Trading factors: Heckscher-Ohlin revisited *

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September 2012

Abstract

The Heckscher-Ohlin model without restrictions on factor trading is shown to be equivalent to an exchange model where the goods are the productive factors and consumer's preferences are derived from consumers' preferences for consumption goods. After having illustrated with Vanek's factor proportion theorem and Leontief's paradox the new perspective brought by this equivalence, this paper is devoted to extending the equilibrium manifold approach from the exchange model to the Heckscher-Ohlin model. This extension yields new properties that deal with the uniqueness of equilibrium, the number of equilibria and the continuity or lack of continuity of equilibrium selections in the Heckscher-Ohlin model. In the case of two consumers (countries), these properties are improved into a complete characterization of economies with a unique equilibrium. Several of these results highlight the important role played by the volume of net trade in factor contents in the Heckscher-Ohlin model.

Keywords: *Heckscher-Ohlin, general equilibrium, factor price equalization, Vanek's factor proportion, Leontief's paradox.*

JEL classification numbers: *D51, F11, F14*

1. Introduction

The main goal of this paper is to study the properties of the Heckscher-Ohlin model without restrictions on factor trading. There are no restrictions in the sense that factors can be freely traded between consumers (countries) and the model can be interpreted as representing an integrated world economy. The equilibria of that model are known in the literature as the "integrated equilibria." Those

*Part of the research reported in this paper was done while visiting Sciences Po, Paris and Boston College, USA during a research leave from the University of York in the first months of 2012. A preliminary version of this paper was delivered as the Gérard Debreu Lecture at the 2012 European General Equilibrium Theory Conference in Exeter, UK and at a seminar at PUC-Rio, Brazil in May-June 2012. I am grateful to the participants at those two events for their comments and remarks and to Sciences Po and Boston College for their hospitality.

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that satisfy Deardorff's "lens condition" for Factor Price Equalization (FPE) ([7]; see also [11], p. 108) are also equilibria of the Heckscher-Ohlin model under the additional restriction that factors cannot be traded between countries.

The existence and efficiency of these "integrated equilibria" follow readily from the very general proofs of these two properties for the general equilibrium model [1, 8, 19, 20]. This is to be contrasted with how little is known beyond these two properties in the general case of the Heckscher-Ohlin model with arbitrary numbers of goods, factors and consumers. The main result of this paper is the equivalence between the unrestricted Heckscher-Ohlin model and an exchange model for productive factors. That equivalence simplifies right away two famous issues of trade theory: Vanek's factor proportion theorem [23] and Leontief's paradox [17]. Also through this equivalence, every property of the smooth (general equilibrium) exchange model has an equivalent formulation within the setup of the unrestricted Heckscher-Ohlin model. The properties highlighted in this paper deal mainly with the equilibrium manifold, the uniqueness and, more generally, the number of equilibria and the continuity or lack of continuity of equilibrium selections. Because of the importance for trade theory of the two country case, this equivalence is also used to give a complete characterization of economies with a unique equilibrium in the two country case. Several of these results if not all underscore the importance of the volume of net trade in factor contents.

This paper is organized as follows. Section 2 is devoted to the main assumptions, definitions and notation regarding the Heckscher-Ohlin model. The factor exchange model whose "goods" are the (productive) factors is defined in Section 3. Section 4 is devoted to the equivalence between the factor exchange model and the Heckscher-Ohlin model with goods and factors. Section 5 deals with the derivation of consumers' "preferences for factors" from their preferences for consumption goods. Several properties of those preferences for factors are so obvious that they are readily applied in the same Section 5 to show the potential of the equivalence property by a new proof of Vanek's factor proportion theorem and a new analysis of the nature of Leontief's famous paradox. The full derivation of the properties of those preferences for factors and their associated demand functions takes place in Section 6. The transcription of the properties of the exchange model dealing with the equilibrium manifold, the uniqueness and non-uniqueness of equilibrium and the continuity or lack of continuity of equilibrium selections to the unrestricted Heckscher-Ohlin model occupies the rest of Section 6. Section 7 is devoted to the characterization in the two country case of the set of economies with a unique equilibrium. Section 8 is devoted to a few concluding comments. The proofs that tend to interfere with the flow of the paper are relegated to a first appendix. A second and very short appendix is about to a very useful sufficient condition for a smooth map to be an embedding.

The mathematical prerequisites for reading this paper are all but standard in this kind of literature. They include multivariate calculus up to the implicit function and the inverse function theorems, some familiarity with convexity, and linear algebra. Knowledge of the definitions of smooth manifolds and mappings, up to embeddings, is recommended though not strictly necessary. A standard reference on the Differential Topology used in economic theory is Milnor's marvelous little book [21]. Valuable geometric insight can also be gained by browsing through Guillemin and Pollack's book [13]. In the current paper as in much of general equilibrium theory, the barrage of Differential Topology is more apparent than real.

2. Definitions, assumptions and notation

2.1. Goods and factors

Consumption goods or, more simply, goods are different from productive or primary factors in the sense that they are arguments of consumers' preferences while factors are not. Therefore, goods are consumed and factors are not. The only use for factors is as inputs in production. To make matters as simple as possible, factors are the only inputs used in production processes. In this paper, goods and factors are freely traded.

The numbers of consumption goods and factors are denoted by k and ℓ respectively. The goods space is $\mathbb{R}^k$ and the factor space $\mathbb{R}^\ell$. The consumption set of every consumer is the strictly positive orthant denoted $X = \mathbb{R}^k_{++}$.

Prices and price sets

Since goods and factors are freely traded, there exist well-defined prices for the goods and factors. The price of good j is denoted by $q_j > 0$. The goods price vector $q = (q_j)$ belongs to $X = \mathbb{R}^k_{++}$. The price of factor κ is denoted by $p_\kappa > 0$. The factor price vector $p = (p_\kappa)$ belongs to $\mathbb{R}^\ell_{++}$. The full price vector is the vector $(q, p) \in X \times \mathbb{R}^\ell_{++}$. In many questions, it is convenient to normalize the full price vector. Unless when the contrary is explicitly stated, all price vectors are numeraire normalized, the numeraire being the ℓ-th factor, which is equivalent to setting $p_\ell = 1$. The normalized factor price space is the set $S = \mathbb{R}^{\ell-1}_{++} \times \{1\}$. The normalized full price set is the Cartesian product $X \times S$.

2.2. Consumers

Preferences and utility functions

There is a finite number m of consumers. Consumer i's preferences, with $1 \leq i \leq m$, are defined by a utility function $u_i : X \to \mathbb{R}$ that satisfies the following properties: 1) Smoothness; 2) Smooth monotonicity, i.e., $Du_i(x_i) \in X$ for $x_i \in X$ where $Du_i(x_i)$ is the gradient vector defined by the first-order derivatives of u_i; 3) Smooth strict quasi-concavity, namely, the restriction of the quadratic form defined by the Hessian matrix $D^2u_i(x_i)$ to the tangent hyperplane to the indifference surface $\{y_i \in X \mid u_i(y_i) = u_i(x_i)\}$ through x_i is negative definite; 4) The indifference surface $\{y_i \in X \mid u_i(y_i) = u_i(x_i)\}$ is closed in $\mathbb{R}^k$ for all $x_i \in X$. (See, for example, [6], Chapter 2.)

Consumers' demand functions

Maximizing the utility $u_i(x_i)$ with $x_i \in X$ subject to the budget constraint $q^T x_i \leq w_i$ (matrix notation) has a unique solution $h_i(q, w_i)$. The function $h_i : X \times \mathbb{R}_{++} \to X$ is consumer i's demand function for consumption goods. The following properties of demand functions used in this paper are well-known:

Proposition 1.

i) (W): Walras law: $q^T h_i(q, w_i) = w_i$ for any $(q, w_i) \in X \times \mathbb{R}_{++}$;

ii) (S): h_i is smooth;

iii) Homogeneity of degree zero: $h_i(\lambda q, \lambda w_i) = h_i(q, w_i)$ for every $\lambda > 0$ and any $(q, w_i) \in X \times \mathbb{R}_{++}$;

iv) (WARP): weak axiom of revealed preferences

$$\left.\begin{array}{r}(q')^T h_i(q, w_i) \leq w_i' \\ h_i(q, w_i) \neq h_i(q', w_i')\end{array}\right\} \quad \Longrightarrow \quad q^T h_i(q', w_i') > w_i;$$

v) (ND): The Slutsky matrix of h_i truncated to its first $\ell - 1$ rows and columns is negative definite at any $(q, w_i) \in X \times \mathbb{R}_{++}$.

Proof. See [6], Section 3.4. □

Factor endowments

Another difference between goods and factors is that consumers are endowed only with factors. Consumer i's endowment vector is denoted by $\omega_i \in \mathbb{R}^\ell_{++}$. The m-tuple $\omega = (\omega_i)$ denotes the distribution of endowments in the economy. The set consisting of all possible endowments, also known as the endowment or parameter space, is denoted by $\Omega = (\mathbb{R}^\ell_{++})^m$. Note that every consumer is endowed with strictly positive quantities of every factor.

2.3. Production

Production functions

There is no joint production of consumption goods. The quantity x^j of the consumption good j that can be produced with the inputs in productive factors represented by the vector $\eta = (\eta^1, \ldots, \eta^\ell) \in \mathbb{R}^\ell_+$ is a function $x^j = F^j(\eta^1, \ldots, \eta^\ell) \geq 0$. The production function F^j is continuous on the non-negative orthant $\mathbb{R}^\ell_+$ and takes the value 0 on its boundary. In addition, on the strictly positive orthant $\mathbb{R}^\ell_{++}$, the function F^j is smooth, monotone (i.e., $\partial F^j/\partial \eta^\kappa > 0$ for $1 \leq \kappa \leq \ell$), homogeneous of degree one and concave (constant returns to scale), with Hessian matrix $D^2F^j(\eta)$ negative semi-definite and of rank $\ell - 1$. All these assumptions regarding production functions are standard. The following properties of production functions are also well-known. Nevertheless, because of the lack of easily accessible references for the general setup of an arbitrary number of factors, proofs are provided for reader's convenience in the Appendix.

Proposition 2.

i) $DF^j(\eta)^T\eta = F^j(\eta) \neq 0$ *for* $\eta \in \mathbb{R}^\ell_{++}$*;*

ii) $\eta^T D^2F^j(\eta) = D^2F^j(\eta)\,\eta = 0$*;*

iii) The kernel of matrix $D^2F^j(\eta)$ *is the one-dimensional subspace generated by* $\eta \in \mathbb{R}^\ell_{++}$*;*

iv) The strict inequality $z^T D^2F^j(\eta)\,z < 0$ *is satisfied for any non-zero vector* $z \in \mathbb{R}^\ell$ *that is not collinear with* η*;*

v) The bordered Hessian matrix $\begin{bmatrix} D^2F^j(\eta) & DF^j(\eta) \\ DF^j(\eta)^T & 0 \end{bmatrix}$ *is invertible.*

Proof. See Appendix, Section A.1. □

Isoquants

The set $\{\eta \in \mathbb{R}^\ell_{++} \mid F^j(\eta) = 1\}$ is the analog in the current setup with ℓ factors of the textbook isoquants for two factors.

Proposition 3.

i) The set $\{\eta \in \mathbb{R}^\ell_{++} \mid F^j(\eta) \geq 1\}$ *is strictly convex;*

ii) Its recession cone is the non-negative orthant $\mathbb{R}^\ell_{+}$*.*

Proof. See Appendix, Section A.2. □

The cost functions

Let $p \in \mathbb{R}^\ell_{++}$ be a (not necessarily normalized) factor price vector. Let η be a factor bundle that minimizes the cost $p^T\eta$ subject to the constraint $F^j(\eta) \geq 1$. Let $\sigma_j(p)$ denote this minimum cost (when it exists). The following proposition describes some properties of these (minimum) cost functions.

