The Division Problem under Constraints*

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Abstract: We characterize axiomatically an extension of the uniform rule proposed to solve the division problem under constraints. This problem consists of allocating a given amount of an homogeneous and perfectly divisible good among a subset of agents with single-peaked preferences on an exogenously given interval of feasible shares. We show that the extended uniform rule is the unique one satisfying efficiency, strategy-proofness, equal treatment of equals, bound monotonicity, and independence of irrelevant coalitions.

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1 Introduction

In the division problem an amount of a perfectly divisible good has to be allocated among a set of agents with single-peaked preferences on the set of all positive amounts of the good. An agent has a single-peaked preference if he considers that there is an amount of the good (the peak) strictly preferred to all other amounts and in both sides of the peak the preference is monotonic, decreasing at its right and increasing at its left. A profile is a vector of single-peaked preferences, one for each agent. It would then be desirable that the chosen vector of allotments of the good depended on the profile. But since preferences are idiosyncratic they have to be elicited by a rule selecting, for each profile of single-peaked preferences, a vector of allotments adding up to the total amount of the good. But in general, the sum of the peaks will be either larger or smaller than the total amount to be allocated. Then, a rule has to solve a positive or negative rationing problem, depending on whether the sum of the peaks exceeds or fails short the amount of the good. Rules differ from each other on how this rationing problem is resolved in terms of its induced properties like the strategic incentives faced by agents, efficiency, fairness, monotonicity, consistency, etc.

The literature on the division problem describes many examples of allocation problems that fits well with this general description. For instance, a group of agents participate in an activity that requires a fixed amount of labor (measured in units of time). Agents have a maximal number of units of time to contribute and consider working as being undesirable. Suppose that labor is homogeneous and the wage is fixed. Then, strictly monotonic and quasi-concave preferences on the set of bundles of money and leisure generate single-peaked preferences on the set of potential allotments where the peak is the amount of working time associated to the optimal bundle. Similarly, a group of agents join a partnership to invest in a project (an indivisible bond with a face value, for example) that requires a fixed amount of money (neither more nor less). Their risk attitudes and wealth induce single-peaked preferences on the amount to be invested. Finally, a group of firms with different sizes have to jointly undertake a unique project of a fixed size. Since they may be involved in other projects their preferences are single-peaked on their respective shares of the project. In all these cases, it is required that a rule solves the rationing problem arising from a vector of peaks that do not add up the needed amount. The uniform rule has emerged as a satisfactory way of solving the division problem. It tries to allocate the good as equally as possible keeping the bounds imposed by efficiency. Sprumont (1991) started a long list of axiomatic characterizations of the uniform rule by showing first that it is the unique efficient, strategy-proof and anonymous rule, and second that anonymity in this characterization can be replaced by envy-freeness. Ching (1992 and 1994), Dagan (1996), Schummer and Thomson (1997), Sönmez (1994), and Thomson (1994a, 1994b, 1995, and 1997) contain alternative characterizations of the uniform rule in the division problem. In the survey on strategy-proofness of Barberà (2010) the division problem and the uniform rule plays a prominent role.

However, in many applications (like those described above), agents' allotments may be constrained by lower and upper bounds. For instance, each agent may only be able to contribute to the activity with an amount of labor, or to invest in the project, if his allotment belongs to a given interval. Therefore, in all these cases the division problem is restricted further by feasibility constraints that are described by a family of closed intervals of non-negative feasible allotments, one for each agent. It is then natural to assume that each agent has a closed interval of feasible allotments and his idiosyncratic preferences are single-peaked on this interval. Moreover, we will be interested in situations where agents' participation is voluntary; namely, each agent has to consider all his strictly positive feasible allotments as being strictly preferred to receive zero (the allotment associated to the prospect of non-participating in the division problem).

Until recently, a large part of the literature on the division problem has assumed implicitly that all allotments were feasible. In this paper we assume that each agent's allotment has to either belong to a given feasible interval of allotments or else be equal to zero. Hence, a division problem under constraints is composed by the set of agents, the amount of the good to be allocated among them, the vectors of lower and upper bounds of their feasible intervals, and their single-peaked preferences on their respective feasible intervals. Given a division problem under constraints, it may be the case that there does not exist a vector of feasible allotments, one for each one of the agents, adding up to the total amount to be allocated. Hence, given any division problem under constraints, a rule has two components. First, the choice of an admissible and non-empty subset of agents among whom it is possible to allocate the amount of the good keeping their feasibility constrains; if there is no such subset, then the rule has to choose the zero allotment for all agents. Second, and given this chosen admissible non-empty subset of agents (called participants), the rule has to assign to each of its members a feasible allotment in such a way that their sum adds up the total amount to be allocated.

Our contribution in this paper is to define an extension of the uniform rule to this class of division problems under constraints and to provide an axiomatic characterization of it by using two classes of desirable properties. The first class is related to the behavior of the rule at a given division problem under constraints. First, *efficiency*. A rule is efficient if it always selects Pareto optimal allocations. Efficiency guarantees that in solving the rationing problem (either positive or negative) no amount of the good is wasted. Second, *equal treatment of equals*. A rule satisfies equal treatment of equals if identical participants receive the same allotment.[1] The second class is related to the restrictions that the properties impose on a rule when comparing its proposal at different division problems under constraints. First, *strategy-proofness*. A rule is strategy-proof if no agent can profitably alter the rule's choice by misrepresenting his preferences. Namely, strategy-proofness guarantees that truth-telling is a weakly dominant strategy in the direct revelation game induced by the rule. Second, *bound monotonicity*. It imposes restrictions on how the rule should change when the upper or the lower bound of the interval of feasible allotments of one agent changes. Assume that the upper bound of an agent decreases. Two situations are possible. Either the allotment of this agent in the initial problem is not larger than the new upper bound or it is strictly larger than the new upper bound. In the first situation bound monotonicity says that both problems must have the same allotment. In the second situation, where the initial allotment is not feasible in the second problem, bound monotonicity says that the agent must receive his new upper bound whereas the rest of the agents can not receive smaller allotments. Symmetric arguments can be applied when the lower bound of an agent increases. Third, *independence of irrelevant coalitions*. A rule satisfies independence of irrelevant coalitions if whenever (i) the set of admissible coalitions in one problem is contained in the set of admissible coalitions in another problem, and (ii) the coalition chosen by the rule in the second and larger problem is admissible for the first and smaller problem then, the rule has to select the same coalition of participants in the two problems.

The paper contains first a preliminary result, Theorem 1, where we show that in the subclass of division problems under constrains with the property that the full set of agents is admissible (*i.e.*, it is possible to allocate the total amount of the good among all agents respecting all feasible constraints), the *feasible uniform rule* is the unique rule satisfying efficiency, strategy-proofness, equal treatment of equals, and bound monotonicity. The feasible uniform rule on this subclass of division problems under constraints tries to allocate the good among all agents in the most egalitarian way respecting not only the bounds imposed by efficiency, but also those imposed by the feasibility constraints. Using this characterization, we show in Theorem 2 that an *extended uniform rule* on the class of all

[1] As we will see in Section 3, this is a weakening of the usual property that requires that equal agents are treated equally. We show that this version is too strong because now it may be necessary that in order to satisfy the feasibility constraints one agent is excluded from the division and receives the zero allotment while an identical agent is included and receives an strictly positive feasible allotment. Hence, the rule will have to specify how this choice among these two identical agents is done.

division problems under constraints selects first, using a monotonic and responsive order on the family of all non-empty and finite subsets of agents, an admissible coalition of participants (if any; otherwise it chooses the zero allotment for all agents) and then it applies the feasible uniform rule to the reduced division problem under constraints obtained by restricting the original problem to this admissible subset of participants.

Several papers are closely related to the present one. First, Bergantiños, Massó and Neme (2012a) studies the division problem with maximal capacity constraints under the assumption that the sum of all upper bounds is larger than the total amount of the good that has to be distributed. Second, Kibris (2003) studies the division problem with maximal capacity constraints assuming free-disposability of the good. Then a rule assigns to each division problem with maximal capacity constraints a vector of shares satisfying the constraints and adding up *less* or equal than the total amount. Kibris (2003) characterizes an extension of the uniform rule to his setting with free-disposability. Third, Bergantiños, Massó and Neme (2012b) considers the division problem when agents' participation is voluntary. Each agent has an idiosyncratic interval of acceptable shares (which, in contrast with our setting here, is private information) where his preferences are single-peaked. Then a rule proposes to each agent either to not participate at all or an acceptable share. Bergantiños, Massó and Neme (2012b) shows that strategy-proofness is too demanding in this setting. Then, they study a subclass of efficient and consistent rules and characterize extensions of the uniform rule that deal explicitly with agents' voluntary participation. Fourth, Kim, Bergantiños and Chun (2012) characterize two families of rules, related with the rules studied in Bergantiños, Massó and Neme (2012b) and this paper, using the separability principle and other properties. Fifth, Manjunath (2012) proposes a division problem where each agent's preferences are characterized by a top and a *minimum* share in such a way that the agent is indifferent between any two quantities that are either below the minimum acceptable share or above the top share. Manjunath (2012) first shows that, under different fairness properties, strategy-proofness and efficiency are incompatible and second, he characterizes axiomatically different rules that solve the rationing problem in his setting. Finally, the division problem with maximal capacity constraints is also considered by Moulin (1999).[2] He characterizes the class of all fixed path mechanisms as the set of rules satisfying efficiency, strategy-proofness, consistency and resource monotonicity. Ehlers (2002a) presents a shorter proof of the main result in Moulin (1999) and Ehlers (2002b) extends it by showing that, for problems with strictly more than two agents, the

[2]In Moulin (1999) the maximal capacity constraints are justified on the basis of technical simplicity in oder to define the priority rationing methods by an ordinary path and to define the duality operator that cuts the main proof in half.

class of all fixed path mechanisms coincides with the set of rules satisfying weak one-sided resource-monotonicity, strategy-proofness and consistency.

The paper is organized as follows. In Section 2 we describe the model. In Section 3 we define several desirable properties that a rule may satisfy. In Section 4 we define the feasible uniform rule for the subclass of division problems under constraints where the grand coalition is admissible. Using the feasible uniform rule on this subclass we define, for all division problems under constraints, the extended uniform rule induced by a monotonic and responsive order on the family of all finite and non-empty subsets of agents and state two axiomatic characterizations. In Theorem 1 we show that (on the subclass of division problems where the grand coalition is admissible) the feasible uniform rule is the unique rule that satisfies efficiency, strategy-proofness, equal treatment of equals, and bound monotonicity. In Theorem 2 we show that a rule (on the class of all division problems under constraints) satisfies efficiency, strategy-proofness, equal treatment of equals, bound monotonicity, and independence of irrelevant coalitions if and only if it coincides with the extended uniform rule induced by a monotonic and responsive order on the family of all non-empty and finite subsets of agents. Section 5 contains some final remarks stating other desirable properties that all extended uniform rules also satisfy. The proofs of the two theorems are in Section 6.

