An Efficient Multi-Item Dynamic Auction with Budget Constrained Bidders[1]

Dolf Talman[2] and Zaifu Yang[3]

Abstract: An auctioneer wishes to sell several heterogeneous indivisible items to a group of potential bidders. Each bidder has valuations over the items but may face a budget constraint and therefore be unable to pay up to his values. In such markets, a Walrasian equilibrium may fail to exist. We develop a novel dynamic auction and prove that the auction always finds a core allocation. In the auction prices that have been increased can be later decreased if they have become too high. The core allocation consists of an assignment of the items and its associated supporting price vector, achieves Pareto efficiency, and is robust against the threat of deviation by any coalition of market participants.

Keywords: Dynamic auction, multi-item auction, budget constraint, core, efficiency.

JEL classification: D44.

1 Introduction

Auction theory typically assumes that all bidders can pay up to their values on the goods for sale. However, in reality buyers often face budget or liquidity constraints and may therefore be unable to afford what the goods are worth to them. Financial constraints arise in a variety of situations, such as less developed countries, business downturns, and financial crises; see e.g., Che and Gale (1998), Laffont and Robert (1996), Maskin (2000), Krishna (2002), Klemperer (2004), and Milgrom (2004). Financial constraints may pose a serious obstacle to the efficient allocation of the goods, resulting in loss of market efficiency. It is known that even when a single item is sold, it is generally impossible to have a mechanism for achieving full market efficiency when bidders face budget constraints, because budget

[1]We are very grateful to the co-editor, associate editor and referees for their comments which helped us to improve the paper considerably. This is a revision of Talman and Yang (2011). The second author gratefully acknowledges the financial support by the Alexander von Humboldt Foundation, the Netherlands Organization for Scientific Research (NWO), CentER of Tilburg University, and the University of York. He wishes to thank the Center for Mathematical Economics (IMW), Bielefeld University, for its hospitality while he was a Humboldt research fellow and finished part of the current paper.

[2]A.J.J. Talman, CentER and Department of Econometrics & Operations Research, Tilburg University, P.O. Box 90153, 5000 LE Tilburg, Netherlands, e-mail: talman@tilburguniversity.edu.

[3]Z. Yang, Department of Economics and Related Studies, University of York, York YO10 5DD, UK, e-mail: zaifu.yang@york.ac.uk.

constraints can fail the existence of a Walrasian equilibrium. Moreover, auctions that do very well without budget constraints, if implemented under budget constraints, often produce highly inefficient outcomes.

Although budget constraints make full market efficiency unattainable due to nonexistence of a Walrasian equilibrium, it is natural to ask whether there exist mechanisms that can achieve an allocation of goods as efficient as possible. To solve this problem, we examine a model in which a finite number of indivisible items are sold to a finite number of budget constrained bidders. Each bidder wants to consume at most one item. When bidders face no budget constraints, the model reduces to the well-known assignment model as studied by Koopmans and Beckmann (1957), Shapley and Shubik (1971), Crawford and Knoer (1981), Demange et al. (1986), and Andersson and Erlanson (2013), among others. We propose a novel dynamic auction and prove that the auction always finds a core allocation in finitely many steps. The core allocation comprises an assignment of items among market participants and its associated supporting price system and achieves Pareto efficiency. The notion of core allocation is more general than that of Walrasian equilibrium and is a widely used solution concept for general exchange economies and nontransferable utility (NTU) games; see Scarf (1967), Shapley and Scarf (1974), and Predtetchinski and Herings (2004). As a prime concept of strategic equilibrium it specifies a stable and appealing distribution of welfare among all the market participants that is immune to the threat of deviation by any coalition.

In early literature, to cite but a few, Che and Gale (1998), Laffont and Robert (1996), Maskin (2000), Krishna (2002), and Zheng (2001) have analyzed various auctions for selling a single item when bidders are financially constrained. Hafalir et al. (2012) have considered a sealed-bid Vickrey auction for selling one divisible good to budget constrained bidders that achieves a near Pareto efficiency, weaker than Pareto efficiency. Moreover, Benoît and Krishna (2001), Brusco and Lopomo (2008), and Pitchik (2009) have studied auctions for selling two items to budget constrained bidders. Recently, van der Laan and Yang (2008) have examined a similar model as the current one and developed an ascending auction that always finds a constrained equilibrium. The constrained equilibrium possesses several interesting properties but does not necessarily yield a core allocation.

Our dynamic auction is most closely related to Ausubel and Milgrom (2002). In their Section 8, Ausubel and Milgrom briefly discuss a dynamic auction with budget constrained bidders after they have extensively examined a family of core-selecting package auctions without budget constrained bidders; see Bernheim and Whinston (1986) for a static setting. The Ausubel-Milgrom auction with budget constrained bidders uses the core of an NTU game as solution and is shown to find a core element in their auction model. However, their auction mechanism cannot be applied to our current model, because their auction requires that every bidder should have a strict preference relation over a finite set of choices, whereas in our model every agent has only a weak preference relation over a continuum of choices and is typically indifferent among many choices, and tie-breaking rules cannot be used. It will be shown that when there are budget constrained bidders, our dynamic auction always finds a core allocation, and furthermore that when no bidder faces budget constraint, the auction always finds a strict core allocation that attains full market efficiency. In contrast to Ausubel and Milgrom's auction, in our auction prices that have been increased can be later decreased if they have become too high. We also point out that Day and Milgrom (2008) and Erdil and Klemperer (2010) have refined and improved the core-selecting package auction of Ausubel and Milgrom (2002) for the case without budget constrained bidders.

The rest of the paper is organized as follows. Section 2 presents the model. Section 3 introduces and analyzes the auction. Section 4 concludes.

2 The Model

Consider an auction model consisting of a seller (i.e., auctioneer) and m potential bidders. The seller (she) has n indivisible goods for sale. Let $N = \{1, \ldots, n\}$ denote the set of items and $M = \{1, 2, \ldots, m\}$ the set of bidders. Item 0 denotes a *dummy good* which has no value and costs nothing for any agent. Let $N_0 = N \cup \{0\}$ denote the set of all items in the market. Every real item $j \in N$ is inherently indivisible and thus can be assigned to at most one bidder. The seller has for each real item $j \in N$ a reservation price $c(j) \in \mathbb{Z}_+$ below which the item will not be sold, where $\mathbb{Z}$ and $\mathbb{Z}_+$ are the set of integers and the set of nonnegative integers, respectively. The reservation price of the dummy good is known to be $c(0) = 0$. Every bidder (he) $i \in M$ demands at most one item and has a (possibly negative) monetary valuation to each item in N_0 given by the function $V^i\colon N_0 \to \mathbb{Z}$ with $V^i(0) = 0$, and is endowed with a budget of $m^i \in \mathbb{Z}_+$ units of money. For each bidder $i \in M$, all values $V^i(j)$, $j \neq 0$, and his budget m^i are private information and thus only bidder i knows these numbers.

