University of York

Department of Economics

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VAR ANALYSIS IN MACROECONOMICS

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Lecture 3

VAR models for non-stationary data

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Causes of non-stationarity

A large number of economic time series are non-stationary.

This includes most national income statistics, and goods and asset prices.

Although there are a large number of non-stationary economic time series, have you ever thought what the sources of the non-stationarity are? There are only three main causes of non-stationarity. Non-stationarity gets transmitted from these sources to all of the other variables.

1. Productivity shocks

These affect output and hence all variables derived from this such as real expenditures.

It is also the cause of balanced economic growth.

2. Money growth

If the money supply has a non-zero growth rate then money is likely to be non-stationary.

For example, if money deviates randomly from a constant rate of growth then the log of the money supply is a random walk with drift.

$$\Delta \ln M_t = \mu + e_t$$

This causes all nominal variables and absolute prices to be non-stationary.

3. Budget constraints

Consider the budget constraint

$$a_{t+1} + c_t = y_t + (1 + r_t)a_t$$

a =assets, c =consumption, y =income, r =interest rate

It follows that a_t is determined by an explosive difference equation:

$$[L^{-1} - (1 + r_t)]a_t = y_t - c_t$$

This has the root

$$L = \frac{1}{1 + r_t} < 1$$

For stationarity we require the root to be greater than unity.

Note:

The feedback from a_t to c_t (or y_t) of the form $c_t = \theta a_t$ may remove the instability as we would have

$$a_{t+1} = y_t + (1 + r_t - \theta) a_t$$

so if $\theta > r_t$ then the asset stock is stable.

For example, in the life-cycle hypothesis with $E_t y_{t+s} = y_t$ we have

$$\begin{aligned} c_t &= r W_t \\ &= r[\frac{y_t}{r} + a_t] \\ &= y_t + r a_t \end{aligned}$$

and hence

$$\begin{aligned} a_{t+1} &= (1 + r_t - r) a_t \\ &\simeq a_t \end{aligned}$$

implying a unit root.

Implications of non-stationarity

1. Model building

Non-stationary series tend to look far more volatile than stationary series

Series with drift terms tend to be volatile around strong trends

If we are seeking to explain a non-stationary variable with drift then we must have a model that is capable of generating such behaviour.

Usually this is achieved by having explanatory variables with equivalent properties.

For example, there is no point in trying to explain GDP using just unemployment - as the Okun model does - as output has a trend and unemployment does not.

Thus if an endogenous variable y_t is I(1) then we need at least one exogenous variable x_t that is also I(1).

We can also have I(0) variables in the model too, and these may affect y_t both in the long run and the short run.

Another example is trying to explain a stationary variable with a non-stationary variable. This won't work either as we need the right-hand side of the equation to be stationary.

Thus we can't explain unemployment using the level of GDP. A better model might be to use the first difference of GDP (or log GDP or capacity utilisation) as this is stationary, or a linear combination of non-stationary variables that is stationary. This takes us into the issue of cointegration which we shall discuss later.

It should be noted it is possible that the internal dynamics of the y_t equation are the cause of its non-stationarity, and not an I(1) exogenous variable.

For example if consumption and income are I(0) - which they are not - then the stock of assets could still be non-stationary due to the dynamic structure of the budget constraint.

2. Econometrics

If data are non-stationary then the sampling distributions of coefficient estimates may not be well approximated by the Normal distribution, as usually assumed.

This is particularly relevant if the series is I(1) but without a drift term.

If a non-stationary series has a drift then the resulting trend will dominate the purely non-stationary component and as a result the Normal distribution will still probably be a good approximation.

The good thing about having non-stationary data is that we get much more precise estimates of some of the coefficients. For example in a regression of y_t on x_t the coefficients of x_t are super-consistent, meaning that they are likely to have much smaller standard errors than if the data were stationary. Roughly, they will be a factor $T^{1/2}$ smaller and the t-statistics will be correspondingly larger.

Also it has been claimed that I(0) variables can be omitted from the model entirely without greatly affecting the estimates of the coefficients of the non-stationary variables.

Actually, in general this is not a good strategy.

There are lots of technicalities in all of this, including how to decide whether a variable is non-stationary. We shall set these aside and ignore such problems in order focus instead on modeling issues.