Proposition 4.

i) The problem of producing good j *at minimal costs has a unique solution* $b_j(p)$ *for any* $p \in \mathbb{R}^\ell_{++}$*;*

ii) *The demand function for factors (associated with the production of good j) $b_j : \mathbb{R}^\ell_{++} \to \mathbb{R}^\ell_{++}$ is smooth and homogeneous of degree zero;*

iii) *The cost function $\sigma_j : \mathbb{R}^\ell_{++} \to \mathbb{R}$ defined by $\sigma_j(p) = p^T b_j(p)$ is smooth and homogeneous of degree one;*

iv) *The cost function σ_j is concave.*

Proof. See Appendix, Section A.3. □

The demand function for factors induced by the production of good j.

The demand function for factors associated with the production of good j satisfies several other properties used in this paper.

Proposition 5.

i) $b_j = D\sigma_j$;

ii) $(p - p')^T(b_j(p) - b_j(p')) \leq 0$ *for* p *and* p' *in* S;

iii) *The Jacobian matrix $Db_j(p)$ defines a negative semidefinite quadratic form;*

iv) *The Jacobian matrix $Db_j(p)$ has rank $\ell - 1$ and $Db_j(p)^T p = 0$;*

v) *Let $p^0 = \lim_{t\to\infty} p^t$ where $p^t \in \mathbb{R}^\ell_{++}$ (non-normalized prices) and $p^0 \neq 0$ has some coordinates equal to 0. Then, $\limsup_{t\to\infty} \|b_j(p^t)\| = +\infty$.*

Proof. See Appendix, Section A.4. □

Production sector's production matrix

Definition 6. *The* **production matrix** *associated with the (non normalized) factor price vector $p \in \mathbb{R}^\ell_{++}$ is the $\ell \times k$ matrix $B(p) = \begin{bmatrix} b_1(p) & b_2(p) & \dots & b_k(p) \end{bmatrix}$. The* **production matrix function** *is the map $B : \mathbb{R}^\ell_{++} \to (\mathbb{R}^\ell_{++})^k$ defined by $p \to B(p)$.*

Here are some properties of this matrix function:

Proposition 7.

i) $DB(p)^T p = 0$;

ii) $D(B(p)^T p) = B(p)^T$;

iii) *Let $p \neq p'$ in S be two (numeraire normalized) factor price vectors. The following strict vector inequality is then satisfied: $p^T B(p) < p^T B(p')$.*

Proof. See Appendix, Section A.5. □

Factor content of a goods bundle

What quantities of factors are required for the production of a goods bundle? Of particular interest are the quantities of factors that minimize those costs.

Proposition 8. *Let $x \in \mathbb{R}^k_{++}$ be a goods bundle. There is a unique factor bundle $y \in \mathbb{R}^\ell_{++}$ that minimizes the total cost of producing the goods bundle x. This factor bundle is equal to $y = B(p)\,x$ (matrix notation).*

Proof. Obvious from the definition of matrix $B(p)$. □

Proposition 8 leads to the following definition:

Definition 9. *The factor content of the goods bundle $x \in \mathbb{R}^k_+$ for the (non-normalized) factor price vector $p \in \mathbb{R}^\ell_{++}$ is the vector $y = B(p)\,x \in \mathbb{R}^\ell_{++}$.*

The factor bunble $y = B(p)\,x \in \mathbb{R}^\ell_{++}$ is also known as the trace or footprint (in the factor space $\mathbb{R}^\ell$) of the goods bundle $x \in \mathbb{R}^k_+$. This factor content depends on factor prices represented by $p \in S$.

2.4. The Heckscher-Ohlin model

Definition

The Heckscher-Ohlin model is defined by the k goods and ℓ factors, the m consumers and the production technology represented by the production matrix function $B : S \to (\mathbb{R}^\ell_{++})^k$. An economy in the Heckscher-Ohlin model is then defined by the specification of consumers' endowments $\omega = (\omega_i)$. This model is often known as the model of an integrated economy in the sense that all countries are integrated within one big country that could be identified to the world economy. Later, in a sequel to this paper, I hope to analyze the case where factors are traded only within the boundaries of each country.

Equilibrium

Given the economy $\omega = (\omega_i) \in \Omega$ and the full price vector $(q, p) \in X \times S$, consumer i's demand for goods is equal to $h_i(q, p^T\omega_i)$. The total demand for goods is then equal to

$$\sum_{1 \le i \le m} h_i\big(q, p^T\omega_i\big).$$

An equilibrium must be such that enough goods are produced in order to satisfy all individual demands for goods. This justifies the following definition:

Definition 10. *The 3-tuple $(q, p, \omega) \in X \times S \times \Omega$ is an* **equilibrium** *of the Heckscher-Ohlin model if there exists a vector $x \in X$ such that the following* *equalities are satisfied:*

$$\sum_{1\le i\le m} h_i(q, p^T\omega_i) = x, \tag{1}$$

$$B(p)\,x = \sum_{1\le i\le m} \omega_i. \tag{2}$$

The component $(q, p) \in X \times S$ is then an **equilibrium price vector** *for the Heckscher-Ohlin economy $\omega = (\omega_i) \in \Omega$. The allocation of goods represented by the m-tuple $\big(h_i(q, p^T\omega_i)\big) \in X^m$ is the corresponding* **equilibrium allocation.**

Proposition 11. *The factor content of the allocation $\big(h_i(q, p^T\omega_i)\big) \in X^m$ associated with the equilibrium (q, p, ω) is the m-tuple $\big(B(p)h_i(q, p^T\omega_i)\big) \in (\mathbb{R}^\ell)^m$.*

Proof. Obvious from Definition 9. □

Equilibrium manifold and the natural projection

The equilibrium manifold for the Heckscher-Ohlin model is the subset $\tilde{E}$ of $X\times S\times\Omega$ consisting of equilibria (q, p, ω). The natural projection $\tilde{\pi} : E \to \Omega$ is the restriction to the equilibrium manifold $\tilde{E}$ of the projection map $(q, p, \omega) \to \omega$.

A program for the direct study of the Heckscher-Ohlin model that would parallel the study of the exchange model along the lines of [2] or [6] would start with the study of the equilibrium manifold $\tilde{E}$, a study where the local and global structures of that set would stand in good place. It would then continue with the natural projection $\tilde{\pi} : \tilde{E} \to \Omega$. A major hurdle that this direct approach would face is the lack of an obvious candidate for the concept of no-trade equilibrium because consumers' endowments consist here of factors that cannot be consumed. The route followed in the current paper bypasses this problem.

3. The factor exchange model

3.1. Definitions

Factors again

The commodity space of the factor exchange model is the Euclidean space of factors $\mathbb{R}^\ell$. Factor ℓ is used as numeraire. The price set is the set $S = \mathbb{R}^{\ell-1} \times \{1\}$ of numeraire normalized factor price vectors previously defined.

Consumers in the factor exchange model

Consumer i's demand function (for factors) in the factor exchange model, with $1 \le i \le m$, is the map $f_i : S \times \mathbb{R}_{++} \to \mathbb{R}^\ell_{++}$ defined by the expression

$$f_i(p, w_i) = B(p)\, h_i(B(p)^T p, w_i). \tag{3}$$

The factor exchange model

It is defined by the ℓ factors (as goods), the m consumers characterized by their demand functions for factors f_i, with $1 \le i \le m$. A factor (exchange) economy is defined by a specific value of the endowment vector $\omega = (\omega_i) \in \Omega$.

Equilibrium in the factor exchange model

The pair $(p, \omega) \in S \times \Omega$ is an equilibrium of the **factor model** if and only if the equilibrium equation

$$\sum_{1 \le i \le m} f_i(p, p^T \omega_i) = \sum_{1 \le i \le m} \omega_i \tag{4}$$

is satisfied. The factor price vector $p \in S$ is then an equilibrium price vector associated with the factor endowment $\omega = (\omega_i) \in \Omega$.

Equilibrium manifold and natural projection for the factor exchange model

The equilibrium manifold for the factor exchange model is the subset E of $S \times \Omega$ consisting of the equilibria (p, ω). The natural projection $\pi : E \to \Omega$ is the restriction to the equilibrium manifold E of the projection map $(p, \omega) \to \omega$. It is a property of the exchange model that its "equilibrium manifold" E is indeed a smooth submanifold of $S \times \Omega$ and is diffeomorphic to a Euclidean space. (See [2] or [6], Proposition 4.9 and 5.8.)

4. Equivalence of the Heckscher-Ohlin and factor models

Equivalence means that the two models have the same properties except that the formulation of these properties may be different from one model to the other. This equivalence will take two forms. An elementary version that shows that the equilibrium equation of one model can be transformed into the equilibrium equation of the other model and conversely. This is the High School approach when solving the quadratic equation $ax^2 + bx + c = 0$ by reducing it to an equation of the form $y^2 = d$. This form of equivalence of equation systems does not suffice, however, to completely identify two different mathematical models and their properties. It is possible to define a stronger form of equivalence by identifying the two models to their natural projection maps and to apply to these two maps the concept of equivalent mappings used in Differential Topology [5, 12].

4.1. Equivalence of the equilibrium equations

In this Section, equivalence is meant in the High School sense:

Proposition 12. *The triple* $(q, p, \omega) \in \mathbb{R}^k_{++} \times S \times \Omega$ *is an equilibrium of the Heckscher-Ohlin model if and only if the pair* $(p, \omega) \in S \times \Omega$ *is an equilibrium of the factor exchange model.*

Proof.
Necessity.
Step 1. Let $(q, p, \omega) \in X \times S \times \Omega$ be an equilibrium of the Heckscher-Ohlin model. The production of commodity j is a zero-profit operation because of the constant returns to scale. This implies the equality $q_j = b_j(p)^T p$ for $1 \leq j \leq k$. These equalities can be rewritten in matrix form as $q = B(p)^T p$.
Step 2. By definition, the two equations

$$\sum_{1 \leq i \leq m} h_i(q, p^T \omega_i) = x, \tag{5}$$

$$B(p)\, x = \sum_{1 \leq i \leq m} \omega_i \tag{6}$$

are satisfied for some $x \in X$. Matrix multiplication of both sides of (5) by $B(p)$ yields

$$\sum_{1 \leq i \leq m} B(p)\, h_i(q, p^T \omega_i) = B(p)\, x.$$

Substituting $B(p)^T p$ to q and applying Equation (6) yields

$$\sum_{1 \leq i \leq m} B(p)\, h_i(B(p)^T p, p^T \omega_i) = \sum_{1 \leq i \leq m} \omega_i, \tag{7}$$

which can be rewritten as

$$\sum_{1 \leq i \leq m} f_i(p, p^T \omega_i) = \sum_{1 \leq i \leq m} \omega_i, \tag{8}$$

the equilibrium equation of the factor exchange model for the factor endowment vector $\omega = (\omega_i) \in \Omega$.

Sufficiency. Let $(p, \omega) \in S \times \Omega$ be an equilibrium of the factor exchange model. Define $q = B(p)^T p \in X$ and let

$$x = \sum_{1 \leq i \leq m} h_i(q, p^T \omega_i). \tag{9}$$

Each vector $h_i(q, p^T \omega_i)$ belongs to $X = \mathbb{R}^k_{++}$ as does the sum $x = \sum_{1 \leq i \leq m} h_i(q, p^T \omega_i)$. Left multiplication by $B(p)$ of equality (9) yields

$$B(p)\, x = B(p) \sum_{1 \leq i \leq m} h_i(q, p^T \omega_i). \tag{10}$$

Since $p \in S$ solves equation (8) and, therefore, equation (7), the right-hand side term of (10) is equal to $\sum_{1 \leq i \leq m} \omega_i$, from which follows the equality

$$B(p)\, x = \sum_{1 \leq i \leq m} \omega_i,$$

which is equation (2) of Definition 10. This proves that the triple (q, p, ω) is an equilibrium of the Heckscher-Ohlin model. □

Application to the factor content of equilibrium allocations in the Heckscher-Ohlin model

Equilibrium allocations in the Heckscher-Ohlin model consist of consumption goods, not factors. The goods allocation associated with the equilibrium $(q, p, \omega) \in \tilde{E}$ is the allocation $x = (x_i) = \big(h_i(q, p^T\omega_i)\big)$, with $1 \leq i \leq m$.

