle \mathbb{1}, e^{(t-s)\tilde{G}}e_n \right\rangle \mathrm{d}s, \quad (24) \end{aligned}$$

using that $e^{\tilde{G}t}e_n = e^{(-n^2+A)t}e_n$ and that, for $n \neq 0$, $\langle e_n, \mathbb{1}\rangle = 0$, we can see that

$$\langle k_D(t), e_n\rangle = e^{-\beta t} \left\langle k_0, e^{(A-n^2)t}e_n \right\rangle = e^{(A-n^2-\beta)t} \langle k_0, e_n\rangle, \qquad n \neq 0$$

If $n = 0$ we have $\langle k_D(t), e_0\rangle = \frac{K(0)}{\sqrt{2\pi}}$ (it follows immediately from Proposition 3.2). Let us we prove now that $k_D(t)$ converges in the L^2-norm to $\frac{K(0)\mathbb{1}}{2\pi}$: consider $\varepsilon \in \left(0, \frac{A(1-\sigma)+\sigma-\rho}{\sigma}\right)$ (it exists thanks to (15)) we have

$$\begin{aligned} \left| k_D(t) - \frac{K(0)\mathbb{1}}{2\pi} \right|^2_{L^2(\mathbb{T})} &= \sum_{n\in\mathbb{Z}} \left\langle \left(k_D(t) - \frac{K(0)\mathbb{1}}{2\pi} \right), e_n \right\rangle^2 \\ &= \sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} e^{2(A-n^2-\beta)t} \langle k_0, e_n\rangle^2 = e^{-2\varepsilon t} \sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} e^{2(A-n^2-\beta+\varepsilon)t} \langle k_0, e_n\rangle^2 \\ &\leq e^{-2\varepsilon t} \sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} \langle k_0, e_n\rangle^2 \leq e^{-2\varepsilon t} |k_0|^2_{L^2(\mathbb{T})} \xrightarrow{t\to\infty} 0, \end{aligned}$$

where we used that, for all $n \neq 0$, $(A - n^2 - \beta + \varepsilon) < 0$. Observe now that the Fourier series of $k(t)$

$$k_D(t)[x] = \frac{K(0)}{2\pi} + \sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} e^{(A-n^2-\beta)t} \langle k_0, e_n\rangle e_n[x] \quad (25)$$

converges uniformly for $t > 0$ so $k_D(t)[\cdot] \in C(\mathbb{T})$ and

$$\sup_{x\in\mathbb{T}} \left| k_D(t)[x] - \frac{K(0)}{2\pi} \right| = \sup_{x\in\mathbb{T}} \left| \sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} e^{(A-n^2-\beta)t} \langle k_0, e_n \rangle e_n[x] \right|$$

$$\leq e^{-\varepsilon t} \left| \sum_{n\in\mathbb{Z}} e^{(A-n^2-\beta+\varepsilon)t} |k_0|_{L^2} \sup_{x\in\mathbb{T}} |e_n[x]| \right| \leq |k_0|_{L^2} e^{-\varepsilon t} \sum_{n\in\mathbb{Z}} e^{(A-n^2-\beta+\varepsilon)t}. \quad (26)$$

Considering $t \geq 1$ and calling $S := \sum_{n\in\mathbb{Z}} e^{(A-n^2-\beta+\varepsilon)} < \infty$, from (26), we have that

$$\sup_{x\in\mathbb{T}} \left| k_D(t)[x] - \frac{K(0)}{2\pi} \right| \leq e^{-\varepsilon t} \left(S|k_0|_{L^2} \right) \xrightarrow{t\to\infty} 0$$

and this proves the uniformly convergence. □