Cointegration

We now follow the analysis in Wickens (1996)

If z_t is a vector of n I(1) variables, then if there exists an $n \times 1$ vector α with no zero elements such that $\alpha' z_t$ is I(0), we say that the variables z_t are COINTEGRATED, and α is the COINTEGRATING VECTOR.

There can be a maximum of $n - 1$ cointegrating vectors for z_t, and a minimum of zero.

The significance of cointegration for modeling is that if $z_t' = (y_t,\, x_t')$ and $\alpha' = (1,\, \beta')$ we can re-write $\alpha' z_t$ as

$$\begin{aligned} \alpha' z_t &= y_t - \beta' x_t = u_t \\ y_t &= \beta' x_t + u_t \end{aligned}$$

where u_t is I(0).

This equation explains y_t in the long run.

It is called a COINTEGRATING REGRESSION

To explain each endogenous variable in a model in the long run we need a different cointegrating vector for each.

u_t may contain additional I(0) variables, but their influence in the long run is likely to be dominated by that of the I(1) variables.

Consider the dynamic model

$$y_t = \alpha y_{t-1} + \beta_1 x_t + \beta_2 x_{t-1} + e_t$$

where y_t and x_t are I(1) and e_t is a stationary disturbance.

We can re-write the equation in many ways:

(i)

$$\begin{aligned} y_t &= \frac{\beta_1 + \beta_2}{1-\alpha} x_t - \frac{\alpha}{1-\alpha} \Delta y_t - \frac{\beta_2}{1-\alpha} \Delta x_t + \frac{1}{1-\alpha} e_t \\ &= \theta x_t + u_t \end{aligned}$$

This is the cointegrating regression.

We note that

$$u_t = -\frac{\alpha}{1-\alpha} \Delta y_t - \frac{\beta_2}{1-\alpha} \Delta x_t + \frac{1}{1-\alpha} e_t$$

As Δy_t and Δx_t are I(0), u_t is I(0) too.

$\theta = \frac{\beta_1+\beta_2}{1-\alpha}$ is the long-run multiplier of x_t on y_t.

Consequently, the cointegrating vector is $(1, -\theta)$

(ii)

$$\Delta y_t = \alpha \Delta x_t - (1-\alpha)(y_{t-1} - \theta x_{t-1}) + e_t$$

This is called an ERROR CORRECTION model.

It shows how y_t responds in the short run to changes in x_t and to deviations from long-run equilibrium $y_{t-1} - \theta x_{t-1}$.

Cointegration analysis

The aim of cointegration analysis is to start with a vector of I(1) variables z_t and to discover how many cointegrating vectors there are. Then to build error correction models for each of these variables. See Johansen (1988).

The analysis starts with a VAR for z_t

$$A(L)z_t = e_t$$

If r cointegrating vectors exist then we can define the $n \times r$ matrices α and β, where β are the cointegrating vectors and α is the loading matrix. Thus

$$\beta' z_t \ \sim I(0)$$

We can now re-write $A(L)$ as

$$\begin{aligned} A(L) &= [I - A^*(L)](I - L) + A(1)L \\ A(1) &= -\alpha\beta' \end{aligned}$$

and hence

$$\Delta z_t = \alpha\beta' z_{t-1} + A^*(L)\Delta z_{t-1} + e_t$$

This is known as a COINTEGRATED VAR or CVAR.

We note that $A(1) = -\alpha\beta'$ is not of full rank as $rk(\alpha\beta') = r$, $(0 \leq r \leq n-1)$.

If we define the cointegrating error

$$w_t = \beta' z_t$$

then the CVAR can also be written as

$$\Delta z_t = \alpha w_{t-1} + A^*(L)\Delta z_{t-1} + e_t$$

As α is an $n \times r$ matrix, in general, each element of w_{t-1} appears in each equation.

In practice, it may turn out that some of the elements of α are zero so that not every cointegrating vector appears in every equation. We shall examine this point later.

Note that each of variables appearing in this equation are I(0), either due to first differencing or to taking linear combinations of variables that are stationary.

Thus if we know the CV β, we are back in the world of modeling stationary data. This is one of the main points of using the CVAR.

If $r = 0$, i.e. there are no cointegrating vectors, then the only way to make the variables stationary is to take first differences. The original model in levels then becomes

$$\Delta z_t = A^*(L)\Delta z_{t-1} + e_t$$

Consequently, an important issue for VAR modeling when some of the variables are I(1) is whether, rather than use a VAR in levels of the original variables, we should use a cointegrated VAR, or a VAR in first differences.

