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Ranking Sets of Characteristics

Abstract This paper uses Lancaster's characteristics approach in order to rank sets of alternative combinations of commodity characteristics. It is assumed that there exists a reference point or reference locus from which the individual evaluates set expansions in north–east direction. The direction of desirable expansion can be the full 90 degree angle, but the angle can also be limited to a particular direction. We also consider the case where the reference level expands to a set of points. We provide axiomatic characterizations for the various cases.

Keywords: ranking of sets, commodity characteristics, reference level

JEL Classification Numbers: D10, D62, D63, D71, D80

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by
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1 Introduction

In microeconomics, sets of alternatives are normally evaluated via families of non–intersecting indifference curves. An immediate question is: whose indifference curves? The answer is more or less unanimous: it should be the indifference curves of the concerned individual and not of somebody who pretends to know better. But there still are some queries. Should the individual's indifference curves stem from an instantaneous or myopic utility function or preference relation or should they come from a long–range utility function which takes account of preference changes and other intertemporal aspects? An answer to this question might be that much depends on what the analysis is used for.

A quite different though related aspect concerning the evaluation of sets of alternatives is the following. It is argued that the point or points chosen via a set of indifference curves do not adequately reflect the richness of opportunities the individual experiences when making his or her choice. The relatively new freedom–of–choice literature focusses on this point and then takes off in various directions.

The standard argument to show that the richness of choice is being neglected in conventional analysis can be made very easily in the following way. Imagine that the individual considered has a simple utility function of the following kind: $u(x) = x_1 \cdot x_2$ for two commodities in quantities x_1 and x_2 (the argument can easily be generalized to higher dimensions). Let us further assume that prices are such that $p_1 = p_2 = 1$ and the budget comprises two units of money. Then the optimal, i.e. utility maximizing allocation is $x^* = (1, 1)$. Now imagine that our consumer does not have the infinitely many options of choice provided by the budget set just described but is simply offered the vector $x = (1, 1)$ for his money. Does this make a difference? The freedom–of–choice literature gave an affirmative answer and emphasized the intrinsic value of choice in great detail.

Once richness of choice is considered, an immediate question is how to measure this richness. If there is a finite number of discrete objects, counting numbers would be a possible option (Pattanaik and Xu (1990)). If richness of opportunities manifests itself in alternative budget sets in an n-dimensional Euclidean space, then a ranking rule that compares opportunity sets according to their volumes appears as a possibility (Xu (2004)). In both approaches, there is no discrimination among objects or, put differently, an equal treatment of objects. In other words, these approaches focus exclusively on the quantity aspect of opportunity sets.

They are non–preference–based. Of course, there are good arguments to consider the quality of alternatives as well (Sen (1991, 1993)). When poor alternatives are added to an already existing set of objects, and these poor alternatives are dominated by one or several of the existing objects in terms of quality with prices being roughly the same, then nothing valuable is added so that the richness of choice has not been increased at all. Therefore, Sen and others have argued in favour of a preference–based approach. Again, the question arises whose preferences and what kind of preference should count.

In this paper we wish to put forward an approach which is based on Lancaster's idea of looking at and evaluating characteristics (Lancaster (1966)). Our analysis will, therefore, be done in n-dimensional characteristics space. In contrast to Lancaster, we shall do without a utility function or preference relation defined on the space of characteristics. We shall require monotonicity with respect to characteristics which later on will be weakened to apply only to certain "directions" within the characteristics space. At the beginning, however, we shall make the standard assumption in microeconomics that component-wise more is preferred to less. The individual who we consider will evaluate alternative sets of opportunities (in terms of characteristics) from a vantage point that we shall call a reference point. So in a certain sense, we are using elements from the concept of boundedly rational behaviour (Simon (1957)). Our consumer views his opportunities from the vantage point of an already realized position that could be interpreted as a status quo or – alternatively – as a point of minimal achievements below which life becomes unpleasant or miserable. The individual then explores his or her possibilities in "north–east" direction where, as just stated, all characteristics which can be attained through a purchase of certain commodities are equally desirable. The expansion north–east should be made as large as possible, given the financial "capabilities" of the individual. If one situation is finally declared to be better than another, certain comparisons among different combinations of characteristics must have been possible and the units among the different characteristics must have been rendered commensurable. This is a basic supposition we have to make. The easiest case of comparison and one that we shall consider while we go along is the one where for a given budget and given commodity prices, one set of characteristics combinations completely lies inside an alternative set so that the latter can undoubtedly be considered as better than the former. If such a situation would always or often come about, "life" would be much easier. And the instruments to make set comparisons would be much simpler. But it is our conviction that such situations will be extremely rare. Therefore, more general cases have to be tackled. Let us now go into *medias res.*