2 Preliminaries

Let $t > 0$ be an amount of an homogeneous and perfectly *divisible good*. A finite set of *agents* is considering the possibility of dividing t among a subset of them, to be determined according to their preferences. We will consider situations where the amount of the good t and the finite set of agents may vary. Let $\mathbb{N}$ be the set of positive integers and let $\mathcal{N}$ be the family of all non-empty and finite subsets of $\mathbb{N}$. The set of agents is then $N \in \mathcal{N}$ with cardinality n. In contrast with Sprumont (1991), we consider decision problems where the amount received by each agent $i \in N$ is constrained either to belong to a given closed interval $[l_i, u_i] \subset [0, +\infty)$, determined by lower and upper exogenous constraints (l_i and u_i, respectively), or to be equal to zero. That is, an agent is either excluded from the division (and receives zero) or else his allotment has to be feasible. We are interested in settings where the participation of the agents in the division problem is voluntary in the sense that all strictly positive feasible allotments are strictly better than receiving zero. Thus, agent i's preferences $\succeq_i$ are defined on the set $\{0\} \cup [l_i, u_i]$, where $[l_i, u_i] \subseteq [0, +\infty]$ is agent i's *interval of feasible allotments*. We assume that $\succeq_i$ is a complete, reflexive, and transitive binary relation on $\{0\} \cup [l_i, u_i]$. Given $\succeq_i$, let $\succ_i$ be the antisymmetric binary relation induced by $\succeq_i$ (*i.e.*, for all $x_i, y_i \in \{0\} \cup [l_i, u_i]$, $x_i \succ_i y_i$ if and only if $y_i \succeq x_i$ does not hold) and let $\sim_i$ be the indifference relation induced by $\succeq_i$ (*i.e.*, for all $x_i, y_i \in \{0\} \cup [l_i, u_i]$, $x_i \sim_i y_i$ if and only if $x_i \succeq_i y_i$ and $y_i \succeq_i x_i$). We will also assume that $\succeq_i$ is single-peaked on $[l_i, u_i]$ and we will denote by $p_i \in [l_i, u_i]$ agent i's *peak*. Formally, agent i's preferences $\succeq_i$ is a complete preorder on the set $\{0\} \cup [l_i, u_i]$ that satisfies the following additional properties:

(P.1) there exists $p_i \in [l_i, u_i]$ such that $p_i \succ_i x_i$ for all $x_i \in [l_i, u_i] \backslash \{p_i\}$;

(P.2) $x_i \succ_i y_i$ for any pair of shares $x_i, y_i \in [l_i, u_i]$ such that either $y_i < x_i \leq p_i$ or $p_i \leq x_i < y_i$; and

(P.3) $x_i \succ_i 0$ for all $x_i \in [l_i, u_i] \backslash \{0\}$.

Observe that agent i's preferences are defined on $\{0\} \cup [l_i, u_i]$ and are independent of t. Moreover, we are admitting the possibilities that $l_i = 0$ and $l_i = p_i = u_i$. Conditions (P.1) and (P.2) are the standard single-peaked restrictions on $[l_i, u_i]$ while condition (P.3) conveys the minimal voluntary participation requirement that all strictly positive allotments in the feasible interval are strictly preferred to the zero allotment. A preference $\succeq_i$ of agent i is (partly) characterized by the triple (l_i, p_i, u_i). There are many preferences of agent i with the same (l_i, p_i, u_i); however, they differ only on how two shares on different sides of p_i are ordered while all of them coincide on the ordering on the shares on each of the sides of p_i. This multiplicity will often be irrelevant. We will assume throughout the paper that for any agent i, the bounds l_i and u_i are fixed and exogenously given while the preference $\succeq_i$ over the interval $[l_i, u_i]$ is idiosyncratic and has to be elicited through a direct revelation mechanism. As we have already discussed in the Introduction, we are interested in division problems where allotments may be restricted by objective feasibility or capacity constraints while every preference $\succeq_i$ satisfying (P.1), (P.2), and (P.3) is a legitimate one for agent i.[3]

Let $N \in \mathcal{N}$ be a set of agents. A *profile* $\succeq_N = (\succeq_i)_{i \in N}$ is an $n-$tuple of preferences satisfying properties (P.1), (P.2) and (P.3) above. Given a profile $\succeq_N$ and agent i's preferences $\succeq'_i$ we denote by $(\succeq'_i, \succeq_{N \backslash \{i\}})$ the profile where $\succeq_i$ has been replaced by $\succeq'_i$ and all other agents have the same preferences. When no confusion arises we denote the profile $\succeq_N$ by $\succeq$.

A division problem under constraints (a *problem* for short) is a 5$-$tuple $(N, t, l, u, \succeq)$ where $N \in \mathcal{N}$ is the finite set of agents, t is the amount of the good to be divided, $l = (l_i)_{i \in N}$

[3] See Bergantiños, Massó, and Neme (2012b) for an analysis of efficient and consistent rules in the division problem when the interval $[l_i, u_i]$ is the set of idiosincratic acceptable allotments for agent i and participation is voluntary.

is the vector of lower constraints, $u = (u_i)_{i \in N}$ is the vector of upper constraints, and $\succeq$ is a profile. Although the vector of lower and upper constraints are part of the definition of the profile $\succeq$, for convenience we explicitly include them in the description of a problem. Let $\mathcal{P}$ be the set of all problems. A problem where all agents have single-peaked preferences on $[0, +\infty)$ is known as the *division problem*; *i.e.*, for all $i \in N$, $l_i = 0$, $u_i = +\infty$, and (P.1) and (P.2) hold.

The set of *feasible allocations* of problem $(N, t, l, u, \succeq)$ is

$$FA\left(N, t, l, u, \succeq\right) = \left\{ (x_1, ..., x_n) \in \mathbb{R}^N_+ \mid \sum_{i \in N} x_i \in \{0, t\} \text{ and, for each } i \in N,\ x_i \in \{0\} \cup [l_i, u_i] \right\}.$$

This set is never empty since the allocation $(0, ..., 0) \in \mathbb{R}^N_+$ is always feasible. Moreover, there are problems for which $(0, ..., 0)$ is the unique feasible allocation; for instance the problem $(N, t, l, u, \succeq)$ where $N = \{1, 2\}$, $t = 10$, and $\succeq_1$ and $\succeq_2$ are characterized by $(l_1, p_1, u_1) = (l_2, p_2, u_2) = (1, 2, 3)$.

A coalition $S \subseteq N$ is *admissible* (at problem $(N, t, l, u, \succeq)$) if either S is empty or it is feasible to divide t among all agents in S; namely, coalition $S \neq \varnothing$ is *admissible* if there exists $x = (x_i)_{i \in S} \in \mathbb{R}^S_+$ such that $\sum_{i \in S} x_i = t$ and $l_i \leq x_i \leq u_i$ for all $i \in S$. Hence, $S \neq \varnothing$ is admissible if and only if $\sum_{i \in S} l_i \leq t \leq \sum_{i \in S} u_i$. We denote by $A\left(N, t, l, u, \succeq\right)$ the set of all admissible coalitions at problem $(N, t, l, u, \succeq)$. The set $A(N, t, l, u, \succeq)$ is non-empty because it always contains the empty coalition.

A *rule* f assigns to each problem in $\mathcal{P}$ a feasible allocation; that is, $f(N, t, l, u, \succeq) \in FA\left(N, t, l, u, \succeq\right)$ for all $(N, t, l, u, \succeq) \in \mathcal{P}$. Hence, a rule f can be seen as a systematic way of assigning to each problem $(N, t, l, u, \succeq) \in \mathcal{P}$ two different but related aspects of the solution of the problem. First, an admissible coalition $c^f\left(N, t, l, u, \succeq\right) \in A(N, t, l, u, \succeq)$ where

$$c^f\left(N, t, l, u, \succeq\right) = \{i \in N \mid f_i\left(N, t, l, u, \succeq\right) \in [l_i, u_i]\}$$

if $c^f\left(N, t, l, u, \succeq\right) \neq \varnothing$. We refer to the agents in $c^f\left(N, t, l, u, \succeq\right)$ as *participants*. Often, and when no confusion arises because the problem $(N, t, l, u, \succeq)$ will be obvious from the context we write c^f instead of $c^f\left(N, t, l, u, \succeq\right)$. Obviously, if $i \notin c^f(N, t, l, u, \succeq)$ then $f_i(N, t, l, u, \succeq) = 0$. Second, how the amount t is divided among the participants; *i.e.*, if $c^f\left(N, t, l, u, \succeq\right) \neq \varnothing$ then,

$$\sum_{i \in c^f} f_i\left(N, t, l, u, \succeq\right) = t.$$

We will see later that to identify rules satisfying appealing properties we may have some freedom when choosing one among all admissible coalitions while the properties will determine a unique way of dividing the amount of the good among the participants.

3 Properties of Rules

In this section we define several properties that a rule may satisfy. The first three are basic and standard properties already used in many axiomatic analysis of the division problem. The last two are bound monotonicity, which restricts how the rule should change when the upper or lower bound of an agent changes, and independence of irrelevant coalitions, which restricts how the participants should be chosen.

A rule is efficient if it always selects a Pareto optimal allocation.

Efficiency (*ef*) For each problem $(N, t, l, u, \succeq) \in \mathcal{P}$ there is no feasible allocation $(y_i)_{i\in N} \in FA(N, \succeq, t, l, u)$ with the property that $y_i \succeq_i f_i(N, t, l, u, \succeq)$ for all $i \in N$ and $y_j \succ_j f_j(N, t, l, u, \succeq)$ for some $j \in N$.

Remark 1 Let f be an efficient rule and let $(N, t, l, u, \succeq) \in \mathcal{P}$ be a problem. Then, $\sum_{i\in c^f} p_i \geq t$ implies that $f_i(N, t, l, u, \succeq) \leq p_i$ for all $i \in c^f$ and $\sum_{i\in c^f} p_i < t$ implies that $f_i(N, t, l, u, \succeq) \geq p_i$ for all $i \in c^f$.

Rules require each agent to report a single-peaked preference on $\{0\} \cup [l_i, u_i]$. A rule is strategy-proof if it is always in the best interest of agents to reveal their preferences truthfully; namely, truth-telling is a weakly dominant strategy in the direct revelation game induced by the rule.