Definition 13. *The factor content of the goods allocation $x = (x_i) \in X^m$ associated with the equilibrium $(q, p, \omega) \in \tilde{E}$ is the factor allocation $(B(p)x_i) \in (\mathbb{R}^\ell_{++})^m$;*

Proposition 14. *Let $(q, p, \omega) \in \tilde{E}$ be an equilibrium of the Heckscher-Ohlin model. The factor content of the equilibrium allocation $x = (x_i) = \big(h_i(q, p^T\omega_i)\big) \in X^m$ is the (equilibrium) allocation $\big(f_i(p, p^T\omega_i)\big) \in (\mathbb{R}^\ell_{++})^m$ in the factor exchange model associated with the equilibrium (p, ω).*

Proof. Follows from $f_i(p, p^T\omega_i) = B(p)h_i(B(p)^T p, p, p^T\omega_i)$. □

Definition 15. *The net trade vector in factor contents in the Heckscher-Ohlin model at the equilibrium $(q, p, \omega) \in \tilde{E}$ is the vector $\big(f_i(p, p^T\omega_i) - \omega_i\big) \in (\mathbb{R}^\ell)^m$.*

Proposition 16. *The net trade vector in factor contents at the equilibrium $(q, p, \omega) \in \tilde{E}$ is equal to 0 if and only if $(p, \omega) \in E$ is a no-trade equilibrium of the factor exchange model.*

Proof. Follows from $f_i(p, p^T\omega_i) - \omega_i = 0$ for $1 \leq i \leq m$. □

4.2. Equivalence of the natural projections

With the Heckscher-Ohlin and factor exchange models represented by their natural projection maps $\tilde{\pi} : \tilde{E} \to \Omega$ and $\pi : E \to \Omega$ respectively, a stronger version of equivalence can be formulated as the equivalence of these two maps in the sense of [5], Definition 5.4.2. or [12], Chapter III, Definition 1.1. For example, this form of equivalence readily implies that the "equilibrium manifold" $\tilde{E}$ of the Heckscher-Ohlin model is a smooth submanifold of $X \times S \times \Omega$ that is diffeomorphic to a Euclidean space since that property is satisfied by the equilibrium manifold E of the factor exchange model. This property could not be derived as easily from the weaker notion of equivalence considered earlier.

Two auxiliary maps

Definition 17. *Define:*

i) $\alpha : X \times S \times \Omega \to S \times \Omega$ *as the projection* $(q, p, \omega) \to (p, \omega)$*;*

ii) $\beta : S \times \Omega \to X \times S \times \Omega$ *by* $\beta(p, \omega) = (B(p)^T p, p, \omega)$*.*

Lemma 18.

i) The maps α and β are smooth;

ii) $\alpha \circ \beta = \mathrm{id}_{S\times\Omega}$*;*

iii) The map $\beta : S \times \Omega \to X \times S \times \Omega$ *is an embedding;*

iv) The image $F = \beta(S \times \Omega)$ *is a smooth submanifold of* $X \times S \times \Omega$ *that is diffeomorphic to* $S \times \Omega$*;*

v) The image $\beta(E)$ *is a smooth submanifold of* $X \times S \times \Omega$ *diffeomorphic to* E*.*

Proof. Obvious for *(i)* and *(ii)*. It follows from (ii) combined with Lemma 43 in Appendix B that β is an embedding, which proves (iii). Then, (iv) follows from (ii) combined with the definition of an embedding. To prove (v), it suffices to combine (iv) with the property that the set E, the equilibrium manifold of the exchange model, is a smooth submanifold of $S \times \Omega$. □

Lemma 19.

i) $\alpha(\tilde{E}) = E$*;*

ii) $\beta(E) = \tilde{E}$*.*

Proof. Follows readily from Proposition 12. □

A commutative diagram

The maps $\tilde{\alpha}$ and $\tilde{\beta}$ are defined by the same formulas as the maps α and β except that their ranges and domains are different. More accurately:

Definition 20. *Define:*

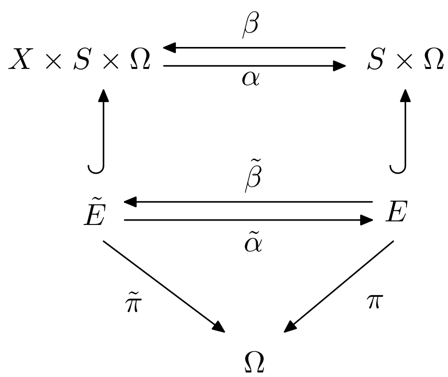
i) $\tilde{\alpha} : \tilde{E} \to E$ *by* $\tilde{\alpha}(q, p, \omega) = \alpha(q, p, \omega)$*;*

ii) $\tilde{\beta} : E \to \tilde{E}$ *by* $\tilde{\beta}(p, \omega) = \beta(p, \omega)$*.*

Proposition 21 (Equivalence of the two models)**.**

i) $\tilde{\alpha} \circ \tilde{\beta} = \mathrm{id}_E$ *and* $\tilde{\beta} \circ \tilde{\alpha} = \mathrm{id}_{\tilde{E}}$*;*

ii) The following diagram is commutative:



iii) The natural projection maps π and $\tilde{\pi}$ are equivalent.

Proof. Properties (i) and ii) follow readily from the formulas defining the maps α, $\tilde{\alpha}$, β, and $\tilde{\beta}$. Property (iii) follows from (ii) and the commutativity of the lower triangle in the diagram. □

5. Two applications to Trade Theory

It follows from Proposition 21 that the properties of the unrestricted Heckscher-Ohlin model can be derived from those of its equivalent factor exchange model. The goal of this Section is to illustrate the potential of this approach with two quick and dirty applications to Trade Theory. Basically, the main ingredient of these two applications is that consumers' "preferences for factors" can be represented by utility functions that are homogeneous of degree one if the corresponding utility functions for goods are themselves homogeneous of degree one. The other usual properties of preferences are assumed to be satisfied, at least provisionally. These issues are to be addressed in Section 6.

5.1. Utility function for factors

We will see that there is a very natural candidate for the indirect utility function for factors. The direct utility (for factors) will then be associated with that indirect utility.

Indirect utility function for factors

Let $\hat{u}_i(q, w_i) = u_i\big(h_i(q, w_i)\big)$ denote consumer i's indirect utility function for goods. Let $p \in \mathbb{R}^{\ell}_{++}$ be a non-normalized factor price vector. The function defined by the formula $\hat{v}_i(p, w_i) = \hat{u}_i(B(p)^T p, w_i)$ is a natural candidate for being an indirect utility function of consumer i for factors. This is confirmed by the following:

Proposition 22. *Consumer i's demand for factors $f_i : S \times \mathbb{R}_{++} \to \mathbb{R}^{\ell}_{++}$ satisfies Roy's identities for the indirect utility function $\hat{v}_i$.*

Proof. See Appendix, Section A.6. □

Remark 1. Roy's identities are necessary and sufficient for the demand function f_i to be associated with the indirect utility function $\hat{v}_i$ if the latter is associated with a strictly quasi-concave utility function v_i. The direct proof of the strict quasi-concavity of v_i presents no major difficulties and is left to the reader. This property can also be seen as resulting from the demand function for factors f_i satisfying the weak axiom of revealed preferences ($WARP$) if h_i satisfies ($WARP$) which, itself, results from the strict quasi-concavity of the direct utility function u_i. See Lemma 34 (i) and also Remark 5.

Direct utility for factors

Proposition 23. *Consumer i's direct utility for factors $v_i : \mathbb{R}^{\ell}_{++} \to \mathbb{R}$ is equal to*

$$v_i(y_i) = \min_{p \in S} u_i\big(h_i(B(p)^T p, p^T y_i)\big).$$

Proof. Follows from the expression of the indirect utility function as $\hat{v}_i(p, p^T y_i) = u_i\big(h_i(B(p)^T p, p^T y_i)$ and from the expression of the direct utility function v_i from the indirect utility function $\hat{v}_i$ as $v_i(y_i) = \min_{p \in S} \hat{v}_i(p, p^T y_i)$. □

5.2. First application: Vanek's factor proportion theorem

Vanek's factor proportion theorem deals with the factor contents of a country's exports and imports in relation to the country's factor endowments in the case of two countries (i.e., $m = 2$) but arbitrarily large numbers ℓ and k of goods and factors, and identical homothetic preferences. There is then no loss of generality in assuming that the utility functions u_1 and u_2 of the two consumers (or countries) are equal and homogeneous of degree one ($u_i(\lambda x) = \lambda u_i(x)$ for $\lambda > 0$, with $i = 1$ or 2).

Proposition 24. *The direct utility functions for factors v_i with $i = 1, 2$ are equal and homogeneous of degree one.*

Proof. Equality $v_1 = v_2$ is obvious. Homogeneity of degree one follows from the formula in Proposition 23 combined with:
i) $\hat{v}_i(p, \lambda w_i) = \lambda \hat{v}_i(p, w_i)$. Follows from:

$$h_i(q, \lambda w_i) = \lambda h_i(q, w_i),$$
$$\hat{u}_i(q, \lambda w_i) = u_i(h_i(q, \lambda w_i)) = u_i(\lambda h_i(q, w_i)) = \lambda \hat{u}_i(q, w_i)$$
$$\hat{v}_i(p, \lambda w_i) = \hat{u}_i(B(p)^T p, \lambda w_i) = \lambda \hat{u}_i(B(p)^T p, w_i) = \lambda \hat{v}_i(p, w_i).$$

ii) $v_i(\lambda y_i) = \lambda v_i(y_i)$. Follows from:

$$v_i(\lambda y_i) = \min_{p \in S} \hat{v}_i(p, p^T \lambda y_i) = \min_{p \in S} \lambda \hat{v}_i(p, p^T y_i) = \lambda \min_{p \in S} \hat{v}_i(p, p^T y_i) = \lambda v_i(y_i).$$ □

Pareto optima and their (unique) supporting factor price vector

Let $r = \omega_1 + \omega_2 \in \mathbb{R}^{\ell}_{++}$ denote the vector of total factor resources. This vector is kept fixed in this Section. Incidentally, this is the usual setup for the Edgeworth box.

Proposition 25. *The set of Pareto optima P in the factor exchange model with fixed total factor resources $r \in \mathbb{R}^{\ell}_{++}$ consists of the allocations $y = (y_1, y_2) = (\lambda r, (1 - \lambda) r) \in \mathbb{R}^{\ell}_{++} \times \mathbb{R}^{\ell}_{++}$ with $\lambda \in [0, 1]$.*

Proof. By homogeneity of degree one of u_i, it comes $Dv_1(\lambda r) = Dv_1(r)$ and $Dv_2((1-\lambda) r) = Dv_2(r)$. From $v_1 = v_2$ comes $Dv_1(r) = Dv_2(r)$, hence $Dv_1(\lambda r) = Dv_2((1 - \lambda) r)$ for any $\lambda \in (0, 1)$. This proves that $y = (y_1, y_2) = (\lambda r, (1 - \lambda) r)$ is a Pareto optimum for any $\lambda \in (0, 1)$. The extension to $\lambda = 0$ and $\lambda = 1$ is obvious. □

The following two corollaries are obvious:

Corollary 26. *Every Pareto optimum* $y = (y_1, y_2) \in \mathbb{R}^\ell_{++} \times \mathbb{R}^\ell_{++}$ *with* $y_1 + y_2 = r$ *is supported by the same (normalized) price vector* $p \in S$*. That price vector is collinear with* $Dv_1(r) = Dv_2(r)$*.*

Corollary 27. *Equilibrium is unique for any* $\omega = (\omega_1, \omega_2)$ *with* $\omega_1 + \omega_2 = r$ *and the equilibrium price vector is equal to* $p \in S$*.*

Remark 2. The Edgeworth box representation of the factor exchange model with an arbitrary number of factors now takes place in a Euclidean space of dimension ℓ. Let O_1 be the origin of coordinates in $\mathbb{R}^\ell$ and O_2 be the point with coordinates the vector of total factor resources $r \in \mathbb{R}^\ell_{++}$. The set of Pareto optima P (associated with the fixed total factor resources $r \in \mathbb{R}^\ell_{++}$) is then the diagonal O_1O_2. The budget hyperplanes (associated with every Pareto optimum) are all parallel to one another. They are also perpendicular to the factor price vector $p \in S$ that is collinear with $Dv_1(r) = Dv_2(r)$.