In section 2, we introduce our basic notation and some definitions. Section 3 discusses certain axiomatic properties that we need for our first characterization results presented in section 4. Section 5 considers asymmetric expansions from the reference point. Section 6 introduces the concept of a reference level that contains more than one point. We end with some concluding remarks in section 7.

2 Basic Notation and Definitions

Let $\mathbb{R}_+$ be the set of all non-negative real numbers, $\mathbb{R}_{++}$ be the set of all positive numbers, $\mathbb{R}^n_+$ be the n-fold Cartesian product of $\mathbb{R}_+$, and $\mathbb{R}^n_{++}$ be the n-fold Cartesian product of $\mathbb{R}_{++}$. The vectors in $\mathbb{R}^n_+$ will be denoted by $x, y, z, a, b, \cdots$, and are interpreted as vectors of characteristics (Lancaster (1966)). For all $x = (x_1, \cdots, x_n), y = (y_1, \cdots, y_n) \in \mathbb{R}^n_+$, define $x \geq y$ as $x_i \geq y_i$ for all $i = 1, \ldots, n$, $x > y$ when $x_i \geq y_i$ for all $i = 1, \cdots, n$ and $x_j > y_j$ for some $j \in \{1, \cdots, n\}$, and $x \gg y$ when $x_i > y_i$ for all $i = 1, \cdots, n$.

There are perhaps several ways to measure the achievements that an agent makes when moving from a vector of characteristics x to another vector y. In this paper, we shall use the notion of a distance function to capture the progress made by the individual. For this purpose, we formally define a distance function between two vectors. For all $x, y \in \mathbb{R}^n_+$, the distance between x and y is a function $d : \mathbb{R}^n_+ \to \mathbb{R}_+$ such that, for all $x, y, z \in \mathbb{R}^n_+$,

(i) $d(x, y) = 0$ iff $x = y$;

(ii) $d(x, y) = d(y, x)$;

(iii) $d(x, z) \leq d(x, y) + d(y, z)$.

To ensure the robustness of our proposed measure of achievement with respect to a chosen distance function, in this paper, we shall consider a class of distance functions. In particular, we shall focus on the class of distance functions, to be denoted by $\mathcal{D}$, such that for each $d \in \mathcal{D}$, d satisfies two additional properties:

(iv) for all $x, y, z \in \mathbb{R}^n_+$, if $x > y > z$ then $d(x, z) \geq d(y, z)$, and if $x \gg y > z$ then $d(x, z) > d(y, z)$;

(v) for all $x \in \mathbb{R}^n_+$ and all $t \geq 0$, the set $\{a \in \mathbb{R}^n_+ | a \geq x, d(a, x) \leq t\}$, to be denoted by $X(x, t, d)$, is closed and bounded.

There are various distance functions that satisfy properties (i)–(v), for example the class $d_p(x, y) = (\sum_{i=1}^n |x_i - y_i|^p)^{1/p}$, with $1 \leq p < \infty$, the commonly used Euclidean distance $d_e(x, y) = \sqrt{\sum_{i=1}^n (x_i - y_i)^2}$ and the distance function $d_1(x, y) = (\sum_{i=1}^n |x_i - y_i|)$ as particular cases. But also the function $d_2(x, y) = \max_{i \in \{1, \ldots, n\}} \{|x_i - y_i|\}$ would be a distance function in our sense.

