University of York

Department of Economics

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VAR ANALYSIS IN MACROECONOMICS

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Lecture 6

Monetary Policy Analysis using a VAR

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1. The analysis of monetary policy shocks

The time-series relation between output and money

There is an extensive literature on the use of VAR models to study the effects of money shocks on GDP

1. FRB St Louis single equation analysis

In the 1960's and 1970's the FRB of St Louis got a lot of attention for their work on this.

Their methodology was to estimate distributed lag dynamic models relating GDP to money of the form

$$y_t = \alpha(L)m_t + e_t$$

where $y = \ln GDP$, $m = \ln M$ and $e =$disturbance. Their finding was that money had a significant effect on output.

Using data from the US 1962-2002 it can be shown that $\alpha(1)$, the long-run effect of money on output is positive and has a t-statistic of 76. (Inflation is also significant when added to this equation; the t-statistic of the long-run coefficient is 5.5.) Using real instead of nominal money balances gives a t-statistic for its long-run coefficient of 31. However, when t is added real money is no longer significant and nominal money has a t-statistic of only 4. Since time is a proxy for technological progress and money is strongly trending, the interpretation of this result is highly problematic.

The prevailing orthodoxy was based on the Keynesian view that GDP depends on government expenditure, not money. The FRBStL showed that in the model

$$y_t = \alpha(L)m_t + \beta(L)g_t + e_t$$

where $g = \ln G$, G=government expenditure, m_t was much more important than g_t, but did not have a permanent affect on y_t.

Note: There is an interesting challenge to the standard interpretation of these models. The standard interpretation is that the model implies that output responds to money with a distributed lag. The alternative interpretation is that in fact there is no lag at all to a change in monetary policy.

If we can write

$$y_t = \theta\varepsilon_t + e_t$$

where ε_t is a money innovation ($E_{t-1}\varepsilon_t$), and if monetary policy is determined by

$$\begin{aligned} m_t &= \gamma(L)m_{t-1} + \varepsilon_t \\ eg \;\; m_t &= m_{t-1} + \varepsilon_t \end{aligned}$$

then

$$\begin{aligned} \varepsilon_t &= m_t - E_{t-1}m_t \\ &= \alpha(L)m_t = [1-\gamma(L)]m_t \end{aligned}$$

If monetary policy changes to say

$$m_t = \eta(L)m_{t-1} + \varepsilon_t$$

and this is fully anticipated then only the money innovation ε_t will affect y_t, and this will impact instantaneously on y_t.

2. Sims VAR analysis

In 1980, Sims introduced VAR analysis to study the relation between money and GDP.

His aim was to free empirical analysis from the use of macroeconometric models based on incredible identifying restrictions, and to introduce more of the transmission mechanism by which money affects GDP than allowed for by the FRBStL approach.

He proposed the use of a VAR in the levels of the logarithms of GDP, the price level (P) and money stock ($M1$ or $M2$) : y_t, p_t, m_t.

Subsequent research argued that additional variables should be included in the VAR, and especially the short-term nominal interest rate the federal funds rate $FFR = i_t$.

Interest rates are an important part of the monetary transmission mechanism, and the interest rate is often the monetary policy instrument, not the money stock, which is demand determined.

For a small open economy it was argued that the exchange rate should also be included as this is an important part of the transmission mechanism.

We will examine all of these issues.

There is also the large question of which of the VAR methodologies should be used.

We may note that y_t, p_t, m_t are all I(1) variables.

Before turning to the specification of a VAR we consider the underlying theoretical framework.

2. Theoretical framework for monetary policy analsysis

Any theoretical framework for the relation between money, output and prices must be centered on

(i) the demand for money function

(ii) the price equation

(iii) whether output is endofenous or exogenous

(iv) the choice of monetary policy instrument

Consider the following highly stylised macroeconomic model

$$\begin{aligned} m_t &= \upsilon + p_t + y_t + \gamma m_{t-1} + e_{mt} \\ \Delta p_t &= \pi + \theta \Delta y_{t-1} + \alpha \Delta p_{t-1} + e_{pt} \\ \Delta y_t &= \mu + e_{yt} \end{aligned}$$

The first equation is a money demand equation, the second is a price equation - both have simple dynamics - and the third assumes that output is a random walk with drift. If e_{yt} is uncorrelated with e_{mt} and e_{pt}, then output is exogenous.

The model can be written as the following SEM

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} y_t \\ p_t \\ m_t \end{bmatrix} = \begin{bmatrix} \mu \\ \pi \\ \nu \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ \theta & 1+\alpha & 0 \\ 0 & 0 & \gamma \end{bmatrix} \begin{bmatrix} y_{t-1} \\ p_{t-1} \\ m_{t-1} \end{bmatrix}$$
$$+ \begin{bmatrix} 0 & 0 & 0 \\ -\theta & -\alpha & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} y_{t-2} \\ p_{t-2} \\ m_{t-2} \end{bmatrix} + \begin{bmatrix} e_{yt} \\ e_{pt} \\ e_{mt} \end{bmatrix}$$

Noting that

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

we can write the SEM as the VAR

$$\begin{bmatrix} y_t \\ p_t \\ m_t \end{bmatrix} = \begin{bmatrix} \mu \\ \pi \\ \mu+\pi+\nu \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ \theta & 1+\alpha & 0 \\ 1+\theta & 1+\alpha & \gamma \end{bmatrix} \begin{bmatrix} y_{t-1} \\ p_{t-1} \\ m_{t-1} \end{bmatrix}$$
$$+ \begin{bmatrix} 0 & 0 & 0 \\ -\theta & -\alpha & 0 \\ -\theta & -\alpha & 0 \end{bmatrix} \begin{bmatrix} y_{t-2} \\ p_{t-2} \\ m_{t-2} \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} e_{yt} \\ e_{pt} \\ e_{mt} \end{bmatrix}$$

If the shocks $e_t = (e_{yt}, e_{pt}, e_{mt})'$ are uncorrelated then error term of the VAR is

$$u_t = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} e_t$$

Note that this has a recursive structure with the logical ordering $z_t = (y_t, p_t, m_t)'$.

The causal structure is that

(i) the shock in e_{yt} affects m_t instantaneously, but p_t with a lag

(ii) the shock e_{pt} affects m_t immediately, but never affects y_t

(iii) the shock e_{mt} affects p_t with a lag, but never affects y_t.

(iv) all shocks affect m_t instantaneously, but only e_{yt} affects y_t at any time.

We also note

- that the shocks e_{yt} and e_{pt} have permanent effects, but the e_{mt} shock is only temporary.

  This implies that all variables will be I(1). In this model, therefore, the monetary shock is better interpreted as a money demand than a money supply shock. This does not prevent m_t from being the monetary instrument.

- The model implies a cointegration relation between the three variables. This can be interpreted as the log velocity of circulation. *If the model is correct*, it suggests that the model is best estimated as a CVAR. Of course, the point about VAR analysis is that it aims to avoid all such structural interpretation. The model would just be estimated unrestrictedly in levels (or first differences to eliminate the non-stationarity of the variables).

- This theoretical framework is only illustrative. Many others could be chosen instead, and some may give different orderings and other implications.

3. Three variable VAR model $(y_t,\ p_t,\ m_t)$

We will focus on the US quarterly data 1959.1-2002.4. Other sample periods could give different results if the VAR is structurally unstable due say to a chnage of policy regime.

$y_t = \ln GDP$, $p_t = \ln GDP$ deflator, $m_t = \ln M1$. All variables are seasonally adjusted.

1. VAR in levels

How large a lag do we need?

What should be the order of these variables?

Which identification scheme should we use: unrestricted, Choleski, orthonormal?

Lag

If we start with 12 lags and then test down we find that the y_t and p_t equations do not depend on m_t or its lags! But m_t does depend on lags in both y_t and p_t.

Also it seems that we need up to 12 lags.

Variable order

Is there any logic to our choice?

We need to decide whether there is likely to be any lag in the response of a variable to shocks in others.

We shall consider the impulse response functions to one standard error of the unrestricted least squares residuals and to Choleski orthogonalised shocks based on three orderings:

(i) $z_t = (y_t,\ p_t,\ m_t)'$

(ii) $z_t = (m_t,\ p_t,\ y_t)'$

(iii) $z_t = (p_t,\ m_t,\ y_t)'$

The first ordering is that indicated by the theoretical model.

The ordering has no effect on the unrestricted residuals.

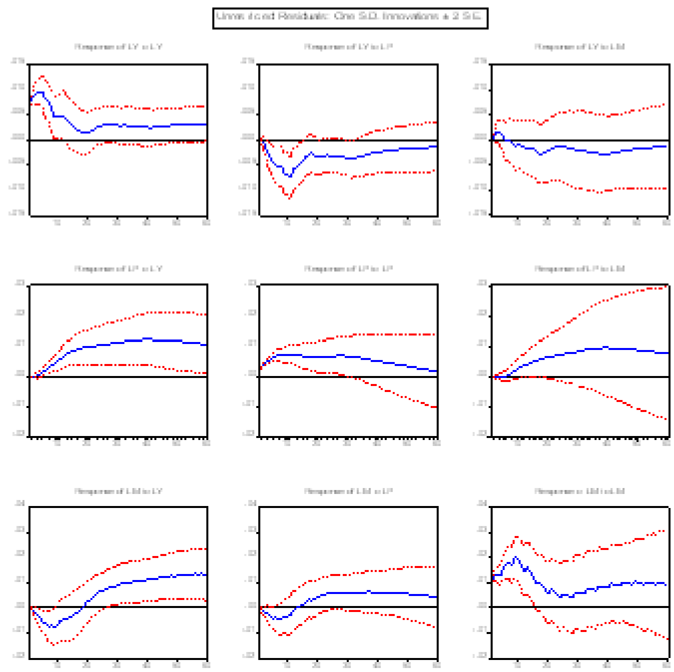
Each row refers to a particular equation (i.e.the response of a variable)

Each column refers to a particular shock

The centre line is the impulse response function and the outer lines are $\pm$ one standard error bands based on the asymptotic distribution.

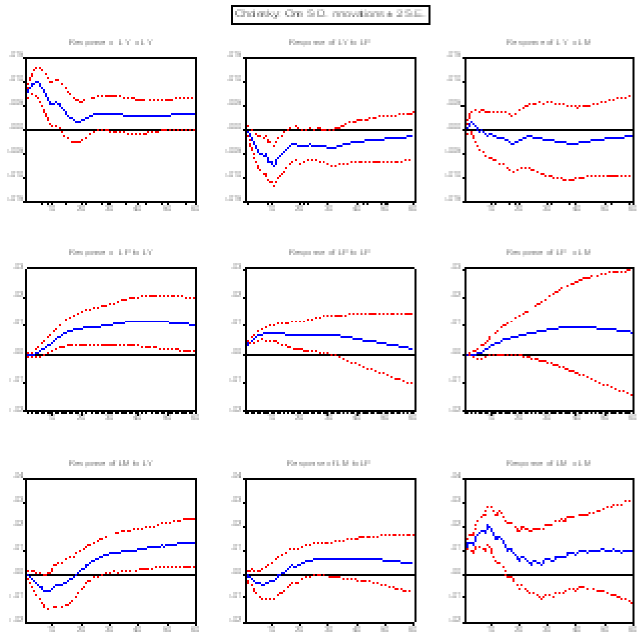
1.Var in levels

Unrestricted residual shocks

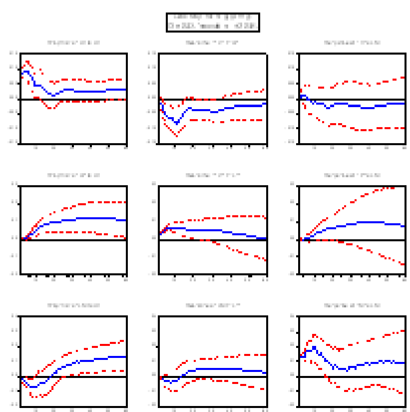


Choleski decomposition

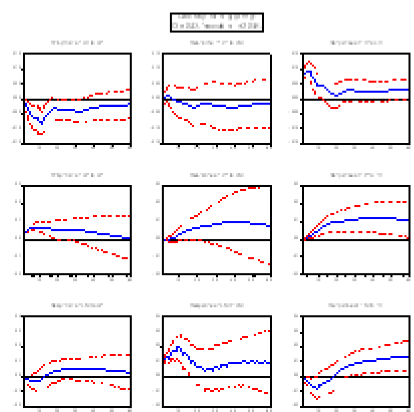
(i) $z_t = (y_t,\, p_t,\, m_t)'$



(ii) $z_t = (m_t,\, p_t,\, y_t)'$



(iii) $z_t = (p_t,\, m_t,\, y_t)'$



It is clear that the ordering has had little effect and the Choleski responses are very similar to the unrestricted responses.

The reason is evident from the correlation matrix of the VAR residuals which is

Residual correlation matrix

	ly	lp	lm
ly	1		
lp	-0.128	1	
lm	-0.002	0.013	1

The correlation between the residuals is very low implying that re-ordering will have little effect on the Choleski transformation matrices.

We also note:

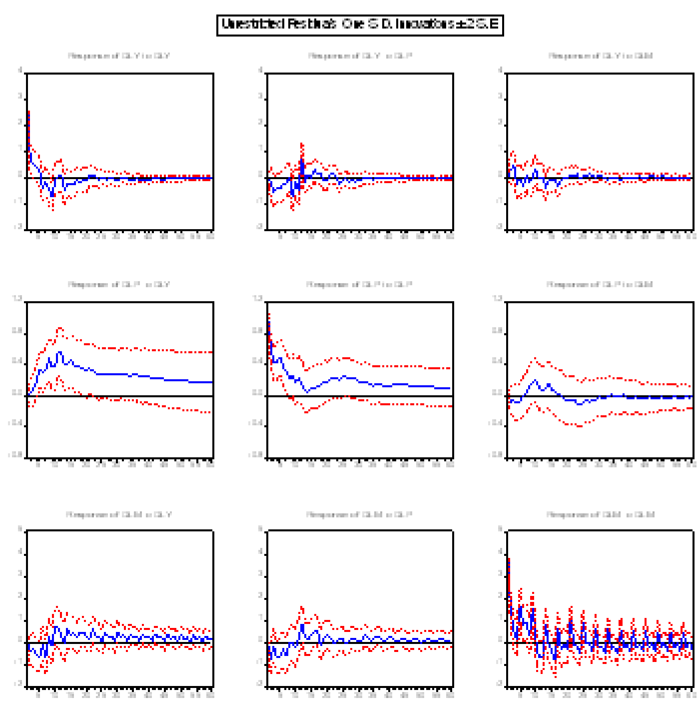
- that the IRF of p to shocks in m and y are persistent