The net imports and exports in factors

There is no loss in generality in choosing the units of the ℓ factors so that the total resources are represented by the vector $r = (1, 1, \dots, 1)$. In addition, those ℓ factors are sorted by assuming $1 > \omega_1^1 > \cdots > \omega_1^\ell > 0$. (Cases of possible equalities are left to the reader.) In other words, Country 1 is relatively more endowed in factor 1 than in factor 2, and more in factor 2 than in factor 3, etc.

Proposition 28 (Vanek [23])**.** *There exists an integer* κ*, with* $2 \le \kappa \le \ell$*, such that* Country 1 *is a net exporter of factor contents* κ' *with* $\kappa' < \kappa$*, and a net importer of factor content* κ'' *for* $\kappa'' \ge \kappa$ *(with the possible exception of factor* κ *itself for which there may be no trade).*

Proof. Let $y = (y_1, y_2)$ be the equilibrium allocation associated with the endowment vector $\omega = (\omega_1, \omega_2)$. Then, $y = (y_1, y_2)$ is a Pareto optimum and, by Proposition 25, is such that $y_1 = \lambda r$ with $0 < \lambda < 1$. This implies

$$y_1^1 = \cdots = y_1^\ell = \lambda.$$

The ranking of factor endowments implies the inequalities

$$\lambda - \omega_1^1 < \lambda - \omega_1^2 < \cdots < \lambda - \omega_1^\ell. \tag{11}$$

It follows from the budget constraint satisfied by consumer 1 that $p^T y_1 = p^T \omega_1$, which can be rewritten as

$$p_1(\lambda - \omega_1^1) + p_2(\lambda - \omega_1^2) + \cdots + p_{\ell-1}(\lambda - \omega_1^{\ell-1}) + (\lambda - \omega_1^\ell) = 0. \tag{12}$$

Some of these terms in Equality (12) must be negative and some positive. This implies $\lambda - \omega_1^1 < 0$ and $\lambda - \omega_1^\ell > 0$. It then follows from Inequalities (11) that there exists some integer κ with $2 \le \kappa \le \ell$ such that

$$\lambda - \omega_1^{\kappa-1} < 0 \le \lambda - \omega_1^\kappa,$$

which is equivalent to

$$\omega_1^\kappa \le \lambda < \omega_1^{\kappa-1}. \qquad (13)$$

All factors up to the $\kappa - 1$-th factor are exported by Country 1. All the other factors are imported, with the possible exception of factor κ. □

The value of λ follows from $p^T y_1 = (p_1 + p_2 + \cdots + p_{\ell-1} + 1)\lambda = p^T \omega_1$. The value of the integer κ can then be determined by inequality (13).

Remark 3. This proof of Vanek's theorem works also in the case where the two utility functions are different but the gradient vectors $Dv_1(r)$ and $Dv_2(r)$ are collinear. The latter property is all that is required for the set of Pareto optima to be the segment O_1O_2 linking the allocations $(r, 0)$ and $(0, r)$ in the Edgeworth box.

5.3. Second application: Leontief's paradox

Assume now that the number of factors is equal to two, i.e., $\ell = 2$. Then, Proposition 28 (Vanek's theorem) tells us that Country 1 that is more endowed in factor 1 than in factor 2 relatively to world resources is going to be a net exporter of factor 1 and importer of factor 2.

Leontief's paradox then takes its origin in the observation made in [17] that the US were a net exporter of labor and a net importer of capital despite having factor endowments relative to the rest of the World higher in capital than in labor. This leads to:

Definition 29 (Leontief's paradox)**.** *Leontief's paradox occurs at equilibrium* (p, ω) *where* $\omega_1^1 > \omega_1^2$ *if Country 1 is a net importer of factor 1 and and a net exporter of factor 2.*

Let $y = (y_1, y_2)$ be the equilibrium (factor) allocation associated with the equilibrium (p, ω). Leontief's paradox is therefore equivalent to the inequality $y_1^1 > \omega_1^1$. It follows from Proposition 28 that a necessary condition for observing Leontief's paradox is that the two countries have different preferences for factors (and therefore for goods too).

Proposition 30. *Let Country 1 and the rest of the World ("Country 2") have homothetic preferences (in addition to the assumptions of Section 2). Leontief's paradox occurs at the equilibrium* (p, ω) *if and only if* $\det(Dv_1(\omega_1), Dv_2(\omega_2)) > 0$.

Leontief's paradox is easy to illustrate in the Edgeworth box. The point N representing the endowment vector $\omega = (\omega_1, \omega_2)$ is located to the South-East of the diagonal O_1O_2. Let M be the point that represents the equilibrium allocation $y = (y_1, y_2)$ associated with the equilibrium (p, ω). It is well-known that the equilibrium allocation M associated with the point N lies in the lens defined by the two indifference curves (in the Edgeworth box) that go through the point N. In addition, the line NM is a budget line. That line is not only tangent at M to the indifference curves through M but is also perpendicular to the equilibrium price vector $p \in S$ associated with $\omega = (\omega_1, \omega_2)$). Its slope is therefore negative.

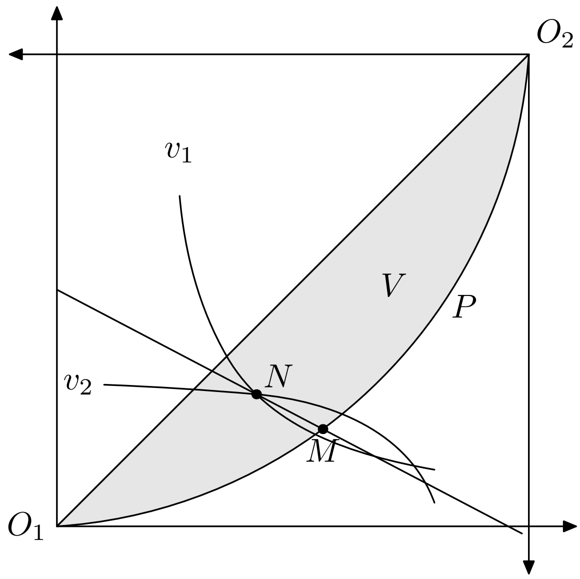


Figure 1: Endowments where Leontief's paradox holds

Proof of Proposition 30. Given the negative slope of the budget line NM, Leontief's paradox becomes equivalent to the endowment point N being to the North-West of the point M. This is in turn equivalent to the point N being the North-West extreme point of the lens determined by the indifference curves through N. This is equivalent to the angle made by the two gradient vectors $Dv_1(\omega_1)$ and $Dv_2(\omega_2)$ being > 0. The latter condition is equivalent to $\det(Dv_1(\omega_1), Dv_2(\omega_2)) > 0$. □

With homothetic preferences, the contract curve P is a smooth curve going from O_1 to O_2 that lies either to the North-West or to the South-East of the diagonal O_1O_2 but does not intersect the interior of the segment O_1O_2. This makes it easier to identify the set of endowments that are conducive to Leontief's paradox.

Proposition 31. *With homothetic preferences, the following two conditions are equivalent:*

i) There exists an endowment vector $\omega = (\omega_1, \omega_2)$ in the South-East half of the Edgeworth box at which the condition $\det(Dv_1(\omega_1), Dv_2(\omega_2)) > 0$ is satisfied;

ii) The contract curve P lies to the South-East half of the Edgeworth box and the condition $\det(Dv_1(\omega_1), Dv_2(\omega_2)) > 0$ is satisfied for all endowment vectors $\omega = (\omega_1, \omega_2)$ in the domain V bounded by the diagonal O_1O_2 and the contract curve P

Proof. Obvious from the geometric representation in the Edgeworth box. □

Corollary 32. *Let V be the interior of the set bounded by the diagonal O_1O_2 and the contract curve P in the Edgeworth box.*

i) If preferences are identical, the set V is empty and Leontief's paradox does not occur.

ii) The set V is non empty and lies at the South-East of the diagonal O_1O_2 if there exists an allocation $\omega = (\omega_1, \omega_2)$ in the Edgeworth box with $\omega_1^1 \geq \omega_1^2$ (the point representing the allocation ω is on or below the diagonal) with $\det\big(Dv_1(\omega_1), Dv_2(\omega_2)\big) > 0$.

iii) If (ii) is satisfied, Leontief's paradox occurs for endowments in the union of the diagonal O_1O_2 and the set V.

Economies with factor endowments in the set V feature the same trade patterns in factor contents as the one observed by Leontief for the USA.

Remark 4. In this section, the only property of the direct utility functions for factors that has been proved is homogeneity of degree one when the utility functions for goods are also homogeneous of degree one. In other words, homothetic preferences for goods imply homothetic preferences for factors. It has been implicit that usual properties like quasi-concavity are satisfied. These issues are addressed in the next section.

6. Properties of the Heckscher-Ohlin model

This section presents several properties of the unrestricted Heckscher-Ohlin model for arbitrary numbers k, ℓ, m of goods, factors and consumers. These properties stand out for their direct relevance to comparative statics. They will result from the equivalence of the unrestricted Heckscher-Ohlin model with some exchange model. (For additional properties of the exchange model that could be reformulated within the setup of the Heckscher-Ohlin model, see [6], Chapter 5 to 8.)

With the assumptions made in Section 2 regarding consumers and production, the first issue is to determine what properties are satisfied by consumers' demand functions for factors.

6.1. Consumers' factor demand functions

Lemma 33. *Consumer i's factor demand function f_i satisfies smoothness (S) (resp. Walras law (W)), if the goods demand function h_i satisfies (S) (resp. (W)).*

Proof. See Section A.7 of the Appendix. □

Lemma 34. *Let (S) and (W) be satisfied by h_i:*

i) h_i satisfies $(WARP) \Longrightarrow f_i$ satisfies $(WARP)$.

ii) h_i satisfies $(ND) \Longrightarrow f_i$ satisfies (ND).

iii) h_i satisfies $(A) \Longrightarrow f_i$ satisfies (A).

Proof. See Section A.8 of the Appendix. □

Proposition 35. *Under the assumptions of Section 2, consumer i's factor demand function* $f_i : S \times \mathbb{R}_{++} \to \mathbb{R}^{\ell}_{++}$ *satisfies (S), (W), (WARP), (ND) and (A) for* $1 \leq i \leq m$.

Proof. Obvious from the two previous Lemmata. □

Remark 5. The weak axiom of revealed preferences ($WARP$) is roughly equivalent to the strict convexity of preferences. When consumer i's preferences are represented by a strictly quasi-concave utility function u_i, Lemma 34 (i) implies that the utility function for factors v_i is also strictly quasi-concave.

It is left to the reader to check that the utility functions for factors v_i satisfy all four properties of smooth consumer theory of Section 2.2 when the latter are satisfied by the utility function u_i.

6.2. The equilibrium manifold for the Heckscher-Ohlin model

Proposition 36. *The equilibrium manifold* $\tilde{E}$ *of the unrestricted Heckscher-Ohlin model is a smooth submanifold of* $X \times S \times \Omega$ *diffeomorphic to the Euclidean space* $\mathbb{R}^{\ell m}$.

Proof. Follows from the same property of the equilibrium manifold E of the factor exchange model. See [2]. (See also [6], Proposition 5.8 for a simpler proof when the parameter space Ω is extended to allow for negative endowments.) □

The diffeomorphism property stated in Proposition 36 implies three different things. First, it states the possibility of parameterizing all the equilibria of the Heckscher-Ohlin model by ℓm parameters. If follows from [6], Chapter 6, Section 6.3 that these parameters can consist of the price vector, the wealth distribution across consumers and the coordinates of the factor net trade vector. Second, the equilibrium manifold is pathconnected. In practice, this means that any two equilibria can be linked by a continuous path in the equilibrium manifold. Third, the natural projection is a smooth map. With the coordinate system for the equilibrium manifold just described, that map can be expressed as a map from $\mathbb{R}^{\ell m}$ into itself. In other words, the natural projection can be identified to a collection of ℓm real-valued functions depending on ℓm real variables. Once that stage has been reached, there is no more need for smooth manifolds and, therefore, for Differential Topology.