At any given point of time, the set of all vectors that may be available to the individual is a subset of $\mathbb{R}^n_+$. Such a set will be called the individual's *characteristics set*. We will use A, B, C, etc. to denote such sets.

Our concern in this paper is to rank different characteristics sets in terms of the achievements that they offer to the individual. In particular, we confine our attention to sets that are

(2.1) *compact*: a set $A \subseteq \mathbb{R}^n_+$ is compact iff A is closed and bounded, (2.2) *convex*: a set $A \subseteq \mathbb{R}^n_+$ is convex iff, for all $x, y \in \mathbb{R}^n_+$ and all $\alpha \in [0,1]$, if $x, y \in A$, then $\alpha x + (1-\alpha)y \in A$,

(2.3) *star-shaped*: a set $A \subseteq \mathbb{R}^n_+$ is star-shaped iff, for all $x \in \mathbb{R}^n_+$ and all $t \in [0,1]$, if $x \in A$, then $tx \in A$.

Let $\mathcal{K}$ be the set of all characteristics sets that are compact, convex and star-shaped. For all $A, B \in \mathcal{K}$, we write $A \subseteq B$ for "A being a subset of B" and $A \subset B$ for "A being a proper subset of B".

For all $A, B \in \mathcal{K}$ and all $x^* \in \mathbb{R}^n_+$, let $A >_{x^*} B$ denote: [whenever $x^* \in B$, for each $d \in \mathcal{D}$, there is a neighborhood, $X(x^*, \epsilon_d, d) = \{x \in \mathbb{R}^n_+ : x \geq x^*, d(x, x^*) \leq \epsilon_d\}$ where $\epsilon_d > 0$ of x^* such that $X(x^*, \epsilon_d, d) \subseteq A$] and [for all $b \in B$ with $b > x^*$, there exists $a \in A$ such that $a \gg b$]. Let $x^0 \in \mathbb{R}^n_{++}$ be a vector of minimal achievements below which the individual's situation is judged to be "poor" or unsatisfactory. We shall call x^0 a reference point. Throughout the following sections, we assume that x^0 is fixed. For all $t \geq 0$ and all $d \in \mathcal{D}$, define

$$X(x^0, t, d) = \{x \in \mathbb{R}^n_+ : x \geq x^0, d(x, x^0) \leq t\}.$$

Scalar t measures the distance between two vectors in characteristics space, according to the distance function $d \in \mathcal{D}$. This, of course, presupposes that, depending on a distance function $d \in \mathcal{D}$, we can quantify each of the characteristics appropriately so that there is a measurement scale common to all characteristics considered.

For all $A \in \mathcal{K}$ and all $d \in \mathcal{D}$, let

$$r(A, d) = \begin{cases} -1 & \text{if } x^0 \notin A, \\ \max_t \{t \in \mathbb{R}_+ : \{x \in \mathbb{R}^n_+ : x \geq x^0, d(x, x^0) \leq t\} \subseteq A\} & \text{if } x^0 \in A. \end{cases}$$

We note that, for all $A \in \mathcal{K}$, if $x^0 \notin A$, then $r(A, d) = -1$ for all $d \in \mathcal{D}$, and if $x^0 \in A$, then $r(A, d) \geq 0$ for all $d \in \mathcal{D}$.

Figure 1 depicts the maximal $t \in \mathbb{R}_+$ for two sets of characteristics A and B when $x^0 \in A \cap B$ and $\mathcal{D}$ contains a single distance function given by the Euclidean distance function.