It is assumed that buying an item $j \in N_0$ against price $p(j)$ by bidder $i \in M$ yields him a utility U^i equal to

$$U^i = \begin{cases} V^i(j) + m^i - p(j) & \text{if } p(j) \leq m^i, \\ -\infty & \text{if } p(j) > m^i. \end{cases}$$

That is, bidders are not allowed to have a deficit of money. By this assumption, no bidder is willing to pay a price for any item above his budget m^i. We say that bidder $i \in M$ is *financially constrained* if $m^i < \max_{j \in N_0} V^i(j)$, i.e., the valuation of bidder i for some items exceeds what he can afford, and that bidder i faces *no financial constraint* otherwise. When no bidder is financially constrained, the model is equivalent to the classical assignment market model as studied by Koopmans and Beckmann (1957), Shapley and Shubik (1971), Crawford and Knoer (1981), Demange et al. (1986), and Andersson and Erlanson (2013), among others.

A price vector $p \in \mathbb{R}^{n+1}$ gives a price $p(j)$ for each item $j \in N_0$. A price vector $p \in \mathbb{R}^{n+1}$ is *feasible* if $p(j) \geq c(j)$ for every $j \in N$ and $p(0) = 0$. The price of the dummy good is always assumed to be zero. Let $M_0 = M \cup \{0\}$ denote the set of all agents, where agent 0 stands for the seller. An assignment is a vector $\pi = (\pi(1), \ldots, \pi(m))$ of items among all bidders in M such that $\pi(i) = \pi(j)$ for $i \neq j$ implies $\pi(i) = 0$. At assignment $\pi = (\pi(1), \ldots, \pi(m))$, each bidder $i \in M$ gets one item, $\pi(i)$, in N_0. The dummy good can be assigned to any number of bidders, while a real item is assigned to at most one bidder. With respect to an assignment π, the set $N_\pi = \{k \in N \mid k \neq \pi(i),\ \forall i \in M\}$ denotes the set of unsold real items, which will be kept by the seller. Let $\mathcal{A}$ denote the collection of all assignments of items in N_0 among all bidders in M.

An assignment $\pi \in \mathcal{A}$ is *fully efficient* if

$$\sum_{i \in M} V^i(\pi(i)) + \sum_{j \in N_\pi} c(j) \geq \sum_{i \in M} V^i(\rho(i)) + \sum_{j \in N_\rho} c(j)$$

for every assignment $\rho \in \mathcal{A}$. A fully efficient assignment maximizes the total social value that can be achieved by allocating the items over all market participants. A mechanism achieves *full market efficiency* if it can find a fully efficient assignment of the market.

At a feasible price vector $p \in \mathbb{R}^{n+1}$, the demand set of bidder $i \in M$ is given by

$$D^i(p) = \{j \in N_0 \mid p(j) \leq m^i, V^i(j) - p(j) = \max\{V^i(k) - p(k) \mid p(k) \leq m^i, k \in N_0\}\}.$$

Observe that for any feasible p, the demand set $D^i(p) \neq \emptyset$, because $p(0) = 0 \leq m^i$ and thus the dummy item is always in the budget set. This means that the bidder has always the possibility not to buy any real item.

A pair (p, π) of a feasible price vector p and an assignment $\pi \in \mathcal{A}$ is said to be *implementable* if $p(\pi(i)) \leq m^i$ for all $i \in M$ and $p(j) = c(j)$ for all $j \in N_\pi$, i.e., every bidder i can afford to buy the item $\pi(i)$ assigned to him, and the price of every unsold item equals its reservation price. A *Walrasian equilibrium* (WE) is an implementable pair (p^*, π^*) such that $\pi^*(i) \in D^i(p^*)$ for all $i \in M$. It is well known from Koopmans and Beckmann (1957), and Shapley and Shubik (1971) that, when there is no financial constraint for any bidder, a Walrasian equilibrium always exists, and that every Walrasian equilibrium (p^*, π^*) yields a fully efficient assignment π^*. Moreover, Crawford and Knoer (1981), Demange et al. (1986), and Andersson and Erlanson (2013) have developed auction mechanisms for finding Walrasian equilibria in such markets.

The following example shows that a Walrasian equilibrium may fail to exist if buyers face financial constraints.

Example 2.1 There are one item and two buyers. The seller's reservation price for the item is zero, $c(1) = 0$. The buyers' valuations are given by $V^1(1) = 4$ and $V^2(1) = 6$ and their budgets by $m^1 = m^2 = 2$. This market has no Walrasian equilibrium, although both buyers have valuations and budgets above the seller's zero reservation price. At price $p(1) \leq 2$ the item is over-demanded, whereas at price $p(1) > 2$ there is no demand for the item.

Although a Walrasian equilibrium may fail to exist, it is interesting to observe in this example that implementable pairs $(p, \pi^1) = ((0,2),(1,0))$ and $(p, \pi^2) = ((0,2),(0,1))$ are core allocations and Pareto efficient. To verify this, consider $(p, \pi^1) = ((0,2),(1,0))$. At (p, π^1), the seller gets utility of $p(1) = 2$, buyer 1 utility of $V^1(1) + m^1 - p(1) = 4$, and buyer 2 utility of $m^2 = 2$. This allocation is stable and thus in the core in the sense that the seller and buyer 2 cannot block it because it is impossible for the seller to achieve a utility of more than 2. The same argument applies to (p, π^2).

If we apply the auction of Demange et al. (1986), it will start with price 0. The item is overdemanded. Then its price is increased to 1. Again, the item is overdemanded, and its price is increased to 2. Once again, the item is overdemanded and its price is increased to 3. Now the auction stops and cannot assign the item to any bidder because no bidder wants the item at price 3. We should remind the reader of the fact that the auction of Demange et al. (1986) was developed for the environment without budget constrained bidders.

The above example shows that when bidders face financial constraints, the Walrasian equilibrium is not guaranteed to exist but the core may be still nonempty. It is well known from Scarf (1967) that an exchange economy or a nontransferable utility game can have a nonempty core under fairly general conditions; see also Shapley and Scarf (1974) and Predtetchinski and Herings (2004). In the sequel, we shall prove the existence of a nonempty core for the auction model with budget constrained bidders. We establish this result by proposing a dynamic auction, which actually finds a core allocation in finitely many steps.