6.3. Regular and singular economies for the Heckscher-Ohlin model

The natural projection $\tilde{\pi} : \tilde{E} \to \Omega$ being a smooth map, one defines a regular (resp. critical) point of that map as an element x of $\tilde{E}$ such that the derivative $D_x\tilde{\pi} : T_x(\tilde{E}) \to T_{\tilde{\pi}(x)}(\Omega)$ is (resp. is not) a bijection. The spaces $T_x(\tilde{E})$ and

$T_{\tilde{\pi}(x)}(\Omega)$ are the tangent spaces to $\tilde{E}$ and Ω at $x \in \tilde{E}$ and $\tilde{\pi}(x) \in \Omega$ respectively. By definition, a singular value $\omega \in \Omega$ of the map $\tilde{\pi} : \tilde{E} \to \Omega$ is the image of a critical point, i.e., there exists $x \in \tilde{E}$ that is a critical point and such that $\tilde{\pi}(x) = \omega \in \Omega$. Let Σ denote the set of singular values of the projection map $\tilde{\pi} : \tilde{E} \to \Omega$. This set is the image by $\tilde{\pi}$ of the set of critical points.

The element $\omega \in \Omega$ is by definition a regular value of the map $\tilde{\pi} : \tilde{E} \to \Omega$ if it is not a singular value. The set of regular values $\mathcal{R}$ of the map $\tilde{\pi}$ is therefore the complement $\Omega \setminus \Sigma$ of the set of singular values Σ.

Remark 6. The maps π and $\tilde{\pi}$ have the same regular and singular values because of the commutativity of the lower triangle in the commutative diagram of Proposition 21. A endowment vector $\omega \in \Omega$ is therefore regular in the Heckscher-Ohlin model if and only if it is regular for the equivalent exchange model.

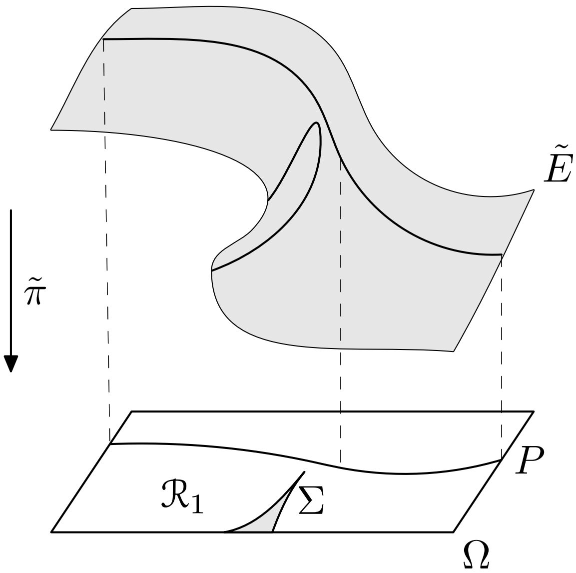


Figure 2: The equilibrium manifold and the natural projection

Proposition 37 (Regular economies for the Heckscher-Ohlin model)**.**

i) The set of regular economies $\mathcal{R}$ is an open and dense subset of the parameter space Ω.

ii) Equilibrium selections are locally unique and smooth in sufficiently small neighborhoods of $\omega \in \mathcal{R}$.

iii) The number of equilibria $\#\tilde{\pi}^{-1}(\omega)$ is constant over every connected component of $\mathcal{R}$.

iv) The modulo 2 degree of the natural projection $\tilde{\pi} : \tilde{E} \to \Omega$ is equal to 1.

Proof. These properties follow readily from the same properties of the exchange model. See Propositions 7.3, 7.5, 7.7, 7.8 and 7.12 in [6] where they are derived from the smoothness and properness of the natural projection. □

Remark 7. In the case of exchange economies, Proposition 37 (i), (ii) and (iii) was first proved by Debreu [9] and (iv) by Dierker [10]. Their proofs use the aggregate excess demand map of an economy. That approach was subsequently extended to production economies by, among others, Kehoe, Mas-Colell and Smale. At variance with other applications of Differential Topology to models with production, the models considered by these authors in [15, 16, 18, 22] encompass the unrestricted Heckscher-Ohlin model. The content of Proposition 37 is therefore included in their results.

Remark 8. Dierker's degree for regular exchange economies and its extension by Kehoe to regular production economies is the topological or Brouwer degree of the natural projection and not the modulo 2 degree considered here. However, these two concepts are intimately related. The Brouwer degree is actually considered in Section 7.6.1 of [6]. An immediate consequence of (iv) is that the number of equilibria at a regular economy (or endowment) is odd.

6.4. The set of factor content of equilibrium allocations

Let P denote the set of equilibrium allocations for the factor exchange model. This set, which is a subset of the endowment set Ω, can also be interpreted in terms of the (unrestricted) Heckscher-Ohlin model.

Proposition 38.

i) The set P of factor contents of equilibrium allocations in the (unrestricted) Heckscher-Ohlin model is the set of equilibrium allocations for the factor exchange model.

ii) The set P is a smooth submanifold of the endowment space $\Omega = (\mathbb{R}^{\ell}_{++})^m$ diffeomorphic to $\mathbb{R}^{m+\ell-1}$.

Proof.

(i): Follows readily from Proposition 16.
(ii): Follows from the structure of the set of equilibrium allocations in the exchange model [6], Corollary 8.5. □

The most important aspect from the perspective of comparative statics of Proposition 38 is the pathconnectedness of the set of factor contents of equilibrium allocations in the Heckscher-Ohlin model.

Remark 9. If total resources are fixed (as in the Edgeworth box), then the set of factor contents of equilibrium allocations is still a smooth submanifold of the parameter space that is diffeomorphic to a Euclidean space, but the dimension is now equal to $m - 1$. In the case of $\ell = 2$ productive factors, $m = 2$ consumers and fixed total resources, this set is the usual contract curve.

6.5. Uniqueness for small factor exchange vectors

The following property relates the number of equilibria to the volume of net trade in factor content:

Proposition 39. *The set P of factor content of equilibrium allocations for the Heckscher-Ohlin model is contained in one connected component of the set of regular economies $\mathcal{R}$. Equilibrium is unique for all endowment vectors ω in that component.*

Proof. Follows from Proposition 8.8 in [6] for the exchange model. □

Let this connected component of the set $\mathcal{R}$ of regular economies, a set that contains the set P, be denoted by $\mathcal{R}_1$. The endowment vector ω then belongs to $\mathcal{R}_1$ when the net trade vector in factor contents $\big(f_i(p, p \cdot \omega_i) - \omega_i\big)$ is sufficiently small.

Remark 10. The inclusion in a single connected component of the set of regular economies $\mathcal{R}$ of the set P of equilibrium allocations in the exchange model and the uniqueness of equilibrium for endowments in that component were first proved by the author in [2].

Remark 11. The properties proved in this section do not require consumers' preferences for goods to be transitive and represented by utility functions that satisfy all the assumptions spelt out in Section 2.2. All that is required is that every consumer's demand function (for goods) satisfies smoothness (S), Walras law (W) and the weak axiom of revealed preferences (WARP), and that the demand function of at least one consumer satisfies (ND) (the negative definiteness of the Slutsky matrix) and desirability (A).

7. Economies with a unique equilibrium: A complete characterization in the case of $m = 2$ consumers

Many applications of the Heckscher-Ohlin model to Trade Theory assume only two consumers, one consumer representing some country and the other one the rest of the world. A sufficient condition for equilibrium to be unique is that the endowment vectors ω belongs to the connected component $\mathcal{R}_1$ of the set of regular economies $\mathcal{R}$ that contains the set of equilibrium allocations P. This condition turns out to be also necessary in the case $m = 2$ considered in Trade Theory.

Proposition 40. *For $m = 2$, the set of regular economies with a unique equilibrium is the open set $\mathcal{R}_1$.*

I gave two different proofs of this property in the case of the exchange model with an arbitrary number of goods (here factors!) [3, 4]. The proof given here is new and shorter.

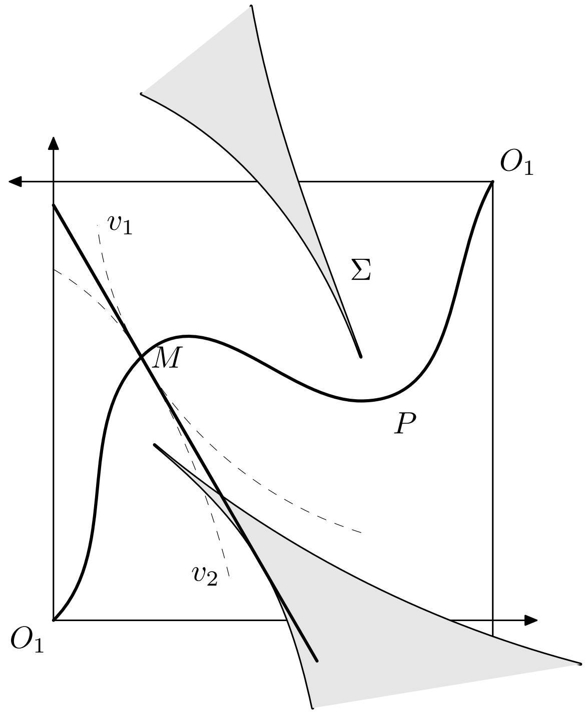


Figure 3: The set $\mathcal{R}_1$: another view.

Total resources are assumed to be constant and preferences represented by utility functions. The set of Pareto optima P is then diffeomorphic to $\mathbb{R}^{m-1} = \mathbb{R}$, hence is a smooth curve. That curve can be parameterized by the utility level of consumer 1 for example. That utility level varies between $v_1(0)$ and $v_1(r)$, where 0 denotes the zero bundle in the factor space $\mathbb{R}^\ell$, with $r \in \mathbb{R}^\ell_{++}$ the total resources in productive factors. Let $M(v_1) = (y_1(v_1), y_2(v_1))$ denote the Pareto optimum associated with consumer 1's utility level $v_1 \in [v_1(0), v_1(r)]$ and $p(v_1)$ the price vector that supports the Pareto optimum $M(v_1)$. Let $w_1(v_1) = p(v_1)^T x_1(v_1)$ be consumer 1's wealth at that Pareto optimum.

Given the endowment vector in productive factors $\omega = (\omega_1, \omega_2)$ (with $\omega_1 + \omega_2 = r$), define $s_1(v_1, \omega)$ by

$$s_1(v_1, \omega) = p(v_1)^T \omega_1 - w_1(v_1).$$

The Pareto optimum $M(v_1)$ is an equilibrium allocation of the exchange model for the endowment vector $\omega = (\omega_1, \omega_2)$ if and only if consumer 1's budget constraint (and therefore of consumer 2's budget constraint) is satisfied, i.e., $s_1(v_1, \omega) = 0$.

The following Lemma gives an alternative characterization of the equilibria of the exchange model associated with $\omega = (\omega_1, \omega_2)$.

Lemma 41. *The equilibria associated with $\omega = (\omega_1, \omega_2)$ correspond to the zeros of the function $v_1 \in [v_1(0), v_1(r)] \to s_1(v_1, \omega) \in \mathbb{R}$.*

Proof. Obvious. Note that this is essentially Negishi's approach to the existence of equilibrium formulated here for the exchange model and in the special case of two consumers. □

For $v_1 = v_1(0)$, the associated Pareto optimum is the allocation $(y_1, y_2) = (0, r)$ and, therefore, $w_1(0) = 0$. It then comes $s_1(v_1(0), \omega) > 0$. Similarly, for $v_1 = v_1(r)$, the associated Pareto optimum is the allocation $(y_1, y_2) = (r, 0)$. It then comes $s_1(v_1(r), \omega) < 0$.

The two inequalities $s_1(v_1(0), \omega) > 0$ and $s_1(v_1(r), \omega) < 0$ imply the existence of a zero by the intermediate value theorem applied to the function $v_1 \to s_1(v_1, \omega)$. But the function $s_1(v_1, \omega)$ also reveals a much more interesting property.

Let the endowment vectors $\omega = (\omega_1, \omega_2)$ and $\omega' = (\omega'_1, \omega'_2)$ have a unique equilibrium price vector $p^* \in S$. In addition, assume that the wealth of consumer 1 (resp. consumer 2) is the same for ω and ω':

$$(p^*)^T \omega_1 = (p^*)^T \omega'_1 \quad , \quad (p^*)^T \omega_2 = (p^*)^T \omega'_2.$$

(In other words, the equilibria (p^*, ω) and (p^*, ω') belong to the same fiber.) Let $v_1^* = v_1(f_1(p^*, (p^*)^T \omega_1))$ denote consumer 1's utility level at the common equilibrium allocation associated with these two equilibria.