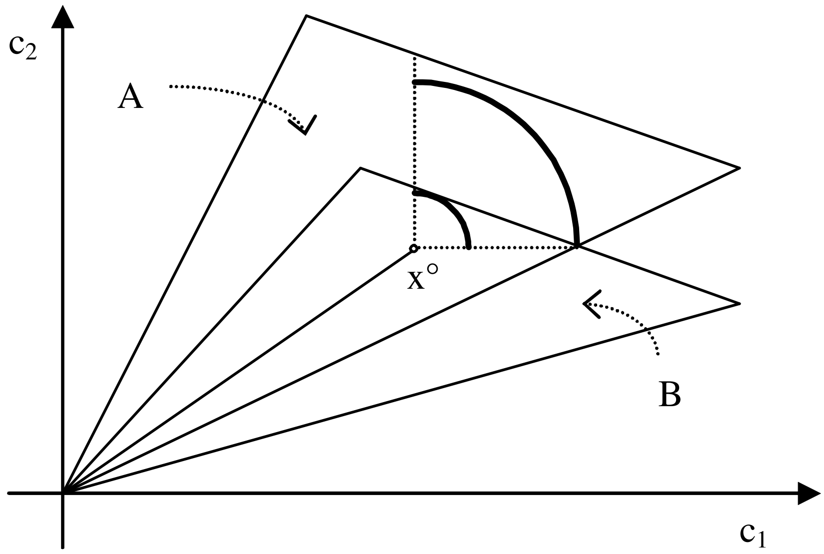

Figure 1: comparison of two characteristics sets A and B

Let $\succeq$ be a binary relation over $\mathcal{K}$ that satisfies *reflexivity:* [for all $A \in \mathcal{K}, A \succeq A$], *transitivity:* [for all $A, B, C \in \mathcal{K}$, if $A \succeq B$ and $B \succeq C$ then $A \succeq C$]. Such a binary relation is called a *quasi-ordering*. When a quasi-ordering satisfies *completeness:* [for all $A, B \in \mathcal{K}$ with $A \neq B, A \succeq B$ or $B \succeq A$], it will be called an *ordering*. The intended interpretation of $\succeq$ is the following: for all $A, B \in \mathcal{K}, [A \succeq B]$ will be interpreted as "*the degree of achievements in terms of characteristics offered by A is at least as great as the degree of achievements offered by B*". $\succ$ and $\sim$, respectively, are the asymmetric and symmetric part of $\succeq$, and are defined as follows: for all $A, B \in \mathcal{K}$, $A \succ B$ iff $A \succeq B$ and not $B \succeq A$, and $A \sim B$ iff $A \succeq B$ and $B \succeq A$.

3 Axiomatic Properties

This section discusses several axioms that are used for characterizing a class of measures of well–being in terms of achievements of characteristics.

Definition 3.1. $\succeq$ over $\mathcal{K}$ satisfies

(3.1.1) **Monotonicity** iff, for all $A, B \in \mathcal{K}$, if $B \subseteq A$ then $A \succeq B$.

(3.1.2) **Betweenness** iff, for all $A, B \in \mathcal{K}$, if $A \succ B$ with $x^0 \in A \cap B$, then there exists $C \in \mathcal{K}$ such that $C >_{x^0} B$ and $A \succ C \succ B$.

(3.1.3) **Dominance** iff, for all $A, B \in \mathcal{K}$, if $x^0 \notin B$, then $A \succeq B$, and furthermore, if $x^0 \in A$, then $A \succ B$.

(3.1.4) **Symmetric Expansion from the Reference Point** iff, for all $A, B \in \mathcal{K}$, if, for each $d \in \mathcal{D}$, there exists $t_d > 0$ such that $X(x^0, t_d, d) \cap A = X(x^0, t_d, d)$, and $B \cap X(x^0, t_d, d) \subset X(x^0, t_d, d)$, then $A \succ B$.

The intuition behind Monotonicity is simple and easy to explain. It requires that whenever B is a subset of A, then A is ranked at least as high as B in terms of achievements. Various versions of Monotonicity have been proposed in the literature; see among others, Pattanaik and Xu (2006), and Xu (2002; 2003). Betweenness requires that when A is judged to offer more achievements than B relative to the reference vector x^0, there must exist a set C such that $C >_{x^0} B$ and A offers more achievements than C, which in turn offers more achievements than B. Dominance requires that whenever the reference vector x^0 is not attainable in B, the level of achievements offered by B cannot be higher than that offered by any other set A, and furthermore, if the reference vector x^0 is attainable under A, then A offers a higher level of achievements than B. Symmetric Expansion from the Reference Point requires that, for two sets A and B, whenever A results from progress made, according to every distance function $d \in \mathcal{D}$, in all dimensions of characteristics vectors, while B does not offer this particular kind of progress, the level of achievements under A is judged to be higher than that offered by B.