Strategy-proofness (*sp*) For each problem $(N, t, l, u, \succeq_N) \in \mathcal{P}$, agent $i \in N$, and single-peaked preference $\succeq'_i$ on $\{0\} \cup [l_i, u_i]$,

$$f_i(N, t, l, u, \succeq_N) \succeq_i f_i\left(N, t, l, u, \left(\succeq'_i, \succeq_{N\setminus\{i\}}\right)\right).$$

Given a problem $(N, t, l, u, \succeq_N)$ we say that agent $i \in N$ *manipulates* f *at profile* $\succeq_N$ *via* $\succeq'_i$ if $f_i\left(N, t, l, u, \left(\succeq'_i, \succeq_{N\setminus\{i\}}\right)\right) \succ_i f_i(N, t, l, u, \succeq_N)$.

A rule satisfies strong equal treatment of equals if identical agents receive the same allotment.

Strong equal treatment of equals (*sete*) For every problem $(N, t, l, u, \succeq) \in \mathcal{P}$ such that there are agents $i, j \in N$, $i \neq j$, and $\succeq_i = \succeq_j$ then, $f_i(N, t, l, u, \succeq) = f_j(N, t, l, u, \succeq)$.

Strong equal treatment of agents is incompatible with efficiency. To see that, consider any problem $(N, t, l, u, \succeq)$ where $N = \{1, 2, 3\}$, $t = 10$, $(l_i, p_i, u_i) = (4, 5, 10)$ for $i = 1, 2, 3$, and $\succeq_1 = \succeq_2 = \succeq_3$. Since the allotment $(\frac{10}{3}, \frac{10}{3}, \frac{10}{3}) \notin FA(N, t, l, u, \succeq)$ any f satisfying strong equal treatment of equals has the property that $c^f = \varnothing$ and $f_i(N, t, l, u, \succeq) = 0$ for all $i = 1, 2, 3$. However, $(0, 5, 5)$ Pareto dominates $(0, 0, 0)$. Thus efficiency and strong equal treatment of equals are incompatible. For this reason, we restrict our attention to the weaker notion of the property requiring that only equal *participants* must be treated equally. The example above suggests that a rule satisfying equal treatment of equal (participants) will have to use some criteria to select among the three allotments $(0,5,5)$, $(5,0,5)$, and $(5,5,0)$ (and corresponding set of participants); but we will deal with that later.

A rule satisfies equal treatment of equals if identical participants receive the same allotment.

Equal treatment of equals (*ete*) For every problem $(N,t,l,u,\succeq) \in \mathcal{P}$ such that there are agents $i,j \in N$, $i \neq j$, $\succeq_i = \succeq_j$, and $i,j \in c^f(N,t,l,u,\succeq)$ then, $f_i(N,t,l,u,\succeq) = f_j(N,t,l,u,\succeq)$.

We note that (*sete*) and (*ete*) coincide with the standard property of equal treatment of equals when they are applied to classical division problems.

We now introduce the property of bound monotonicity, which imposes restrictions on how the rule changes when the upper or lower bounds of the interval of feasible allotments of one agent changes. Take a problem $(N,t,l,u,\succeq)$ where the upper bound of agent k decreases to $u'_k < u_k$ without changing his preferences (*i.e.*, $\succeq'_k$ coincides with $\succeq_k$ on $[l_k, u'_k]$). A natural notion of bound monotonicity says the following. First, assume that $f_k(N,t,l,u,\succeq) \leq u'_k$ then, $f(N,t,l,u,\succeq)$ is also feasible in $(N,t,l,(u'_k,u_{-k}),(\succeq'_k,\succeq_{-k}))$. Bound monotonicity says that f selects the same allocation in both problems (*i.e.*, $f(N,t,l,(u'_k,u_{-k}),(\succeq'_k,\succeq_{-k})) = f(N,t,l,u,\succeq)$). Second, assume that $u'_k < f_k(N,t,l,u,\succeq)$ then, $f(N,t,l,u,\succeq)$ is not feasible in $(N,t,l,(u'_k,u_{-k}),(\succeq'_k,\succeq_{-k}))$. If we can divide t in $(N,t,l,(u'_k,u_{-k}),(\succeq'_k,\succeq_{-k}))$ among the same set of agents as in $(N,t,l,u,\succeq)$ (*i.e.*, $c^f(N,t,l,u,\succeq) \in A(N,t,l,(u'_k,u_{-k}),(\succeq'_k,\succeq_{-k}))$) then, bound monotonicity says that agent k receives his new upper bound ($f_k(N,t,l,(u'_k,u_{-k}),(\succeq'_k,\succeq_{-k})) = u'_k$) and the rest of agents receive something not smaller than in $(N,t,l,u,\succeq)$ (*i.e.*, $f_i(N,t,l,(u'_k,u_{-k}),(\succeq'_k,\succeq_{-k})) \geq f_i(N,t,l,u,\succeq)$ for all $i \in N\backslash\{k\}$). If u'_k is so small that we can not divide t in $(N,t,l,(u'_k,u_{-k}),(\succeq'_k,\succeq_{-k}))$ among the same set of agents as in $(N,t,l,u,\succeq)$ (*i.e.*, $c^f(N,t,l,u,\succeq) \notin A(N,t,l,(u'_k,u_{-k}),(\succeq'_k,\succeq_{-k}))$) then, bound monotonicity says nothing. We apply the same ideas to the lower bound.

We now define the property of bound monotonicity formally.

Bound monotonicity (*bm*)

(bm.1) Let $(N,t,l,u,\succeq), (N,t,l,u',\succeq') \in \mathcal{P}$ be two problems such that for some agent $k \in N$, $u'_k < u_k$, $\succeq'_k$ coincides with $\succeq_k$ on $[l_k, u'_k]$, $u_i = u'_i$ and $\succeq_i = \succeq'_i$ for all $i \in N\backslash\{k\}$, and $c^f(N,t,l,u,\succeq) \in A(N,t,l,u',\succeq')$. Then, $c^f(N,t,l,u',\succeq') = c^f(N,t,l,u,\succeq)$ and

$$f_i(N,t,l,u',\succeq') \geq \min\{f_i(N,t,l,u,\succeq), u'_i\} \text{ for each } i \in N. \quad (1)$$

(bm.2) Let $(N,t,l,u,\succeq), (N,t,l',u,\succeq') \in \mathcal{P}$ be two problems such that for some agent $k \in N$, $l_k < l'_k$, and $\succeq'_k$ coincides with $\succeq_k$ on $[l'_k, u_k]$, $l_i = l'_i$ and $\succeq_i = \succeq'_i$ for all $i \in N \setminus \{k\}$, and $c^f(N,t,l,u,\succeq) \in A(N,t,l',u,\succeq')$. Then, $c^f(N,t,l',u,\succeq') = c^f(N,t,l,u,\succeq)$ and

$$f_i(N,t,l',u,\succeq') \leq \max\{f_i(N,t,l,u,\succeq), l'_i\} \text{ for each } i \in N. \tag{2}$$

To clarify the definition assume that $f_k(N,t,l,u,\succeq) \leq u'_k < u_k$ and $c^f(N,t,l,u,\succeq) \in A(N,t,l,u',\succeq')$. Then, $c^f(N,t,l,u',\succeq') = c^f(N,t,l,u,\succeq)$ and (1) implies that $f(N,t,l,u,\succeq) = f(N,t,l,u',\succeq')$. Now suppose that $u'_k < f_k(N,t,l,u,\succeq)$ and $c^f(N,t,l,u,\succeq) \in A(N,t,l,u',\succeq')$. Then, $c^f(N,t,l,u',\succeq') = c^f(N,t,l,u,\succeq)$ and (1) implies that $f_k(N,t,l,u',\succeq') = u'_k$ and $f_i(N,t,l,u',\succeq') \geq f_i(N,t,l,u,\succeq)$ for all $i \in N\setminus\{k\}$. Thus, condition (1) can be rewritten as

$$\begin{aligned}
f_k(N,t,l,u',\succeq') &\geq \min\{f_k(N,t,l,u,\succeq), u'_k\} \text{ and} \\
f_i(N,t,l,u',\succeq') &\geq f_i(N,t,l,u,\succeq) \text{ for all } i \in N\setminus\{k\}.
\end{aligned}$$

Similarly, condition (2) can be rewritten as

$$\begin{aligned}
f_k(N,t,l',u,\succeq') &\leq \max\{f_k(N,t,l,u,\succeq), l'_k\} \text{ and} \\
f_i(N,t,l',u,\succeq') &\leq f_i(N,t,l,u,\succeq) \text{ for all } i \in N\setminus\{k\}.
\end{aligned}$$

A rule satisfies independence of irrelevant coalitions if the following requirement holds. Consider two problems where the set of admissible coalitions of the first one is contained in the set of admissible coalitions of the second one. Assume that the coalition chosen by the rule in the second problem is admissible for the first one. Then, the rule chooses the same coalition of participants in the two problems.

Independence of irrelevant coalitions (*iic*) For any two problems $(N,t,l,u,\succeq)$, $(N',t',l',u',\succeq') \in \mathcal{P}$ such that $c^f(N,t,l,u,\succeq) \in A(N',t',l',u',\succeq') \subset A(N,t,l,u,\succeq)$ then,

$$c^f(N',t',l',u',\succeq') = c^f(N,t,l,u,\succeq).$$

4 The Uniform Principle: Two Characterizations

In this section we present the two main results of the paper.

We first define the feasible uniform rule for the subclass of division problems under constraints with the property that the set of all agents is an admissible coalition. The uniform rule U on problems without constraints (see Sprumont (1991)) tries to allocate the good as equally as possible, keeping the efficiency bounds binding (all agents have to be rationed in the same direction). The feasible uniform rule U^F, on the subclass of division problems under constraints with the property that the set of all agents is an admissible coalition, does the same than U but it takes also into account the feasibility constraints. We show in Theorem 1 that the feasible uniform rule U^F is the unique rule satisfying efficiency, strategy-proofness, equal treatment of equals, and bound monotonicity on the subclass of problems where the set of all agents is an admissible coalition.

Let P be the set of division problems under constraints with the property that the set of all agents is an admissible coalition; namely,

$$P = \left\{ (N, t, l, u, \succeq) \in \mathcal{P} \mid \sum_{i \in N} l_i \leq t \leq \sum_{i \in N} u_i \right\}.$$

Feasible uniform rule The *feasible uniform rule* U^F on P is defined as follows. For each $(N, t, l, u, \succeq) \in P$ and $i \in N$,

$$U_i^F (N, t, l, u, \succeq) = \begin{cases} \min\{p_i, \max\{l_i, \alpha\}\} & \text{if } \sum_{j \in N} p_j \geq t \\ \max\{p_i, \min\{u_i, \alpha\}\} & \text{if } \sum_{j \in N} p_j < t, \end{cases}$$

where α is the unique number satisfying $\sum_{j \in N} U_j^F (N, t, l, u, \succeq) = t$.