To introduce the precise notion of core, we first give several definitions. At an implementable pair (p, π), the utilities that the bidders and the seller achieve are given by

$$U^i(p, \pi) = V^i(\pi(i)) + m^i - p(\pi(i)), \;\; i \in M,$$

and

$$U^0(p, \pi) = \sum_{i \in M} p(\pi(i)) + \sum_{j \in N_\pi} c(j),$$

respectively. An implementable pair (p, π) is *Pareto efficient* if there does not exist another implementable pair (p', π') such that $U^i(p', \pi') > U^i(p, \pi)$ for all $i \in M_0$. A nonempty subset of M_0 is called a coalition. For a coalition $S \subseteq M_0$, a *feasible assignment* ρ^S is an assignment in $\mathcal{A}$ such that $\rho^S(i) = 0$ for every $i \in M \setminus S$. That is, at ρ^S, every bidder in S receives at most one real item and every bidder outside S receives the dummy good. A pair (q, ρ^S) of a feasible price vector $q \in \mathbb{R}^{n+1}_+$ and a feasible assignment ρ^S for S is *implementable* if $q(\rho^S(i)) \leq m^i$ for all $i \in S$ and if $q(j) = c(j)$ for every unsold item $j \in N_{\rho^S}$. An implementable pair (p, π) is a *core allocation* if there does not exist any implementable pair (q, ρ^S) for some coalition $S \subseteq M_0$ such that $U^i(q, \rho^S) > U^i(p, \pi)$ for all $i \in S$. Clearly, a core allocation is Pareto efficient. Also a core allocation (p, π) ensures individual rationality, that is, $U^i(p, \pi) \geq m^i$ for every bidder $i \in M$ and $U^0(p, \pi) \geq \sum_{h \in N} c(h)$ for the seller. Observe that when we consider any coalition of two or more agents, we only need to concentrate on coalitions that contain at least the seller. Coalitions consisting of only two or more bidders can be excluded.

A stronger core notion is the strict core. An implementable pair (p, π) is a *strong core* or *strict core allocation* if there does not exist any implementable pair (q, ρ^S) for some coalition $S \subseteq M_0$ such that $U^i(q, \rho^S) \geq U^i(p, \pi)$ for all $i \in S$ with at least one strict inequality. In Example 2.1, the core allocation (p, π^2) is a strict core allocation but the core allocation (p, π^1) is not. In general, if players do not necessarily have strict preferences, the core exists but the strict core may not. This is why Scarf (1967) and Shapley and Scarf (1974) use the notion of core rather than strict core. The following example is a slight modification of Example 2.1 and shows that the strict core can be empty.

Example 2.2 There are one item and two buyers. The seller's reservation price is zero, $c(1) = 0$. The buyers' valuations are given by $V^1(1) = 6$ and $V^2(1) = 6$ and their budgets by $m^1 = m^2 = 2$. As in the previous example this market has no Walrasian equilibrium. The implementable pairs $(p, \pi^1) = ((0, 2), (1, 0))$ and $(p, \pi^2) = ((0, 2), (0, 1))$ are core allocations but not strict core allocations. For instance, (p, π^1) can be weakly blocked by the seller and bidder 2.

In the previous examples we demonstrate that a Walrasian equilibrium may not exist if bidders are budget constrained. In both examples all bidders have equal budget. One may wonder if the nonexistence of Walrasian equilibrium has anything to do with equal budget. The following example shows that with more than one item the nonexistence of Walrasian equilibrium depends crucially on whether bidders are budget constrained or not, regardless of whether their budgets are equal or not.

Example 2.3 There are two items and four bidders. The seller's reservation prices are all zero, $c(1) = c(2) = 0$. The bidders' valuations and budgets are given in the Table 1. Observe that budgets of the bidders are not all the same. This market has no Walrasian equilibrium. To see this, first notice that there are two fully efficient assignments,

$\pi^1 = (0,0,1,2)$ and $\pi^2 = (0,0,2,1)$. Suppose there would be a Walrasian equilibrium with supporting prices $p(1)$ and $p(2)$. For bidder 1, we then have $3 - p(1) \leq 0$; for bidder 2 we have $4 - p(1) \leq 0$; for bidder 3 we have $9 - p(1) = 11 - p(2) \geq 0$; and for bidder 4 we have $7 - p(1) = 9 - p(2) \geq 0$. These inequalities imply $p(1) \geq 4$ and $p(2) \geq 6$. However, bidder 4 is not able to pay any equilibrium price of either item 1 or 2, because $m^4 = 3 < 4 = \min\{p(1), p(2)\}$.

It is easy to show that implementable pairs $(p, \pi^1) = ((0,4,3),(0,0,1,2))$ and $(q, \pi^2) = ((0,3,4),(0,0,2,1))$ are strict core allocations. We remark that if we change the budget of bidder 1 or 2 to 0 or 2, this market still does not have a Walrasian equilibrium.

Table 1: Bidders' valuations and budgets.

Bidder	Item 0	Item 1	Item 2	Budget
1	0	3	0	1
2	0	4	0	1
3	0	9	11	4
4	0	7	9	3

3 The dynamic auction

In this section we establish the existence of a core allocation for the auction model with budget constrained bidders. This result can be seen as a generalization of the classic existence theorem for the assignment markets without budget constraints to the more general case which permits budget constraints. While in the classical model (see Shapley and Shubik (1971)) the set of Walrasian equilibrium allocations coincides with the (strong) core, in the current model the core will be shown to be non-empty and can be strictly larger than the set of Walrasian equilibrium allocations if the latter set is nonempty.

Theorem 3.1 *There exists at least one core allocation in the auction model with budget constrained bidders.*

We shall design a dynamic auction that can actually find in finitely many steps a core allocation, thus yielding a constructive proof for this theorem. Briefly speaking, the auction works as follows. Every bidder submits for each item a price, his bid, that he is willing and able to pay. Taking all bids and her own reservation prices into account the auctioneer chooses an assignment that gives her the highest revenue and offers a spot market price for each item, which in case the item is not assigned anymore can be lower than the spot market price for the item in the previous round. Then every bidder updates his bid for each item according to the new spot market prices, his previous bids, his budget, and his valuations. Repeat this process until no bidder is willing to make any higher bid.