Let $[\omega, \omega'] = \{(1 - t)\omega + t\omega' \mid t \in [0, 1]\}$ denote the line segment with extremities ω and ω'. It then comes:

Lemma 42. *Every $\omega'' \in [\omega, \omega']$ has a unique equilibrium price vector. That equilibrium price vector is equal to $p^* \in S$.*

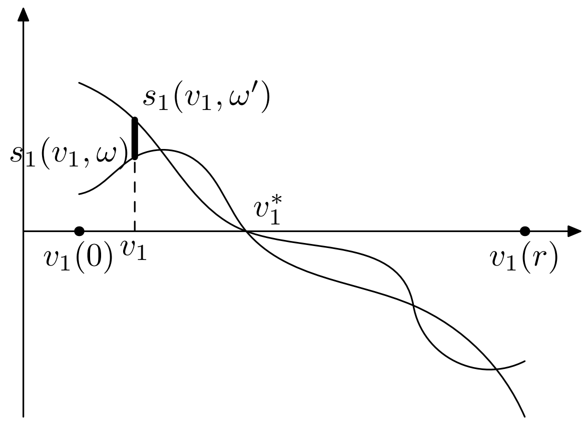


Figure 4: Graph of the functions $v_1 \to s_1(v_1, \omega)$ and $v_1 \to s_1(v_1, \omega')$.

Proof. The graphs of the functions $v_1 \to s_1(v_1, \omega))$ and $v_1 \to s_1(v_1, \omega'))$ intersect the horizontal axis at v_1^*. Both graphs are above the horizontal axis for $v_1 < v_1^*$ and below for $v_1 > v_1^*$. By the linearity of the function $s_1(v_1, \omega) = p(v_1)^T \omega_1 - w_1(v_1)$ with respect to ω_1, it comes

$$s_1(v_1, \omega'') = (1 - t)s_1(v_1, \omega) + ts_1(v_1, \omega').$$

This implies that $s_1(v_1, \omega'')$ belongs to the interval determined by $s_1(v_1, \omega)$ and $s_1(v_1, \omega')$. Therefore, $s_1(v_1, \omega'')$ is > 0 for $v_1 < v_1^*$ and < 0 for $v_1 > v_1^*$. □

Proof of Proposition 40. Let N represent in the Edgeworth box the point $\omega = (\omega_1, \omega_2)$ with the unique equilibrium price vector $p^* \in S$. The point $M(v_1^*)$ representing the Pareto optimum or equilibrium allocation associated with the endowment $\omega = (\omega_1, \omega_2)$ has the same equilibrium price vector p^* and belongs to the same fiber as ω. Therefore, all points of the segment $NM(v_1^*)$ are contained in the set of regular economies with a unique equilibrium by Lemma 42. To complete the proof, it suffices to observe that the set of equilibrium allocations (or Pareto optima) P is also contained in the connected component $\mathcal{R}_1$ of the set of regular economies with a unique equilibrium. □

Remark 12. One key argument in the proof of Proposition 40 is the pathconnectedness of the set P of equilibrium allocations in the exchange model. This property is readily seen to be satisfied for variable total resources even in the case where preferences are not transitive and, therefore, not representable by utility functions. For fixed total resources and transitive preferences represented by utility functions, the pathconnectedness of the set P of equilibrium allocations then easily follows from its identification with the set of Pareto optima, a set that is pathconnected as being diffeomorphic to a Euclidean space. This is the approach followed here. Nevertheless, I have recently extended to the case of fixed total resources and non transitive preferences the property for the set P of equilibrium allocations of being diffeomorphic to a Euclidean space (publication to follow shortly). This new result proves that the characterization of the set of regular endowments (or economies) with a unique equilibrium requires nothing more than the assumptions spelled out in Remark 11 for consumers' demand functions.

8. Concluding comments

This paper provides among other results the theoretical justification that has been missing so far of the importance, theoretical and applied, of the volume of net trade in factor contents as a tool for the analysis of international trade. The accelerating trend towards more specialization and globalization that has been going on in the world economy since a few decades has led to increasingly large volumes of net trade in those factor contents. These large volumes are non only suggestive of but also conducive to multiple equilibria and discontinuous equilibrium selections. Those properties are so deeply ingrained in the equilibrium equation of a beautiful and simple model, a model that requires nothing more than widely accepted assumptions on preferences and production, that it is difficult not to take them seriously, however surprising, disturbing and challenging they may be. The Physicist Dirac whose equations led to the discovery of the positron would certainly not have disagreed with that view.

In the simple version of the Heckscher-Ohlin model considered here, all countries have access to the same technologies. That assumption does simplify the mathematics to the point that, as shown in this paper, the Heckscher-Ohlin model becomes equivalent to an exchange model. Accommodating different technologies in different countries and including the possibility of increasing returns to scale considered by the New and New New Trade models adds layers of complexity that prevent the simple and nice mathematical equivalence with an exchange model to remain true. Nevertheless, a majority of properties of the Heckscher-Ohlin model, including those dealing with the volume of net trade in factor contents, are likely to remain true in these more general versions. The current paper is a first but necessary step in a study of these more general models along the lines laid down by the study of smooth exchange economies.

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A. Proofs

A.1. Proof of Proposition 2

i) $DF^j(\eta)^T\eta = F^j(\eta) \neq 0$ *for* $\eta \in \mathbb{R}^\ell_{++}$. By homogeneity of degree one, it comes $F^j(\lambda\eta) = \lambda F^j(\eta)$ with $\lambda \in \mathbb{R}$. It then suffices to take the derivative with respect to λ (Euler's identity). The inequality $DF^j(\eta)^T\eta \neq 0$ for $\eta \in \mathbb{R}^\ell_{++}$ follows from $F^j(\eta) \neq 0$.

ii) $\eta^T D^2F^j(\eta) = D^2F^j(\eta)\eta = 0$. The first order partial derivatives of F^j are homogeneous of degree zero. It then suffices to apply Euler's identity to these derivatives.

iii) Kernel of matrix $D^2F^j(\eta)$ *collinear with* $\eta \in \mathbb{R}^{\ell}_{++}$. The rank of $D^2F_j(\eta)$ is equal to $\ell - 1$. Its kernel is therefore one dimensional. One concludes by observing that the kernel also contains the vector $\eta = (\eta^1, \dots, \eta^{\ell}) \neq 0$.

iv) The strict inequality $z^T D^2F^j(\eta)\, z < 0$ *is satisfied for any non-zero vector* $z \in \mathbb{R}^{\ell}$ *not collinear with* η. All the ℓ eigenvalues of the symmetric matrix $D^2F^j(\eta)$ are real. A set of ℓ two by two orthogonal eigenvectors corresponds to these eigenvalues. Furthermore, the vector η can be chosen as the eigenvector associated with the eigenvalue 0. The $\ell - 1$ remaining eigenvalues are then strictly negative because of the rank assumption. Their associated eigenvectors generate a hyperplane that is orthogonal to the vector η. The restriction to that hyperplane of the quadratic form defined by matrix $D^2F^j(\eta)$ is therefore negative definite.

The vector $z \in \mathbb{R}^{\ell}$ is the sum $z = z' + z''$ of its orthogonal projections z' and z'', with z' in the vector space generated by η and z'' in the hyperplane orthogonal to η. It comes $z^T D^2F^j(\eta)z = (z'')^T D^2F^j(\eta)z''$ since z' is collinear with the vector η. The strict inequality $(z'')^T D^2F^j(\eta)z'' < 0$ then follows from $z'' \neq 0$ whenever z is not collinear with η.

v) Bordered Hessian matrix $\begin{bmatrix} D^2F^j(\eta) & DF^j(\eta) \\ DF^j(\eta)^T & 0 \end{bmatrix}$ *invertible.* Assume the contrary. There exists a vector $z = (\bar{z}, z^{\ell+1}) \neq 0 \in \mathbb{R}^{\ell} \times \mathbb{R}$ such that

$$\begin{bmatrix} D^2F^j(\eta) & DF^j(\eta) \\ DF^j(\eta)^T & 0 \end{bmatrix} \begin{bmatrix} \bar{z} \\ z^{\ell+1} \end{bmatrix} = 0.$$

This equality can be rewritten as

$$D^2F^j(\eta)\bar{z} + z^{\ell+1} DF^j(\eta) = 0 \tag{14}$$

$$\bar{z}^T DF^j(\eta) = 0. \tag{15}$$

Left multiplication of (14) by $\bar{z}^T$ yields, given (15),

$$\bar{z}^T D^2F^j(\eta)\, \bar{z} = 0.$$

By (iii), the vector $\bar{z}$ is therefore collinear with η: $\bar{z} = \lambda\eta$ with $\lambda \in \mathbb{R}$. By (ii), $D^2F^j(\eta)\,\bar{z} = 0$, so that (14) becomes $z^{\ell+1}DF^j(\eta) = 0$. It then follows from $DF^j(\eta) \neq 0$, itself a consequence of (i), that $z^{\ell+1} = 0$. This implies $z = \lambda(\eta, 0) \in \mathbb{R}^{\ell+1}$. Equation (15) becomes $\lambda\eta^T DF^j(\eta) = 0$, which is equivalent to $\lambda DF^j(\eta)^T\eta = 0$. Combined with (i), this implies $\lambda = 0$, a contradiction with $z \neq 0$.

A.2. Proof of Proposition 3

i) Strict convexity of $\{\eta \in \mathbb{R}^{\ell}_{++} \mid F^j(\eta) \geq 1\}$. Let η and η' in $\mathbb{R}^{\ell}_{++}$ be such that $F^j(\eta) = F^j(\eta') = 1$. The vector η and η' are not collinear. Otherwise, assume $\eta' = \lambda\eta$ with $\lambda \neq 1$. Then, we would have $1 = F^j(\eta') = F^j(\lambda\eta) = \lambda F^j(\eta) = \lambda$, a contradiction with $\lambda \neq 1$.

The second derivative of the function $t \in [0, 1] \to F^j((1-t)\eta + t\eta')$ is equal to $(\eta' - \eta)^T D^2F^j((1-t)\eta + t\eta')(\eta' - \eta)$ and is strictly negative by Proposition 2, (iv) because $\eta' - \eta$ is not collinear with η. This implies the strict concavity of the function $t \in [0, 1] \to F^j((1-t)\eta + t\eta')$, hence the strict inequality $F^j((1-t)\eta + t\eta') > 1$ for $t \in (0, 1)$ and, therefore, the strict convexity of the set $\{\eta \in \mathbb{R}^{\ell}_{++} \mid F^j(\eta) \geq 1\}$.

ii) Recession cone. The vector $d \in \mathbb{R}^{\ell}$ defines a direction of recession for the set $\{\eta \in \mathbb{R}^{\ell}_{++} \mid F_j(\eta) \geq 1\}$ if, for some η^* in that set, the set $\{\eta^* + \alpha d \mid \alpha \geq 0\}$ is also contained in that set. This is equivalent to having $F_j(\eta^* + \alpha d) \geq 1$ for $\alpha \geq 0$. This is obviously satisfied by the monotonicity and continuity of F_j for $d \in \mathbb{R}^\ell_+$, which proves that the recession cone contains the non-negative orthant.

Conversely, let $d \in \mathbb{R}^\ell$ with at least one strictly negative coordinate. There is no loss of generality in assuming $d^1 < 0$. Let $\alpha > 0$ be defined by $\eta^{*1} + \alpha d^1 = 0$. Let $\alpha^t > \alpha$ be a sequence with $\lim_{t\to\infty} \alpha^t = \alpha$. Then, $\lim_{t\to\infty} F^j(\eta^* + \alpha^t d) = 0$, which contradicts the inequality $F^j(\eta^* + \alpha^t d) \geq 1$ and d cannot be a direction of recession.