Remark 2 Consider the problem $(N, t, l, u, \succeq) \in P$ and a division problem without constraints $(N, t, \succeq')$ (*i.e.*, $l_i' = 0$ and $u_i' = +\infty$ for all $i \in N$) such that for each $i \in N$, $\succeq_i'$ coincides with $\succeq_i$ on $[l_i, u_i]$ and $U(N, t, \succeq') \in FA(N, t, l, u, \succeq)$. Then, $U(N, t, \succeq') = U^F (N, t, l, u, \succeq)$. Thus, the feasible uniform rule U^F can be considered as an extension of the uniform rule U from division problems without constraints to P. Observe that the extension of the uniform rule to problems with voluntary participation when the bounds are idiosyncratic presented in Bergantiños, Massó, and Neme (2012b) does not have this property. Let us clarify it with an example. Suppose that $N = \{1, 2\}$, $t = 10$, $l = (1, 3)$, $u = (8, 8)$ and $p = (6, 6)$. Thus, $U^F (N, t, l, u, \succeq) = (5, 5)$ whereas the rule in Bergantiños, Massó, and Neme (2012b) chooses $(4, 6)$; namely, it increases uniformly the shares starting from l.

Theorem 1 *The feasible uniform rule U^F on P is the unique rule satisfying efficiency, strategy-proofness, equal treatment of equals, and bound monotonicity. Besides, the four properties are independent.*

Proof See Subsection 6.1.

We now consider the general case. We first extend the feasible uniform rule to the class of all problems in $\mathcal{P}$. Let $(N, t, l, u, \succeq)$ be a problem in $\mathcal{P}$. The extended uniform rule selects at $(N,t,l,u,\succeq)$ the feasible set of participants by maximizing a given order ρ (a complete, antisymmetric and transitive binary relation) on $\mathcal{N}$, restricted to the family of admissible coalitions $A(N,t,l,u,\succeq) \subseteq 2^N$, and then it applies the feasible uniform rule to this selected set of participants to choose their allotments.

Extended Uniform Rule Let ρ be an order on $\mathcal{N}$. The *extended uniform rule on $\mathcal{P}$ induced by the order ρ on $\mathcal{N}$*, denoted by $U^{F,\rho}$, is defined as follows. For each $(N,t,l,u,\succeq) \in \mathcal{P}$ and $i \in N$,

$$U_i^{F,\rho}(N,t,l,u,\succeq) = \begin{cases} U_i^F(c^{U^{F,\rho}}, t, (l_j)_{j \in c^{U^{F,\rho}}}, (u_j)_{j \in c^{U^{F,\rho}}}, (\succeq_j)_{j \in c^{U^{F,\rho}}}) & \text{if } i \in c^{U^{F,\rho}} \\ 0 & \text{if } i \notin c^{U^{F,\rho}}, \end{cases}$$

where $c^{U^{F,\rho}} \in A(N,t,l,u,\succeq)$ and $c^{U^{F,\rho}} \rho S$ for all $S \in A(N,t,l,u,\succeq) \backslash c^{U^{F,\rho}}$.

Obviously, the family of extended uniform rules on $\mathcal{P}$ is large. We are interested in the subfamily of rules that satisfy efficiency, strategy-proofness, equal treatment of equals, bound monotonicity and independence of irrelevant coalitions. To identify it we restrict the order ρ on $\mathcal{N}$ to satisfy the properties of monotonicity and responsiveness.

Definition 1 We say that an order ρ on $\mathcal{N}$ is

(i) *monotonic* if for all $S \in \mathcal{N}$ and $i \notin S$, $(S \cup \{i\}) \rho S$; and

(ii) *responsive* if for all $S, T \in \mathcal{N}$ and $i \notin S \cup T$, $S \rho T$ implies $(S \cup \{i\}) \rho (T \cup \{i\})$.

Theorem 2 below characterizes the set of extended uniform rules with the property that they choose the admissible coalition according to a monotonic and responsive order on $\mathcal{N}$ as the class of rules satisfying efficiency, strategy-proofness, equal treatment of equals, bound monotonicity, and independence of irrelevant coalitions.

Theorem 2 *Let f be a rule on $\mathcal{P}$. Then, f satisfies efficiency, strategy-proofness, equal treatment of equals, bound monotonicity, and independence of irrelevant coalitions if and only if $f = U^{F,\rho}$ for some monotonic and responsive order ρ on $\mathcal{N}$. Besides, the five properties are independent.*

Proof See Subsection 6.2.

5 Final Remarks

In this section we present some other properties that the uniform rule satisfies in the classical division problem. While some of them are satisfied by the extended uniform rule in our setting some others are not. Nevertheless, if we proceed by weakening such properties as we did with the principle of equal treatment of equals, the extended uniform rule satisfies the new formulation of the corresponding weaker principles.

A rule is *non-bossy* if whenever an agent receives the same share at two problems that are identical except for the preferences of this agent then, the shares of all the other agents also coincide at the two problems. Formally,

NON-BOSSY For each problem $(N,t,l,u,\succeq)$, each $i \in N$, and each $\succeq_i'$ such that $f_i(N,t,l,u,\succeq) = f_i\left(N,t,l,u\left(\succeq_i', \succeq_{N\setminus\{i\}}\right)\right)$ then, $f_j(N,t,l,u,\succeq) = f_j\left(N,t,l,u\left(\succeq_i', \succeq_{N\setminus\{i\}}\right)\right)$ for all $j \in N\setminus\{i\}$.

A rule is consistent if the following requirement holds. Apply the rule to a given problem and assume that a subset of agents leave with their corresponding shares. Consider the new problem formed by the set of agents that remain with the same preferences that they had in the original problem and the total amount of the good minus the sum of the shares received by the subset of agents that already left. Then, the rule does not require to reallocate the shares of the remaining agents.

CONSISTENCY For each problem $(N,t,l,u,\succeq)$, each non-empty subset of agents $S \subset N$, and each $i \in S$,

$$f_i(N,t,l,u,\succeq) = f_i\left(S, t - \sum_{j \in c^f(N,t,l,u,\succeq)\setminus S} f_j(N,t,l,u,\succeq), (l_i)_{i\in S}, (u_i)_{i\in S}, (\succeq_i)_{i\in S}\right).$$

A rule is own-peak monotonic if when the peak of an agent increases and the rest of the problem remains the same, this agent does not receive less.

OWN-PEAK MONOTONICITY For all $(N,t,l,u,\succeq), (N,t,l,u,\succeq') \in \mathcal{P}$, $p_i' \leq p_i$ implies $f_i(N,t,l,u,(\succeq_i', \succeq_{-i})) \leq f_i(N,t,l,u,\succeq)$.

A rule is tops-only when it depends only on the peaks of the preferences.

TOPS-ONLY For all $(N,t,l,u,\succeq), (N,t,l,u,\succeq') \in \mathcal{P}$, $p_i = p_i'$ for all $i \in N$ implies $f(N,t,l,u,\succeq) = f(N,t,l,u,\succeq')$.

A rule satisfies maximality if the set of participants is always maximal according to set-wise inclusion.

MAXIMALITY For any $T \subset N$ such that $c^f(N,t,l,u,\succeq) \subsetneq T$, T is not an admissible coalition for $(N,t,l,u,\succeq)$.

To show that any extended uniform rule on $\mathcal{P}$ induced by a monotonic and responsive order ρ on $\mathcal{N}$ satisfies the above properties is straightforward. We state this without proof as Proposition 1 below.

Proposition 1. *For each monotonic and responsive order ρ on $\mathcal{N}$, the extended uniform rule $U^{F,\rho}$ is non-bossy, consistent, own-peak monotonic, tops-only, and satisfies maximality.*

We now introduce some properties that in the strong version (as in classical division problems) no extended uniform rule on $\mathcal{P}$ does satisfy. Nevertheless, a weaker version of them (obtained by weakening the properties as we did with equal treatment of equals) are satisfied by every extended uniform rule on $\mathcal{P}$ induced by a monotonic and responsive order ρ on $\mathcal{N}$. In all cases, when applied to classical division problems, the strong and the weak versions coincide.

The basic principle under envy-freeness is that no agent can strictly prefer the share received by another agent.

Strong envy freeness For each $(N, t, l, u, \succeq) \in \mathcal{P}$ and each pair of agents $i, j \in N$, $f_i(N, t, l, u, \succeq) \succeq_i f_j(N, t, l, u, \succeq)$.

We weaken this notion in two ways. First, we only require to compare allotments of participants (as in the case of *ete*). Secondly, we admit unfeasible envies (when agent i envies the allocation of agent j but agent i's allocation is not feasible for agent j).

Envy freeness For each $(N, t, l, u, \succeq) \in \mathcal{P}$ and each pair of agents $i, j \in c^f$ such that $f_j(N, t, l, u, \succeq) \succ_i f_i(N, t, l, u, \succeq)$, then the vector of shares $x = (x_k)_{k \in c^f}$, where $x_i = f_j(N, t, l, u, \succeq)$, $x_j = f_i(N, t, l, u, \succeq)$, and $x_k = f_k(N, t, l, u, \succeq)$ for all $k \in c^f \backslash \{i, j\}$ has the property that $x \notin FA(N, t, l, u, \succeq)$.

A rule is strong individually rational from equal division if all agents receive a share that is at least as good as the equal division share.

Strong individual rationality from equal division For each $(N, t, l, u, \succeq) \in \mathcal{P}$ and each $i \in N$,

$$f_i(N, t, l, u, \succeq) \succeq_i \frac{t}{n}.$$

We now weaken this principle by applying it only when the equal share is feasible.

Individual rationality from equal division For each $(N, t, l, u, \succeq) \in \mathcal{P}$ for which $(\frac{t}{n}, ..., \frac{t}{n}) \in FA(N, t, l, u, \succeq)$ then, for all $i \in N$,

$$f_i(N, t, l, u, \succeq) \succeq_i \frac{t}{n}.$$

One-sided resource-monotonicity says that if the good is scarce, an increase of the amount to be shared should make all agents better off. Symmetrically, if the good is too abundant, a decrease of the amount to be shared should make all agents better off.

Strong one-sided resource-monotonicity For all $(N,t,l,u,\succeq),(N,t',l,u,\succeq) \in \mathcal{P}$ with the property that either $t \leq t' \leq \sum_{i\in N} p_i$ or $\sum_{i\in N} p_i \leq t' \leq t$ then, $f_i(N,t',l,u,\succeq) \succeq_i f_i(N,t,l,u,\succeq)$ for all $i \in N$.

We weaken the principle by applying it only when, after changing the amount to be divided, the set of admissible coalitions does not change.

One-sided resource-monotonicity For all $(N,t,l,u,\succeq),(N,t',l,u,\succeq) \in \mathcal{P}$ with the property that $A(N,t,l,u,\succeq) = A(N,t',l,u,\succeq)$ and either $t \leq t' \leq \sum_{i\in N} p_i$ or $\sum_{i\in N} p_i \leq t' \leq t$ then, $f_i(N,t',l,u,\succeq) \succeq_i f_i(N,t,l,u,\succeq)$ for all $i \in N$.