In each round $t \in \mathbb{Z}_+$ of the auction, every bidder $i \in M$ offers a (feasible) bidding function $p_t^i : N_0 \to \mathbb{Z}$ with $p_t^i(0) = 0$ and $p_t^i(k) \leq \min\{m^i, V^i(k)\}$ for every $k \in N$. That is, no bidder is willing to bid above his budget or his value for any item. Let $P_t = (p_t^1, \ldots, p_t^m)$ be the bidding system at round t. Since the auctioneer wishes to achieve the highest revenue, her choice set at round t is determined by

$$S(P_t) = \{\pi \in \mathcal{A} \mid \sum_{k \in N_\pi} c(k) + \sum_{i \in M} p_t^i(\pi(i)) = \max_{\rho \in \mathcal{A}} (\sum_{k \in N_\rho} c(k) + \sum_{i \in M} p_t^i(\rho(i)))\}.$$

We now give a detailed description of the auction.

The dynamic auction

Step 1: Every bidder $i \in M$ offers a bidding function p_0^i. Set $t = 0$ and go to Step 2.

Step 2: Based on the current bidding system $P_t = (p_t^1, \ldots, p_t^m)$, the auctioneer announces an assignment $\pi_t \in S(P_t)$ and a spot market price vector $\bar{p}_t \in \mathbb{R}^{n+1}$ as follows. If $t = 0$ or $\pi_{t-1} \notin S(P_t)$, take π_t to be any element of $S(P_t)$, and set $\bar{p}_t(0) = 0$, $\bar{p}_t(\pi_t(i)) = p_t^i(\pi_t(i))$ when $\pi_t(i) \in N$ for some $i \in M$, and $\bar{p}_t(k) = c(k)$ when $k \in N_{\pi_t}$, and go to Step 3. If $t > 0$ and $\pi_{t-1} \in S(P_t)$, then set $\pi_t = \pi_{t-1}$, $\bar{p}_t(0) = 0$, for any $\pi_t(i) \in N$ for some $i \in M$, set $\bar{p}_t(\pi_t(i)) = \bar{p}_{t-1}(\pi_t(i)) + 1$ when, at round $t-1$, some bidder j increased his bid for the item $\pi_t(i)$ to $\bar{p}_{t-1}(\pi_t(i)) + 1$, and set $\bar{p}_t(k) = \bar{p}_{t-1}(k)$ otherwise, and go to Step 3.

Step 3: Each bidder $i \in M$ updates his bids by setting $\tilde{p}_t^i(k) = \min\{p_t^i(k), \bar{p}_t(k)\}$ for all $k \in N$. For any bidder $i \in M$, if there exists some item $k \in N$ such that $V^i(k) - \tilde{p}_t^i(k) > V^i(\pi_t(i)) - \tilde{p}_t^i(\pi_t(i))$ and $\tilde{p}_t^i(k) < m^i$, bidder i updates his bidding function by setting $p_{t+1}^i(k) = \tilde{p}_t^i(k) + 1$ for one such k, and setting $p_{t+1}^i(h) = \tilde{p}_t^i(h)$ for any other item $h \in N$. Every other bidder i sets $p_{t+1}^i = \tilde{p}_t^i$. When $p_{t+1}^i \neq \tilde{p}_t^i$ for some $i \in M$, then set $t = t + 1$ and go back to Step 2. Otherwise, setting $p_t^i = \tilde{p}_t^i$ for every $i \in M$, the auction stops and the output is (P_t, π_t).

In each round of the auction, every bidder is announcing his bid for each of the items. Based on these bids the auctioneer determines an assignment and spot market prices. In the first round or if the assignment in the previous round is not maximizing the seller's revenues, then the auctioneer chooses a new assignment that maximizes her revenues and sets the spot market price of an item equal to the price announced by the bidder who is assigned the item and if the item is not assigned, the spot market price becomes the reservation price. This implies that spot prices of items that are not assigned anymore may fall back to their reservation prices. If the previous assignment is still optimal for the auctioneer, then she sticks to the assignment and increases the price of any assigned item with one when in the previous round there is at least one bidder who increased his bid for the item above its spot market price. Then each bidder considers for every item the minimum of the spot market price and the price he announced before. If at these accounting prices there is an item which gives a bidder more utility than the item that was assigned to him and its price is below his budget, then he will announce for one such item in the next round a higher price and for the other items the accounting price. The other bidders announce their accounting prices for each item. As soon as no bidder updates his price anymore the auction stops.

This auction bears some similarity with pay-as-you-bid auctions used on the Internet or more traditionally by governments for selling treasury bills, but it differs from them in three crucial aspects: First, in Step 2, the auctioneer does not merely announce the spot price for each item but more importantly she adjusts the spot price upwards when she observes that some bidder increases his bid. Second, in Step 3, unlike the existing pay-as-you-bid auctions in which bidders are not allowed to decrease their bids, the current auction permits bidders to reduce their bids in order to avoid overbidding. Third, in Step 3, the current auction has a flexible rule for the bidders to adjust their bids for those items which they can afford and give them higher profits, whereas the existing auctions typically require the bidders to adjust their bids for those items which give the bidders the highest profits.

The current auction differs from the ascending auction introduced by van der Laan and Yang (2008) (LY-auction in short). In the latter auction in each round the prices of the items in some minimal over-demanded set are increased with one and as soon as a set of items becomes under-demanded one of the items in this set is assigned to an agent who demands the item. Such an item cannot be demanded later on in the auction, while this may occur in the new auction, in particular this may happen when a price is decreased. In this respect the auction also differs from the well-known ascending auction of Demange et al. (1986) (DGS-auction in short). We remark that the LY-auction is a generalization of the DGS-auction from the case with no budget constraints to the case with budget constraints.

Here we use the Example 2.3 to illustrate the differences between the current auction on the one hand and the DGS-auction and the TY-auction on the other hand. When the DGS-auction or LY-auction starts with the reservation prices zero, item 1 is demanded by both bidders 1 and 2 and is minimally over-demanded, which is still the case if the price of item 1 is equal to 1. If now the price of item 1 would increase to 2, then no bidder demands item 1 anymore because bidders 1 and 2 cannot afford to buy item 1 because of their insufficient budgets, and bidders 3 and 4 still prefer item 2 to item 1. The DGS-auction gets stuck because item 1 becomes under-demanded at price 2 and the LY-auction assigns item 1 to either bidder 1 or bidder 2 and continues with the remaining bidders and item 2 by increasing its price as long as both bidders 3 and 4 demand item 2. When the price of item 2 rises to 4, the item will be assigned to bidder 3, because the price exceeds bidder 4's budget. This allocation, however, is not a core allocation because bidders 3 and 4 and the seller can jointly do better.