A fundamental identification problem for the CVAR

When econometricians first began to realise the profound consequences for economic modeling, estimation and inference of having non-stationary data the initial response was that it helped to explain various anomalous results not previously explicable.

For example, for thirty years one of the main concerns in econometrics was to devise a better estimator for simultaneous equation models than OLS as it was widely acknowledged that OLS had major faults: it was biased and inconsistent.

However, a careful look at the results often showed that the new estimates based on 2SLS, 3SLS, IV or FIML were very similar to the OLS estimates. Why was this?

The answer lies in the super-consistency property of models with I(1) variables.

An implication is that in theory it is possible to omit all I(0) from the model and just carry out estimation using the I(1) variables - in levels. The resulting estimates are super-consistent despite the misspecification.

The aim of using a simultaneous equation estimator is to take account of the correlation between an explanatory variable and the equation disturbance term so that it does not cause inconsistency. We can think of the problem as arising due to an "errors in variables" phenomenon. A typical endogenous variable in a system is affected by all of the shocks to the system and hence is correlated with the disturbances in each equation which are functions of the shocks.

It transpires, however, that these shocks are all I(0) and hence any correlation between the disturbances and the I(1) endogenous variables can therefore be ignored. This means that we can return to using OLS without causing too much deterioration in the properties of the estimates.

A similar revelation seemed to flow from the notion of cointegration.

A CV contains important information about long-run relations and this helps model building and estimation.

In particular, it was widely thought that one could use the CVs to uncover long-run economic relations without the need to model the short-run dynamics fully. And it is the long run where economic theory has most to say.

This follows from the two ways above of writing a dynamic model, the use of cointegrating regressions which can omit the dynamics, and from cointegration analysis.

All one had to do was to find (or state) which variables were I(1) and then carry out a cointegration analysis to determine how many long-run relations can be derived and what these relations are. Just press the button!

This reversed the previous methodology where first much effort was put into getting the short-run dynamics right. It was then possible to derive the long-run properties.

However, Wickens (1996) pointed out that there is a fundamental identification problem with cointegration analysis. Although cointegration analysis can tell us how many CVs there are, the problem is that the CVs are not in general long-run economic relations due to an identification problem.

Although the CVs are unique in a statistical sense, they are not unique in an economic sense.

The reason for this is that α and β are not of full rank. As result it is possible to define an infinite number of non-singular $r \times r$ matrices H such that

$$
\begin{aligned}
-A(1) &= \alpha\beta' \\
&= \alpha H^{-1} H \beta' \\
&= \alpha^* \beta^{*\prime}
\end{aligned}
$$

where

$$\beta^* = \beta H'$$

is another set of CVs that satisfies the model.

In other words we could write the model as

$$\begin{aligned}\Delta z_t &= \alpha^* \beta^{*\prime} z_{t-1} + A^*(L)\Delta z_{t-1} + e_t \\ &= \alpha^* w^*_{t-1} + A^*(L)\Delta z_{t-1} + e_t\end{aligned}$$

where

$$w^*_t = \beta^{*\prime} z_t$$

is another set of long run relations (cointegrating residuals) consistent with the data.

Worse still, the computer program chooses an H matrix but does not reveal what it is. So the estimates reported are scrambled already.

The question now is how can we avoid this and obtain CVs that we know are the long run economic relations we are looking for?

The answer is that we must impose this knowledge *a priori*.

The SEM and the CVAR

Suppose we start with a structural equation system that incorporates our economic knowledge

$$\begin{aligned} B(L)y_t + C(L)x_t &= u_t \\ y_t, x_t &\sim I(1),\ u_t \sim i.i.d(0, \Sigma) \end{aligned}$$

The long-run model is then

$$\begin{aligned} B(1)y_t + C(1)x_t &= \bar{u}_t \\ \bar{u}_t &\sim I(0) \end{aligned}$$

and $\begin{bmatrix} B(1) & C(1) \end{bmatrix}$ are the long-run structural coefficients

Note: $B(1) = \sum_i B_i$.