Proposition 2 *Let ρ be a monotonic and responsive order on $\mathcal{N}$. Then, the extended uniform rule $U^{F,\rho}$ does not satisfy strong envy freeness, strong individual rationality from equal division, and strong one-sided resource-monotonicity. Nevertheless, $U^{F,\rho}$ satisfies envy freeness, individual rationality from equal division, and one-sided resource-monotonicity.*

Proof Let ρ be any monotonic and responsive order on $\mathcal{N}$. It is easy to check that $U^{F,\rho}$ satisfies the weak versions of the properties.

To show that $U^{F,\rho}$ does not satisfy strong envy freeness, let $(N,t,l,u,\succeq) \in \mathcal{P}$ be such that $N = \{1,2\}$, $t = 10$, $l = (7,0)$, $u = (9,9)$ and for each $x \in [1,3]$ and $y \in [7,9]$ we have that $y \succeq_2 x$. The set of feasible allocations is

$$FA(N,t,l,u,\succeq) = \{(x_1, 10 - x_1) \mid x_1 \in [7,9]\} \cup \{(0,0)\}.$$

Since $U^{F,\rho}$ is efficient, $U^{F,\rho}(N,t,l,u,\succeq) \neq (0,0)$, which means that $U^{F,\rho}$ does not satisfy strong envy freeness.

To show that $U^{F,\rho}$ does not satisfy strong individual rationality from equal division, let $(N,t,l,u,\succeq) \in \mathcal{P}$ be such that $N = \{1,2\}$, $t = 10$, $l = (1,2)$, $u = (3,8)$ and $p_2 = 5$. The set of feasible allocations is

$$FA(N,t,l,u,\succeq) = \{(x_1, 10 - x_1) \mid x_1 \in [2,3]\} \cup \{(0,0)\},$$

which means that $U^{F,\rho}$ does not satisfy strong individual rationality from equal division.

To show that $U^{F,\rho}$ does not satisfy strong one-sided resource-monotonicity, let $(N,t,l,u,\succeq) \in \mathcal{P}$ be such that $N = \{1,2,3\}$, $t = 10$, $t' = 14$, $l = (1,1,12)$, $u = (6,6,20)$, $p = (5,5,15)$, and for each $i \in \{1,2\}$ and each $x,y \in [1,6]$ $x \succeq_i y$ if and only if $|x-5| \leq |y-5|$. Now,

$$\begin{aligned}
FA(N,t,l,u,\succeq) &= \{(x_1, 10-x_1, 0) \mid x_1 \in [4,6]\} \cup \{(0,0,0)\}. \\
FA(N,t',l,u,\succeq) &= \{(x_1, 0, 14-x_1) \mid x_1 \in [1,2]\} \cup \{(0, x_2, 14-x_2) \mid x_2 \in [1,2]\} \\
&\quad \cup \{(1,1,12)\} \cup \{(0,0,15)\} \cup \{(0,0,0)\}.
\end{aligned}$$

Since $U^{F,\rho}$ is efficient, for each $i \in \{1,2\}$, $f_i(N,t,l,u,\succeq) \in [4,6]$ and $f_i(N,t',l,u,\succeq) \leq 2$. Thus, $U^{F,\rho}$ does not satisfy strong one-sided resource-monotonicity. ■

The proof of Proposition 2 establishes the following Corollary.

Corollary 1 *There is no rule on $\mathcal{P}$ that satisfies strong individual rationality from equal division. Moreover, let f be an efficient rule. Then, f neither satisfies strong envy freeness nor strong one-sided resource-monotonicity on $\mathcal{P}$.*

6 Proofs

We present the proofs of the main results of the paper.

6.1 Proof of Theorem 1

To prove that U^F satisfies efficiency, strategy-proofness, equal treatment of equals, and bound monotonicity on P is not difficult and therefore omitted.

We prove that U^F is the unique rule satisfying the four properties on P. We do it by proving the following five lemmata.

Lemma 1.1 *Let f be an (ef) and (bm) rule on P and let $(N,t,l,u,\succeq) \in P$. If $c^f(N,t,l,u,\succeq) \subsetneq S \subseteq N$ then, $S \notin A(N,t,l,u,\succeq)$.*

Proof To obtain a contradiction suppose that there exists $j \in S \in A(N,t,l,u,\succeq)$ and $j \notin c^f(N,t,l,u,\succeq)$. Hence, $f_j(N,t,l,u,\succeq) = 0$ and $f_j(N,t,l,u,\succeq) \notin [l_j,u_j]$. Thus, $0 < l_j$. Let $(N,t,l^1,u^1,\succeq^1) \in P$ be such that $(N,t,l^1,u^1,\succeq^1)$ coincides with $(N,t,l,u,\succeq)$ except that $l_i^1 = u_i^1 = f_i(N,t,l,u,\succeq)$ for all $i \in c^f(N,t,l,u,\succeq)$. By (bm),

$$f(N,t,l,u,\succeq) = f\left(N,t,l^1,u^1,\succeq^1\right).$$

Since $\sum_{i\in c^f} f_i(N,t,l,u,\succeq) = t$ and $f_i(N,t,l,u,\succeq) = 0$ for all $i \notin c^f(N,t,l,u,\succeq)$, we have that $\sum_{i\in S} f_i(N,t,l,u,\succeq) = t > 0$. Let $k \in c^f(N,t,l,u,\succeq)$ be such that $f_k(N,t,l,u,\succeq) > 0$. Let $(N,t,l^2,u^2,\succeq^2) \in P$ be such that it coincides with $(N,t,l^1,u^1,\succeq^1)$ except that $l_j > l_j^2 = \varepsilon > 0$ for ε sufficiently small, $u_j^2 = t$ and $\succeq_j^2$ coincides with $\succeq_j^1$ on $[l_j^1,u_j^1]$. By (bm),

$$f\left(N,t,l^2,u^2,\succeq^2\right) = f\left(N,t,l^1,u^1,\succeq^1\right).$$

Let $(N,t,l^3,u^3,\succeq^3) \in P$ be such that it coincides with $(N,t,l^2,u^2,\succeq^2)$ except that $0 < l_k^3 = p_k^3 = f_k(N,t,l,u,\succeq) - \varepsilon$ and $\succeq_k^3$ coincides with $\succeq_k^2$ on $[l_k^2,u_k^2]$. By (bm)

$$f\left(N,t,l^3,u^3,\succeq^3\right) = f\left(N,t,l^2,u^2,\succeq^2\right).$$

Let $x = (x_i)_{i\in N}$ be such that $x_j = \varepsilon$, $x_k = f_k(N,t,l,u,\succeq) - \varepsilon$, and $x_i = f_i(N,t,l^3,u^3,\succeq^3)$ for all $i \in N\backslash\{j,k\}$, if any. But $x \in FA(N,t,l^3,u^3,\succeq^3)$ and x Pareto dominates $f(N,t,l^3,u^3,\succeq^3)$, a contradiction with the efficiency of f. ■

An immediate consequence of Lemma 1.1 is that if $N \in A(N,t,l,u,\succeq)$ then, $c^f(N,t,l,u,\succeq) = N$. Hence, by (ete), for all $(N,t,l,u,\succeq) \in \mathcal{P}$ such that $N \in A(N,t,l,u,\succeq)$, $\succeq_i = \succeq_j$ implies $f_i(N,t,l,u,\succeq) = f_j(N,t,l,u,\succeq)$.

Lemma 1.2 *Let f be an (ef) and (sp) rule on P. Then, f is own-peak monotonic.*

Proof Let $(N,t,l,u,\succeq)$, $(N,t,l,u,\succeq') \in \mathcal{P}$ and $j \in N$ be such that $p'_j \leq p_j$. To obtain a contradiction, assume

$$f_j(N,t,l,u,\succeq) < f_j(N,t,l,u,(\succeq'_j,\succeq_{-j})). \tag{3}$$

We consider two cases.

Case 1: $\sum_{i\in N} p_i \leq t$. By (ef), $p_i \leq f_i(N,t,l,u,\succeq)$ for all $i \in N$. Hence,

$$p'_j \leq p_j \leq f_j(N,t,l,u,\succeq) < f_j(N,t,l,u,(\succeq'_j,\succeq_{-j})),$$

which implies, by (P.2), that $f_j(N,t,l,u,\succeq) \succ'_j f_j(N,t,l,u,(\succeq'_j,\succeq_{-j}))$, a contradiction with (sp).

Case 2: $\sum_{i\in N} p_i > t$. By (ef),

$$f_i(N,t,l,u,\succeq) \leq p_i \tag{4}$$

for all $i \in N$. We consider two subcases.

Subcase 2.1: $t \leq \sum_{i\neq j} p_i + p'_j$. By (ef), for all $i \neq j$; $f_i(N,t,l,u,(\succeq'_j,\succeq_{-j})) \leq p_i$ and $f_j(N,t,l,u,(\succeq'_j,\succeq_{-j})) \leq p'_j$. Hence, by (3),

$$f_j(N,t,l,u,\succeq) < f_j(N,t,l,u,(\succeq'_j,\succeq_{-j})) \leq p'_j \leq p_j,$$

which implies, by (P.2), that $f_j(N,t,l,u,(\succeq'_j,\succeq_{-j})) \succ_j f_j(N,t,l,u,\succeq)$, a contradiction with (sp).

Subcase 2.2: $\sum_{i\neq j} p_i + p'_j < t$. By (ef), for all $i \neq j$; $p_i \leq f_i(N,t,l,u,(\succeq'_j,\succeq_{-j}))$ and $p'_j \leq f_j(N,t,l,u,(\succeq'_j,\succeq_{-j}))$. Thus, $p'_j \leq f_j(N,t,l,u,\succeq)$; otherwise, by (4),

$$t = \sum_{i\in N} f_i(N,t,l,u,\succeq) < \sum_{i\neq j} p_i + p'_j$$

a contradiction. Hence,

$$p'_j \leq f_j(N,t,l,u,\succeq) < f_j(N,t,l,u,(\succeq'_j,\succeq_{-j})),$$

which implies, by (P.2), that $f_j(N,t,l,u,\succeq) \succ'_j f_j(N,t,l,u,(\succeq'_j,\succeq_{-j}))$, a contradiction with (sp). ■

Lemma 1.3 *Let f be an (ef) and (sp) rule. Then, for all $j \in N$:*

(a) *If $p_j < f_j(N,t,l,u,\succeq)$ and $\succeq'_j$ satisfies $0 \leq p'_j \leq f_j(N,t,l,u,\succeq)$ then, $f_j(N,t,l,u,(\succeq'_j,\succeq_{-j})) = f_j(N,t,l,u,\succeq)$.*

(b) *If $f_j(N,t,l,u,\succeq) < p_j$ and $\succeq'_j$ satisfies $f_j(N,t,l,u,\succeq) \leq p'_j \leq t$, then $f_j(N,t,l,u,(\succeq'_j,\succeq_{-j})) = f_j(N,t,l,u,\succeq)$.*

Proof Let f be an efficient and strategy-proof rule and j be an agent.