- that the IRF of m to shocks in p and y are persistent

2. VAR in first differences

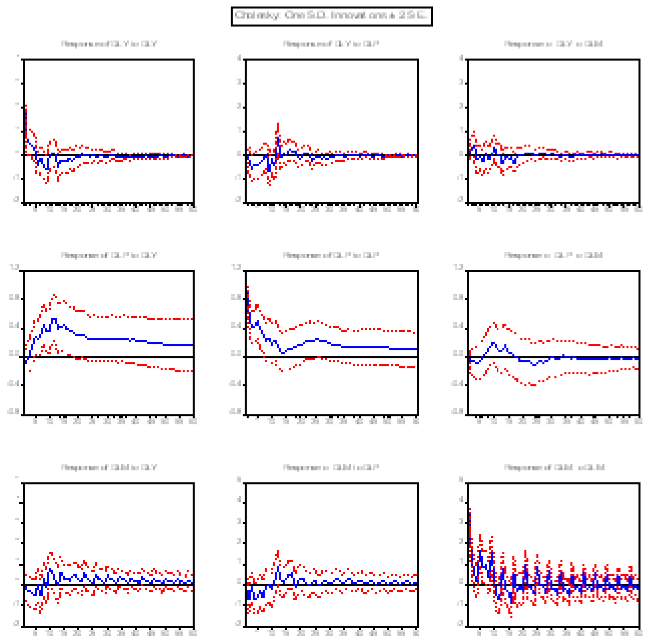
Using a VAR in Δz_t

Unrestricted residuals

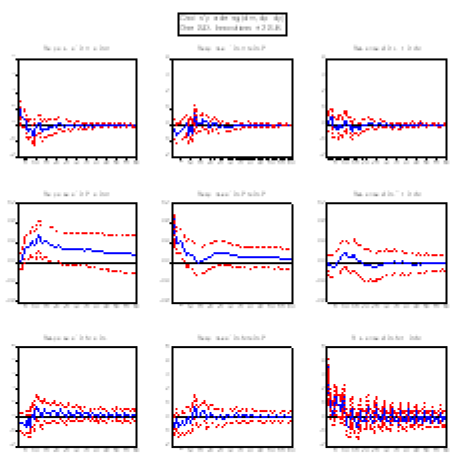


Choleski decomposition

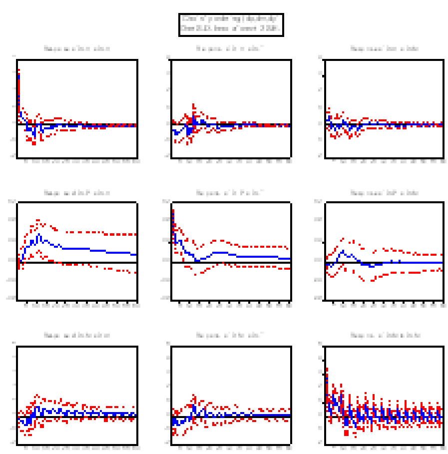
(i) $z_t = (y_t,\, p_t,\, m_t)'$



(ii) $z_t = (m_t,\, p_t,\, y_t)'$



(iii) $z_t = (p_t,\, m_t,\, y_t)'$



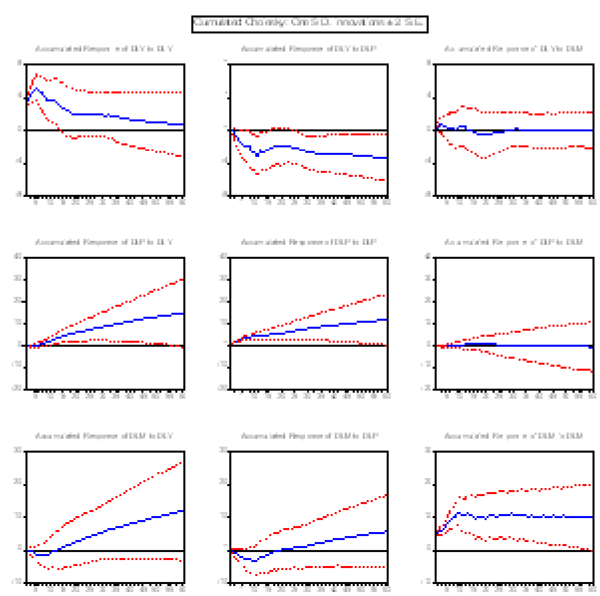
Once again the four sets of impulse response functions are similar. And again the correlation matrix of residuals is close to diagonal:

Residual correlation matrix

	ly	lp	lm
ly	1		
lp	-0.120	1	
lm	0.034	-0.004	1

The IRFs for differences look quite different from those in levels. This is because the shocks are temporary not permanent.

To make the comparison we need to cumulate the shocks for the differenced model. The following are the cumulated impulse response functions for the Choleski decomposition with the ordering $z_t = (y_t,\, p_t,\, m_t)'$. These can be compared to the corresponding levels results. Although the levels shocks do show evidence of permament shocks especially for p and m, cumulating shocks gives much stronger effects.

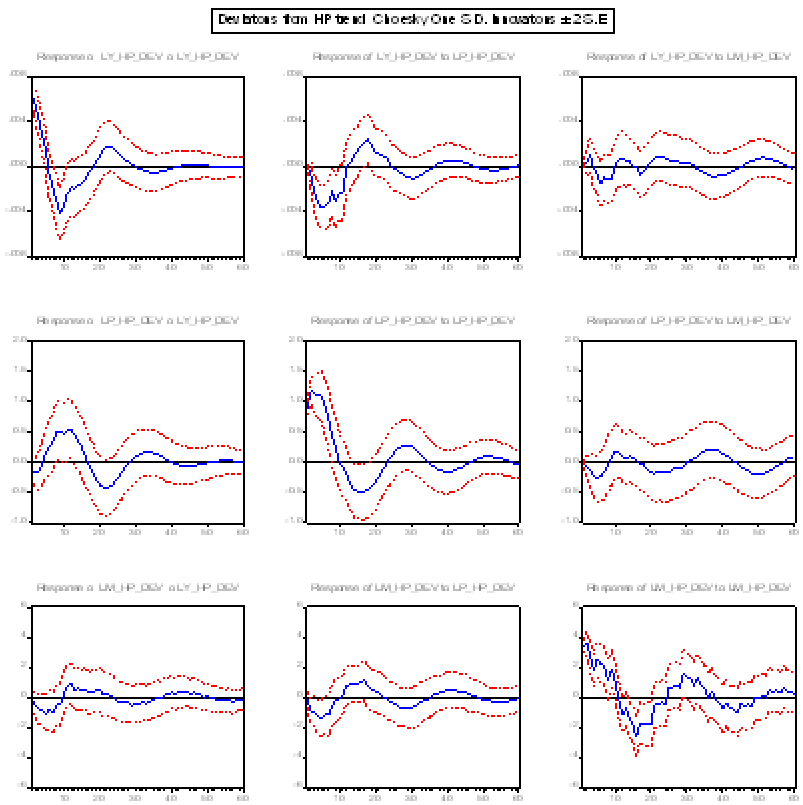


Apart from the reponse of m_t the response functions are similar.

The model gives a reason why the response to the cumulative money shock is different. In the VAR in differences the money shock is permanent whereas in the model the money shock is temporary as it is a money demand shock. This using a VAR in differences may incorrectly impose an extra permanent shock.

3. VAR using the HP filter

Using the order $z_t = (y_t,\, p_t,\, m_t)$ but using the de-trended HP filtered series with $\lambda = 1600$ instead of the original series gives the following IRFs



These IRFs are very different from those of the original variables in levels. In the long run we note that all shocks die out. And in the short run the greatest differences are in the responses of p_t; deviations from the long run are much less persistent for the filtered series. This probably reflects the fact that we are now dealing with variables that have no long-run trend.

4. Using a cointegrated VAR

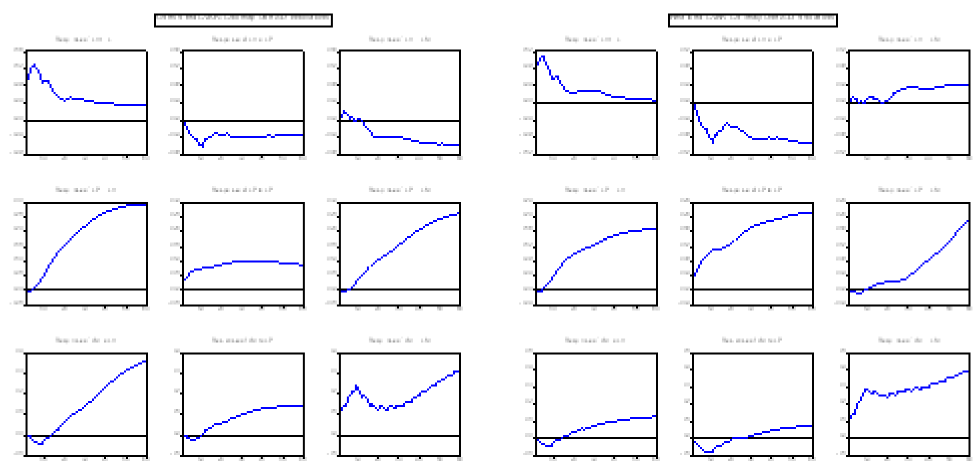
Tests show that for $z_t = (y_t, p_t, m_t)$ there is just one cointegrating vector. We examine IRFs for unrestricted estimates of the cointegrating vectors and the restriction that the cointegrating vector is associated with the money demand function and satisfies

$$w_t = (-1, -1, 1)z_t \sim I(0)$$

In each case, compared with the VAR in first differences implies, we are including an extra variable in each equation, namely w_{t-1}. And compared with the levels VAR we are taking account of the cointegrating restrictions.

For the unrestricted CVAR the cointegrating vector is $(1.00, -6.22, 4.28)$ which makes little economic sense.

The IRFs for the unrestricted and restricted CVs and based on a Choleski decomposition with the ordering as in z_t are



The two sets of results are fairly similar, and are similar to the cumulative IRFs of the first differenced model, but they differ much more from the earlier results for the levels VAR.

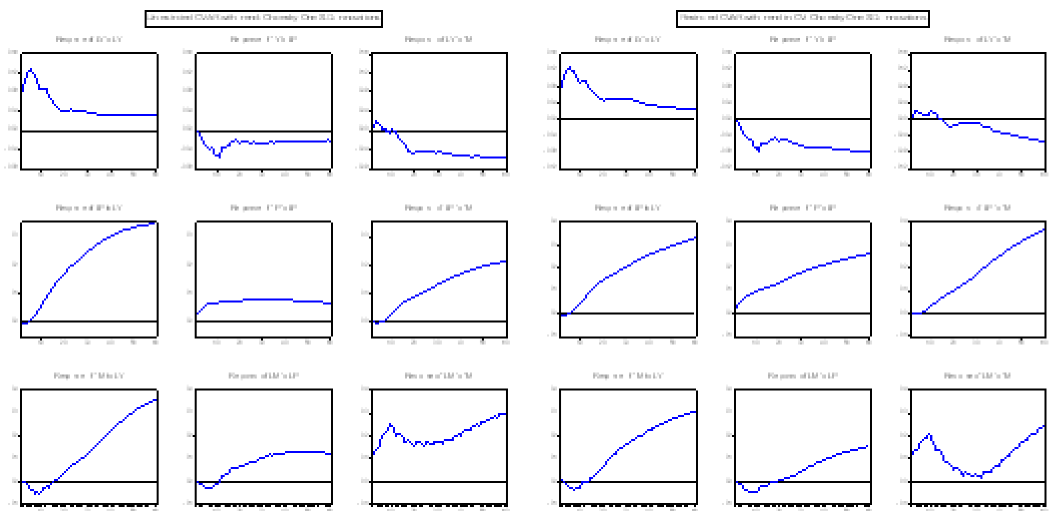
(i) Permanent shocks are now much more in evidence particularly to p_t and m_t.

(ii) The response of p_t shows the greatest discrepancy particularly to its own shock which is now permanent.

(iii) Another difference is the response of y_t to the shock in m_t; it appears now to have a small positive permanent effect whereas previously it was positive at all horizons except the very short run.

There is however clear evidence that the log velocity of circulation has a trend. If allowance is made for this in the CV by including a linear trend (the $t-statistic$ for the trend in the unrestricted CVAR is -2.4 and in the restricted CVAR is -11.6) then the CV for the unrestricted model becomes $(1.00, -1.04, 0.79)$.

The IRFs for the two CVARs are



The main effect of this amendment is to to the restricted model. It alters the long-run effect on y_t of the m_t shock which switches to being negative as in unrestricted model - and in the levels results.

Conclusions so far

1. The CVAR results seem to be the most plausible. They are fairly similar to the *cumulated* response functions of the differenced data, but are much more different from the levels results - especially for p_t and m_t.

2. The levels results fail to pick up the permanent shocks adequately.

3. Neither the the ordering of the variables nor the choice of identification scheme in deriving the shocks from the residuals has a large effect, but this probably reflects the lack of correlation of the VAR residuals.

4. The HP filter produces very different results. In particular, there are no permanent effects. This reflects the removal of all trends in the data. The implication is that using de-trended data provides very different information. It shows the path back to trend or equilibrium following a shock, but does not reveal whether or not the shock has a permanent effect.

4. Four variable VAR monetary model (y_t, p_t, m_t, i_t)

1. VAR in levels

This is the model that has attracted the most attention in the literature

We shall now focus more on the economics.

In general the argument for including additional variables is that they are involved in the transmission mechanism of monetary shocks to GDP

However, for the sole purpose of obtaining a VAR model, it is not necessary. To see this consider the VAR in $z_t' = (z_{1t}', z_{2t}')$. For convenience we write it as

$$\begin{aligned} z_t &= A(L)z_{t-1} + e_t \\ \begin{bmatrix} z_{1t} \\ z_{2t} \end{bmatrix} &= \begin{bmatrix} A_{11}(L) & A_{12}(L) \\ A_{21}(L) & A_{22}(L) \end{bmatrix} \begin{bmatrix} z_{1,t-1} \\ z_{2,t-1} \end{bmatrix} + \begin{bmatrix} e_{1t} \\ e_{2t} \end{bmatrix} \end{aligned}$$

and the corresponding VMA is

$$\begin{aligned} z_t &= C(L)e_t \\ \begin{bmatrix} z_{1t} \\ z_{2t} \end{bmatrix} &= \begin{bmatrix} C_{11}(L) & C_{12}(L) \\ C_{21}(L) & C_{22}(L) \end{bmatrix} \begin{bmatrix} e_{1t} \\ e_{2t} \end{bmatrix} \end{aligned}$$

Hence, we can replace z_{2t} in the VAR equation for z_{1t} to get

$$\begin{aligned} z_{1t} &= A_{11}(L)z_{1,t-1} + [I + A_{12}(L)C_{21}(L)L]e_{1t} + A_{12}(L)C_{22}(L)L]e_{2t} \\ &= A_{11}(L)z_{1,t-1} + B(L)u_t \end{aligned}$$

This can be inverted as the VAR

$$\begin{aligned} B(L)^{-1}[I - A_{11}(L)L]z_{1t} &= u_t \\ D(L)z_{1t} &= u_t \end{aligned}$$

Thus

1. omitting variables from a VAR does not imply that we cannot represent the remaining sub-set of variables as a VAR, or capture the transmission mechanism.

2. However, the order of the lag needed for $D(L)$ will be greater than for $A(L)$.

3. But most important, the shocks in the smaller VAR are linear combinations of the larger VAR and will therefore not be the same shocks as in the larger VAR. This could lead to incorrect economic conclusions.

Order of the variables

Most studies use a Choleski decomposition, but there is a lot of disagreement over the order of the variables when it is introduced. To see whether it is likely to matter we note that the residuals of a VAR in levels has the correlation matrix where i_t is the Federal Funds rate.