The reduced form is

$$y_t + \Pi_1(L)y_{t-1} + \Pi_2(L)x_t = v_t$$

In the long run it is

$$\begin{aligned} y_t + \Pi x_t &= \bar{v}_t \\ \Pi &= B(1)^{-1}C(1) \end{aligned}$$

We can separate the long-run from the short-run structure of the SEM by re-writing it as

$$B^*(L)\Delta y_t + C^*(L)\Delta x_t + B(1)y_{t-1} + C(1)x_{t-1} = u_t$$

or as

$$B^*(L)\Delta y_t + C^*(L)\Delta x_t + B(1)(y_{t-1} + \Pi x_{t-1}) = u_t$$

We now add equations for the weakly exogenous variables x_t

$$D(L)\Delta x_t + E(L)\Delta y_{t-1} = \varepsilon_t$$

Notice that all of the variables are first differences; there are no variables in levels. This is because cointegrating relations imply a long-run structure, but being exogenous variables they can have no long-run structure otherwise they would have had to have been treated as structural equations and be determined by the model.

The two sets of equations can be written as a single system

$$\begin{bmatrix} B^*(L) & C^*(L) \\ E(L)L & D(L) \end{bmatrix} \begin{bmatrix} \Delta y_t \\ \Delta x_t \end{bmatrix} + \begin{bmatrix} B(1) & C(1) \\ 0 & 0 \end{bmatrix} \begin{bmatrix} y_{t-1} \\ x_{t-1} \end{bmatrix} = \begin{bmatrix} u_t \\ \varepsilon_t \end{bmatrix}$$

This can be written as a VAR in levels or a CVAR

$$\begin{aligned} A(L)z_t &= e_t \\ A^*(L)\Delta z_t + Az_{t-1} &= e_t \\ A^*(L)\Delta z_t &= \alpha\beta' z_{t-1} + e_t \end{aligned}$$

where

$$\begin{aligned} A(L) &= A^*(L)(1-L) + AL \\ A &= \begin{bmatrix} B(0) & C(0) \\ 0 & I \end{bmatrix}^{-1} \begin{bmatrix} B(1) & C(1) \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} I + \Pi_1(1) & \Pi_2(1) \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} I \\ 0 \end{bmatrix} \begin{bmatrix} I + \Pi_1(1) & \Pi_2(1) \end{bmatrix} \\ &= -\alpha\beta' \end{aligned}$$

This shows why it is that the matrix A is not of full rank n but has rank r:

it is the product of

(i) an $n \times r$ matrix of known coefficients $\begin{bmatrix} I \\ 0 \end{bmatrix}$, and

(ii) an $r \times n$ matrix of long-run reduced-form coefficients $\begin{bmatrix} I + \Pi_1(1) & \Pi_2(1) \end{bmatrix}$ which form the CVs.

These matrices are not uniquely defined as we can write

$$\begin{aligned} A &= -\alpha H^{-1} H \beta' \\ &= \begin{bmatrix} H^{-1} \\ 0 \end{bmatrix} H \begin{bmatrix} I + \Pi_1(1) & \Pi_2(1) \end{bmatrix} \end{aligned}$$

If we choose

$$H = B(0)$$

then

$$\begin{aligned} A &= \begin{bmatrix} B(0)^{-1} \\ 0 \end{bmatrix} B(0) \begin{bmatrix} I + \Pi_1(1) & \Pi_2(1) \end{bmatrix} \\ &= \begin{bmatrix} B(0)^{-1} \\ 0 \end{bmatrix} \begin{bmatrix} B(1) & C(1) \end{bmatrix} \end{aligned}$$

The CVs are now the long-run structural coefficients.

To repeat, any choice of H gives a valid CVAR, but only particular choices enable us to interpret the resulting CVs. In practice the Johansen program selects an H matrix on statistical grounds and in consequence the CVs will be neither the long-run structural or reduced-form coefficients, but an unknown (and unknowable) linear combination of them.

The only exception is where $r = 1$, i.e. there is only one CV. In this case, the CV is unique.

Conclusion

1. We can use a CVAR to discover how many CVs there are in the data.

2. We must then pre-specify the long-run structure of the model and either

(i) impose these restrictions when obtaining the FIML estimates

(ii) or obtain prior estimates of the long-run coefficients and use these to construct w_t.

We can then estimate the model by OLS.

The CVAR and system shocks

One of the main aims of VAR analysis in macroeconomics is to identify the shocks affecting the variables and to compute the effects of these shocks on the variables, i.e. the impulse response functions.

We can show, however, that even if we have identified the CVs, we still cannot identify the shocks.