(a) Assume $p_j < f_j(N,t,l,u,\succeq)$ and let $\succeq'_j$ be such that $0 \leq p'_j \leq f_j(N,t,l,u,\succeq)$. By (ef), Remark 1 implies that for all $i \in c^f(N,t,l,u,\succeq)$,

$$p_i \leq f_i(N,t,l,u,\succeq). \tag{5}$$

Since $p'_j \leq f_j(N,t,l,u,\succeq)$, (5) implies

$$\sum_{i \in c^f \setminus \{j\}} p_i + p'_j \leq \sum_{i \in c^f} f_i(N,t,l,u,\succeq) = t.$$

We now show that $f_j(N,t,l,u,(\succeq'_j,\succeq_{-j})) = f_j(N,t,l,u,\succeq)$. To obtain a contradiction, assume otherwise and consider two cases.

<u>Case 1</u>: $f_j(N,t,l,u,\succeq) < f_j(N,t,l,u,(\succeq'_j,\succeq_{-j}))$. Then

$$p'_j \leq f_j(N,t,l,u,\succeq) < f_j(N,t,l,u,(\succeq'_j,\succeq_{-j})),$$

which implies, by (P.2), that

$$f_j(N,t,l,u,\succeq) \succ'_j f_j(N,t,l,u,(\succeq'_j,\succeq_{-j})),$$

contradicting (sp).

<u>Case 2</u>: $f_j(N,t,l,u,(\succeq'_j,\succeq_{-j})) < f_j(N,t,l,u,\succeq)$. We consider two subcases.

<u>Subcase 2.1</u>: $p_j \leq f_j(N,t,l,u,(\succeq'_j,\succeq_{-j}))$. Then

$$p_j \leq f_j(N,t,l,u,(\succeq'_j,\succeq_{-j})) < f_j(N,t,l,u,\succeq).$$

Hence, by (P.2),

$$f_j(N,t,l,u,(\succeq'_j,\succeq_{-j})) \succ_j f_j(N,t,l,u,\succeq),$$

contradicting (sp).

Subcase 2.2: $f_j(N,t,l,u,(\succeq'_j,\succeq_{-j})) < p_j$. Then, $p'_j > 0$ and

$$f_j(N,t,l,u,(\succeq'_j,\succeq_{-j})) < p_j = p'_j < f_j\left(N,t,l,u,\succeq\right). \tag{6}$$

If $f_j(N,t,l,u,(\succeq'_j,\succeq_{-j})) \notin [l_j,u_j]$ then, $f_j(N,t,l,u,(\succeq'_j,\succeq_{-j})) = 0$. By (P.3), j manipulates f at $(N,t,l,u,(\succeq'_j,\succeq_{-j}))$ via $\succeq_j$, contradicting (*sp*). Assume $f_j(N,t,l,u,(\succeq'_j,\succeq_{-j})) \in [l_j,u_j]$ and let $\succeq''_j$ be such that $p''_j = p_j$ and

$$f_j(N,,t,l,u,(\succeq'_j,\succeq_{-j})) \succ''_j f_j\left(N,t,l,u,\succeq\right). \tag{7}$$

By Lemma 1.2, f is own-peak monotonic. Hence,

$$f_j(N,t,l,u,(\succeq''_j,\succeq_{-j})) = f_j\left(N,t,l,u,\succeq\right).$$

By (7),

$$f_j(N,,t,l,u,(\succeq'_j,\succeq_{-j})) \succ''_j f_i(N,t,l,u,(\succeq''_j,\succeq_{-j})),$$

contradicting (*sp*).

(b) We omit the proof since it follows a symmetric argument to the one used to prove (a).

■

A consequence of Lemma 1.3 is that if f is (*ef*) and (*sp*) on P then, f is tops-only on P.

Lemma 1.4 *Let f be a rule satisfying (ete) and (bm) on P, and assume $(N,t,l,u,(\succeq'_{\{i,j\}},\succeq_{N\setminus\{i,j\}})) \in P$ is such that $u_k = t$ for all $k \in N$ and $\succeq'_i$ coincides with $\succeq'_j$ on $[\max\{l_i,l_j\},t]$. Then, it is not possible that simultaneously*

$$f_i(N,t,l,u,(\succeq'_{\{i,j\}},\succeq_{N\setminus\{i,j\}})) \leq U_i^F(N,t,l,u,(\succeq'_{\{i,j\}},\succeq_{N\setminus\{i,j\}}))$$

and

$$f_j(N,t,l,u,(\succeq'_{\{i,j\}},\succeq_{N\setminus\{i,j\}})) > U_j^F(N,t,l,u,(\succeq'_{\{i,j\}},\succeq_{N\setminus\{i,j\}}))$$

hold.

Proof We consider separately the three possible cases.

Case 1: $l_i = l_j$. The statement follows since $\succeq'_i = \succeq'_j$ and f and U^F satisfy (*ete*).

Case 2: $l_i < l_j$. Assume

$$f_j(N,t,l,u,(\succeq'_{\{i,j\}},\succeq_{N\setminus\{i,j\}})) > U_j^F(N,t,l,u,(\succeq'_{\{i,j\}},\succeq_{N\setminus\{i,j\}})). \tag{8}$$

We want to show that $f_i(N,t,l,u,(\succeq'_{\{i,j\}},\succeq_{N\setminus\{i,j\}})) < U_i^F(N,t,l,u,(\succeq'_{\{i,j\}},\succeq_{N\setminus\{i,j\}}))$ does not hold. By (8), $l_j \leq U_j^F(N,t,l,u,(\succeq'_{\{i,j\}},\succeq_{N\setminus\{i,j\}})) < f_j(N,t,l,u,(\succeq'_{\{i,j\}},\succeq_{N\setminus\{i,j\}}))$. Let

$l_j^* = l_i$ and consider the preference $\succeq_j^*$ of agent j on $[l_j^*, u_j]$ that coincides with $\succeq_i'$ on $[l_i, u_i] = [l_j^*, u_j]$. By (*bm*),

$$f(N,t,(l_j^*, l_{-j}), u, (\succeq_i', \succeq_j^*, \succeq_{N\setminus\{i,j\}})) = f(N,t,l,u,(\succeq_{\{i,j\}}', \succeq_{N\setminus\{i,j\}})). \tag{9}$$

Since $\succeq_j^* = \succeq_i'$, by (*ete*), $f_j(N,t,(l_j^*, l_{-j}), u, (\succeq_i', \succeq_j^*, \succeq_{N\setminus\{i,j\}})) = f_i(N,t,(l_j^*, l_{-j}), u, (\succeq_i', \succeq_j^*, \succeq_{N\setminus\{i,j\}}))$. By (9),

$$f_j(N,t,l,u,(\succeq_{\{i,j\}}', \succeq_{N\setminus\{i,j\}})) = f_i(N,t,l,u,(\succeq_{\{i,j\}}', \succeq_{N\setminus\{i,j\}})). \tag{10}$$

By (*ete*),

$$U_j^F(N,t,(l_j^*, l_{-j}), u, (\succeq_i', \succeq_j^*, \succeq_{N\setminus\{i,j\}})) = U_i^F(N,t,(l_j^*, l_{-j}), u, (\succeq_i', \succeq_j^*, \succeq_{N\setminus\{i,j\}})). \tag{11}$$

By (*rm*),

$$U^F(N,t,(l_j^*, l_{-j}), u, (\succeq_i', \succeq_j^*, \succeq_{N\setminus\{i,j\}})) = U^F(N,t,l,u,(\succeq_{\{i,j\}}', \succeq_{N\setminus\{i,j\}})).$$

By (11),

$$U_j^F(N,t,l,u,(\succeq_{\{i,j\}}', \succeq_{N\setminus\{i,j\}})) = U_i^F(N,t,l,u,(\succeq_{\{i,j\}}', \succeq_{N\setminus\{i,j\}})).$$

By (8) and (10),

$$f_i(N,t,l,u,(\succeq_{\{i,j\}}', \succeq_{N\setminus\{i,j\}})) > U_i^F(N,t,l,u,(\succeq_{\{i,j\}}', \succeq_{N\setminus\{i,j\}})).$$

Thus,

$$f_i(N,t,l,u,(\succeq_{\{i,j\}}', \succeq_{N\setminus\{i,j\}})) \leq U_i^F(N,t,l,u,(\succeq_{\{i,j\}}', \succeq_{N\setminus\{i,j\}}))$$

does not hold.

Case 3: $l_i > l_j$. The argument is similar to the one used to prove Case 2 and omitted.■

Lemma 1.5. *Let f be a rule satisfying (ef), (sp), (ete), and (bm) on P. Then, $f = U^F$.*

Proof Let $(N,t,l,u,\succeq) \in P$ be arbitrary. We want to show that $f(N,t,l,u,\succeq) = U^F(N,t,l,u,\succeq)$. Assume that $\sum_{i\in N} p_i \geq t$. The case $\sum_{i\in N} p_i < t$ is similar and omitted. By (*ef*), $f_i(N,t,l,u,\succeq) \leq p_i$ for all $i \in N$.

Let $u_1^1 = +\infty$ and consider any $\succeq_1^1$ defined on $[l_1, +\infty]$ that coincides with $\succeq_1$ on $[l_1, u_1]$. For each $i \in N\setminus\{1\}$ define $\succeq_i^1 = \succeq_i$ and $u_i^1 = u_i$. Applying (*bm*) to $(N,t,l,u,\succeq)$ and $(N,t,l,u^1,\succeq^1)$ we obtain that for each $i \in N$,

$$\begin{aligned} f_i(N,t,l,u,\succeq) &\leq \max\left\{f_i\left(N,t,l,u^1,\succeq^1\right), l_i\right\} \\ &= f_i\left(N,t,l,u^1,\succeq^1\right). \end{aligned}$$

By (ef), $f_i\left(N,t,l,u^1,\succeq^1\right)=f_i\left(N,t,l,u,\succeq\right)$.

Let $u_2^2=+\infty$ and consider any $\succeq_2^2$ defined on $[l_2,+\infty]$ that coincides with $\succeq_2^1$ on $[l_2,u_2^1]$. For each $i\in N\backslash\{2\}$ define $\succeq_i^2=\succeq_i^1$ and $u_i^2=u_i^1$. Proceeding as in the previous case we obtain that for each $i\in N$, $f_i\left(N,t,l,u^2,\succeq^2\right)=f_i\left(N,t,l,u^1,\succeq^1\right)$.