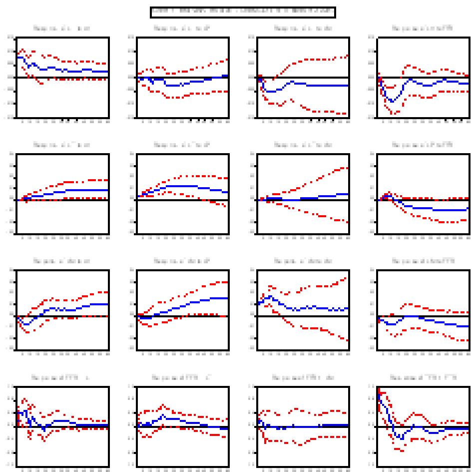
	y	p	m	i
y	1			
p	-0.113	1		
m	0.027	0.102	1	
i	0.332	0.204	-0.031	1

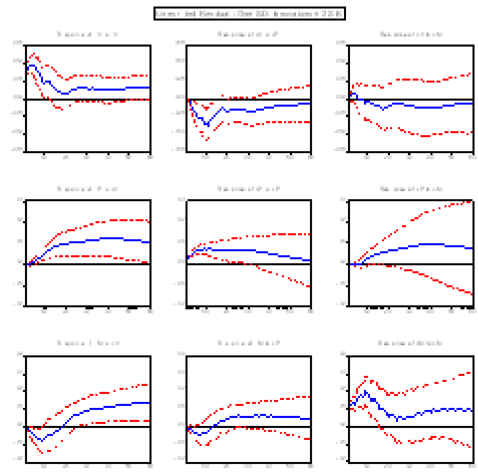
Thus the correlations between the residuals of the equation for i_t with those from the equations for y_t and p_t are higher than previous correlations. This suggests that the order of the variables, and especially of i_t, may be of greater importance.

VAR in levels

1. Unrestricted residual IRFs

The IRFs for the unrestricted residuals which are unaffected by the ordering are given below together with the corresponding IRFs for the three-variable model for comparison.

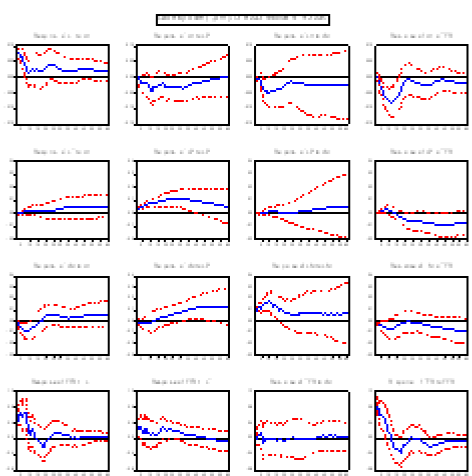


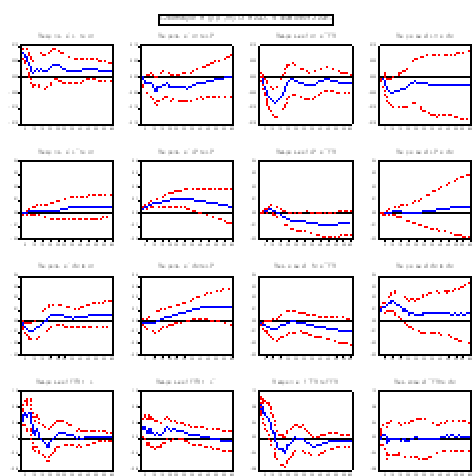


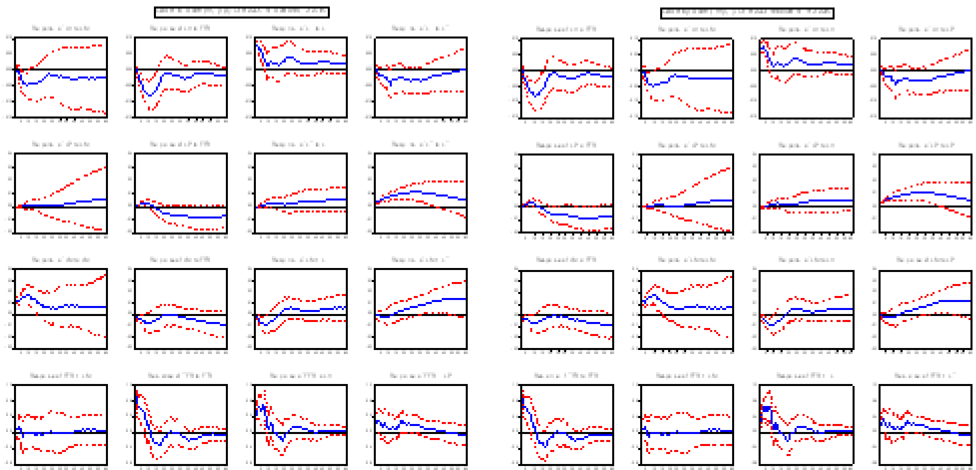
The top 3×3 block corresponds to those for the three-variable VAR. These IRFs seem little changed even though lags of i_t are significant in each equation.

2. Choleski IRFs

We compare the following orderings







The results are again very similar to each other suggesting that the order is not very important.

Leeper, Sims and Zha (1997) estimate a levels VAR in the order $z_t = (p_t, y_t, i_t, m_t)'$

One of their findings is that the interest rate falls initially, following a positive money shock rather than rise, and then rise later. The rise is called the "LIQUIDITY PUZZLE".

Liquidity Puzzle

- The liquidity *effect* is that an increase in the money supply shifts the money supply function to the right and hence results in a new money market equilibrium in which the rate of interest is lower.

- the liquidity *puzzle* is that interest rates increase, not fall.

An alternative argument one could put is that an increase in the rate of growth of the money supply will fuel inflationary expectations and result in an increase in nominal interest rates via the Fisher equation.

Thus it might be important to distinguish between an increase in the level of the money supply and an increase in its rate of growth.

We could also note that if the money shock were a money demand shock then the expected response of interest rates is a rise.

So it is also important to distinguish between a money demand and a money supply shock

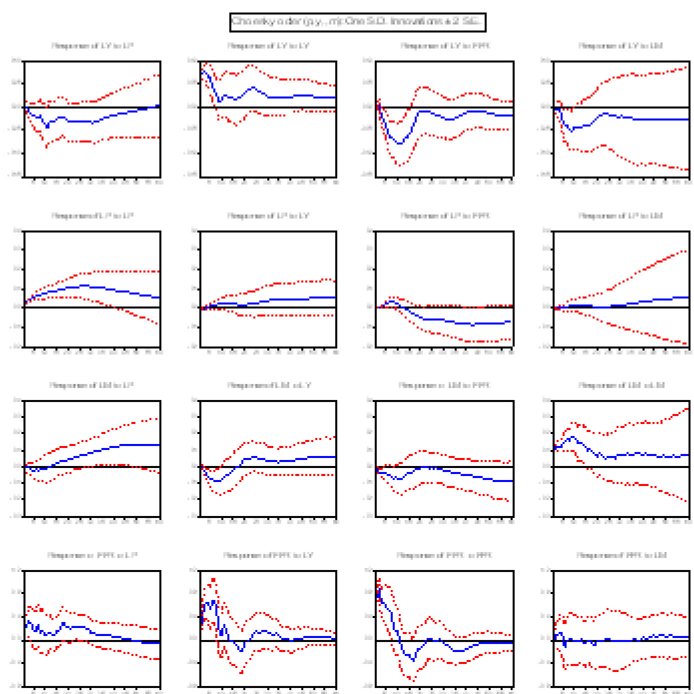
Leeper, Sims and Zha find that the price level increases substantially following a positive interest rate shock. This is called the "PRICE PUZZLE".

Price Puzzle

- prices increase after a rise in interest rates

- One might expect that a rise in interest rates would reduce aggregate demand and hence reduce inflationary pressures

For the data we have been using the results are



The liquidity effect is given in the last graph (the graph at position $4,4$). The price effect is given by the $(2,3)$ graph. We observe both a liquidity puzzle and a price puzzle in the short run, but neither is very strong. Similar results are also present in the previous impulse response functions.

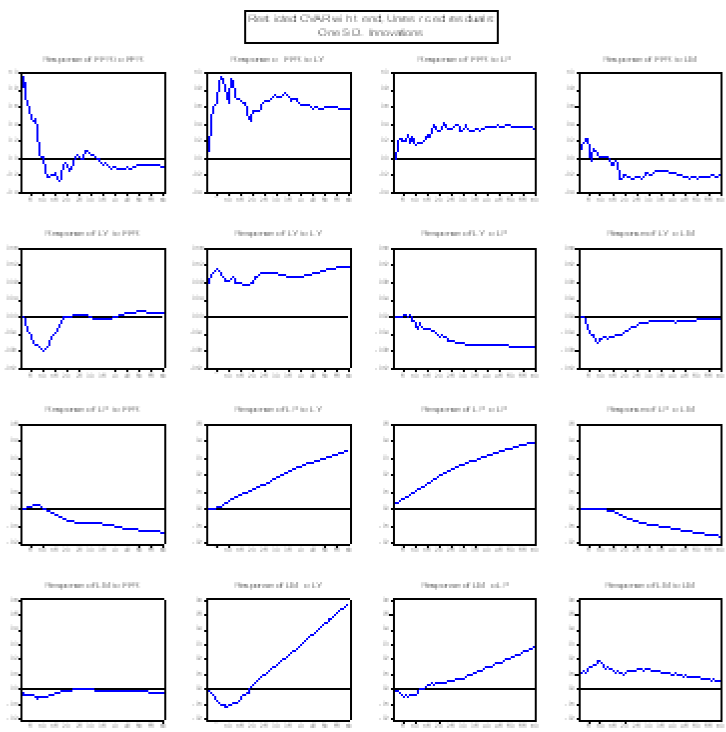
2. Using a CVAR with long-run restrictions

As the Federal Funds rate is borderline I(1) we assume for the present that is I(1).

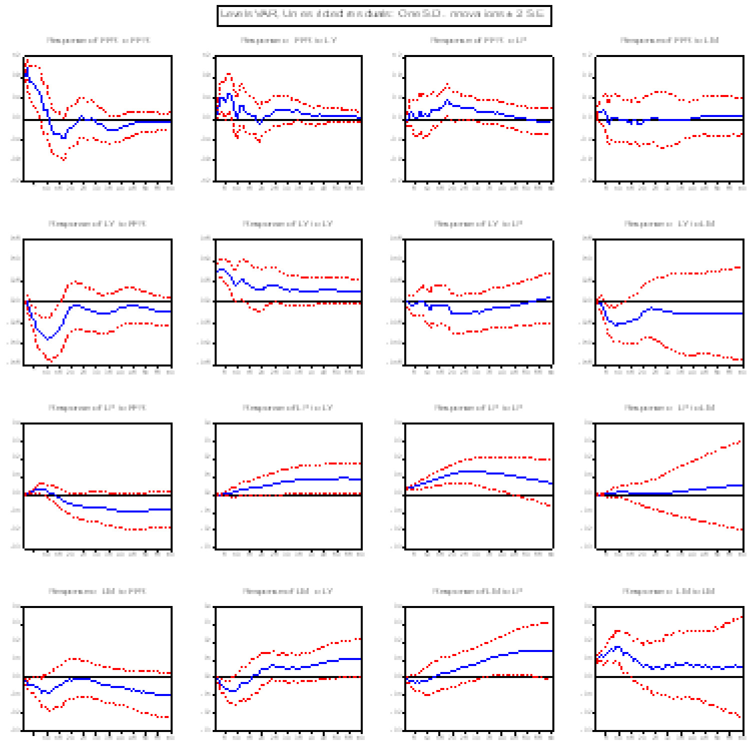
There is a single cointegrating vector which we assume is the long-run demand for money. We also assume that this is given by relating the log velocity to the interest rate and a trend. We therefore estimate a restricted CVAR with a trend in the CV.

We estimate the CVAR with the variables in the order $z_t = (i_t, y_t, p_t, m_t)'$

The IRFs for the unrestricted residuals are

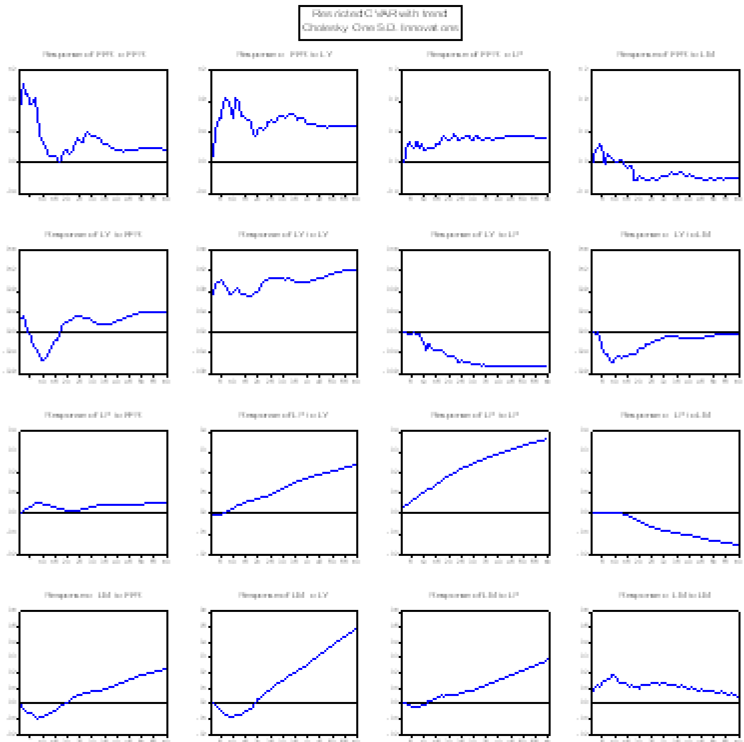


For comparison we give the corresponding levels VAR in these variables



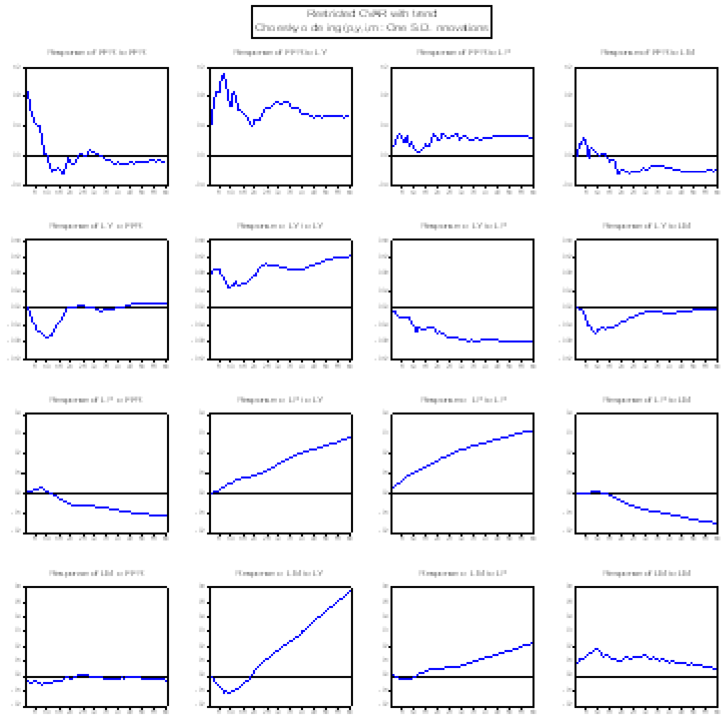
The are a number of differences in the IRFs. In the restricted CVAR, output, price and money shocks all seem to have a permanent effect on i_t, but not in the levels VAR.

Using a Choleski decomposition with the same ordering we obtain



Compared with the unrestricted residuals the main differences are to the i_t IRFs.

Using the ordering of Leeper, Sims and Zha, i.e. $z_t = (p_t, y_t, i_t, m_t)'$ we obtain



The results show little difference.