In order to analyse the effects of the shocks in a VAR based on the variables z_t we need to invert the VAR and represent it as a vector moving average (VMA). We need to do the same in the case of a CVAR. This is known as the Granger Representation Theorem.

There is, however, an important difference: the VMA is in terms of Δz_t and not z_t.

The CVAR can be written

$$A(L)z_t = e_t$$

Inverting this using the Granger Representation Theorem gives the VMA

$$\Delta z_t = R(L)e_t$$

This can be re-written as

$$\begin{aligned} \Delta z_t &= R(1)e_t + R^*(L)\Delta e_t \\ &= \gamma\theta' e_t + R^*(L)\Delta e_t \end{aligned}$$

where the $n \times n$ matrix $R(1)$ is of rank $n - r$ as it can be factorized into

$$R(1) = \gamma\theta'$$

where γ and θ are $n \times (n - r)$ matrices.

For the same reason that α and β are not uniquely defined, neither are γ and θ as there are an infinite number of $(n-r) \times (n-r)$ matrices Q that satisfy

$$R(1) = \gamma Q^{-1} Q \theta'$$

If we define

$$\Delta \tau_t = e_t$$

then τ_t is I(1) and we can write

$$\Delta z_t = \gamma \theta' \Delta \tau_t + R^*(L) \Delta e_t$$

Integrating to get back to z_t gives

$$z_t = z_0 + \gamma \theta' \tau_t + R^*(L) e_t$$

Thus we have decomposed z_t into two components:

(i) $z^T = z_0 + \gamma \theta' \tau_t$ which is I(1)

(ii) $R^*(L) e_t$ which is I(0).

The first component z_t^T is the trend in z_t. It is not, however, deterministic but stochastic.

Each element of z_t consists of a linear combination of $n - r$ I(1) variables $\theta' \tau_t$.

These are known as the COMMON STOCHASTIC TRENDS (CSTs).

They capture the effects of productivity and money supply shocks etc.

They are also the permanent shocks to z_t. It can be shown that there are $n - r$ of them.

The presence of permanent shocks explains why z_t is I(1).

The second component explains deviations of z_t from its stochastic trend

They are the cyclical component of z_t.

These are the temporary shocks to z_t. It can be shown that there are r of them.

These shocks are I(0).

We now look more closely at the permanent and temporary shocks and prove how many of each there are and what properties they possess.

The SEM and the shocks

We base our analysis on the structural system stated above.

Wickens (1996) shows that

1. There is no unique way to obtain γ and θ from α and β.
2. Thus the CSTs are not uniquely defined.
3. The CSTs are linear functions of the shocks to the exogenous variables.
4. Thus the source of non-stationarity is non-stationary exogenous variables (this excludes internal dynamics).

It is possible to write the A matrix defined for the CVAR as the product of three $n \times n$ matrices:

$$A = \begin{bmatrix} B(0) & C(0) \\ 0 & I \end{bmatrix}^{-1} \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} B(1) & C(1) \\ J & K \end{bmatrix}$$

where J and K are arbitrary (i.e. any) matrices (K must be non-singular)

It can be shown that this implies that $R(1)$ in the VMA can be written

$$\begin{aligned} R(1) &= \begin{bmatrix} B(1) & C(1) \\ J & K \end{bmatrix}^{-1} \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} B(0) & C(0) \\ 0 & I \end{bmatrix} \\ &= \{ \begin{bmatrix} B(1) & C(1) \\ J & K \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ I \end{bmatrix} \} \begin{bmatrix} 0 & I \end{bmatrix} \\ &= \gamma\theta' \end{aligned}$$

Hence, $\theta' = \begin{bmatrix} 0 & I \end{bmatrix}$ is an $n \times (n - r)$ matrix.

The CSTs are therefore

$$\theta' \tau_t = Q \begin{bmatrix} 0 & I \end{bmatrix} \tau_t = Q \tau_{2t} = Q \textstyle\sum_1^t \varepsilon_s$$

where ε_t are the shocks to the exogenous variables and there $n - r$ of them (the number of exogenous variables).

We have therefore shown that the CSTs derive solely from the exogenous variables. The non-stationarity of the endogenous variables y_t is due to the effects on them of the exogenous variables.

Further, the permanent shocks to the endogenous variables are also due to the shocks to the exogenous variables. The exogenous shocks also have temporary effects.