Repeating this argument we obtain that $f\left(N,t,l,u^n,\succeq^n\right)=f\left(N,t,l,u,\succeq\right)$. Thus, we can assume that $u_i=+\infty$ for all $i\in N$.

Without loss of generality assume that $p_1\geq p_2\geq ...\geq p_n$. To obtain a contradiction, assume that $U^F\left(N,t,l,u,\succeq\right)\neq f\left(N,t,l,u,\succeq\right)$. Then, there exists $i^1\in N$ such that

$$U_{i^1}^F\left(N,t,l,u,\succeq\right)<f_{i^1}\left(N,t,l,u,\succeq\right)\leq p_{i^1}\leq p_1.$$

<u>Step 1</u>: Take $\succeq_{i^1}'$ defined on $[l_{i^1},u_{i^1}]$ and that it coincides with $\succeq_1$ on $[\max\{l_1,l_{i^1}\},t]$. By Lemma 1.3,

$$U_{i^1}^F(N,t,l,u,(\succeq_{i^1}',\succeq_{N\backslash\{i^1\}}))=U_{i^1}^F\left(N,t,l,u,\succeq\right).$$

By Lemma 1.2, f is own-peak monotonic. Since $p_{i^1}'\leq p_{i^1}$,

$$f_{i^1}\left(N,t,l,u,\succeq\right)\leq f_{i^1}(N,t,l,u,(\succeq_{i^1}',\succeq_{N\backslash\{i^1\}})).$$

Thus,

$$U_{i^1}^F(N,t,l,u,(\succeq_{i^1}',\succeq_{N\backslash\{i^1\}}))<f_{i^1}(N,t,l,u,(\succeq_{i^1}',\succeq_{N\backslash\{i^1\}})).$$

<u>Step 2</u>: Then, there exists $i^2\in N\backslash\{i^1\}$ such that

$$f_{i^2}(N,t,l,u,(\succeq_{i^1}',\succeq_{N\backslash\{i^1\}}))<U_{i^2}^F(N,t,l,u,(\succeq_{i^1}',\succeq_{N\backslash\{i^1\}}))\leq p_{i^2}.$$

Take $\succeq_{i^2}'$ defined on $[l_{i^2},u_{i^2}]$ and that it coincides with $\succeq_1$ on $[\max\{l_1,l_{i^2}\},t]$. By Lemma 1.3,

$$f_{i^2}(N,t,l,u,(\succeq_{\{i^1,i^2\}}',\succeq_{N\backslash\{i^1,i^2\}}))=f_{i^2}(N,t,l,u,(\succeq_{i^1}',\succeq_{N\backslash\{i^1\}})).$$

By Lemma 1.2, f is own-peak monotonic. Since $p_{i^2}'\leq p_{i^2}$,

$$U_{i^2}^F(N,t,l,u,(\succeq_{i^1}',\succeq_{N\backslash\{i^1\}}))\leq U_{i^2}^F(N,t,l,u,(\succeq_{\{i^1,i^2\}}',\succeq_{N\backslash\{i^1,i^2\}})).$$

Thus,

$$f_{i^2}(N,t,l,u,(\succeq_{\{i^1,i^2\}}',\succeq_{N\backslash\{i^1,i^2\}}))<U_{i^2}^F(N,t,l,u,(\succeq_{\{i^1,i^2\}}',\succeq_{N\backslash\{i^1,i^2\}})).$$

By Lemma 1.4,

$$f_{i^1}(N,t,l,u,(\succeq_{\{i^1,i^2\}}',\succeq_{N\backslash\{i^1,i^2\}}))\leq U_{i^1}^F(N,t,l,u,(\succeq_{\{i^1,i^2\}}',\succeq_{N\backslash\{i^1,i^2\}})).$$

Step 3: Then, there must exist $i^3 \in N \backslash \{i^1, i^2\}$ such that

$$U^F_{i^3}(N, t, l, u, (\succeq'_{\{i^1,i^2\}}, \succeq_{N\backslash\{i^1,i^2\}})) < f_{i^3}(N, t, l, u, (\succeq'_{\{i^1,i^2\}}, \succeq_{N\backslash\{i^1,i^2\}})).$$

Take $\succeq'_{i^3}$ defined on $[l_{i^3}, u_{i^3}]$ that it coincides with $\succeq_1$ on $[\max\{l_1, l_{i^3}\}, t]$. By Lemma 1.3,

$$U^F_{i^3}(N, t, l, u, (\succeq'_{\{i^1,i^2,i^3\}}, \succeq_{N\backslash\{i^1,i^2,i^3\}})) = U^F_{i^3}(N, t, l, u, (\succeq'_{\{i^1,i^2\}}, \succeq_{N\backslash\{i^1,i^2\}})).$$

By Lemma 1.2,

$$f_{i^3}(N, t, l, u, (\succeq'_{\{i^1,i^2\}}, \succeq_{N\backslash\{i^1,i^2\}})) \leq f_{i^3}(N, t, l, u, (\succeq'_{\{i^1,i^2,i^3\}}, \succeq_{N\backslash\{i^1,i^2,i^3\}})).$$

Thus,

$$U^F_{i^3}(N, t, l, u, (\succeq'_{\{i^1,i^2,i^3\}}, \succeq_{N\backslash\{i^1,i^2,i^3\}})) < f_{i^3}(N, t, l, u, (\succeq'_{\{i^1,i^2,i^3\}}, \succeq_{N\backslash\{i^1,i^2,i^3\}}).$$

By applying Lemma 1.4 twice, we obtain that

$$U^F_{i^1}(N, t, l, u, (\succeq'_{\{i^1,i^2,i^3\}}, \succeq_{N\backslash\{i^1,i^2,i^3\}})) \leq f_{i^1}(N, t, l, u, (\succeq'_{\{i^1,i^2,i^3\}}, \succeq_{N\backslash\{i^1,i^2,i^3\}})).$$

and

$$U^F_{i^2}(N, t, l, u, (\succeq'_{\{i^1,i^2,i^3\}}, \succeq_{N\backslash\{i^1,i^2,i^3\}})) \leq f_{i^2}(N, t, l, u, (\succeq'_{\{i^1,i^2,i^3\}}, \succeq_{N\backslash\{i^1,i^2,i^3\}})).$$

Continuing with this procedure, at Step n, we obtain that either

$$\begin{array}{rcl} U^F_{i^n}\left(N, t, l, u, \succeq'_N\right) & < & f_{i^n}\left(N, t, l, u, \succeq'_N\right) \text{ and for all } j \in N\backslash\left\{i^n\right\}, \\ U^F_{i^j}\left(N, \succeq'_N, t, l, u\right) & \leq & f_{i^j}\left(N, \succeq'_N, t, l, u\right) \end{array}$$

or else

$$\begin{array}{rcl} f_{i^n}\left(N, t, l, u, \succeq'_N\right) & < & U^F_{i^n}\left(N, t, l, u, \succeq'_N\right) \text{ and for all } j \in N\backslash\left\{i^n\right\} \\ f_{i^j}\left(N, t, l, u, \succeq'_N\right) & \leq & U^F_{i^j}\left(N, t, l, u, \succeq'_N\right). \end{array}$$

In both cases we have a contradiction because

$$\sum_{i \in N} f_i\left(N, \succeq'_N, t, l, u\right) = \sum_{i \in N} U^F_i\left(N, \succeq'_N, t, l, u\right) = t.$$

■

We now prove that the four properties are independent.

- (ef) is independent of the other three properties.

  We define the rule f^1 as follows. Let $(N, t, l, u, \succeq) \in P$. For each $i \in N$,

$$f_i^1 (N, t, l, u, \succeq) = median \{l_i, \alpha, u_i\},$$

  where α is such that $\sum_{i \in N} f_i (N, u, \succeq, t) = t$. Then, f^1 satisfies (sp), (ete), and (bm) but fails (ef).

- (sp) is independent of the other three properties.

  We define the rule f^2 as follows. Let $(N, t, l, u, \succeq) \in P$. For each $i \in N$,

$$f_i^2 (N, t, l, u, \succeq) = \begin{cases} p_i + \min \{\alpha, u_i - p_i\} & \text{if } \sum\limits_{i \in N} p_i < t \\ U_i^F (N, t, l, u, \succeq) & \text{if } \sum\limits_{i \in N} p_i \geq t, \end{cases}$$

  where α is such that $\sum_{i \in N} f_i (N, t, l, u, \succeq) = t$. Then, f^2 satisfies (ef), (ete), and (bm) but fails (sp).

- (ete) is independent of the other three properties.

  We define f^3 as the priority rule given by the order $(1, 2, ..., n)$ applied to the set of efficient allocations. Namely, let $(N, t, l, u, \succeq) \in P$. We define f^3 formally, by considering separately the two following cases.

  1. $\sum_{i \in N} p_i \geq t$. Take k as the unique agent satisfying that $\sum_{i=1}^{k} p_i + \sum_{i=k+1}^{n} l_i \leq t < \sum_{i=1}^{k+1} p_i + \sum_{i=k+2}^{n} l_i$. For each $i \in N$,

$$f_i^3 (N, t, l, u, \succeq) = \begin{cases} p_i & \text{if } i \leq k \\ t - \sum_{i=1}^{k} p_i - \sum_{i=k+2}^{n} l_i & \text{if } i = k+1 \\ l_i & \text{if } i \geq k+2. \end{cases}$$

  2. $\sum_{i \in N} p_i < t$. Take k as the unique agent satisfying that $\sum_{i=1}^{k+1} p_i + \sum_{i=k+2}^{n} u_i \leq t < \sum_{i=1}^{k} p_i + \sum_{i=k+1}^{n} u_i$. For each $i \in N$,

$$f_i^3 (N, t, l, u, \succeq) = \begin{cases} p_i & \text{if } i \leq k \\ t - \sum_{i=1}^{k} p_i - \sum_{i=k+2}^{n} u_i & \text{if } i = k+1 \\ u_i & \text{if } i \geq k+2. \end{cases}$$

  Then, f^3 satisfies (ef), (sp), and (bm) but fails (ete).

- (bm) is independent of the other properties.

  We define the rule f^4 inspired by the Constant Equal Losses rule used in bankruptcy problems. Let $(N,t,l,u,\succeq) \in P$. For each $i \in N$,

$$f_i^4(N,t,l,u,\succeq) = \begin{cases} \max\{u_i - \alpha, p_i\} & \text{if } \sum_{i\in N} p_i < t \\ \min\{\max\{l_i, u_i - \alpha\}, p_i\} & \text{if } \sum_{i\in N} p_i \geq t, \end{cases}$$

  where α is such that $\sum_{i\in N} f_i(N,t,l,u,\succeq) = t$. Then, f^4 satisfies (ef), (sp), and (ete) but fails (bm). ■

6.2 Proof of Theorem 2

Let ρ be any monotonic and responsive order on $\mathcal{N}$. To prove that $U^{F,\rho}$ satisfies (ef), (sp), (ete), (bm), and (iic) on $\mathcal{P}$ is not difficult and therefore omitted.