In the short run all of these results display both a liquidity puzzle and a price puzzle.

3. The Wickens and Motto approach

The proposal of Wickens and Motto (2001) is to include two additional pieces of information into the CVAR analysis

(i) knowledge of the long-run structure between the variables

(ii) knowledge of which variables are endogenous and which are exogenous.

They propose the following stylised model1960.1-1998.4

$$i_t = \rho + \Delta p_t + \xi_t$$

$$m_t - p_t = \upsilon + y_t - \lambda i_t + e_t$$

$$\Delta y_t = \gamma + \alpha \Delta y_{t-1} + \varepsilon_{yt}$$

$$\Delta m_t = \mu + \theta \Delta m_{t-1} + \varepsilon_{mt}$$

where

i=federal funds rate - I(0)

p=log GDP price deflator - I(1)

y=log GDP - I(1)

m=log M1 - I(1)

Long-run structure: $m_t - p_t - y_t$ is I(0)

Thus

e_t = money demand shock

ε_t = shock to the rate of growth of the money supply

ε_{yt} = output or productivity shock

ξ_t = interest rate shock

The model can be written in matrix form as

$$\begin{bmatrix} 1 & -1 & 0 & 0 \\ -\lambda & 1 & 1 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \Delta z_t^* = \begin{bmatrix} \rho \\ -\upsilon \\ \gamma \\ \mu \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ -\lambda & 1 & 1 & -1 \end{bmatrix} z_{t-1}^*$$

$$+ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & \mu \end{bmatrix} \Delta z_{t-1}^* + v_t$$

$z_t^{*\prime} = \{i_t \; p_t \; y_t \; m_t\}$ and $v_t' = \{\xi_t \; -e_t \; \varepsilon_{yt} \; \varepsilon_{mt}\}$

It is clear from the matrix of coefficients for Δz_t^* in the CVAR that were the model to be written as a levels VAR it would not have a recursive structure. This is due to the presence of $-\lambda$ in the first column of the second row. To be recursive there would need to be a one period delay in the impact of interest rates on money demand - an implausible restriction.

- Only one coefficient, λ, requires prior estimation
- A standard Johansen test does not reject this long-run structure.

Equations for the endogenous variables

- $$\begin{bmatrix} \Delta i_t \\ \Delta p_t \end{bmatrix} = B^*(0)^{-1} \begin{bmatrix} \rho \\ \upsilon \end{bmatrix} - B^*(0)^{-1} \begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \Delta y_t \\ \Delta m_t \end{bmatrix}$$
$$+B^*(0)^{-1} w_{t-1}^{\#} + B^*(0)^{-1} \begin{bmatrix} \xi_t \\ -e_t \end{bmatrix}$$

where

$$B^*(0) = \begin{bmatrix} 1 & -1 \\ -\lambda & 1 \end{bmatrix}$$

$$w_t^{\#} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -\lambda & 1 & 1 & -1 \end{bmatrix} z_t^*$$

- In the model actually estimated

  - its short-run dynamics were generalized

  - $B^*(0)$ was not restricted

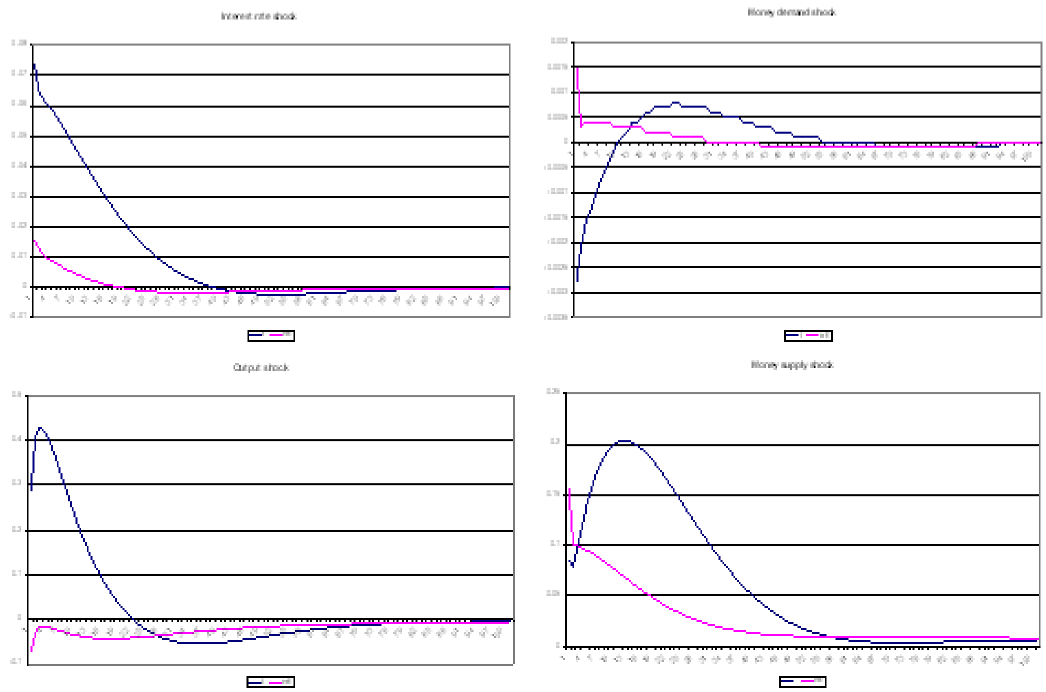
  - two lags in Δz_t^* were added

- An estimate of $\lambda = 8.9$ was obtained from equation (**??**)

Results

CVAR

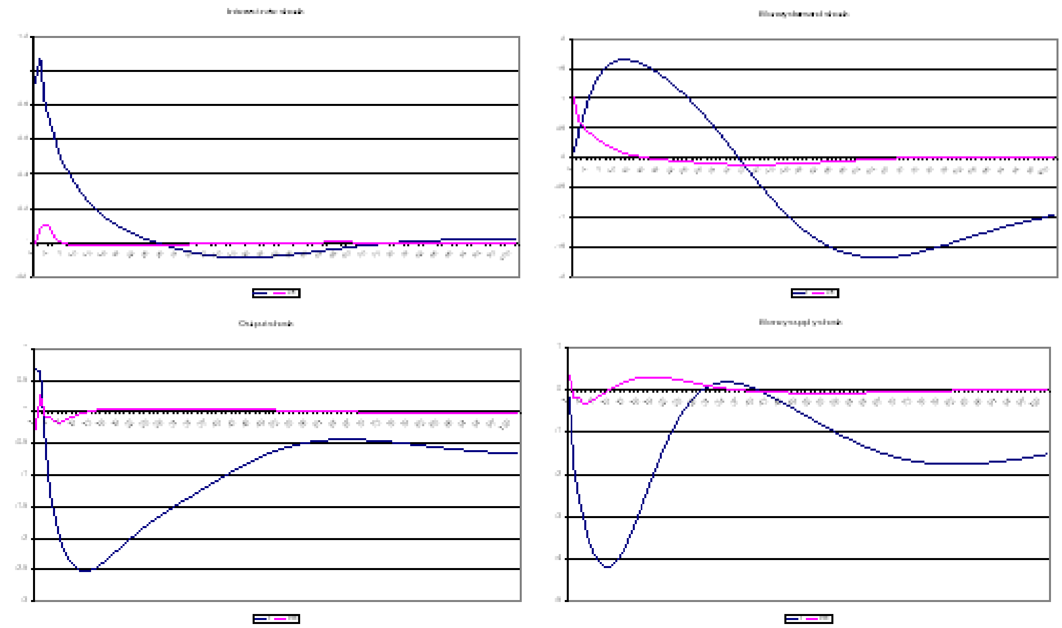
Figures 10-13



In interpreting these results, we note that the measure of the money demand shock reported is the *negative* of the true money demand shock. We find, therefore, that i_t increases both as a result of a positive money demand shock, and a shock to the rate of growth of the money stock. This is what economic theory would lead us to expect.

Levels VAR

Figures 14-17



Summary of findings

- Impulse response functions - computed here as dynamic multpliers

  Figures 10-13 show responses of the federal funds rate and the inflation rate

- shocks may be interpreted as an interest rate, a money demand, an output and a money supply shock, respectively

- each shock has a temporary effect on i_t and Δp_t

- but the output and money supply shocks have a permanent effect on p_t, the price level

- Fig 10 displays the price puzzle (an increase in the FFR causes the price level to rise)

- Money supply shock

- has a positive effect on the interest rate

- sign reflects the increase in the rate of growth of the money stock

- this increases inflation and in turn causes the nominal interest rate to rise

- Money demand shock

  (NB: the Figures report the effects of $-e_t$, a negative demand shock.)

- has a negative effect on the interest rate

- has a positive impact effect on the price level

- Can compare with a VAR in levels that uses a Choleski decomposition

- impulse response functions for the interest rate and inflation are displayed in Figures 14-17

- the shocks are labelled as above to denote the ordering of the variables but do have the same meaning unless the shocks either have the appropriate recursive structure or are orthogonal.

- the first shock (labelled interest rate shock) has the same shaped

 impulse response function as the previous interest rate shock

- but the other three shocks do not look like the remaining structural shocks

Conclusions

The impulse response functions

(i) are very different from those obtained from a standard unrestricted VAR in levels

and using a Choleski decomposition

(ii) do not suffer from the liquidity effect

Alternative approach based on a VAR

As noted in a previous lecture, an alternative approach is to achieve stationary variables by imposing the long-run cointegrating relation and to difference the remaining I(1) variables- recall that we are assuming i_t is I(0). The model can then be estimated as a VAR.

The cointegrating relation is the log velocity

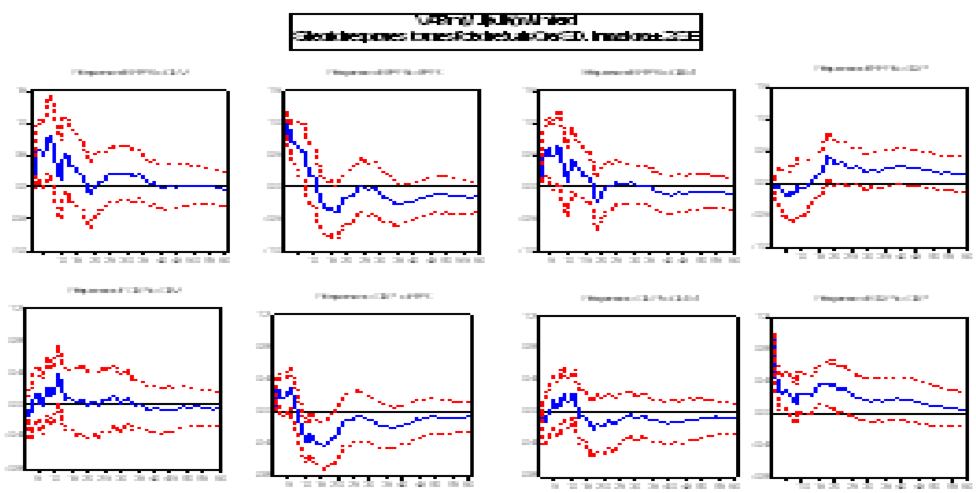
$$v_t = p_t + y_t - m_t$$

We assume that the money demand function has a trend so this is included in the VAR.

We must then include any two of $\Delta p_t, \Delta y_t$ and Δm_t, together with i_t.

Unrestricted residual IRFs for i_t and Δp_t are reported below for the VAR in $(v_t, i_t, \Delta p_t, \Delta m_t)$ that includes a trend.

The shock to v_t is a negative money demand shock and the shock to Δm_t is a money supply shock.



The results show that

(i) all of the shocks are temporary on these variables

(ii) i_t increases due to a negative money demand shock and to a positive money supply shock.

(iii) i_t increases following an inflation shock, but only after a lag

(ii) Δp_t initially increases a little following an interest rate shock, but then falls even more.

This suggests that, from a policy point of view, there is a delay in the impact of interest rates on inflation.

5. An open economy VAR with a floating exchange rate

We consider briefly the effect of taking account of a floating exchange rate in the above VARs. We are particularly interested in the possible role of the exchange rate in the transmission mechanism from interest rates to inflation.

According to the previous discussion of the open economy, the VAR will depend on whether the money supply or the interest rate is the policy instrument. If it is the latter then money is simply an output of the process and money supply shocks are irrelevant - money demand shocks will still be relevant as they can affect monetary policy.

The following is the rational expectations monetary model of the exchange rate with sticky prices. It is related to the Dornbusch model of the exchange rate.

$$\begin{aligned}
\Delta p_t &= \alpha\left(s_t + p_t^* - p_{t-1}\right) + \beta E_t[\Delta p_{t+1}] \\
i_t &= i_t^* + E_t[\Delta s_{t+1}] \\
m_t &= p_t + y_t - \lambda i_t
\end{aligned}$$

We have previously concluded that

(i) If m_t is the policy instrument then the solution is obtained from

$$\begin{aligned}
\beta E_t[p_{t+1}] + (1+\beta)p_t - (1-\alpha)p_{t-1} - \alpha s_t &= \alpha p_t^* \\
p_t - \lambda E_t[s_{t+1}] + \lambda s_t &= m_t - y_t + \lambda i_t^*
\end{aligned}$$

Assuming that i_t and i_t^* are I(1), this would suggest a CVAR in $(p_t, s_t, y_t, m_t, p_t^*, i_t^*)$ with three CVs

$$p_t - p_t^* - s_t$$

$$i_t - i_t^*$$

$$m_t - p_t - y_t - \lambda i_t$$

If i_t and i_t^* are I(0) there would be only two CVs

$$p_t - p_t^* - s_t$$

$$m_t - p_t - y_t$$

(ii) If i_t is the policy instrument then the solution is obtained from

$$\begin{aligned} \beta E_t[p_{t+1}] + (1+\beta)p_t - (1-\alpha)p_{t-1} - \alpha s_t &= \alpha p_t^* \\ i_t - E_t[s_{t+1}] + s_t &= i_t^* \end{aligned}$$

This would suggest a CVAR in (p_t, s_t, p_t^*, i_t^*) with two CVs

$$p_t - p_t^* - s_t$$

$$i_t - i_t^*$$

or,if i_t and i_t^* are I(0), just the first CV.

Treating the UK as the small economy of interest and the US as the foreign country and using monthly data from 1975.1 -1997.12 we cannot reject the hypothesis that both interest rates are I(1).

We construct a VAR in 8 variables: $(p_t, s_t, i_t, m_t, y_t, y_t^*, p_t^*, i_t^*)$. We presume that only the first 4 of these are endogenous and the rest are exogenous, although we ignore this distinction.