In contrast, the shocks to the structural equations only have temporary effects.

Non-stationary variables without cointegration

It is possible that the variables are I(1) but there are no cointegrating vectors. In this case no long-run relations are defined. In effect, we can set $\beta = 0$. The CVAR then becomes (including a possible constant)

$$A(L)\Delta z_t = \mu + e_t, \qquad A_0 = I,\ e_t \sim i.i.d(0, \Sigma)$$

Thus the appropriate model is a VAR in first differences of the variables in which the roots of $|A(L)| = 0$ all lie outside the unit circle.

We can re-write this as the VMA

$$\begin{aligned}\Delta z_t &= A(1)^{-1}\mu + A(L)^{-1}e_t = \nu + C(L)e_t \\ &= \upsilon + C(1)e_t + C^*(L)\Delta e_t\end{aligned}$$

where now $C(1)$ is of full rank.

If $\Delta\tau_t = e_t$ then we can solve for z_t as

$$z_t = z_0 + \nu t + C(1)\tau_t + C^*(L)e_t$$

where

$$z_t^T = z_0 + \nu t + C(1)\tau_t$$

is the trend in z_t. The first two terms in the trend form a linear trend; the last term is the stochastic trend. In this case there are now as many independent stochastic trends as there are variables.

De-trend (filter) the data to transform them to stationarity

The CVAR deals with the non-stationarity of the data by transforming the variables into stationary variables either by taking a linear combination of them that is stationary (the cointegrating residuals) or by taking first differences.

Whilst first differences are in common use, surprisingly, perhaps, instead of using cointegration, it is more usual to transform the data to stationarity by de-trending them. The resulting data are then deviations about trend.

Such a transformation of the data is called filtering the data.

Taking first differences is also an example of applying a stationary inducing filter.

A popular de-trending scheme is the Hodrick-Prescott (HP) filter. This is now standard on most econometric packages.

The aim is to decompose z_t into two components: a trend μ_t and a "cycle" or deviation from trend c_t. Thus

$$z_t = \mu_t + c_t$$

It can be shown that μ_t is I(1) and c_t is I(0).

The HP filter is a two-sided symmetric moving average of the original data, z_t.

If μ_t denotes the HP trend then

$$\begin{aligned}\mu_t &= \theta(L)z_t = \textstyle\sum_{s=-n}^{n}\theta_s z_{t-s}, \quad \sum\theta_s = 1, \quad \theta_s = \theta_{-s} \\ \theta(L) &= \textstyle\sum_{s=-n}^{n}\theta_s L^s = \theta_0 + \sum_{s=1}^{n}\theta_s(L + L^{-1})\end{aligned}$$

The HP filter solves the following problem

$$\min_{\mu_t} \textstyle\sum_{t=1}^{T}[(z_t - \mu_t)^2 + \lambda(\Delta^2\mu_{t+1})^2]$$

where λ is a pre-specified number.

If $\lambda = 0$ then $\mu_t = z_t$.

As $\lambda \to \infty$ we find $\mu_t \to$ a linear trend.

Thus by making λ smaller, μ_t is made to follow z_t more closely.

The choices for λ that are usually recommended, although these are completely arbitrary, are 100 for annual data, 1600 for quarterly data and 14400 for monthly data.

The trend μ_t is the solution of the MA

$$z_t = \lambda\mu_{t+2} - 4\lambda\mu_{t+1} + (1+6\lambda)\mu_t - 4\lambda\mu_{t-1} + \lambda\mu_{t-2}$$

This can be re-written as

$$z_t - \mu_t = \lambda(1-L)^4\mu_{t+2} = \lambda\Delta^4\mu_{t+2}$$

Thus the trend is I(1) and the de-trended series is I(0) and is proportional to the 4^{th} difference of the trend. In other words, μ_t is a 4^{th} difference even though z_t may only be I(1). z_t is therefore over-differenced. This induces cycles in the de-trended series.

It follows that c_t is

$$\begin{aligned} c_t &= y_t - \mu_t \\ &= \lambda\Delta^4(z_{t+2} - c_{t+2}) \\ &= \lambda(\Delta^4 z_{t+2} - \Delta^4 c_{t+2}) \end{aligned}$$

or

$$\Delta^4 z_t = \Delta^4 c_t + \lambda c_{t-2} = \varphi(L)c_t$$

Thus employing an HP filter entails specifying a time series model for z_t that requires 4^{th} differencing.