Let f be a rule satisfying (ef), (sp), (ete), (bm), and (iic). We prove that there exists a monotonic and responsive order ρ on $\mathcal{N}$ for which $f = U^{F,\rho}$.

We first define (using f) a binary relation ρ on $\mathcal{N}$. Let $S, S' \in \mathcal{N}$. Three cases are possible.

Case 1: $S \supset S'$. Then, set $S\rho S'$.

Case 2: $S' \supset S$. Then, set $S'\rho S$.

Case 3: There exist agents $j \in S\backslash S'$ and $j' \in S'\backslash S$. Consider any problem $(N,t,l,u,\succeq) \in \mathcal{P}$ where $S, S' \subseteq N$ and for each $i \in N$, $l_i = p_i = u_i$, and

$$p_i = \begin{cases} \varepsilon & \text{if } i \in S \cap S' \\ \varepsilon^2 & \text{if } i \in S\backslash(S' \cup \{j\}) \\ t - \varepsilon|S\cap S'| - \varepsilon^2|S\backslash(S'\cup\{j\})| & \text{if } i = j \\ \varepsilon^3 & \text{if } i \in S'\backslash(S \cup \{j'\}) \\ t - \varepsilon|S\cap S'| - \varepsilon^3|S'\backslash(S\cup\{j'\})| & \text{if } i = j' \\ \varepsilon^4 & \text{if } i \in N\backslash(S \cup S'). \end{cases}$$

Moreover, choose $\varepsilon > 0$ small enough to make sure that $0 < p_i < t$ for all $i \in N$ and $A(N,t,l,u,\succeq) = \{S, S'\}$. Observe that such $\varepsilon > 0$ exists. Since f is efficient, $c^f(N,t,l,u,\succeq) \in \{S,S'\}$. Then, if $c^f(N,t,l,u,\succeq) = S$ set $S\rho S'$ and if $c^f(N,t,l,u,\succeq) = S'$ set $S'\rho S$.

Since f satisfies (iic), the order ρ does not depend on the particular chosen problem $(N,t,l,u,\succeq) \in \mathcal{P}$. Namely, let $(N,t',l',u',\succeq') \in \mathcal{P}$ be such that $A(N,t',l',u',\succeq') = \{S,S'\}$. Then, $c^f(N,t',l',u',\succeq') = c^f(N,t,l,u,\succeq)$. Thus, ρ is well defined.

It is immediate to see that the binary relation ρ on $\mathcal{N}$ is complete, antisymmetric, monotonic and responsive. Using similar arguments to those used in the proof of Lemma 13 in Bergantiños, Massó, and Neme (2012b) it is possible to show that ρ is transitive.

Lemma 2.1 *Let f be a rule satisfying (ef), (sp), (ete), (bm), and (iic) and let ρ be its corresponding complete, antisymmetric, transitive, monotonic and responsive order on $\mathcal{N}$ defined as in Cases 1, 2, and 3 above. Then, $f(N,t,l,u,\succeq) = U^{F,\rho}(N,t,l,u,\succeq)$ for all $(N,t,l,u,\succeq) \in \mathcal{P}$.*

Proof Let $(N,t,l,u,\succeq) \in \mathcal{P}$ be arbitrary and suppose that f and ρ satisfy the hypothesis of Lemma 2.1. If $A(N,t,l,u,\succeq) = \varnothing$ then, $c^f(N,t,l,u,\succeq) = c^{U^{F,\rho}}(N,t,l,u,\succeq)$ and $f(N,t,l,u,\succeq) = U^{F,\rho}(N,t,l,u,\succeq) = (0,...,0)$. Assume $A(N,t,l,u,\succeq) \neq \varnothing$. By (*ef*), $c^f(N,t,l,u,\succeq)$ and $c^{U^{F,\rho}}(N,t,l,u,\succeq)$ are non-empty. Since $S \in A(S,t,(l_i)_{i\in S},(u_i)_{i\in S},(\succeq_i)_{i\in S})$ implies $S \in A(N,t,l,u,\succeq)$, we have that $A(S,t,(l_i)_{i\in S},(u_i)_{i\in S},(\succeq_i)_{i\in S}) \subseteq A(N,t,l,u,\succeq)$. In particular, $c^f(N,t,l,u,\succeq) \equiv c^f \in A(c^f,t,(l_i)_{i\in c^f},(u_i)_{i\in c^f},(\succeq_i)_{i\in c^f}) \subseteq A(N,t,l,u,\succeq)$. Hence, by *(iic)*, $c^f(c^f,t,(l_i)_{i\in c^f},(u_i)_{i\in c^f},(\succeq_i)_{i\in c^f}) = c^f(N,t,l,u,\succeq)$. Since $(c^f,t,(l_i)_{i\in c^f},(u_i)_{i\in c^f},(\succeq_i)_{i\in c^f}) \in P$ and f satisfies (*ef*), (*sp*), (*ete*), and (*bm*), by Theorem 1,

$$f_i(N,t,l,u,\succeq) = \begin{cases} U_i^F(c^f,t,(l_j)_{j\in c^f},(u_j)_{j\in c},(\succeq_j)_{j\in c^f}) & \text{if } i \in c^f \\ \\ 0 & \text{if } i \notin c^f. \end{cases} \tag{12}$$

We show that $c^f \rho S$ for all $S \in A(N,t,l,u,\succeq)\backslash c^f$ by considering separately the following three cases.

<u>Case 1</u>: $S \subsetneq c^f$. Then, by Case 1 in the definition of ρ, $c^f \rho S$.

<u>Case 2</u>: $c^f \subsetneq S$. We obtain a contradiction. Since $S \in A(N,t,l,u,\succeq)$, $S \in A(S,t,(l_i)_{i\in S},(u_i)_{i\in S},(\succeq_i)_{i\in S})$. Thus, $(S,t,(l_i)_{i\in S},(u_i)_{i\in S},(\succeq_i)_{i\in S}) \in P$. By Theorem 1, $f(S,t,(l_i)_{i\in S},(u_i)_{i\in S},(\succeq_i)_{i\in S}) = U^F(S,t,(l_i)_{i\in S},(u_i)_{i\in S},(\succeq_i)_{i\in S})$. Hence, $c^f(S,t,(l_i)_{i\in S},(u_i)_{i\in S},(\succeq_i)_{i\in S}) = S$. Since $c^f \subsetneq S$, $c^f \in A(S,t,(l_i)_{i\in S},(u_i)_{i\in S},(\succeq_i)_{i\in S})$. Moreover, $A(S,t,(l_i)_{i\in S},(u_i)_{i\in S},(\succeq_i)_{i\in S}) \subseteq A(N,t,l,u,\succeq)$. By *(iic)*, $c^f = c^f(S,t,(l_i)_{i\in S},(u_i)_{i\in S},(\succeq_i)_{i\in S}) = S$, a contradiction.

<u>Case 3</u>: $c^f\backslash S \neq \varnothing$ and $S\backslash c^f \neq \varnothing$. Let $(N',t',l',u',\succeq') \in \mathcal{P}$ be as in the definition of ρ where $S' = c^f$. Thus, $A(N',t',l',u',\succeq') = \{c^f,S\} \subseteq A(N,t,l,u,\succeq)$. By *(iic)*, $c^f(N',t',l',u',\succeq') = c^f(N,t,l,u,\succeq)$. Hence, by the definition of ρ, $c^f \rho S$.

Thus, $c^f(N,t,l,u,\succeq) = c^{U^{F,\rho}}(N,t,l,u,\succeq)$. By (12), $f(N,t,l,u,\succeq) = U^{F,\rho}(N,t,l,u,\succeq)$. ■

We now prove that the five properties are independent.

- (*ef*) is independent of the other four properties.

Let f^1 be defined as in the independence of the properties of Theorem 1. We extend f^1 to problems where N is not admissible as we did with the uniform rule. Namely, let ρ be a monotonic and responsive order on $\mathcal{N}$. We define $f^{1,\rho}$ as follows. For any $(N, t, l, u, \succeq) \in \mathcal{P}$ and $i \in N$,

$$f_i^{1,\rho}(N, t, l, u, \succeq) = \begin{cases} f_i^1(c^{c^{f^{1,\rho}}}, t, (l_j)_{j \in c^{f^{1,\rho}}}, (u_j)_{j \in c^{f^{1,\rho}}}, (\succeq_j)_{j \in c^{f^{1,\rho}}}) & \text{if } i \in c^{f^{1,\rho}} \\ 0 & \text{if } i \notin c^{f^{1,\rho}}, \end{cases}$$

where $c^{f^{1,\rho}} \in A(N, t, l, u, \succeq)$ and $c^{f^{1,\rho}} \rho S$ for all $S \in A(N, t, l, u, \succeq) \backslash c^{f^{1,\rho}}$.

It is not difficult to prove that $f^{1,\rho}$ satisfies all properties but (ef).

- (sp) is independent of the other four properties.

  Let f^2 be defined as in the independence of the properties of Theorem 1. Let ρ be a monotonic and responsive order on $\mathcal{N}$. We define $f^{2,\rho}$ from f^2 as we did with $f^{1,\rho}$.

  It is not difficult to prove that $f^{2,\rho}$ satisfies all properties but (sp).

- (ete) is independent of the other four properties.

  Let f^3 be defined as in the independence of the properties of Theorem 1. Let ρ be a monotonic and responsive order on $\mathcal{N}$. We define $f^{3,\rho}$ from f^3 as we did with $f^{1,\rho}$.

  It is not difficult to prove that $f^{3,\rho}$ satisfies all properties but (ete).

- (bm) is independent of the other four properties.

  Let f^4 be defined as in the independence of the properties of Theorem 1. Let ρ be a monotonic and responsive order on $\mathcal{N}$. We define $f^{4,\rho}$ from f^4 as we did with $f^{1,\rho}$.

  It is not difficult to prove that $f^{4,\rho}$ satisfies all properties but (bm).

- *(iic)* is independent of the other four properties. Let ρ and ρ' be two different monotonic and responsive orders on $\mathcal{N}$. We define f^5 as follows.

$$f^5(N, t, l, u, \succeq) = \begin{cases} U^{F,\rho}(N, t, l, u, \succeq) & \text{if } 1 \in N \\ U^{F,\rho'}(N, t, l, u, \succeq) & \text{if } 1 \notin N. \end{cases}$$

It is not difficult to prove that f^5 satisfies all properties but *(iic).* ■

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