In the proposed auction if all bidders start to bid zero prices the auctioneer chooses any assignment, after which bidders 1 or 2 or both announce price 1 for item 1 and bidders 3 or 4 or both announce price 1 for item 2. In the next round the auctioneer sets both spot market prices equal to one and assigns provisionally item 1 to one of the bidders 1 and 2 who increased his price for item 1 and assigns provisionally item 2 to one of the bidders 3 and 4 who increased his price for item 2. Bidders 1 and 2 are not updating their prices any further because they reached their budget constraint, but the bidder 3 or 4 or both who were not provisionally assigned item 2 announce a higher price for item 2. At price 3 for item 2 and still price 1 for item 1 bidder 3 becomes indifferent between the two items and he may announce price 2 for item 1 if he is not provisionally assigned an item or do that in the next round when the price of item 2 has become equal to 3. The auctioneer finally assigns item 2 to bidder 3 and item 1 to bidder 4 and no items anymore to bidders 1 or 2. This allocation is indeed a core element. Unlike the LY-auction, the current auction only provisionally assigns item 1 in the beginning to one of the bidders 1 and 2, but in a later round assigns this item to a bidder who is willing to pay a higher price. This will occur when in later rounds of the auction the price of another item for which the bidder has a higher valuation becomes too high.

Next we compare the proposed auction with Ausubel and Milgrom (2002). As we have pointed out earlier, our auction is most closely related to theirs; see also Milgrom (2004, Section 8.3.2). Although they focus their analysis on a family of package auctions without budget constrained bidders, they also briefly discuss a dynamic auction with budget constrained bidders in their Section 8. That auction (with budget constrained bidders) uses the core of an NTU game as the solution and is shown to find a core element in their auction model. In their model, bidders are allowed to demand several items. However, their auction mechanism cannot be applied to our current model, because their auction requires every bidder to have a strict preference relation over a finite set of choices, whereas in our model every agent has only a weak preference relation over a continuum of choices $N_0 \times \mathbb{R}$ and is typically indifferent among several choices. Next is an example to show why tie-breaking rules cannot be used to deal with weak preferences.

Example 3.2 There are two items and three bidders. The parameters are given by $V^i(j) = 1$, $V^i(0) = 0$, $m^i = 3$, and $c(j) = 0$ for all $i = 1, 2, 3$ and all $j = 1, 2$. Bidders are indifferent between the two items. We break ties as follows. When a bidder is indifferent among bundles, he prefers the bundle with item 1 to the bundle with item 2 and the latter one to the bundle with the dummy good. It is easy to see that there exists a Walrasian equilibrium (p, π) where $p(1) = p(2) = 1$ and $\pi(1) = 1$, $\pi(2) = 2$, and $\pi(3) = 0$. At $p^0 = (c(0), c(1), c(2)) = (0, 0, 0)$ we have $D^i(p^0) = \{1\}$ for all $i \in M$. We adjust prices from p^0 to $p^1 = (0, 1, 0)$ because item 1 is over-demanded. At p^1 we have $D^i(p^1) = \{2\}$ for all $i \in M$. Prices are adjusted to $p^2 = (0, 1, 1)$, because item 2 is over-demanded. At p^2 we have $D^i(p^2) = \{1\}$ for all $i \in M$. Prices are adjusted to $p^3 = (0, 2, 1)$ since item 1 is over-demanded. At p^3 we have $D^i(p^3) = \{2\}$ for all $i \in M$. Prices are adjusted to $p^4 = (0, 2, 2)$ since item 2 is over-demanded. The auction now ends up with $D^i(p^4) = \{0\}$ for all $i \in M$. But this is not an equilibrium state and thus the auction terminates in disequilibrium, although there is an equilibrium.

When the auction stops with some (P_t, π_t), it is possible that a budget-constrained bidder $i \in M$ is assigned an item $\pi_t(i)$ which is not contained in his demand set $D^i(p_t^i)$. In this case, it holds that $p_t^i(k) = m^i$ for any $k \in N$ satisfying $V^i(k) - p_t^i(k) > V^i(\pi_t(i)) - p_t^i(\pi_t(i))$, i.e. any item that this bidder strictly prefers has a price equal to his budget.

Lemma 3.3 *In every round t of the auction, it holds that $\pi_t \in S(\tilde{P}_t)$.*

Proof. In case of $\pi_{t-1} \notin S(P_t)$, by definition of $\tilde{P}_t$ it holds that

$$\sum_{k \in N_\rho} c(k) + \sum_{i \in M} p_t^i(\rho(i)) \geq \sum_{k \in N_\rho} c(k) + \sum_{i \in M} \tilde{p}_t^i(\rho(i))$$

for all $\rho \in \mathcal{A}$ and

$$\sum_{k \in N_{\pi_t}} c(k) + \sum_{i \in M} \tilde{p}_t^i(\pi_t(i)) = \sum_{k \in N_{\pi_t}} c(k) + \sum_{i \in M} p_t^i(\pi_t(i)).$$

Moreover, $\pi_t \in S(P_t)$ implies

$$\sum_{k \in N_{\pi_t}} c(k) + \sum_{i \in M} p_t^i(\pi_t(i)) \geq \sum_{k \in N_\rho} c(k) + \sum_{i \in M} p_t^i(\rho(i))$$

for all $\rho \in \mathcal{A}$. Therefore, it holds that

$$\sum_{k \in N_{\pi_t}} c(k) + \sum_{i \in M} \tilde{p}_t^i(\pi_t(i)) \geq \sum_{k \in N_\rho} c(k) + \sum_{i \in M} \tilde{p}_t^i(\rho(i))$$

for all $\rho \in \mathcal{A}$. Similarly, we can show for the case $\pi_{t-1} \in S(P_t)$ that $\pi_t = \pi_{t-1} \in S(\tilde{P}_t)$. $\square$

The existing auctions in the literature are either ascending or descending. It is relatively easy to prove their finite convergence because of monotonicity. However, this is not the case for the current auction, because the bidders may also decrease some of their bids. This feature makes it not possible to use familiar arguments. Instead, we have to explore a different monotonicity argument which makes use of the bids of all bidders and the revenues of the seller.