4 First Characterization Results

In this section, we present an axiomatic characterization of a class of achievement rankings defined as below:

For all $A, B \in \mathcal{K}$,
if $[r(A, d) \geq r(B, d)$ for all $d \in \mathcal{D}]$ then not$(B \succ A)$;
if $[r(A, d) > r(B, d)$ for all $d \in \mathcal{D}]$ then $A \succ B$;
if $A = B$ or $[r(A, d) = r(B, d) = -1$ for all $d \in \mathcal{D}]$ then $A \sim B$.

We shall denote this class of achievement rankings as $\mathcal{B}^r$. It may be noted that, when $\mathcal{D}$ contains a single distance function, the set $\mathcal{B}^r$ is a singleton containing one ordering which is given by the following:
Let $\mathcal{D} = \{d\}$, then for all $A, B \in \mathcal{K}$,

$$A \succeq^r B \Leftrightarrow r(A, d) \geq r(B, d).$$

Theorem 4.1. Suppose $\succeq$ over $\mathcal{K}$ is a quasi-ordering. Then, $\succeq$ satisfies Monotonicity, Betweenness, Dominance, and Symmetric Expansion from the Reference Point if and only if $\succeq \in \mathcal{B}^r$.
Proof. It can be checked that each $\succeq \in \mathcal{B}^r$ is a quasi-ordering and satisfies

Monotonicity, Betweenness, Dominance and Symmetric Expansion from the Reference Point. We now show that if $\succeq$ over $\mathcal{K}$ satisfies Monotonicity, Betweenness, Dominance and Symmetric Expansion from the Reference Point, then $\succeq \in \mathcal{B}^r$.

(i) We first show that, for all $A, B \in \mathcal{K}$, if $A = B$ or $[r(A,d) = r(B,d) = -1$ for all $d \in \mathcal{D}]$, then $A \sim B$. Let $A, B \in \mathcal{K}$. When $A = B$, by reflexivity of $\succeq$, $A \sim B$ follows easily. Consider next that $[r(A,d) = r(B,d) = -1$ for all $d \in \mathcal{D}]$; that is, $x^0 \notin A$ and $x^0 \notin B$. Since $x^0 \notin A$, by Dominance, it follows that $B \succeq A$. Similarly, by Dominance and from $x^0 \notin B$, it follows that $A \succeq B$. Therefore $A \sim B$.

(ii) Second, we show that for all $A, B \in \mathcal{K}$, if $r(A,d) \geq 0 > r(B,d) = -1$, then $A \succ B$. Note that, in this case, it must be the case that $x^0 \in A$ and $x^0 \notin B$. By Dominance, $A \succ B$ follows easily.

(iii) Third, we show that for all $A, B \in \mathcal{K}$, if $[r(A,d) > r(B,d) \geq 0$ for all $d \in \mathcal{D}]$, then $A \succ B$. Let $A, B \in \mathcal{K}$ be such that $[r(A,d) > r(B,d) \geq 0$ for all $d \in \mathcal{D}]$. Then, for each $d \in \mathcal{D}$, there exists $t_d > 0$ such that $X(x^0, t_d, d) \cap A = X(x^0, t_d, d)$ and $B \cap X(x^0, t_d, d) \subset X(x^0, t_d, d)$. By Symmetric Expansion from the Reference Point, we obtain $A \succ B$.