A.3. Proof of Proposition 4.

i) Existence and uniqueness of the solution to the constrained cost minimization problem. Let $\eta^* \in \mathbb{R}^\ell_{++}$ be such that $F^j(\eta^*) \geq 1$. Adding the constraint $p^T\eta \geq p^T\eta^*$ has no effect on the solutions of the constrained cost minimization problem. But the set defined by the two constraints is not only closed as the intersection of two closed sets, it is also bounded for obvious reasons. It follows from the compactness of that set and the continuity of the cost function that a solution exists to the constrained cost minimization problem.

The constraint $F^j(\eta) \geq 1$ is obviously binding by the monotonicity of F^j. The proof that the solution is unique is standard and proceeds by contradiction. Let $\eta \neq \eta'$ be two different solutions. By definition, the equalities $p^T\eta = p^T\eta'$ and $F^j(\eta) = F^j(\eta') = 1$ are satisfied. Let $\eta'' = (\eta + \eta')/2$. It follows from the strict concavity of the production function F^j restricted to a line not going through the origin that the strict inequality $F^j(\eta'') > 1$. It then suffices $\eta''' = \lambda\eta''$ such that $F^j(\eta''') = 1$ to get a contradiction to the definition of η and η' as cost minimizers.

ii) Homogeneity and smoothness of the factor demand functions. Homogeneity of degree zero of the demand function for factor $p \to b_j(p)$ is obvious. Smoothness follows from the application of the implicit function theorem to the first order conditions. These conditions take the form

$$\begin{cases} DF^j(\eta) - \mu q = 0, \\ F^j(\eta) - 1 = 0. \end{cases}$$

It is well-known that they are necessary and sufficient given the concavity of F^j. Smoothness then follows from Proposition 2, (v).

iii) The cost function σ_j is smooth and homogeneous of degree one. Homogeneity is obvious. Smoothness follows from the cost function being the product of two smooth functions.

iv) The cost function σ_j is concave. Let $p \neq p'$ in $\mathbb{R}^\ell_{++}$. For $t \in [0, 1]$, the two inequalities

$$\sigma_j(p) = p^T b_j(p) \leq p^T b_j\big((1-t)p + tp'\big) \quad , \quad \sigma_j(p') = (p')^T b_j(p') \leq (p')^T b_j\big((1-t)p + tp'\big)$$

follow from the definitions of $\sigma_j(p)$ and $\sigma_j(p')$ as cost minimizing for p and p' respectively. Multiplication by $(1-t)$ and t of the first and second inequalities respectively followed by adding them up yields

$$(1-t)\sigma_j(p) + t\sigma_j(p') \leq \big((1-t)p + tp'\big)^T b_j\big((1-t)p + tp'\big) = \sigma_j\big((1-t)p + tp'\big). \qquad (16)$$

Remark 13. The above proof can easily be adapted to show that if p and p' are not collinear and $t \in (0, 1)$, then, inequality (16) is strict.

A.4. Proof of Proposition 5

iii) $Db_j(p)^T p = 0$. Follows readily from Euler's identity applied to $b_j(p)$, a function that is homogeneous of degree zero by (ii).

iv) $(p - p')^T(b_j(p) - b_j(p')) \leq 0$ *for* p *and* p' *in* S. The inequalities $p^T(b_j(p) - b_j(p')) \leq 0$ and $(p')^T(b_j(p') - b_j(p)) \leq 0$ follow from the definitions of $b_j(p)$ and $b_j(p')$. It then suffices to add up these two inequalities.

v) The Jacobian matrix $Db_j(p)$ *defines a negative semidefinite quadratic form.* The derivation of that property from *(iv)* is standard.

vi) The Jacobian matrix $Db_j(p)$ *has rank* $\ell - 1$ *and its kernel is collinear with the factor price vector* p. The idea of the proof is to show that any vector $v \neq 0 \in \mathbb{R}^\ell$ in the kernel of $Db_j(p)$ is collinear to the factor price vector $p \in \mathbb{R}^\ell_{++}$. From the first order conditions, it comes $DF^j(b_j(p)) - \mu(p)p = 0$ where $\mu(p) \neq 0$.

Taking the derivative of this equality with respect to the price vector p yields

$$D^2F^j(b_j(p))Db_j(p) = \mu(p)I + p(D\mu)^T$$

with I being the $\ell \times \ell$ identity matrix. Right multiplication of this equality by $v \neq 0$ in the kernel of $Db_j(p)$ yields

$$\mu(p)v = -p\big((D\mu)^T v\big)$$

where $(D\mu)^T v$ is a real number. This equality implies that the vector $v \neq 0$ is necessary collinear with the factor price vector p.

vii) $\limsup_{t\to\infty} \|b_j(p^t)\| = +\infty$ *for* $p^0 = \lim_{t\to\infty} p^t$ *and* $p^0 \neq 0$ *but some coordinates equal to* 0.
Step 1. One sees readily that The set $\{y \in \mathbb{R}^\ell_{++} \mid F^j(y) = 1 \text{ and } y \leq (A, \dots, A)\}$ is bounded away from zero for $A > 0$.
Step 2. The proof now proceeds by contradiction. Let $y^t = b_j(p^t)$ and assume $\limsup_{t\to\infty} \|y^t\| < +\infty$. This is equivalent to the sequence $\|y^t\|$ being bounded. There exists a real number $A > 0$ such that the inequalities $0 \leq y^t \leq (A, A, \dots, A)$ are satisfied for all t. Recall that $F^j(y^t) = 1$. Therefore, there exists by Step 1 $y_A \in \mathbb{R}^\ell_{++}$ such that $y_A \leq y^t \leq A$ for all t. By considering if necessary a subsequence, there is no loss in generality by assuming that the sequence y^t converges to some y^0 that satisfies the inequalities $y_A \leq y^0 \leq A$. By continuity, it comes $F^j(y^0) = 1$. In addition, the price vector p^t is collinear with the gradient vector $DF^j(y^t)$, i.e., there exists $\lambda^t > 0$ such that $p^t = \lambda^t DF^j(y^t)$. The sequences p^t and $DF^j(y^t)$ are bounded from above and bounded away from zero: therefore, the sequence λ^t is also bounded from above and away from zero. Considering once more and if necessary a subsequence, the sequence λ^t converges to some $\lambda^0 > 0$. It then follows from the continuity of DF^j that, at the limit, $p^0 = \lambda^0 DF^j(y^0)$. The contradiction comes from some coordinates of p^0 being equal to zero while each partial derivative of the production function F^j is different from zero.

A.5. Proof of Proposition 7

i) $(DB(p))^T p = 0$. Follows readily from Proposition 5, (v).

ii) $D\big(B(p)^T p\big) = B(p)^T$. The derivative of the matrix product $B(p)^T p$ is equal to

$$D(B(p)^T p) = B(p)^T + DB(p)^T p.$$

One concludes by applying (i).

iii) $p^T B(p) < p^T B(p')$ *for* $p \neq p' \in S$. Follows from Proposition 5, (iii) applied to each $b_j(p)$ add $b_j(p')$.

A.6. Proof of Proposition 22

Roy's identity is obtained by taking the first order derivative of the equality relating the direct and indirect utility functions v_i and $\hat{v}_i$ and the associated demand function f_i:

$$\hat{v}_i(p, w_i) = v_i\big(f_i(p, w_i)\big). \tag{17}$$

Using the column matrix notation for the vector $f_i(p, w_i)$ and $\partial_p \hat{v}_i(p, w_i)$, Roy's identity can be written as

$$\partial_{w_i} \hat{v}_i(p, w_i)\, f_i(p, w_i) = -\partial_p \hat{v}_i(p, w_i). \tag{18}$$

Proving that equality (18) is satisfied starts by relating the terms of this equality to the direct utility function u_i. Taking the derivative of $\hat{v}_i(p, w_i) = \hat{u}_i(B(p)^T p, w_i) = \hat{u}_i(q, w_i)$ with $q = B(p)^T p$ with respect to w_i yields

$$\partial_{w_i} \hat{v}_i(p, w_i) = \partial_{w_i} \hat{u}_i(B(p)^T p, w_i) = \partial_{w_i} \hat{u}_i(q, w_i).$$

Similarly, the derivative with respect to p combined with the chain rule yields

$$\partial_p \hat{v}_i(p, w_i)^T = \partial_q \hat{u}_i(q, w_i)^T \; D(B(p)^T p).$$

The equality $D(B(p)^T p) = B(p)^T$ of Proposition 7, (ii) implies the equality

$$\partial_p \hat{v}_i(p, w_i)^T = \partial_q \hat{u}_i(q, w_i)^T \, B(p)^T$$

and, after taking the transpose,

$$\partial_p \hat{v}_i(p, w_i) = B(p)\, \partial_q \hat{u}_i(q, w_i)\,.$$

Roy's identity (18) now takes the form

$$\partial_{w_i} \hat{u}_i(q, w_i)\, f_i(p, w_i) = -B(p)\, \partial_q \hat{u}_i(q, w_i). \tag{19}$$

It follows from the definition of $h_i(q, w_i)$ as the demand function associated with the utility function u_i that it satisfies Roy's identity with respect to the indirect utility function $\hat{u}_i(q, w_i)$, which, in matrix notation, takes the form:

$$\partial_{w_i} \hat{u}_i(q, w_i) h_i(q, w_i) = -\partial_q \hat{u}_i(q, w_i).$$

Left multiplication by $B(p)$ yields

$$\partial_{w_i} \hat{u}_i(q, w_i) B(p)\, h_i(q, w_i) = -B(p)\, \partial_q \hat{u}_i(q, w_i),$$

which, after substituting $f_i(p, w_i) = B(p) h_i(q, w_i)$, is exactly equation (18). (Note that the partial derivative $\partial_{w_i} \hat{u}_i(q, w_i)$ is a real number so that $\partial_{w_i} \hat{u}_i(q, w_i) B(p) = B(p) \partial_{w_i} \hat{u}_i(q, w_i)$.)

A.7. Proof of Lemma 33

Smoothness (S). Let h_i be smooth. The production matrix function $p \to B(p)$ is smooth. Therefore, the demand function $f_i(p, w_i) = B(p)\, h_i(B(p)^T p, w_i)$ is smooth by composition of smooth functions, which proves (S).

Walras law (W). Let h_i satisfy (W). It then comes

$$p^T f_i(p, w_i) = p^T B(p)\, h_i(B(p)^T p, w_i) = q^T h_i(q, w_i) = w_i.$$

A.8. Proof of Lemma 34

i) (WARP) for $h_i \Longrightarrow$ (WARP) for f_i.

Let (p, w_i) and (p', w_i') with $f_i(p, w_i) \neq f_i(p', w_i')$ be such that $(p')^T f_i(p, w_i) \leq w_i'$. If $p = p'$, inequality $(p')^T f_i(p, w_i) \leq w_i'$ becomes by (W) $w_i \leq w_i'$. The assumption $f_i(p, w_i) \neq f_i(p, w_i')$ implies $w_i \neq w_i'$, from which follows $w_i < w_i'$. Then, it follows from $p^T f_i(p', w_i') = w_i'$ by (W) that the strict inequality $p^T f_i(p', w_i') > w_i$ is satisfied.

If $p \neq p'$, the strict inequalities $(p')^T B(p') < (p')^T B(p)$ and $p^T B(p) < p^T B(p')$ are satisfied. The inequality $(p')^T f_i(p, w_i) \leq w_i'$ can be spelled out as

$$(p')^T B(p)\, h_i(B(p)^T p, w_i) \leq w_i'. \tag{20}$$

For $q = B(p)^T p$ and $q' = B(p')^T p'$, the positivity of matrices $B(p)$ and $B(p')$ and of the demand vector $h_i(q, w_i) \in X$ combined with the (strict) inequality $(p')^T B(p') < (p')^T B(p)$ implies the strict inequality

$$(p')^T B(p')\, h_i(q, w_i) < (p')^T B(p)\, h_i(q, w_i),$$

which, combined with (20), yields

$$(q')^T h_i(q, w_i) < w_i'. \tag{21}$$

This strict inequality implies the inequality $h_i(q, w_i) \neq h_i(q', w_i')$. (Assume $h_i(q, w_i) = h_i(q', w_i')$, then it comes $(q')^T h_i(q, w_i) = (q')^T h_i(q', w_i') = w_i'$ by (W) which in combination with the strict inequality (21) yields $w_i' < w_i'$, a contradiction.)