We assume that there are 3 cointegrating vectors

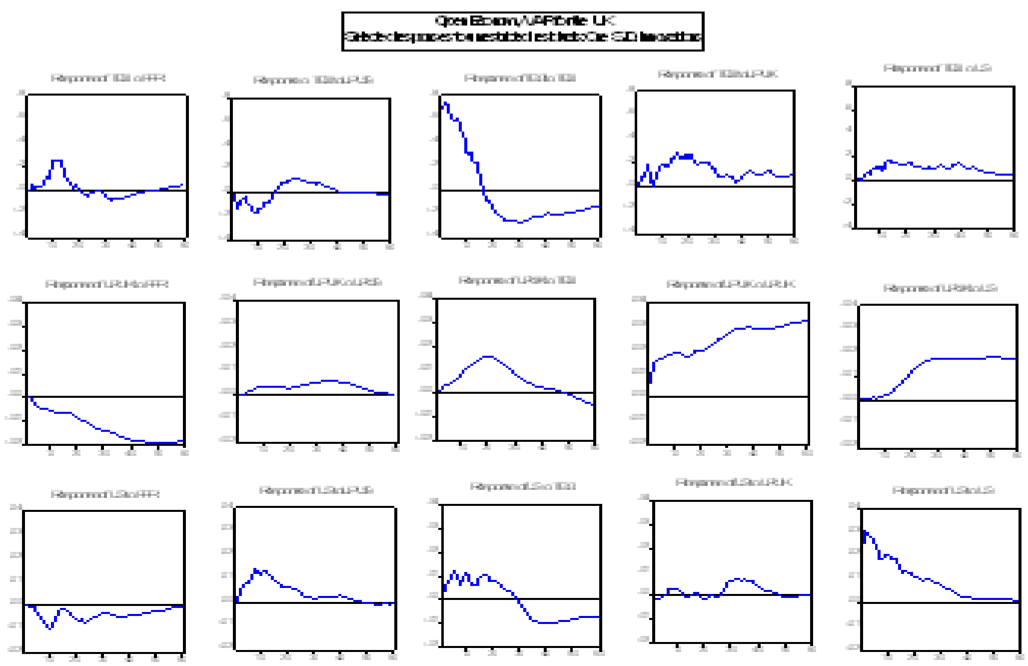
$$p_t - p_t^* - s_t$$

$$i_t - i_t^*$$

$$m_t - p_t - y_t - \lambda i_t$$

In addition, we allow the long-run demand for money function to include a time trend which is significant. Also, we impose all of the restrictions implied by these CVs.

We report just the responses of (i_t, p_t, s_t) to shocks in $(i_t^*, p_t^*, i_t, p_t, s_t)$.



Imposing a Choleski decomposition produces almost identical results.

An interesting feature of these results is that a shock to the US price level initially affects the exchange rate and not UK prices. Later UK prices respond.

An increase in the UK interest rate also initially affects the exchange rate more than the price level.

And the exchange rate seems to respond more strongly to changes in the US than the UK price level , and to the UK than the US interest rate .

6. Monetary Policy Analysis with a VAR

We consider three issues:

1. What are the implications for the identification a VAR arising from a monetary policy context?

2. Are the sort of models commonly formulated for monetary policy analysis capable of being represented as a VAR?

3. Can a VAR be used to analyse either old or new monetary policies?

Order of variables in a VAR

Christiano, Eichenbaum and Evans (1999) discuss the implications of a policy context for the order of variables appropriate to a recursive structure in a VAR.

They argue that

(i) the variables can be classified into three types so that $z_t' = (y_t', u_t, x_t')$

- y_t are the target of policy variables whose response we are interested in

- u_t is the policy instrument

- x_t are other variables that affect the transmission mechanism from u_t to y_t

(ii) The VAR takes the block recursive form (y shocks are correlated among themselves, but not with u or x shocks)

$$\begin{aligned} A(L)z_t &= e_t, \quad E(e_t e_t') = \Lambda, \text{ a diagonal matrix} \\ A_0 &= \begin{bmatrix} A_{11} & 0 & 0 \\ A_{21} & A_{22} & 0 \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \end{aligned}$$

(iii) u_t is the monetary policy variable (money supply, interest rate etc) and monetary policy and it is determined by

$$u_t = -A_{21}y_t + B(L)z_{t-1} + e_{ut}$$

where e_{ut} is the monetary policy shock.

(iv) As $A_{12} = 0$ and $A_{13} = 0$, y_t is contemporaneously unaffected by e_{ut} and e_{xt}.

(v) The x_t are variables not seen by the monetary authority when setting policy hence $A_{23} = 0$. But the y_t variables are seen when setting policy, hence $A_{21} \neq 0$.

(vi) The x_t variables have no current effect on y_t but they are part of the longer term transmission mechanism of monetary policy shocks to y_t.

To achieve identification, if there is more than one variable in each block, we still need restrictions on A_{11}, A_{22} and A_{33}.

For the four variables considered previously (y_t, p_t, m_t, i_t) it might be reasonable to assume that

(i) all variables affect y_t with a lag

(ii) That y_t is not known when setting monetary policy, but p_t is known.

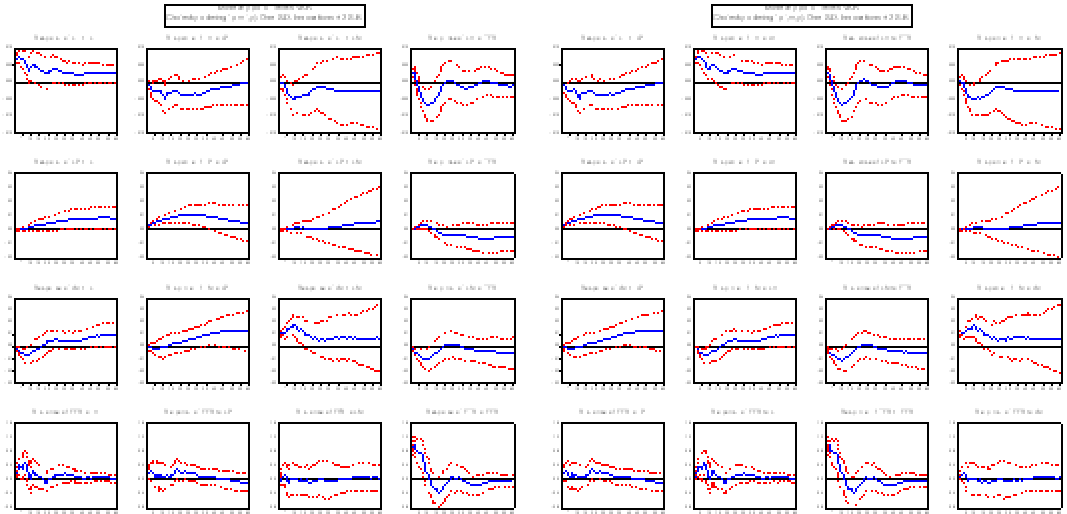
(iii) If m_t is the policy variable then i_t is known to the policy maker, and $vice - versa$.

Possible orderings for the two monetary policy instruments m_t and i_t are therefore

$$\begin{aligned} m_t \quad &: \quad z_t' = (p_t, i_t, m_t, y_t) \\ i_t \quad &: \quad z_t' = (p_t, m_t, i_t, y_t) \end{aligned}$$

The impulse response functions that are obtained (Figs 18 and 19) are virtually identical:

Monetary policy instrument are m and i



There is clearly very little difference.

Christiano , Eichenbaum and Evans consider a VAR with 7 variables: GDP, GDP price deflator, commodity prices, federal funds rate, total reserves, non-borrowed reserves and either M1 or M2. See their Figure 2 p86.

Is the identification scheme of Christiano, Eichenbaum and Evans useful?

Is this ordering necessarily appropriate?

Try a new model

Consider the following reasonably plausible model

$$
\begin{aligned}
\Delta m_t &= \upsilon + p_t + y_t - \lambda i_t - \gamma m_{t-1} + e_{mt} \\
\Delta p_t - \pi &= \theta(y_t - y_t^*) + \varphi(\Delta p_{t-1} - \pi) + e_{pt} \\
y_t - y_t^* &= \rho - \phi(i_{t-1} - \Delta p_{t-1}) + e_{yt} \\
\Delta y_t^* &= \mu + \varepsilon_t \\
i_t &= \rho + \pi + \alpha(\Delta p_t - \pi) + \beta(y_t - y_t^*) + \xi_{it} \\
\Delta m_t &= \pi + \xi_{mt}
\end{aligned}
$$

where we assume that the disturbances are mutually uncorrelated.

y_t^* is capacity output, and is a random walk with drift.

The last two equations are alternative representations of the monetary policy instrument. Only one of these equations is included in the model.

If monetary policy sets the rate of growth of the money supply then we use the last equation.

If monetary policy instrument is the interest rate then the penultimate equation is used instead and is a Taylor rule. If $\alpha = \beta = 0$, then interest rate policy is discretionary and doesn't follow a rule.

If the money supply is the policy instrument then the interest rate is determined by the money demand equation (i.e. in the money market).

The model has the long-run solution

$$
\begin{aligned}
m_t &= \upsilon + p_t + y_t - \lambda i_t \\
\Delta p_t &= \pi \\
y_t &= y_t^* \\
y_t^* &= y_0^* + \mu t + \textstyle\sum_{s=1}^{t} \varepsilon_s \\
i_t &= r + \pi \\
r &= \frac{\rho}{\phi}
\end{aligned}
$$

y_t^* can be eliminated from the model to give

$$
\begin{aligned}
\Delta m_t &= \upsilon + p_t + y_t - \lambda i_t - \gamma m_{t-1} + e_{mt} \\
\Delta^2 p_t &= -\theta\mu + \theta\Delta y_t + \varphi\Delta^2 p_{t-1} + \Delta e_{pt} - \theta\varepsilon_t \\
\Delta y_t &= \mu - \phi(\Delta i_{t-1} - \Delta^2 p_{t-1}) + \Delta e_{yt} + \varepsilon_t \\
\Delta i_t &= -\beta\mu + \alpha\Delta^2 p_t + \beta\Delta y_t + \Delta\xi_{it} - \beta\varepsilon_t \\
\Delta m_t &= \pi + \xi_{mt}
\end{aligned}
$$

Noting that if a_t and b_t are independent *i.i.d.* processes then we can write

$$\Delta a_t + \theta b_t = u_t - \lambda u_{t-t}$$

where u_t is another *i.i.d.* process. The model written in levels has the form

$$A_0 z_t = A^*(L) z_{t-1} + B_0 e_t + B_1 e_{t-1}$$

To produce a standard recursive VAR we require that A_0 and B_0 are (lower) triangular.

(i) If money is the monetary policy instrument then, using the CEE ordering of variables we have

$$
\begin{aligned}
z_t &= (y_t, m_t, p_t, i_t) \\
A_0 &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -\theta & 1 & 0 \\ -1 & 1 & -1 & \lambda \end{bmatrix} \\
B_0 &= I
\end{aligned}
$$

Thus A_0 has a recursive structure and we obtain a recursive VAR.

We can see that if there were no delay in the effect of the real interest rate on output then the VAR would not be recursive.

(ii) if the interest rate is the policy instrument then the VAR satisfies

$$
\begin{aligned}
z_t &= (y_t, i_t, p_t, m_t) \\
A_0 &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ -\beta & 1 & -a & 0 \\ 0 & -\theta & 1 & 0 \\ -1 & \lambda & -1 & 1 \end{bmatrix} \\
B_0 &= I
\end{aligned}
$$

Thus A_0 is not triangular due to the feedback from current output onto the interest rate. If $\alpha = 0$, then A_0 would then be recursive.

Estimates of the impulse response functions for these two orderings are again virtually identical.

Using a VAR to analyse monetary policy

It is common to find that after estimation, a VAR is used to study the effects of policy.

When is this permissable?

In the model above monetary policy would be represented by the shocks ξ_{mt} or ξ_{it}.

Although we estimate ξ_{mt}, we do not obtain an estimate of ξ_{it}, but only of a combination of ξ_{it} and ε_t.

(i) money is the instrument

In the first case we could simulate the effects of monetary policy by estimating the IRFs for the ξ_{mt} shock. This would imply that the shock is temporary.

This would be equivalent to computing the dynamic multiplier for m_t in a model formed from the output, price and money demand equations.

If the monetary authority decided to raise the rate of money growth then the shock would be permanent and we would need to examine the effect of changing the level of π.

This could be achieved by looking at the cumulative response function to ξ_{mt}.

(ii) the interest rate is the instrument

We would now like to do the same for the interest rate shock ξ_{it}, but we do not have an estimate of it and therefore cannot simulate the policy on the basis of this model. All we have instead is an estimate of $\Delta\xi_{it} - \beta\varepsilon_t$.

(iii) Simulating a change of policy

Suppose that we estimate a VAR on the basis of money being the policy instrument, but would like to know what would happen if we switched to using the interest rate as the instrument.

Clearly we do not have an estimate of the VAR based on the new policy regime and looking at the IRFs for e_{mt} would not be appropriate.

In order to carry out this analysis we would need to have an estimate of the original structural model. We could then append any equation we wanted, or look at the appropriate dynamic multipliers.

This ignores what many see as a major flaw: the Lucas critique.

According to the Lucas critique policy analysis of this sort can only be carried out if have a model that involves the deep structural parameters. And even the structural model above fails this test. A VAR is even more vulnerable.

The reason is that a policy change involves altering the deep structural parameters. The parameters of the SEM and the VAR are functions of the deep structural parameters and would therefore change following the policy switch.

This analysis is particularly relevant for rational expectations models, such as dynamic stochastic general equilibrium (DSGE) macroeconomic models.

The Euler equations and the technological relations (eg the production function) contain the deep structural parameters and the SEM is a backward-looking projection of the DSGE model.

We now propose a method for analysing a change in policy using a VAR that is not subject to the Lucas critique.

Formulating (monetary) policy from a VAR

We consider a VAR in $\{\Delta p_t, x_t, i_t\}$. Or, if there are other variables w_t, that may be independent sources of shocks or part of the transmission mechanism from interest rates to inflation and output, a VAR in $\{\Delta p_t, x_t, w_t, i_t\}$. If we define $z_t' = (\Delta p_t, x_t, w_t'; i_t)$, we write the VAR as

$$z_t = A(L)z_{t-1} + e_t$$

There is a major problem with using a VAR like this to analyse a change in policy. If the equation in the VAR for the policy instrument is changed to reflect the new rule, in general this will affect all of the other VAR equations, and in a way that is unknown. Thus it would not be correct simply to change the VAR equation of the policy instrument and use the resulting model to analyse the effects of the policy change on inflation and output. So what would be the correct way to proceed?

We partition z_t as $z_t' = (z_{1t}', z_{2t}')$ where z_{1t} are the response variables (they include $\Delta p_t, x_t$) and z_{2t} are the policy instruments, and we partition e_t conformably as $e_t' = (e_{1t}', e_{2t}')$. We can now write the VAR in partitioned form as

$$\begin{bmatrix} z_{1t} \\ z_{2t} \end{bmatrix} = \begin{bmatrix} A_{11}(L) & A_{12}(L) \\ A_{21}(L) & A_{22}(L) \end{bmatrix} \begin{bmatrix} z_{1,t-1} \\ z_{2,t-1} \end{bmatrix} + \begin{bmatrix} e_{1t} \\ e_{2t} \end{bmatrix}$$

The problem can now be reformulated as being due to e_{1t} and e_{2t} being contemporaneously correlated. This implies that if the VAR equations for the policy instruments are changed then the correlation structure of the VAR errors will change too. If the errors were uncorrelated there would be no problem. We therefore seek a way of replacing the equation for the policy instrument so that the correlation structure of the VAR errors is unaffected. This can be accomplished if we transform the VAR equations for z_{1t} into a VAR that is conditional on the current value of the policy instrument.

To do this we define the linear function

$$e_{1t} = \varepsilon_t + G e_{2t}$$

where $E(\varepsilon_t e_{2t}) = 0$, i.e. ε_t is the component of e_{1t} that is uncorrelated with e_{2t}. As result

$$e_t = \begin{bmatrix} e_{1t} \\ e_{2t} \end{bmatrix} = \begin{bmatrix} I & G \\ 0 & I \end{bmatrix} \begin{bmatrix} \varepsilon_t \\ e_{2t} \end{bmatrix}$$

In other words, we are applying a block Choleski decomposition to the the original VAR residuals.