Lemma 3.4 *The auction terminates in finitely many rounds.*

Proof. Let $R(\pi, P) = \sum_{k \in N_\pi} c(k) + \sum_{i \in M} p^i(\pi(i))$ denote the revenue of the seller at (π, P), and let $R(P) = \max\{R(\pi, P) \mid \pi \in \mathcal{A}\}$ be the highest revenue at P. In each round t, we have $R(\pi_t, P_t) = R(P_t)$. We need to consider two cases. In case of $\pi_{t-1} \notin S(P_t)$, we have $R(P_t) = R(\pi_t, P_t) > R(\pi_{t-1}, P_t)$, by the auction rule we have $R(\pi_{t-1}, P_t) \geq R(\pi_{t-1}, \tilde{P}_{t-1})$, and from Proposition 3.3 it follows that $R(\pi_{t-1}, \tilde{P}_{t-1}) = R(\pi_{t-1}, P_{t-1}) = R(P_{t-1})$. This proves $R(P_t) > R(P_{t-1})$.

In case of $\pi_{t-1} \in S(P_t)$, we have $R(P_t) = R(P_{t-1})$, because $p^i_t(\pi_t(i)) = p^i_{t-1}(\pi_{t-1}(i))$ for all $i \in M$. In this case, at least one bidder increases his bid and none of the bidders decreases any of his bids.

The two arguments imply that in each round either the revenue of the seller is strictly increasing or the revenue of the seller remains the same while no bidder is bidding less and at least one bidder is bidding more. Therefore, due to the finiteness of all values and budgets, the auction must stop in finitely many rounds. □

Let $\pi^* = \pi_t$ and $p^* = \bar{p}_t$ when the auction stops at round t. The following theorem shows that the outcome (p^*, π^*) is a core allocation.

Theorem 3.5 *When bidders are budget constrained, the outcome (p^*, π^*) found by the auction is in the core and thus Pareto efficient.*

Proof. Suppose to the contrary that (p^*, π^*) is not in the core. Clearly, the pair (p^*, π^*) is individually rational. Then there exist a coalition S consisting of the seller and at least one bidder and an implementable pair (q, ρ^S) for S such that $U^i(q, \rho^S) > U^i(p^*, \pi^*)$ for all $i \in S$. For the seller this implies

$$\begin{aligned}
\textstyle\sum_{j \in N} q(j) &= \textstyle\sum_{i \in S} q(\rho^S(i)) + \sum_{j \in N_{\rho^S}} c(j) \\
&= U^0(q, \rho^S) \\
&> U^0(p^*, \pi^*) \\
&= \textstyle\sum_{i \in M} p^*(\pi^*(i)) + \sum_{j \in N_{\pi^*}} c(j) \\
&= \textstyle\sum_{j \in N} p^*(j).
\end{aligned}$$

Then there exists $j^* \in N$ such that $q(j^*) > p^*(j^*)$. This means that some bidder $i^* \in S$ is assigned item j^* at ρ^S, i.e., $\rho^S(i^*) = j^*$, because $\min\{p^*(j), q(j)\} \geq c(j)$ for all $j \in N$ and

$q(j) = c(j)$ for every unassigned item $j \in N_{\rho^S}$. Since (q, ρ^S) is implementable for S and $U^{i^*}(q, \rho^S) > U^{i^*}(p^*, \pi^*)$, we have $V^{i^*}(j^*) - q(j^*) > V^{i^*}(\pi^*(i^*)) - p^*(\pi^*(i^*))$ and $q(j^*) \leq m^{i^*}$. From $q(j^*) > p^*(j^*)$ and by definition of $\tilde{P}_t$ it then follows that

$$\begin{aligned} V^{i^*}(j^*) - p^*(j^*) &> V^{i^*}(j^*) - q(j^*) \\ &> V^{i^*}(\pi^*(i^*)) - p^*(\pi^*(i^*)) \\ &= V^{i^*}(\pi^*(i^*)) - \tilde{p}_t^{i^*}(\pi^*(i^*)), \end{aligned}$$

where t is the round in which the auction stops. Because $\tilde{p}_t^i(k) \leq \bar{p}_t(k) = p^*(k)$ for all $i \in M$ and $k \in N$, this implies

$$\begin{aligned} V^{i^*}(j^*) - \tilde{p}_t^{i^*}(j^*) &\geq V^{i^*}(j^*) - p^*(j^*) \\ &> V^{i^*}(\pi^*(i^*)) - \tilde{p}_t^{i^*}(\pi^*(i^*)) \end{aligned}$$

and $\tilde{p}_t^{i^*}(j^*) \leq \bar{p}_t(j^*) = p^*(j^*) < q(j^*) \leq m^{i^*}$. But then, at prices $\tilde{p}_t^{i^*}$ bidder i^* would make a new offer, and therefore the auction could not have stopped at round t. □

The next theorem exhibits another attractive feature of the proposed auction that when no bidder is budget constrained, the auction finds a strict core allocation, whose assignment is also fully efficient. This property is certainly what a good auction could best have but it is not an easy observation and in fact it is a natural outcome of an elaborate design of the auction, as the following proof indicates. We point out that although the first part of the proof is similar to that of Theorem 3.5, we will have to explore the crucial fact that no bidder is budget constrained. This fact, however, cannot be used in the proof of Theorem 3.5 in the presence of budget constrained bidders.

Theorem 3.6 *When no bidder faces a budget constraint, the outcome (p^*, π^*) found by the auction is a strict core allocation and the assignment π^* is fully efficient.*

Proof. We first show that (p^*, π^*) is a strict core allocation. Suppose to the contrary that (p^*, π^*) is not in the strict core. Clearly, the pair (p^*, π^*) is individually rational. Then there exist a coalition S consisting of the seller and at least one bidder and an implementable pair (q, ρ^S) for S such that $U^i(q, \rho^S) \geq U^i(p^*, \pi^*)$ for all $i \in S$ with at least one strict inequality. Let t be the round in which the auction stops. We need to consider two cases separately.

Case 1. The seller can be strictly improved, i.e.,

$$\begin{aligned} \textstyle\sum_{j \in N} q(j) &= \textstyle\sum_{i \in S} q(\rho^S(i)) + \sum_{j \in N_{\rho^S}} c(j) \\ &= U^0(q, \rho^S) \\ &> U^0(p^*, \pi^*) \\ &= \textstyle\sum_{i \in M} p^*(\pi^*(i)) + \sum_{j \in N_{\pi^*}} c(j) \\ &= \textstyle\sum_{j \in N} p^*(j). \end{aligned}$$