The consequence of replacing z_t by the trend-free process (or cycle c_t) is to induce cycles into the data where none may have existed before.

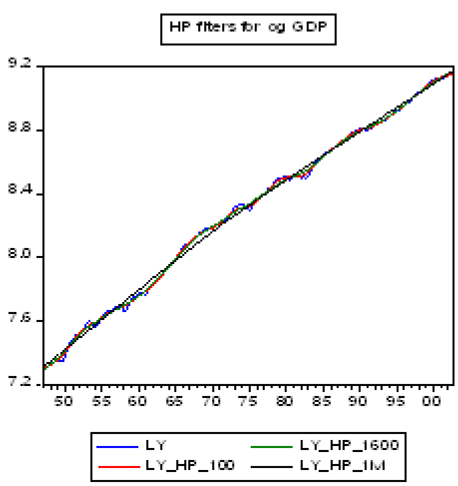
This means that the impulse response functions of data de-trended by the HP filter may look different from those de-trended by other means such as a simple low-order polynomial of time.

Another disadvantage of the HP filter compared with a polynomial in time is that the HP filter cannot be extrapolated out of sample.

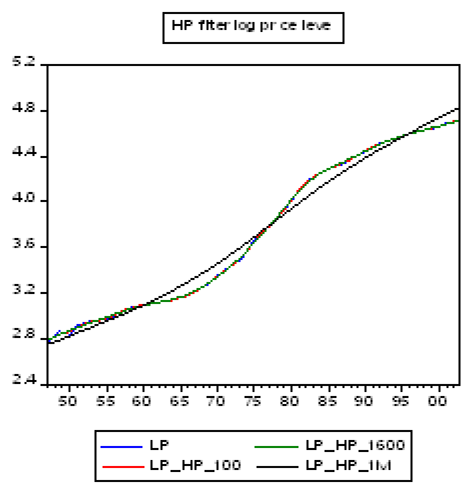
Example

We examine the use of the HP filter for quarterly data on log US GDP and the log US price deflator 1947.1- 2002.4.

The following graphs compare the HP trends when $\lambda = 100$, 1600 and 1000000.



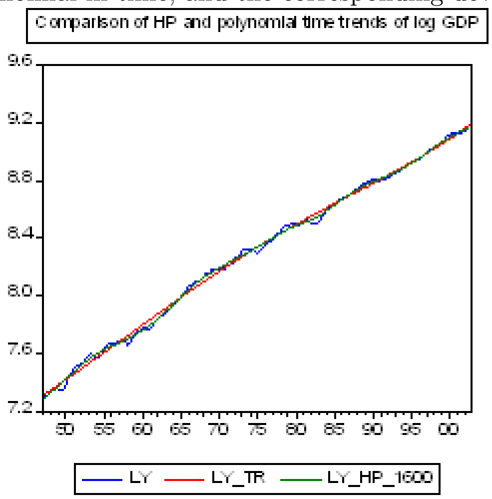
logGDP

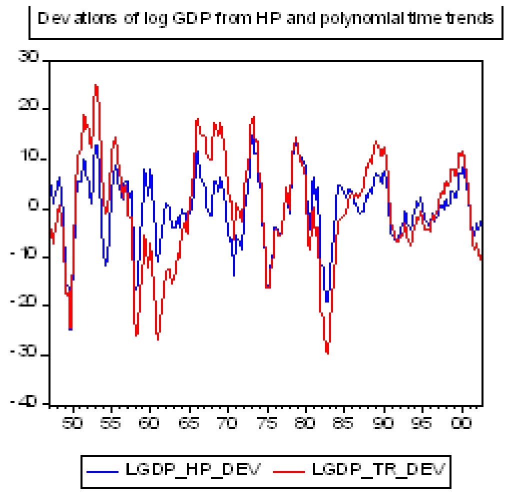


logP

The HP filters with $\lambda = 100$ and 1600 track the original series much more closely than that with $\lambda = 1000000$. It is especially noticeable for Log GDP that the HP filter with $\lambda = 100$ tracks the original series too closely to be a satisfactory trend.