We may derive G from Σ, the covariance matrix of the VAR errors as follows. G is defined such that

$$\begin{aligned} \Sigma &= E[e_t e_t'] \\ &= \begin{bmatrix} I & G \\ 0 & I \end{bmatrix} \begin{bmatrix} \Sigma_{\varepsilon\varepsilon} & 0 \\ 0 & \Sigma_{22} \end{bmatrix} \begin{bmatrix} I & 0 \\ G' & I \end{bmatrix} \\ &= \begin{bmatrix} \Sigma_{\varepsilon\varepsilon} + G\Sigma_{22}G' & G\Sigma_{22} \\ \Sigma_{22}G' & \Sigma_{22} \end{bmatrix} \end{aligned}$$

where $E[\varepsilon_t \varepsilon_t'] = \Sigma_{\varepsilon\varepsilon}$. Hence,

$$G = \Sigma_{12}\Sigma_{22}^{-1}$$

Thus G can easily be estimated from the covariance matrix of VAR residuals.

Denoting

$$H = \begin{bmatrix} I & G \\ 0 & I \end{bmatrix}$$

we pre-multiply the VAR by

$$H^{-1} = \begin{bmatrix} I & -G \\ 0 & I \end{bmatrix}$$

with the result that the disturbances associated with z_{1t} are uncorrelated with those of z_{2t}

$$H^{-1} z_t = H^{-1} A(L) z_{t-1} + H^{-1} e_t$$

In partitioned form this is

$$
\begin{aligned}
z_{1t} &= [A_{11}(L) - GA_{21}(L)]z_{1,t-1} + Gz_{2t} + [A_{12}(L) - GA_{22}(L)]z_{2,t-1} + \varepsilon_t \\
z_{2t} &= A_{21}(L)z_{1,t-1} + A_{22}(L)z_{2t} + e_{2t}
\end{aligned}
$$

The fact that z_{2t} appears in this z_{1t} equation is a reflection of the fact that the correlation between e_{1t} and e_{2t} implies that z_{2t} affects z_{1t} contemporaneously. The new equation for z_{1t} can be described as a conditional VAR as it is a VAR in which z_{2t} is exogenous.

Only at this stage do we replace the equation for z_{2t} by the new policy rule. Suppose this takes the general form

$$Fz_{1t} + z_{2t} = A_{21}^*(L)z_{1,t-1} + A_{22}^*(L)z_{2t} + e_{2t}^*$$

A Taylor rule, for example, is non-stochastic and has no lagged dynamics, and so $A_{21}^*(L)$, $A_{22}^*(L)$ and e_{2t}^* would all be zero.

The new complete model can now be written in structural form as

$$\begin{bmatrix} I & -G \\ F & I \end{bmatrix} \begin{bmatrix} z_{1t} \\ z_{2t} \end{bmatrix} = \begin{bmatrix} A_{11}(L) - GA_{21}(L) & A_{12}(L) - GA_{22}(L) \\ A_{21}^*(L) & A_{22}^*(L) \end{bmatrix} \begin{bmatrix} z_{1,t-1} \\ z_{2,t-1} \end{bmatrix} + \begin{bmatrix} \varepsilon_t \\ e_{2t}^* \end{bmatrix}$$

or as the VAR

$$
\begin{aligned}
\begin{bmatrix} z_{1t} \\ z_{2t} \end{bmatrix} &= \begin{bmatrix} I & -G \\ F & I \end{bmatrix}^{-1} \begin{bmatrix} A_{11}(L) - GA_{21}(L) & A_{12}(L) - GA_{22}(L) \\ A_{21}^*(L) & A_{22}^*(L) \end{bmatrix} \begin{bmatrix} z_{1,t-1} \\ z_{2,t-1} \end{bmatrix} \\
&\quad + \begin{bmatrix} I & -G \\ F & I \end{bmatrix}^{-1} \begin{bmatrix} \varepsilon_t \\ e_{2t}^* \end{bmatrix}
\end{aligned}
$$

We have now constructed a new VAR that can be used for policy analysis. We can, for example, perform impulse response analysis on this VAR in the usual way. We can forecast under the policy change, and we can carry out counter-factual analysis examining how the economy would have behaved in the past under a change of policy.

We note that the response of z_{1t} to ε_t in the new VAR must take account of the fact that we have carried out a transformation of the disturbances. Thus in the original VAR $\frac{\partial z_{1t}}{\partial \varepsilon_t} = I$ but in the new VAR $\frac{\partial z_{1t}}{\partial \varepsilon_t} = I - F(I + FG)^{-1}G$. Thus under the new policy rule the response of z_{1t} to ε_t is different. We also note that now z_{2t} will in general respond to ε_t.

To summarize, if we wish to analyze the effect of a change in policy rule within a VAR framework, we construct an estimate of the VAR of the response variables that is conditional on the policy instrument and then combine this with the new policy rule to form a complete system. The conditional VAR can be constructed as a linear transformation of the original VAR. The transformation matrix is estimated from the covariance matrix of the original VAR. We can then derive a new VAR from the completed model. Later we consider how to derive the policy rule optimally from the VAR.

Rational expectations monetary models

It is becoming fashionable to use a new Keynesian model to analyse interest rate policy. This can be derived formally from a DSGE model - see for example Cooley and Dwyer J of Ectcs (1998), Woodford (2002).

The following is a typical model used by Clarida, Gali and Gertler (1998) :

$$\begin{aligned}
\Delta p_t &= \lambda E_t \Delta p_{t+1} + \gamma x_t + e_{pt} \\
x_t &= \theta E_t x_{t+1} - \phi(i_t - E_t \Delta p_{t+1}) + e_{xt} \\
i_t &= \rho + \alpha(\Delta p_t - \pi) + \beta x_t + e_{it}
\end{aligned}$$

where x_t = the output gap measured as the percentage deviation of output from full capacity output.

A variant used by Taylor (1999) is to use the forward-looking Taylor rule:

$$i_t = \rho + \alpha(E_t \Delta p_{t+1} - \pi) + \beta E_t x_{t+1} + e_{it}$$

A theoretically questionable feature of the IS equation is that the real interest rate is expected to affect deviations of output from trend in the long run. This would imply that deviations could stay permanently above or below trend. Real interest rates might be expected to affect output demand and output capacity (and through this the trend), but not deviations from trend which measure the net effect (i.e. excess demand). Even in a short-run model, such misspecifications may be harmful for the empirics. New Keynesian analysis often seems to be willing to tolerate such solecisms.

Due to the presence of a Taylor rule, the Clarida, Gali and Gertler (1998) model has no exogenous variables. The solution can be written

$$A_0 w_t = \delta + A_1 E_t w_{t+1} + A_1 w_{t-1} + u_t$$

$$
\begin{bmatrix} 1 & \phi & \phi \\ \gamma & 1+\lambda & 0 \\ -\alpha & -\beta & 1 \end{bmatrix}\begin{bmatrix} x_t \\ p_t \\ i_t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \rho \end{bmatrix} + \begin{bmatrix} \theta & \phi & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & 0 \end{bmatrix} E_t \begin{bmatrix} x_{t+1} \\ p_{t+1} \\ i_{t+1} \end{bmatrix}
$$

$$
+ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\alpha & 0 \end{bmatrix}\begin{bmatrix} x_{t-1} \\ p_{t-1} \\ i_{t-1} \end{bmatrix} + \begin{bmatrix} e_{xt} \\ e_{pt} \\ e_{it} \end{bmatrix}
$$

The model must now be written in companion form as

$$
\begin{aligned}
\begin{bmatrix} A_0 & -A_1 \\ I & 0 \end{bmatrix}\begin{bmatrix} w_t \\ w_{t-1} \end{bmatrix} &= \begin{bmatrix} \delta \\ 0 \end{bmatrix} + \begin{bmatrix} A_{-1} & 0 \\ 0 & I \end{bmatrix} E_t \begin{bmatrix} w_{t+1} \\ w_t \end{bmatrix} + \begin{bmatrix} u_t \\ 0 \end{bmatrix} \\
\begin{bmatrix} w_t \\ w_{t-1} \end{bmatrix} &= \begin{bmatrix} A_0 & -A_1 \\ I & 0 \end{bmatrix}^{-1}\begin{bmatrix} A_{-1} & 0 \\ 0 & I \end{bmatrix} E_t \begin{bmatrix} w_{t+1} \\ w_t \end{bmatrix} \\
&\quad + \begin{bmatrix} A_0 & -A_1 \\ I & 0 \end{bmatrix}^{-1}\begin{bmatrix} u_t \\ 0 \end{bmatrix} \\
v_t &= \eta + BE_t v_{t+1} + a_t \\
&= (I-B)^{-1}\eta + \sum\nolimits_{s=0} B^s E_t a_{t+s} \\
&= (I-B)^{-1}\eta + a_t
\end{aligned}
$$

Hence

$$
E_t v_{t+1} = (I-B)^{-1}\eta
$$

- a constant!

We extract $E_t \Delta p_t$, $E_t x_{t+1}$ and $E_t i_{t+1}$ from $E_t v_{t+1}$. All are expected to be constant in the future for this model. This is not an unreasonable outcome for this model as we would exxpect that in the long run inflation will be non-zero, the output gap will be zero and the interest rate will equal the real rate plus inflation.

We can therefore write the model as a structural VAR(1) in w_t

$$A_0 w_t = \psi + A_1 w_{t-1} + u_t$$

Note that A_0 is not triangular and so we have an identification problem.

More generally, the rational expectations will not be constant.

If, for example, there were higher order lags in the original model then the rational expectations would take the form

$$E_t w_{t+1} = G(L) w_t$$

Again we can write the original model as a VAR, but in general there will be identification problems to confront.

Leeper and Zha (2000)

This paper is an example of this sort of model. We consider their results in some detail.

Their basic model is new Keynesian consisting of three equations: an IS function, an aggregate supply function (AS) and a monetary policy rule (MP)

$$\begin{aligned}
IS &: \quad x_t = -\frac{1}{\sigma}(i_t - E_t \Delta p_{t+1} - r) + \kappa(1-\theta) E_t x_{t+1} + \varepsilon_t^{IS} \\
AS &: \quad \Delta p_t = \lambda_0 x_t + \lambda_1 x_{t-1} + \psi \Delta p_{t-1} + (1-\psi)\beta E_t \Delta p_{t+1} + \varepsilon_t^{AS} \\
MP &: \quad i_t = \gamma_0 + \gamma_{\pi 1}(\Delta p_t - \bar{\pi}) + \gamma_{\pi 2}(\Delta p_{t-1} - \bar{\pi}) \\
& \qquad + \gamma_{x1} x_t + \gamma_{x2} x_{t-1} + \gamma_{m1}(\Delta M_t - \bar{\mu}) + \omega i_{t-1} + \varepsilon_t^{MP}
\end{aligned}$$

By contrast the Taylor rule is

$$MP : i_t = \gamma_0 + \gamma_{\pi 1}(\Delta p_t - \bar{\pi}) + \gamma_{x1} x_t + \varepsilon_t^{MP}$$

Leeper and Zha say that the reduced form for this model takes the form

$$A(L) z_t = \varepsilon_t$$

where $A(L)$ has a lag of 2.

In order to consider identification issues they examine the simpler model due to Taylor (1999)

$$\begin{aligned} IS &: \quad x_t = -\frac{1}{\sigma}(i_t - \Delta p_t - r) + \varepsilon_t^{IS} \\ AS &: \quad \Delta p_t = \lambda_1 x_{t-1} + \Delta p_{t-1} + \varepsilon_t^{AS} \\ MP &: \quad i_t = \gamma_0 + \gamma_{\pi 1}\Delta p_t + \gamma_{x1} x_t + \varepsilon_t^{MP} \end{aligned}$$

Imposing certain restrictions Taylor obtains the following estimates

$$\begin{aligned} IS &: \quad x_t = -0.024 + 0.795(i_t - \Delta p_t) \\ AS &: \quad \Delta p_t = 0.077 x_{t-1} + \Delta p_{t-1} \\ MP &: \quad i_t = 0.032 + 0.820\Delta p_t - 0.597 x_t \end{aligned}$$

These are not at all consistent with the theory - e.g. the signs of $i_t - \Delta p_t$ and x_t.

Leeper and Zha then consider using two alternative restrictions:

(i) that $\sigma = 0.5$

(ii) that the strict Taylor rule holds when $\gamma_{\pi 1} = 1.5$ and $\gamma_{x1} = 0.5$.

Their model estimates are then

(i)

$$\begin{aligned} IS &: \quad x_t = 0.063 - 2i_t + 1.658\Delta p_t \\ AS &: \quad \Delta p_t = 0.077 x_{t-1} + \Delta p_{t-1} \\ MP &: \quad i_t = 0.030 + 1.027\Delta p_t + 1.535 x_t \end{aligned}$$

(ii)

$$\begin{aligned} IS &: \quad x_t = 0.024 - 0.732 i_t + 0.545\Delta p_t \\ AS &: \quad \Delta p_t = 0.077 x_{t-1} + \Delta p_{t-1} \\ MP &: \quad i_t = 0.009 + 1.5\Delta p_t + 0.5 x_t \end{aligned}$$

- Both models have highly serially correlated errors

- and exogenous shifts in policy have unrealistically large impacts on output, but virtually no impact on inflation

This implies there is little role for monetary policy in controlling inflation.

Leeper and Zha then attempt to reformulate this model along the more general lines of the new Keynesian model above in order to give more plausible results.

Essentially, they include different restrictions from Taylor.

Taylor's restrictions on $A(L)$ take the form

$$A_0 = \begin{bmatrix} \times & 0 & \times \\ \times & \times & \times \\ \times & 0 & \times \end{bmatrix},\ A_1 = \begin{bmatrix} \times & 0 & 0 \\ \times & \times & \times \\ 0 & 0 & 0 \end{bmatrix},\ A_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \times & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

L&Z suggest changing one coefficient $A_{0,12}$

$$A_0 = \begin{bmatrix} \times & \times & \times \\ \times & \times & \times \\ \times & 0 & \times \end{bmatrix},\ A_1 = \begin{bmatrix} \times & 0 & 0 \\ \times & \times & \times \\ 0 & 0 & 0 \end{bmatrix},\ A_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \times & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Essentially they free-up the lag structure a little and obtain the estimates

(i)

$$\begin{aligned} IS &: x_t = 0.060 - 2i_t + 1.126\Delta p_t + 1.495x_{t-1} \\ AS &: \Delta p_t = 0.025x_{t-1} + 0.962\Delta p_{t-1} \\ MP &: i_t = 0.023 + 0.690\Delta p_t + 0.517x_t \end{aligned}$$

(ii)

$$\begin{aligned} IS &: \; x_t = 0.027 - 0.555 i_t + 0.069 \Delta p_t + 1.090 x_{t-1} \\ AS &: \; \Delta p_t = 0.053 x_{t-1} + 0.979 \Delta p_{t-1} \\ MP &: \; i_t = -0.011 + 1.5 \Delta p_t + 0.5 x_t \end{aligned}$$

These two sets of estimates give a much more stable model.

AS shocks continue to be the dominant source of inflation in the short run

But in the longer run AD shocks are as important as AS.

Comment

This analysis illustrates the problem of approaching the data with highly simplified models.

(i) The coefficient estimates are often implausible due to possible biases.

(ii) Other properties, such as impulse response functions are then affected

(iii) The estimates are very unstable to small changes in specification

(iv) It might be better to take more account of the restrictions arising from economic theory and of alternative econometric approaches to estimating the new Keynesian model.