Then there exists $j^* \in N$ such that $q(j^*) > p^*(j^*)$. This means that some bidder $i^* \in S$ is assigned item j^* at ρ^S, i.e., $\rho^S(i^*) = j^*$, because $\min\{p^*(j), q(j)\} \geq c(j)$ for all $j \in N$ and $q(j) = c(j)$ for every unassigned item $j \in N_{\rho^S}$. Since (q, ρ^S) is implementable for S and $U^{i^*}(q, \rho^S) \geq U^{i^*}(p^*, \pi^*)$, we have $V^{i^*}(j^*) - q(j^*) \geq V^{i^*}(\pi^*(i^*)) - p^*(\pi^*(i^*))$ and $q(j^*) \leq m^{i^*}$. From $q(j^*) > p^*(j^*)$ and by definition of $\tilde{P}_t$ it then follows that

$$\begin{aligned} V^{i^*}(j^*) - p^*(j^*) &> V^{i^*}(j^*) - q(j^*) \\ &\geq V^{i^*}(\pi^*(i^*)) - p^*(\pi^*(i^*)) \\ &= V^{i^*}(\pi^*(i^*)) - \tilde{p}_t^{i^*}(\pi^*(i^*)). \end{aligned}$$

Because $\tilde{p}_t^i(k) \leq \bar{p}_t(k) = p^*(k)$ for all $i \in M$ and $k \in N$, this implies

$$\begin{aligned} V^{i^*}(j^*) - \tilde{p}_t^{i^*}(j^*) &\geq V^{i^*}(j^*) - p^*(j^*) \\ &> V^{i^*}(\pi^*(i^*)) - \tilde{p}_t^{i^*}(\pi^*(i^*)) \end{aligned}$$

and $\tilde{p}_t^{i^*}(j^*) \leq \bar{p}_t(j^*) = p^*(j^*) < q(j^*) \leq m^{i^*}$. But then, at prices $\tilde{p}_t^{i^*}$ bidder i^* would make a new offer, and therefore the auction could not have stopped at round t.

Case 2. The seller is just weakly improved and at least one bidder is strictly improved. For the seller, we have

$$\begin{aligned} \textstyle\sum_{j \in N} q(j) &= \textstyle\sum_{i \in S} q(\rho^S(i)) + \sum_{j \in N_{\rho^S}} c(j) \\ &= U^0(q, \rho^S) \\ &\geq U^0(p^*, \pi^*) \\ &= \textstyle\sum_{i \in M} p^*(\pi^*(i)) + \sum_{j \in N_{\pi^*}} c(j) \\ &= \textstyle\sum_{j \in N} p^*(j). \end{aligned}$$

There are two subcases. In Subcase 1, there exists $j^* \in N$ such that $q(j^*) > p^*(j^*)$. Then the proof goes the same line as Case 1 above. In Subcase 2, $q(j) = p^*(j)$ for all $j \in N$. Since (q, ρ^S) is implementable for S and $U^{i^*}(q, \rho^S) > U^{i^*}(p^*, \pi^*)$ holds for some bidder $i^* \in S$ who is assigned some item $j^* \in N$ by ρ^S, we have $V^{i^*}(j^*) - q(j^*) > V^{i^*}(\pi^*(i^*)) - p^*(\pi^*(i^*))$. By the auction rule, we have $V^i(\pi^*(i)) - p^*(\pi^*(i)) \geq 0$ for every bidder $i \in M$. This implies $V^{i^*}(j^*) - q(j^*) > 0$. Because $m^{i*} \geq \max_{h \in N_0} V^{i*}(h)$ and $V^{i^*}(j^*) - q(j^*) > 0$, we must have $q(j^*) < m^{i*}$. From $q(j^*) = p^*(j^*)$ and by definition of $\tilde{P}_t$ it then follows that

$$\begin{aligned} V^{i^*}(j^*) - p^*(j^*) &= V^{i^*}(j^*) - q(j^*) \\ &> V^{i^*}(\pi^*(i^*)) - p^*(\pi^*(i^*)) \\ &= V^{i^*}(\pi^*(i^*)) - \tilde{p}_t^{i^*}(\pi^*(i^*)). \end{aligned}$$

Because $\tilde{p}_t^i(k) \leq \bar{p}_t(k) = p^*(k)$ for all $i \in M$ and $k \in N$, this implies

$$\begin{aligned} V^{i^*}(j^*) - \tilde{p}_t^{i^*}(j^*) &\geq V^{i^*}(j^*) - p^*(j^*) \\ &> V^{i^*}(\pi^*(i^*)) - \tilde{p}_t^{i^*}(\pi^*(i^*)) \end{aligned}$$

and $\tilde{p}_t^{i^*}(j^*) \le \bar{p}_t(j^*) = p^*(j^*) = q(j^*) < m^{i^*}$. But then, at prices $\tilde{p}_t^{i^*}$ bidder i^* would make a new offer, and therefore the auction could not have stopped at round t.

Next we prove that π^* is fully efficient. Recall that when the auction terminates with (P_t, π_t) at round t, it holds that $p_t^i = \tilde{p}_t^i$ for every $i \in M$. We will show that when no bidder has budget constraint, i.e., $m^i \ge \max_{h \in N_0} V^i(h)$ for every bidder $i \in M$, we have

$$\pi_t(i) \in D^i(p_t^i), \ \ i \in M \tag{3.1}$$

and

$$\pi_t \in S(P_t). \tag{3.2}$$

The latter follows immediately from the auction rule. To prove the first relation above, suppose that $\pi_t(i) \notin D^i(p_t^i)$ for some $i \in M$. Then there exists $k \in N$ such that

$$p_t^i(k) \le m^i \text{ and } V^i(k) - p_t^i(k) > V^i(\pi_t(i)) - p_t^i(\pi_t(i)) \ge 0.$$

So $V^i(k) - p_t^i(k) > 0$. If $p_t^i(k) = m^i$, then we have $V^i(k) > p_t^i(k) = m^i$, contradicting $m^i \ge \max_{h \in N_0} V^i(h)$. Hence, $p_t^i(k) < m^i$. Consequently, at prices $\tilde{p}_t^i$ bidder i would make a new offer and thus the auction could not have stopped at round t.