Now compare the trends of log GDP based on an HP filter with $\lambda = 1600$ and a 4^{th}-order polynomial in time, and the corresponding deviations from trend





The polynomial time trend tracks log GDP somewhat less closely than the HP trend. As a result, the deviations from trend of the polynomial have a slightly larger amplitude. The correlations between the trends is 0.999 and between the deviations is 0.798. When $\lambda = 1000000$, the correlation between the deviations with the polynomial time trend is 0.958. Arguably, therefore, an HP filter with a much high value of λ than 1600 might be a better representation of the trend in the sense that it tracks short-run movements less closely yet still captures the long run.

An aside on the benefits of stabilisation v. growth

Much of macroeconomics, and especially macroeconomic policy, is concerned with stabilisation. How much benefit does stabilisation really bring?

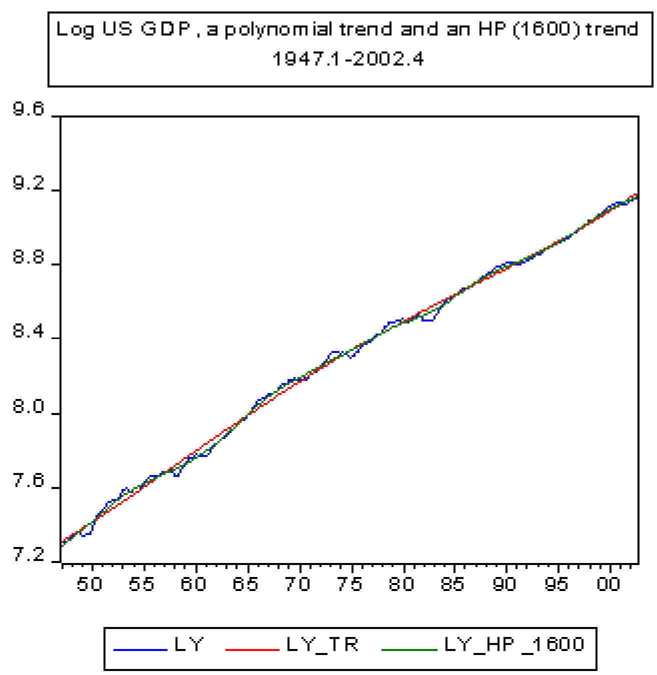
A classical decomposition of an economic time series such as log GDP is:

$$\ln GDP_t = trend + cycle + seasonal + irregular\ error$$

How much of the variance of log GDP is explained by each of these?

Stabilisation is important if the cycle explains a large proportion.

As the data are deseasonalised we exclude the seasonal component.



These components can be estimated from the equation ($y_t = \ln GDP_t$):

$$\begin{aligned} y_t &= \beta_0 + \beta_1 trend_t + \Sigma_{i=1}^{12} \Delta y_{t-i} + e_t \\ & \quad polynomial \\ trend_t &= HP\ filter \\ & \quad y_{t-1} \end{aligned}$$

The variance decomposition of y_t is:

Variance decomposition of log US GDP (%)

Period	1947.1 − 2002.4	1970.1 − 2002.4
	Polynomial trend	
Trend	99.73	99.34
Cycle	0.12	0.46
Error	0.15	0.20
Total	100.00	100.00
	HP 1600 trend	
Trend	99.90	99.66
Cycle	0.06	0.24
Error	0.04	0.10
Total	100.00	100.00
	Stochastic trend	
Trend	99.96	99.91
Cycle	0.01	0.09
Error	0.03	0.00
Total	100.00	100.00

Clearly, there is very little output gain in removing the cycle entirely.

A much larger output gain would result from a higher long-run rate of growth.

Some words of caution about the methodology.

1. It might be better to compare the loss of cumulative output over time as a result of not operating the economy at full capacity. For the UK, the benefits of stabilisation in which the economy then operates at full capacity are about 5% of cumulative output.

2. The long-run rate of growth might be affected by the presence of cycles;

- for example, cycles and hence uncertainty cause a reluctance to invest.

3. In the long run log GDP has no finite variance.

Conclusions

1. There is a fundamental lack of identification of the shocks in both VAR and CVAR analysis.

2. This makes it very difficult to give an economic interpretation to the estimated shocks.

3. New problems arise in trying to cope with non-stationary data.

4. What to do depends in part on whether the data are cointegrated or not,

and whether one wishes to take account of long-run relations between variables implied by cointegration, or deal with non-stationarity by filtering the data.

We examine this issue in more detail when we consider numerical examples of VAR modelling later in the course.