A full rational expectations solution of the Taylor/Leeper and Zha model

Given our earlier discussion about solving and estimating RE models, we can examine how one might go about estimating the Taylor model considered by Leeper and Zha.

We can write the model as

$$\begin{aligned}\Delta p_t &= \lambda E_t \Delta p_{t+1} + \gamma x_t + e_{pt} \\ x_t &= \theta E_t x_{t+1} - \phi(i_t - E_t \Delta p_{t+1}) + e_{xt}\end{aligned}$$

and define $y_t' = (\Delta p_t, x_t) \sim I(0)$. If we ignore the Taylor rule, there is one exogenous variable i_t. Thus

$$\begin{bmatrix} 1 & -\gamma \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \Delta p_t \\ x_t \end{bmatrix} = \begin{bmatrix} \lambda & 0 \\ \phi & \theta \end{bmatrix} E_t \begin{bmatrix} \Delta p_{t+1} \\ x_{t+1} \end{bmatrix} + \begin{bmatrix} 0 \\ -\phi \end{bmatrix} i_t + \begin{bmatrix} e_{pt} \\ e_{xt} \end{bmatrix}$$

or

$$\begin{bmatrix} \Delta p_t \\ x_t \end{bmatrix} = \begin{bmatrix} \lambda + \gamma\phi & \gamma\theta \\ \phi & \theta \end{bmatrix} E_t \begin{bmatrix} \Delta p_{t+1} \\ x_{t+1} \end{bmatrix} + \begin{bmatrix} -\gamma\phi \\ -\phi \end{bmatrix} i_t + \begin{bmatrix} 1 & \gamma \\ 0 & 1 \end{bmatrix} \begin{bmatrix} e_{pt} \\ e_{xt} \end{bmatrix}$$

The auxiliary equation is

$$|I - \begin{bmatrix} \lambda + \gamma\phi & \gamma\theta \\ \phi & \theta \end{bmatrix} L| = 0$$

This has two roots, λ_1 and λ_2, say. One will be greater than unity and one less than unity. The solution is therefore a saddlepath and can be written as

$$\begin{aligned}\begin{bmatrix} \Delta p_t \\ x_t \end{bmatrix} &= \lambda_2 \begin{bmatrix} -\gamma\phi \\ -\phi \end{bmatrix} \textstyle\sum_{s=0}^{\infty} \lambda_1^{-s} E_t i_{t+s} + (1-\lambda_2) \begin{bmatrix} \Delta p_{t-1} \\ x_t - 1 \end{bmatrix} \\ &+ \begin{bmatrix} 1 & \gamma \\ 0 & 1 \end{bmatrix} \begin{bmatrix} e_{pt} \\ e_{xt} \end{bmatrix}\end{aligned}$$

If policy is purely discretionary then the private sector will form expectations about current and future interest rates.

Suppose that the monetary policy authority announces that the interest rate will take the constant value i for all periods, and this is believed, then the solution would be

$$\begin{bmatrix} \Delta p_t \\ x_t \end{bmatrix} = \frac{\lambda_1\lambda_2}{1-\lambda_1}\begin{bmatrix} \gamma\phi \\ \phi \end{bmatrix} i + (1-\lambda_2)\begin{bmatrix} \Delta p_{t-1} \\ x_{t-1} \end{bmatrix} + \begin{bmatrix} 1 & \gamma \\ 0 & 1 \end{bmatrix}\begin{bmatrix} e_{pt} \\ e_{xt} \end{bmatrix}$$

If the shocks e_t were observable in time t, then this policy would probably not be credible because it would not be time consistent. When next period's shocks e_{t+1} occur it may be optimal to change the interest rate.

Alternatively, the private sector might fit a univariate time-series model to i_t and use this to form $E_t i_{t+s}$. Writing the forecasts as

$$E_t i_{t+s} = \delta_s(L) i_t$$

the solution would be

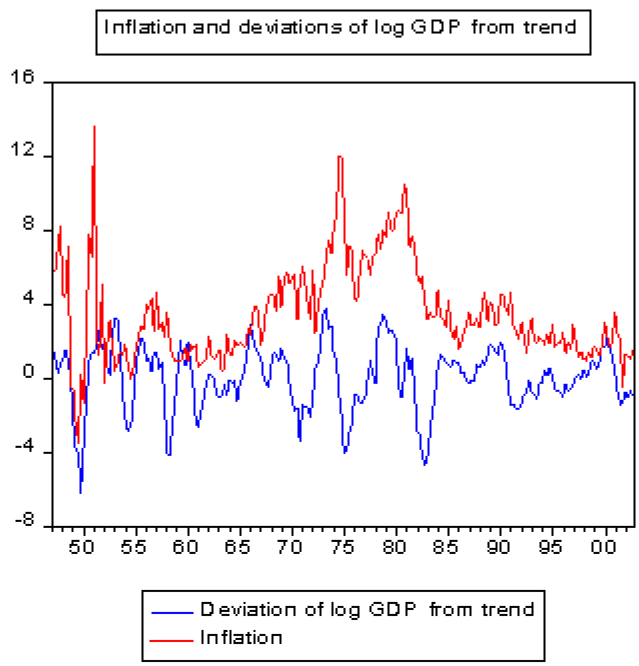
$$\begin{aligned}
\begin{bmatrix} \Delta p_t \\ x_t \end{bmatrix} &= \lambda_2 \begin{bmatrix} -\gamma\phi \\ -\phi \end{bmatrix} \textstyle\sum_{s=0}^{\infty} \lambda_1^{-s}\delta_s(L) i_t + (1-\lambda_2)\begin{bmatrix} \Delta p_{t-1} \\ x_{t-1}1 \end{bmatrix} \\
&\quad + \begin{bmatrix} 1 & \gamma \\ 0 & 1 \end{bmatrix}\begin{bmatrix} e_{pt} \\ e_{xt} \end{bmatrix} \\
&= \lambda_2 \begin{bmatrix} -\gamma\phi \\ -\phi \end{bmatrix} \varphi(L) i_t + (1-\lambda_2)\begin{bmatrix} \Delta p_{t-1} \\ x_{t-1} \end{bmatrix} \\
&\quad + \begin{bmatrix} 1 & \gamma \\ 0 & 1 \end{bmatrix}\begin{bmatrix} e_{pt} \\ e_{xt} \end{bmatrix}
\end{aligned}$$

This is an example of the solution in the form of equation (**??**).

Applying this to our data we find that i_t is an AR(8).

Some new empirical evidence

The first issue is what is the broad relation between inflation and the output gap. We use the HP filter to measure the output gap with $\lambda = 1600$.



The theory predicts that an increase in the output gap causes inflation to increase. Implictly, therefore, the theory assumes that the economy is subject only to demand shocks. A supply shock, such as an oil price shock, would be expected to raise inflation but to decrease output and hence the ouput gap.

The graph shows periods when the correlation was negative (from about 1965-73) and periods when it was positive (mainly after 1973). It also suggests that there is a lag in response of inflation to the output gap.

The following table gives the mean values of the output gap measure, inflation for each $5-year$ period from 1956-2000 and the correlation between the two.

	x_t	Δp_t	$correl(x_t, \Delta p_t)$
$1956-1960$	-0.40	2.27	0.06
$1961-1965$	-1.94	1.42	0.58
$1966-1970$	-1.59	4.19	-0.40
$1971-1975$	-1.16	6.57	-0.13
$1976-1980$	3.16	7.31	0.40
$1981-1985$	-3.07	4.58	0.09
$1986-1990$	3.03	3.30	0.43
$1991-1995$	-2.72	2.37	-0.32
$1996-2000$	1.94	1.72	0.08
$1956-2000$	0.05	3.75	0.09

The table confirms the variability in the correlation and shows that there was also a period of negative correlation over the period 1991-1995. We also note that the periods of negative correlation were all periods when output was below trend ($x_t < 0$). However; not all periods when $x_t < 0$ had a negative correlation.

We estimate $E_t \Delta p_{t+1}$ and $E_t x_{t+1}$ using a VAR in $(\Delta p_t, x_t, i_t)$ with 12 lags. The fitted values are used as the forecasts $E_t \Delta p_{t+1}$ and $E_t x_{t+1}$.

We now consider estimates of a slightly more general version of the model which has additional dynamics

$$
\begin{aligned}
\Delta p_t &= \underset{(0.2)}{0.018} + \underset{(9.1)}{0.614} E_t \Delta p_{t+1} + \underset{(6.3)}{0.383}\, \Delta p_{t-1} - \underset{(0.3)}{0.013} x_t + e_{pt} \\
x_t &= -\underset{(0.95)}{0.108} + \underset{(19.8)}{0.592} E_t x_{t+1} + \underset{(15.1)}{0.434} x_{t-1} + \underset{(1.4)}{0.016} r_t \\
&\quad + \underset{(6.11)}{0.181} \Delta r_{t-1} - \underset{(3.8)}{0.108} \Delta r_{t-2} + \underset{(3.2)}{0.090}\, \Delta r_{t-3} + e_{xt}
\end{aligned}
$$

If the lagged dependent variable were omitted, then both equations would give estimates of the expectations variable that are very close to, and not significantly different from, 1. As it stands, the sum of the inflation coefficients on the right-hand side add to unity. These both imply a unit root as the inflation equation can be written approximately as

$$\Delta p_t - \Delta p_{t-1} = 1.5(E_t \Delta p_{t+1} - \Delta p_t) - 0.03x_t + 2.5e_{pt}$$

As this is an unstable equation we re-write it as

$$\Delta p_t - \Delta p_t = 0.67(\Delta p_{t-1} - \Delta p_{t-2}) + 0.02x_{t-1} - 1.7e_{p,t-1} + \xi_{pt}$$

where $\xi_{pt} = \Delta p_t - E_{t-1}\Delta p_t$. This implies a backward looking autoregression in the *change* in inflation.

In the original price equation x_t has a negative coefficient, not positive as the theory predicts, but it is not significant. In contrast, the coefficient of x_{t-1} is positive in the implied autoregression which is correct. An important implication follows from this. In examining the sign of an effect, especially in the long run, it is necessary to take account of the dynamic structure of a model. We conclude, therefore, that an increase in the output gap does tend to raise inflation, but does so only temporarily; it has no long-run effect on inflation. The lack of significance of x_t, however, suggests that it doesn't even have a temporary effect on inflation.

These results call into question in a dramatic way a key part of the transmission mechanism of inflation targeting. They imply that interest rate targeting does not work as the theory predicts.

The dynamics of the output equation also imply a unit root. The real interest rate r_t in the output gap equation is of only marginal significance but lags of changes in the real interest are significant. So these results are not consistent with the model either.

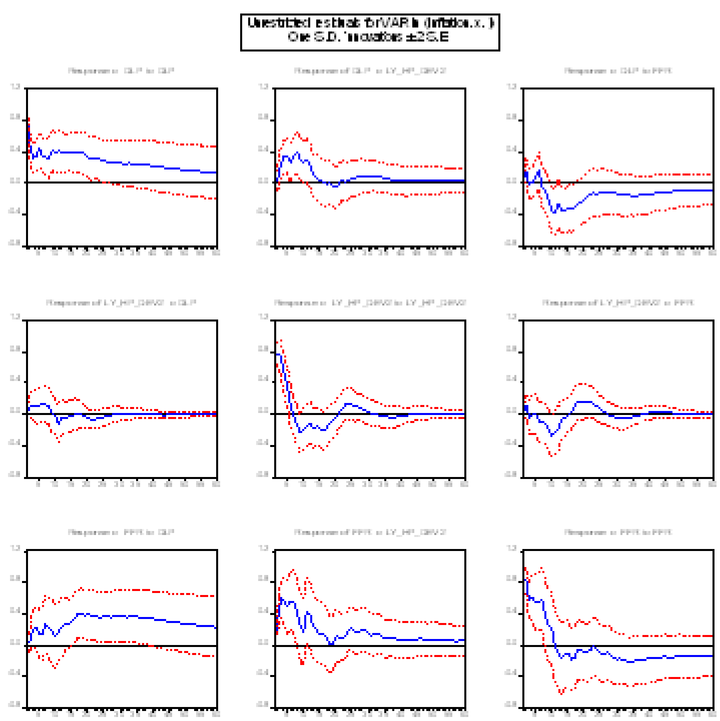
Given the dynamic structure of the estimated equations, they are virtually identical to a corresponding model written in terms of differences in inflation, the output gap and the real interest rate as the model can re-written as the VAR

$$\begin{aligned}
\Delta\pi_t &= 0.67\Delta\pi_{t-1} + 0.02\Delta x_t - 1.7e_{p,t-1} + \xi_t \\
\Delta x_t &= 0.67\Delta x_{t-1} - 0.3\Delta r_{t-1} + 0.16(\Delta r_{t-2} - \Delta r_{t-3}) - 1.7e_{xt} + \xi_{xt}
\end{aligned}$$

where $\xi_{xt} = \Delta x_t - E_{t-1}\Delta x_t$.

Thus the real interest rate no longer has a permanent effect on the output gap, but only a temporary effect. From a theoretical point of view this seems more reasonable. Further, the shocks e_{pt} and e_{xt} have permanent effects on inflation and the output gap. This is not an attractive implication of the model from a theoretical perspective as we would expect that inflation and the output gap are mean reverting processes.

It is of interest to compare these findings with a VAR in $(\Delta p_t, x_t, i_t)$ with 12 lags. The impulse response functions can be obtained are



(i) Clearly all of the shocks are temporary, even the interest rate shock on output deviations. Thus adding a lag structure seems to have rescued the model by allowing the lags to offset what would otherwise be a permament effect according to the model.

(ii) An output shock raises inflation as predicted by the model, but does so with a slight lag.

(iii) An interest rate shock takes time to feed through to output deviations and then, with a further lag, to inflation.

(iv) Initially, an output shock affects the interest rate more than an inflation shock, but the latter has a much longer-lasting effect on the interest rate.

Inflation targeting rules using a VAR

For an extensive analysis and comparison of the use of different types of rule - ad hoc, Taylor and optimal rules - for inflation targeting see Rudebusch and Svensson (1999) in Taylor (1999) *Monetary Policy Rules*.

Rudebusch and Svensson contrast the use of what they call *instrument* and *targeting* rules in a simple two equation VAR, specifically

$$\begin{aligned} \Delta p_{t+1} &= \alpha_p(L)\Delta p_t + \alpha_x x_t + \varepsilon_{t+1} \\ x_{t+1} &= \beta_x(L)x_t - \beta_i(L)(i_t - \Delta p_t) + \eta_{t+1} \end{aligned}$$

where x_t =output gap and $\beta_i(L) = \beta_i \frac{1}{4}(1 + L + L^2 + L^3)$, i.e. a pre-specified four-quarter moving average distributed lag function with coefficient β_i.

Instrument rule

The policy instrument (here the interest rate) is expressed as an explicit function of the available information. This information may be past values of variables in the VAR or forecast future values. A general formulation would be

$$i_t = \theta_i(L)i_{t-1} + \theta_{p1}(L^{-1})E_t\Delta p_{t+1} + \theta_{p2}(L)\Delta p_t + \theta_x(L)x_t$$

The first term on the right-hand side is an interest rate smoothing function. The second involves expected future inflation which makes the rule forward looking.

Rudebusch and Svensson consider 9 different different instrument rules of this sort, varying from the simple to the more cpmplicated. For transparency reasons, R&S prefer as simple a rule as possible. The Taylor rule is also a special case of this general formulation.