Because $m^i \ge \max_{h \in N_0} V^i(h)$, we can write bidder i's demand set as

$$D^i(p^i) = \{j \in N_0 \mid V^i(j) - p^i(j) = \max_{h \in N_0}\{V^i(h) - p^i(h)\}\}.$$

To see this, consider any item $j \in N$ with price $p^i(j) > m^i$. We have

$$\begin{array}{rcl} V^i(j) - p^i(j) & < & V^i(j) - m^i \\ & \le & \max_{h \in N_0} V^i(h) - m^i \\ & \le & m^i - m^i \\ & = & V^i(0) - p^i(0) \\ & \le & \max_{h \in N_0}(V^i(h) - p^i(h)) \end{array}$$

Take any assignment $\rho \in \mathcal{A}$. By (3.1), for any bidder $i \in M$, we have

$$V^i(\pi^*(i)) - p^i(\pi^*(i)) \ge V^i(\rho(i)) - p^i(\rho(i)).$$

It follows that

$$\begin{array}{l} \sum_{j \in N_{\pi^*}} c(j) + \sum_{i \in M} V^i(\pi^*(i)) - \Big(\sum_{j \in N_\rho} c(j) + \sum_{i \in M} V^i(\rho(i))\Big) \\ \ge \sum_{j \in N_{\pi^*}} c(j) + \sum_{i \in M} p^i(\pi^*(i)) - \Big(\sum_{j \in N_\rho} c(j) + \sum_{i \in M} p^i(\rho(i))\Big). \end{array}$$

By (3.2), we have

$$\sum_{j \in N_{\pi^*}} c(j) + \sum_{i \in M} p^i(\pi^*(i)) - \Big(\sum_{j \in N_\rho} c(j) + \sum_{i \in M} p^i(\rho(i))\Big) \ge 0.$$

Consequently, we have

$$\sum_{j \in N_{\pi^*}} c(j) + \sum_{i \in M} V^i(\pi^*(i)) - \Big(\sum_{j \in N_\rho} c(j) + \sum_{i \in M} V^i(\rho(i))\Big) \ge 0$$

for all $\rho \in \mathcal{A}$, i.e., π^* is fully efficient. □

4 Concluding remarks

In this article we propose a dynamic auction for finding a core allocation in a setting where bidders are budget constrained and each bidder demands at most one item. When no bidder is budget constrained, the auction always finds a strict core allocation whose assignment attains full market efficiency. It is worth pointing out that the auctions developed by Kelso and Crawford (1982), Gul and Stacchetti (2000), Milgrom (2000), Ausubel and Milgrom (2002), Perry and Reny (2005), Ausubel (2004, 2006), and Sun and Yang (2009) allow bidders to demand multiple items, albeit in the absence of budget constraint. This more general but also more difficult case remains to be explored when bidders face budget constraints.

References

[1] T. Andersson and A. Erlanson (2013), "Multi-item Vickrey-English-Dutch auctions," *Games and Economic Behavior* 81, 116-129.

[2] L. Ausubel (2004), "An efficient ascending-bid auction for multiple objects," *American Economic Review* 94 (5), 1452-1475.

[3] L. Ausubel (2006), "An efficient dynamic auction for heterogeneous commodities", *American Economic Review* 96, 602-629.

[4] L. Ausubel and P. Milgrom (2002), "Ascending auctions with package bidding," *Frontiers of Theoretical Economics* 1 (1), 1-42.

[5] J.P. Benoît and V. Krishna (2001), "Multiple-object auctions with budget constrained bidders," *Review of Economic Studies* 68, 155-179.

[6] D. Bernheim and M. Whinston (1986), "Menu auctions, resource allocation and economic influence," *Quarterly Journal of Economics* 101, 1-31.

[7] S. Brusco and G. Lopomo (2008), "Budget constraints and demand reduction in simultaneous ascending-bid auctions," *The Journal of Industrial Economics* 56, 113-142.

[8] Y.K. Che and I. Gale (1998), "Standard auctions with financially constrained bidders," *Review of Economic Studies* 65, 1-21.

[9] V.P. Crawford and E.M. Knoer (1981), "Job matching with heterogeneous firms and workers," *Econometrica* 49, 437-450.

[10] R. Day and P. Milgrom (2008), "Core-selecting package auctions," *International Journal of Game Theory* 36, 393-407.

[11] G. Demange, D. Gale, and M. Sotomayor (1986), "Multi-item auctions," *Journal of Political Economy* 94, 863-872.

[12] A. Erdil and P. Klemperer (2010), "A new payment rule for core-selecting package auctions," *Journal of the European Economic Association* 8, 537-547.

[13] F. Gul and E. Stacchetti (2000), "The English auction with differentiated commodities," *Journal of Economic Theory* 92, 66-95.

[14] I. Hafalir, R. Ravi, and A. Sayedi (2012), "A near Pareto optimal auction with budget constraints," *Games and Economic Behavior* 74, 699-708.

[15] A. Kelso and V.P. Crawford (1982), "Job matching, coalition formation, and gross substitutes," *Econometrica* 50, 1483-1504.

[16] P. Klemperer (2004), *Auctions: Theory and Practice*, Princeton University Press, Princeton.

[17] T.C. Koopmans and M. Beckmann (1957), "Assignment problems and the location of economic activities," *Econometrica* 25, 53-76.

[18] V. Krishna (2002), *Auction Theory*, Academic Press, New York.

[19] J.J. Laffont and J. Robert (1996), "Optimal auction with finanically constrained bidders," *Economics Letters* 52, 181-186.

[20] G. van der Laan and Z. Yang (2008), "An ascending multi-item auction with financially constrained bidders," Tinbergen Institute Discussion Paper No. 08-017, VU University Amsterdam, Amsterdam.

[21] E. Maskin (2000), "Auctions, development, and privatization: efficient auctions with liquidity-constrained buyers," *European Economic Review* 44, 667-681.

[22] P. Milgrom (2000), "Putting auction theory to work: The simultaneous ascending auction," *Journal of Political Economy* 108, 245-272.

[23] P. Milgrom (2004), *Putting Auction Theory to Work*, Cambridge University Press, London.

[24] M. Perry and P. Reny (2005), "An efficient multi-unit ascending auction," *Review of Economic Studies* 72, 567-592.

[25] C. Pitchik (2009), "Budget-constrained sequential auctions with incomplete information," *Games and Economic Behavior* 66, 928-949.

[26] A. Predtetchinski and P.J.J. Herings (2004), "A necessary and sufficient condition for the non-emptiness of the core of a non-transferable utility game," *Journal of Economic Theory* 116, 84-92.

[27] H. Scarf (1967), "The core of an n-person game," *Econometrica* 35, 50-69.

[28] L.S. Shapley and H. Scarf (1974), "On cores and indivisibilities," *Journal of Mathematical Economics* 1, 23-37.

[29] L.S. Shapley and M. Shubik (1971), "The assignment game I: The core," *International Journal of Game Theory* 1, 111-130.

[30] N. Sun and Z. Yang (2009), "A double-track adjustment process for discrete markets with substitutes and complements," *Econometrica* 77, 933-952.

[31] A.J.J. Talman and Z. Yang (2011), "An efficient multi-item dynamic auction with budget constrained bidders," CentER DP No. 2011-96, Tilburg University, Tilburg.

[32] C.Z. Zheng (2001), "High bids and broke winners," *Journal of Economic Theory* 100, 129-171.