By (WARP) that is satisfied by h_i, inequality (21) implies the strict inequality

$$q^T h_i(q', w_i') > w_i, \tag{22}$$

which can be rewritten as

$$p^T B(p)\, h_i(q', w_i') > w_i. \tag{23}$$

Inequality $p^T B(p) \leq p^T B(p')$ then implies the inequality

$$p^T B(p) h_i(q', w_i') \leq p^T B(p') h_i(q', w_i').$$

This inequality can be rewritten as

$$q^T h_i(q', w_i') \leq p^T B(p') h_i(q', w_i') = p^T f_i(p', w_i').$$

Combining this inequality with the strict inequality (22) yields

$$p^T f_i(p', w_i') > w_i,$$

which proves that (WARP) is satisfied by f_i.

ii) (ND) for $h_i \Longrightarrow$ (ND) for f_i.

The price vector $p \in \mathbb{R}^{\ell}_{++}$ is not normalized in this proof because the computation of Slutsky matrices requires taking derivatives with respect to the price p_ℓ of the numeraire good. Without price normalization, the factor demand function $f_i(p, w_i)$ is homogeneous of degree zero. The $\ell \times \ell$ matrix $\partial_p f_i(p, w_i)$ consists of the first order derivatives of f_i with respect to the (coordinates of the) price vector p. Similarly, let $\partial_q h_i(q, w_i)$ denote the $k \times k$ matrix of partial derivatives for the goods demand function h_i with respect to the price vector $q \in X$. Let $q = B(p)^T p$ in the following developments.

Step 1: Negative definiteness of the restriction of the quadratic form associated with matrix $\partial_q h_i(q, w_i)$ to the hyperplane $\{z \in \mathbb{R}^k \mid z^T h_i(q, w_i) = 0\}$. It follows from Hildenbrand and Jerison [14] that (ND) for h_i is equivalent to the restriction of the quadratic form

$$z \in \mathbb{R}^k \to z^T \partial_q h_i(q, w_i)\, z$$

to the hyperplane $h_i(q, w_i)^\perp = \{z \in \mathbb{R}^k \mid z^T h_i(q, w_i) = 0\}$ being negative definite.
Step 2: $\partial_p f_i(p, w_i) = DB(p)\, h_i(B(p)^T p, w_i) + B(p)\, \partial_p h_i(B(p)^T p, w_i)$. Follows from taking the derivative of the product $f_i(p, w_i) = B(p) h_i(B(p)^T p, w_i)$ with respect to the price vector $p \in \mathbb{R}^\ell_{++}$.

Step 3: $\partial_p h_i(B(p)^T p, w_i) = \partial_q h_i(q, w_i)\, B(p)^T$. Application of the chain rule yields

$$\partial_p h_i(B(p)^T p, w_i) = \partial_q h_i(q, w_i)\, D\big(B(p)^T p\big).$$

It then suffices to apply Proposition 7, (ii).
Step 4: $\partial_p f_i(p, w_i) = DB(p)\, h_i(q, w_i) + B(p)\, \partial_q h_i(q, w_i)\, B(p)^T$ *with* $q = B(p)^T p$. It suffices to substitute the expression obtained in Step 3 in the formula of Step 2.

Step 5: The quadratic form defined $v \in \mathbb{R}^\ell \to v^T DB(p)\, h_i(q, w_i) v$ *with* $q = B(p)^T p$ *is negative semi-definite, with rank* $\ell - 1$ *and* $v^T DB(p)\, h_i(q, w_i) v < 0$ *for* v *not collinear with* p. The column matrix $B(p)\, h_i(q, w_i)$ is equal to

$$B(p)\, h_i(q, w_i) = \sum_{1 \le j \le \ell} b_j(p)\, h_i^j(q, w_i).$$

Its partial derivative with respect to p (q is considered as fixed) is the $\ell \times \ell$ matrix

$$DB(p)\, h_i(q, w_i) = \sum_{1 \le j \le \ell} (Db_j(p))\, h_i^j(q, w_i).$$

Each square matrix $Db_j(p)$ defines a quadratic form that is negative semidefinite, with rank $k - 1$ and kernel collinear with p by Proposition 5, (iv) and (v).

The linear combination of these negative semidefinite quadratic forms with the strictly positive coefficients $h_i^j(q, w_i)$, with $1 \le j \le k$, is negative semidefinite and takes a value different from zero and, therefore, is strictly negative for any vector $v \in \mathbb{R}^\ell$ that is not collinear with the price vector $p \in \mathbb{R}^\ell_{++}$.
Step 6: $v \in f_i(p, w_i)^\perp$ *implies* $B(p)^T v \in h_i(B(p)^T p, w_i)^\perp$. The relation $v \in f_i(p, w_i)^\perp$ is equivalent to $v^T f_i(p, w_i) = v^T B(p) h_i(B(p)^T p, w_i) = 0$. This relation is equivalent to $z = B(p)^T v \in h_i(B(p)^T p, w_i)^\perp$.
Step 7: $p^T f_i(p, w_i) \neq 0$ *for any* $p \in \mathbb{R}^\ell_{++}$. Follows readily from Walras law: $p^T f_i(p, w_i) = w_i \neq 0$.
Step 8: $v^T \partial_p f_i(p, w_i)\, v < z^T \partial_q h_i(q, w_i)\, z$ *for any* $v \neq 0 \in f_i(p, w_i)^\perp$, $q = B(p)^T p$ *and* $z = B(p)^T v$. Let $v \neq 0 \in h_i(q, w_i)^\perp$. The vector v is not collinear with p. Assume the contrary. There exists $\lambda \neq 0$ with $v = \lambda p$. Then, it comes $v^T f_i(p, w_i) = \lambda p^T f_i(p, w_i) \neq 0$ by Step 7, a contradiction.

The strict inequality

$$v^T DB(p)\, h_i(q, w_i)\, v < 0$$

then follows from Step 5. The combination with Step 4 implies

$$v^T \partial_p f_i(p, w_i)\, v = v^T DB(p)\, h_i(q, w_i)\, v + v^T B(p)\, \partial_q h_i(q, w_i)\, B(p)^T v,$$

from which follows the strict inequality

$$v^T \partial_p f_i(p, w_i)\, v < z^T \partial_q h_i(q, w_i)\, z. \tag{24}$$

Step 9: The restriction of the quadratic form defined by $\partial_p f_i(p, w_i)$ *to the hyperplane* $f_i(p, w_i)^\perp$ *is negative definite.* Follows readily from the strict inequality (24) combined with the negative definiteness of the restriction of the quadratic form defined by $\partial_q h_i(q, w_i)$ to the hyperplane $h_i(q, w_i)^\perp$ proved in Step 1.
Step 10: f_i *satisfies (ND).* Follows readily from the equivalence of the property stated in Step 9 with (ND) for f_i by [14].

iii) (A) for $h_i \Longrightarrow$ *(A) for* f_i. Again, factor price vectors are not normalized. Let $(p^t, w_i^t) \in \mathbb{R}^\ell_{++}$ be a sequence of non-normalized price and income vectors converging to $(p^0, w_i^0) \in \mathbb{R}^\ell_+ \times \mathbb{R}_{++}$, with some but not all coordinates of the price vector p^0 equal to zero. It follows from Proposition 5 (v) that, for each good j , with $1 \leq j \leq k$, there is at least one factor κ whose demand $b_j^\kappa(p^t)$ tends to $+\infty$. In other words, in order to produce one unit of good j at factor prices p^t that tends to p^0, there is at least one production factor κ, with $1 \leq \kappa \leq \ell$ whose demand $b_j^\kappa(p^t)$ tends to $+\infty$.

Let $q^t = B(p^t)^T p^t$. By continuity, the sequence q^t tends to a limit $q^0 \in \mathbb{R}_+^k$, where some coordinates of q^0 may be equal to 0. If none of these coordinates are equal to 0, continuity implies $\lim_{t\to\infty} h_i(q^t, w_i^t) = h_i(q^0, w_i^0) \in \mathbb{R}_{++}^k$. It then follows from

$$f_i(p^t, w_i^t) = B(p^t)\, h_i(B(p^t)^T p^t, w_i^t) = B(p^t)\, h_i(q^t, w_i^t)$$

that $\lim_{t\to\infty} \|f_i(p^t, w_i^t)\|$ is equal to $+\infty$ by the result of the previous paragraph. If some coordinates of q^0 are equal to 0, it follows from Property (A) that is satisfied by h_i that $\limsup_{t\to\infty} \|h_i(q^t, w_i^t)\| = +\infty$, i.e., there exists some consumption good j such that the demand $h_i^j(q^t, w_i^t)$ tends to $+\infty$. This implies that the demand for the production factor h to produce the quantity $h_i^j(q^t, w_i^t)$ of consumption good j also tends to $+\infty$.

B. A lemma about embeddings

An embedding $\phi : X \to Y$ is a smooth map between two smooth manifolds X and Y that is an immersion (its derivative map $D\phi(x) : T_xX \to T_{f(x)}Y$ between the tangent spaces T_xX and $T_{f(x)}Y$ is into, i.e., an injection) and also a homeomorphism between its domain X and its image $\phi(X)$. A very nice feature of embeddings is that the image $\phi(X)$ is then also a smooth submanifold of the range Y. Embeddings provide a very convenient way of proving that some subset $\phi(X)$ of the smooth manifold Y is actually a smooth submanifold of Y. The global structure of the smooth submanifold $\phi(X)$ as homeomorphic to X then comes as a courtesy. The application of the following lemma requires little more than the computation of derivatives (i.e., Jacobian matrices).

Lemma 43. *Let $\phi : X \to Y$ and $\psi : Y \to X$ be two smooth mappings between smooth manifolds with:*

i) $\psi : Y \to X$ is onto (i.e., a surjection);

ii) $\phi \circ \psi = \mathrm{id}_Y$.

Then, $Z = \phi(X))$ is a smooth submanifold of Y diffeomorphic to X.

Proof. The strategy is to show that the smooth map $\phi : X \to Y$ is an embedding, which therefore implies that its image $Z = \phi(X)$ is a submanifold of Y diffeomorphic to X.

To prove the homeomorphism part, we first observe that ϕ, viewed as a map from X to $Z = \phi(X)$, is a surjection. To prove that ϕ is an injection, assume $\phi(x) = \phi(x')$. Since $\psi : Y \to X$ is onto, there exist y and y' with $x = \psi(y)$ and $x' = \psi(y')$. It comes $\phi(x) = \phi \circ \psi(y) = y$ and $\phi(x') = \phi \circ \psi(y') = y'$, hence $y = y'$. This proves that ϕ viewed as a map from X to Z is a continuous bijection.

Let $\psi\,|\,Z$ denote the restriction of the map ψ to the subset Z of Y. The relation $\psi \circ \phi = \mathrm{id}_Y$ implies $(\psi\,|\,Z) \circ \phi = \mathrm{id}_Y$ from which follows that the inverse map of ϕ (as a map between X and Z) is $\psi\,|\,Z$. The maps $\phi : X \to Y$ and $\psi : Y \to X$ are continuous (in fact, smooth). It follows readily from the definition of the induced topology of Z that the restriction $\psi\,|\,Z : Z \to X$ is also continuous as well as the map (also denoted by) $\phi : X \to Z = \phi(X)$. (Note that the fact that Z is simply a subset of Y equipped with the induced topology does not make it a "nice" subset of Y yet, which prevents us from using the above argument to infer that $\psi\,|\,Z : Z \to X$ and $\phi : X \to Z$ are smooth mappings.) At the moment, these two maps are just continuous. They therefore define inverse homeomorphism between X and Z.

To prove the immersion part, take $y \in Y$. Let $x = \psi(y)$. The relation $\phi \circ \psi = \mathrm{id}_Y$ yields, by taking its derivative,

$$D\phi(x) \circ D\psi(y) = \mathrm{id}_{T_y(Y)}$$

where $T_y(Y)$ denotes the tangent space to the manifold Y at y. This relation implies that the linear map between tangent spaces $D\psi(y) : T_y(Y) \to T_x(X)$ is an injection. The map $\phi : X \to Y$ is therefore an immersion. In combination with the homeomorphism part above, this proves that the map $\phi : X \to Y$ is an embedding. □