The problem with an instrument rule is how to choose the coefficients.

Targeting rule

This is the rule that emerges from optimal policy - i.e. maximising an objective function (or minimising a loss function) subject to constraints (the model of the economy, i.e. the VAR). In other words, in a targeting rule the aim is to make an optimal choice of the coefficients of the instrument rule.

A typical loss function is

$$\mathcal{L} = E_t(1-\beta)\sum_{s=0}\beta^s[(\Delta p_{t+s}-\pi)^2+\alpha x_{t+s}^2]$$

Here the objective is to penalise deviations from target. The target for inflation is π and the target output gap is zero. If $\beta = 1$ an equal weight is given to each period. This is equivalent to having the single period objective function

$$\mathcal{L} = (\Delta p_{t+s}-\pi)^2+\alpha x_{t+s}^2$$

Rudebusch and Svensson compare the losses arising from the use of different rules by adding the interest rate rule to their VAR and calculating the consequences for inflation and the output gap. They vary not only the coefficients in the rule, but also the weights in the objective function.

Their conclusion is that the optimal targeting rule performs best, but there is little choose between this and two of the instrument rules. The simplest of these involves the average forecast inflation rate over the next two years

$$i_t = 0.74 i_{t-1} + 1.423\left(\frac{1}{8}\sum_{s=0}^{7} E_t\Delta p_{t+s}\right) + 0.16 x_t$$

Optimal monetary policy

We now consider how to derive an optimal rule.

The objective function

Since there is strong evidence that monetary authorities prefer to make a number of small changes to interest rates rather than a few large changes, we may need a more general objective function than that above. We therefore allow the monetary authority to have a target interest rate (for example, to keep interest rates close to their long-run equilibrium value $\bar{i} = \pi + r$) and to penalise changes in the interest rate. This can captured by the loss function

$$\mathcal{L} = E_t(1-\beta)\sum_{s=0}\beta^s[(\Delta p_{t+s} - \pi)^2 + \alpha x_{t+s}^2 + \gamma(\Delta i_{t+s})^2 + \delta(i_{t+s} - \bar{i})^2]$$

Under simple strict inflation targeting $\alpha = \gamma = \delta = 0$ and $\beta = 1$.

We suppose the model of the economy is a VAR in $\{\Delta p_t, x_t, i_t\}$. The objective function is maximised subject to this constraint.

The VAR model constraint

As previously noted, a major problem with the use of a VAR in this way is that if we estimate an equation for the policy instrument i_t, and then replace it by the optimal policy rule, if the two are different then the VAR will have changed as a result of optimising policy. The old VAR will then no longer be the correct model to use to analyse the effects of monetary policy on inflation and output.

Our solution is to use instead a VAR that is conditioned on the interest rate. This can be obtained as previously by transforming the original VAR by a Choleski transformation. Thus, in the notation used before, the model constraint can be written as

$$z_{1t} = [A_{11}(L) - GA_{21}(L)]z_{1,t-1} + Gz_{2t} + [A_{12}(L) - GA_{22}(L)]z_{2,t-1} + \varepsilon_t$$

or, more compactly, as

$$z_{1t} = \alpha(L)z_{1,t-1} + \beta(L)z_{2t} + \varepsilon_t$$

where $z_{1t} = (\Delta p_t, x_t)'$, $z_{2t} = i_t$, $\alpha(L) = \sum_{s=0}\alpha_s L^s$, $\beta(L) = \sum_{s=0}\beta_s L^s$ and $\beta_0 = G$.

Before proceeding to solve the optimal problem we introduce more notation. We re-write the conditional VAR in companion form as a first-order conditional VAR:

$$y_t = Ay_{t-1} + bi_t + u_t$$

where

$$\begin{aligned}
y_t' &= [z_t', z_{t-1}', ..., z_{t-m+1}'] \\
z_t' &= (z_{1t}', i_t, \Delta i_t) \\
u_t' &= [\varepsilon_t', 0, 0, 0..., 0] \\
b' &= [G', 1, 1, ..., 0] \\
\Delta p_t &= l_p' y_t, \qquad l_p' = (1, 0, 0, 0, ..., 0) \\
x_t &= l_x' y_t, \qquad l_x' = (0, 1, 0, 0, ...0) \\
i_t &= l_i' y_t, \qquad l_i' = (0, 0, 1, 0, ..., 0) \\
\Delta i_t &= l_\Delta' y_t, \qquad l_\Delta' = (0, 0, 1, 0, ..., 0)
\end{aligned}$$

we have included the identity $\Delta i_t = i_t - i_{t-1}$ as an additional equation in this system and

$$A = \begin{bmatrix}
\alpha_0 \ \beta_0 & \alpha_1 \ \beta_1 & . & . & \alpha_{m-1} \ \beta_{m-1} \\
0 : -1, 0, ..0 & 0 & . & . & . \\
I & 0 & . & . & . \\
0 & I & . & . & . \\
. & . & . & . & . \\
. & . & 0 & I & 0
\end{bmatrix}$$

1. The solution as a problem in optimal control

We can now re-state the optimal control problem following Chow (1975, pp 152-160) as

$$\max E_t \textstyle\sum_{s=0}^{\infty} (y_{t+s} - a_{t+s})' K_{t+s} (y_{t+s} - a_{t+s})$$

subject to the conditional VAR. In the present case a_{t+s} is a constant a say, and K_{t+s} is a diagonal matrix dependent only on s (and not t) through the discount factor $(1-\beta)\beta^s$. Thus

$$\begin{aligned} K_{t+s} &= diag\{J_s\} \\ J_s &= (1-\beta)\beta^s J \\ J &= diag\{1, \alpha, 0', \delta, \gamma\} \end{aligned}$$

Using the results of Chow the solution to this problem is the control rule

$$i_{t+s} = m_{t+s} + M_{t+s} y_{t+s-1}$$

where

$$\begin{aligned} M_{t+s} &= -(b'H_{t+s}b)^{-1}b'H_{t+s}A \\ m_{t+s} &= (b'H_{t+s}b)^{-1}b'h_{t+s} \\ H_{t+s-1} &= K_{t+s-1} + A'H_{t+s}(A + bM_{t+s}) \\ h_{t+s-1} &= K_{t+s-1}a + (A + bM_{t+s})'h_{t+s} \end{aligned}$$

Thus for $s = 0$

$$\begin{aligned} i_t &= m_t + M_t y_{t-1} \\ M_t &= -[(G', 1, 1)J(G', 1, 1)']^{-1}(G', 1, 1)J \\ m_t &= [(G', 1, 1)J(G', 1, 1)']^{-1}(G', 1, 1)h_t \end{aligned}$$

Since it is always optimal for the monetary authority to re-optimise each period, in practice only this first-period rule will be implemented. Consequently, the optimal rule will have fixed coefficients and take the form

$$i_t = m + My_{t-1}$$

2. A direct solution

As it is not possible to get more intuition into what this rule entails without specifying the model in more detail, we consider a different approach which does not exploit the optimal control literature. This involves minimising $\mathcal{L}$ with respect to i_t directly.

First, we note the following results

$$\begin{aligned}
y_{t+s} &= A^s y_t + \textstyle\sum_{j=1}^{s} A^{s-j} b i_{t+j} + \sum_{j=1}^{s} A^{s-j} u_{t+j} \\
\frac{dy_{t+s}}{di_t} &= A^s b, \quad s = 0, 1, 2, ... \\
\frac{d(\Delta p_{t+s})}{di_t} &= l_p' \frac{dy_{t+s}}{di_t} = l_p' A^s b \\
\frac{dx_{t+s}}{di_t} &= l_x' \frac{dy_{t+s}}{di_t} = l_x' A^s b
\end{aligned}$$

Minimising $\mathcal{L}$ with respect to i_t gives the first-order conditions for $s = 0$

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial i_t} &= 2E_t(1-\beta) \textstyle\sum_{s=0} \beta^s [(l_p' A^s b)(\Delta p_{t+s} - \pi) + \alpha(l_x' A^s b) x_{t+s}] \\
&\quad + 2(1-\beta)\gamma \Delta i_t - 2(1-\beta)\beta\gamma E_t \Delta i_{t+1} + 2\delta(i_t - \bar{i}\,) \\
&= 0
\end{aligned}$$

Hence the optimal interest rate setting is

$$\begin{aligned}
i_t &= \frac{\delta}{\delta + \gamma(1+\beta)} \bar{i} + \frac{\beta\gamma}{\delta + \gamma(1+\beta)} E_t i_{t+1} + \frac{\gamma}{\delta + \gamma(1+\beta)} i_{t-1} \\
&\quad - \frac{1}{\delta + \gamma(1+\beta)} E_t \textstyle\sum_{s=0} \beta^s [(l_p' A^s b)(\Delta p_{t+s} - \pi) + \alpha(l_x' A^s b) x_{t+s}]
\end{aligned}$$

or, written more compactly, is

$$i_t = \theta_0 + \theta_1 E_t i_{t+1} + \theta_2 i_{t-1} + \textstyle\sum_{s=0} \theta_{3s} E_t \Delta p_{t+s} + \sum_{s=0} \theta_{4s} E_t x_{t+s}$$

Thus the interest rate depends on deviations of current and expected future inflation from target and the output gap from zero. If the monetary authority has an interest rate target ($\delta \neq 0$) then there is a target level of interest rates, and if it cares about smoothing interest rate changes

($\gamma \neq 0$) then interest rate dynamics affect the policy rule. If the monetary authority does not care about smoothing then $\gamma = 0$ and we obtain the purely forward looking rule

$$i_t = \theta_0 + \sum_{s=0} \theta_{3s} E_t \Delta p_{t+s} + \sum_{s=0} \theta_{4s} E_t x_{t+s}$$

in which lagged values of i_t only affect the current choice through their information about future inflation and output.

Implementing this solution

In order to implement this rule, the monetary authority must substitute the current values and forecasts of future inflation and the output gap. These could be obtained from the conditional VAR. We note that for $s = 1, 2$,...

$$\begin{aligned} E_t(\Delta p_{t+s}) &= l_p' E_t(y_{t+s}) = l_p'[A^s y_t + \textstyle\sum_{j=1}^{s} A^{s-j} b E_t(i_{t+j})] \\ E_t(x_{t+s}) &= l_x' E_t(y_{t+s}) = l_x'[A^s y_t + \textstyle\sum_{j=1}^{s} A^{s-j} b E_t(i_{t+j})] \end{aligned}$$

and for $s = 0$

$$\begin{aligned} \Delta p_t &= l_p' y_t \\ x_t &= l_x' y_t \end{aligned}$$

Thus, the interest rate rule can be re-written in terms of the current value y_t as

$$\begin{aligned} i_t = \; & \frac{\delta}{\delta + \gamma(1+\beta)} \bar{i} + \frac{\beta\gamma}{\delta + \gamma(1+\beta)} E_t i_{t+1} + \frac{\gamma}{\delta + \gamma(1+\beta)} i_{t-1} \\ & - \frac{1}{\delta + \gamma(1+\beta)} \textstyle\sum_{s=0} \beta^s [(l_p' A^s b) l_p' + \alpha (l_x' A^s b) l_x'] A^s y_t \\ & - \frac{1}{\delta + \gamma(1+\beta)} \textstyle\sum_{s=1} \beta^s [(l_p' A^s b) l_p' + \alpha (l_x' A^s b) l_x'] \textstyle\sum_{j=1}^{s} A^{s-j} b E_t(i_{t+j}) \\ & + \frac{1}{\delta + \gamma(1+\beta)} \textstyle\sum_{s=0} \beta^s (l_p' A^s b) \pi \end{aligned}$$

or, more compactly, as

$$i_t = \phi_0 + \phi_1 i_{t-1} + \phi_2 y_t + \sum_{s=1} \phi_{3s} E_t i_{t+s}$$

The optimal interest rate rule now no longer depends explicitly on future values of Δp_t and x_t.

The forward-looking component of the optimal interest rate rule is now captured by expected optimal settings of future interest rates. We also note that through y_t, the current information set, i_t may be dependent both on other variables such as asset prices and - even if $\gamma = 0$ - on past interest rates.

The constant interest rate assumption

It is common for the monetary authorities to assume that future interest rates will stay constant. It is clear from the second version of the rule why this is an attractive expedient. In effect, the rule could then be written formally as

$$i_t = \frac{\phi_0}{1 - \sum_{s=1} \phi_{3s}} + \frac{\phi_1}{1 - \sum_{s=1} \phi_{3s}} i_{t-1} + \frac{\phi_2}{1 - \sum_{s=1} \phi_{3s}} y_t$$

where we recall that y_t also contains i_t. Re-grouping the variables enables this equation to be written in terms of the original variables as

$$i_t = \gamma_0 + \gamma_1(L)\Delta p_t + \gamma_2(L)x_t + \gamma_3(L)i_{t-1}$$

Evaluating the loss function

This can now be combined with the conditional VAR to form a complete model in which all of the variables are endogenous. But this model can be transformed into a VAR in $(\Delta p_t, x_t, i_t)$ which may be used for impulse response analysis and forecasting. It can also be used to evaluate the loss function. By altering the interest rate equation and re-forming the VAR it would be possible to analyse the welfare costs of using different rules. In this way the Taylor rule, in which only current inflation and the output gap are present, can be compared with the optimal rule. The benefits of including an interest rate target in the loss function, and of caring about interest rate smoothing can also be examined in this way.

This solution can be related to that obtained using the optimal control theory results. This rule only had lagged values of variables on the right-hand side. If we eliminate Δp_t and x_t by substituting from the VAR then this rule becomes a purely backward looking rule.

Some special cases

If the monetary authority is not concerned about the level or changes in interest rates so that $\delta = \gamma = 0$, or if the monetary authority is a strict inflation targeter so that $\alpha = 0$, then there will be a different policy rule. If $\delta = \gamma = 0$ then $\theta_0 = \theta_1 = \theta_2 = 0$ and the policy rule must satisfy

$$E_t \sum_{s=0} \beta^s [(l_p' A^s b)(\Delta p_{t+s} - \pi) + \alpha(l_x' A^s b) x_{t+s}] = 0$$

or

$$\begin{aligned} & \sum_{s=0} \beta^s [(l_p' A^s b) l_p' + \alpha(l_x' A^s b) l_x'] A^s y_t \\ & + \sum_{s=1} \beta^s [(l_p' A^s b) l_p' + \alpha(l_x' A^s b) l_x'] \sum_{j=1}^{s} A^{s-j} b E_t(i_{t+j}) \\ = \; & \sum_{s=0} \beta^s (l_p' A^s b) \pi \end{aligned}$$

This may be written as

$$\Lambda y_t + \sum_{s=1} \Phi_s E_t(i_{t+s}) = \pi^*$$

where i_t is an element of y_t. Thus the rule can be written as

$$i_t = \phi_0 + \phi_1(L)\Delta p_t + \phi_2(L) x_t + \phi_3(L^{-1}) E_t(i_{t+1}) + \phi_4(L) i_{t-1}$$

Additional variables

Finally, we note that other variables that might influence inflation or output such as asset prices and the exchange rate do not appear in the interest rate rule. It would require these variables to be part of the loss function for them to appear in the rule. But if they were included in the VAR, then they would form part of the information set used to forecast future values of inflation an output etc. As a result they would also appear in the optimal rule derived from control theory.

Future research

A more detailed examination of the whole question of optimal policy using a VAR is part of my current research agenda. It can also be used for fiscal policy. The present value budget constraint would then need to be part of the model constraints in the optimisation.