(iv) We next show that, for all $A, B \in \mathcal{K}$, if $[r(A,d) \geq r(B,d)$ for all $d \in \mathcal{D}]$, then not$(B \succ A)$. Let $A, B \in \mathcal{K}$, and $[r(A,d) \geq r(B,d)$ for all $d \in \mathcal{D}]$. Note that if $r(B,d) = -1$ for some $d \in \mathcal{D}$, then $x^0 \notin B$. As a consequence, $r(B,d) = -1$ for all $d \in \mathcal{D}$. We have already dealt with these situations in (i) and (ii). Therefore, we assume that $r(B,d) \geq 0$ for all $d \in \mathcal{D}$. Hence, $x^0 \in A \cap B$. To show that not$(B \succ A)$ holds in this case, we use the proof by contradiction. Suppose not, that is, suppose that $B \succ A$ holds true. Then, by Betweenness, there exists $C \in \mathcal{K}$ such that $C >_{x^0} A$ and $B \succ C \succ A$. Since $C >_{x^0} A$, there exists $C' \in \mathcal{K}$ such that $C' \subseteq C$ and [for each $d \in \mathcal{D}$, there exists $t_d > 0$ such that $X(x^0, t_d, d) \cap C' = X(x^0, t_d, d)$ and $X(x^0, t_d, d) \cap A \subset X(x^0, t_d, d)$]. Then, for each $d \in \mathcal{D}$, $r(C', d) > r(A, d)$. Note that $r(A,d) \geq r(B,d)$ for all $d \in \mathcal{D}$. Therefore, $r(C',d) > r(B,d)$ for all $d \in \mathcal{D}$. From (iii) above, $C' \succ B$. On the other hand, from $C' \subseteq C$, by Monotonicity, $C \succeq C'$. The transitivity of $\succeq$ implies that $C \succ B$, a contradiction with $B \succ C$, which was established a few lines earlier. Therefore, not$(B \succ A)$.

(i) – (iv) complete the proof of Theorem 4.1. ■

When $\mathcal{D}$ contains a single distance function and $\succeq$ is an ordering, we have the following result, which is a corollary to Theorem 4.1.

Theorem 4.2. Suppose that $\succeq$ over $\mathcal{K}$ is an ordering and that $\mathcal{D}$ contains a single distance function d. Then, $\succeq$ satisfies Monotonicity, Betweenness, Dominance, and Symmetric Expansion from the Reference Point if and only if $\succeq = \succeq^r$.

As Theorems 4.1 and 4.2 demonstrate, we have allowed for both a quasi ordering approach and a complete ordering approach.

5 A Modification: Asymmetric Expansion

Axiom 3.1.4 declares every expansion north–east of size t_d exactly as desirable as every other expansion north–east of the same size. This makes perfect sense in the case of elementary characteristics which are vital for subsistence. But there are also situations where characteristics in certain proportions are viewed as better than other combinations. Consider a well–balanced nutrition for children or older people that is designed to establish certain proportions of various vitamins together with certain quantities of protein and calcium, let's say. Or imagine an individual who ponders over the "right" mode of transportation where factors such as speed, punctuality, comfort and accessibility should be adequately combined. Or consider alternative vacation projects that for one person should combine properties such as being adventurous, exclusive and offering exquisite food, while another person would aspire to achieve a certain proportion of nature, culture and access to healthy food. All these different aspects are indeed characteristics in Lancaster's sense.

More technically, we shall consider expansions north–east inside certain angles δ. Central to this is the set

$$X_\delta(x^0, t, d) = \{x_\delta \epsilon \mathbb{R}^n_+ : x_\delta \geq x^0, d(x_\delta, x^0) \leq t\} \ ,$$

where the symbol δ restricts the relation "at least as large as" to points which lie in angle δ. If δ is equal to 90 degrees, we are back to our analysis in the preceding sections. If $\delta = 0$, then there is a unique direction. For purposes of illustration and given that $\mathcal{D}$ contains a single distance function, viz., the Euclidean distance function, Figure 2 shows two possible angles of expansion. As will be clear in a moment, maximal expansion will again be our criterion. In Figure 2, the direction of the desired angle does not matter in a set comparison between A and B. Given the position of the reference point x^0, set A is always better than set B. This is not so in Figure 3, where the direction of the angle matters in a set comparison between A and B. Figure 4 depicts a situation, where the angle has shrunk to a unique direction. If vectors x_A and x_B in this figure either lie on a line perpendicular to x^0 or on a line horizontal to x^0, the person considered would only be interested in one of the characteristics, given x^0. One could interpret this as partial satiation, i.e., satiation with respect to one of the two characteristics.

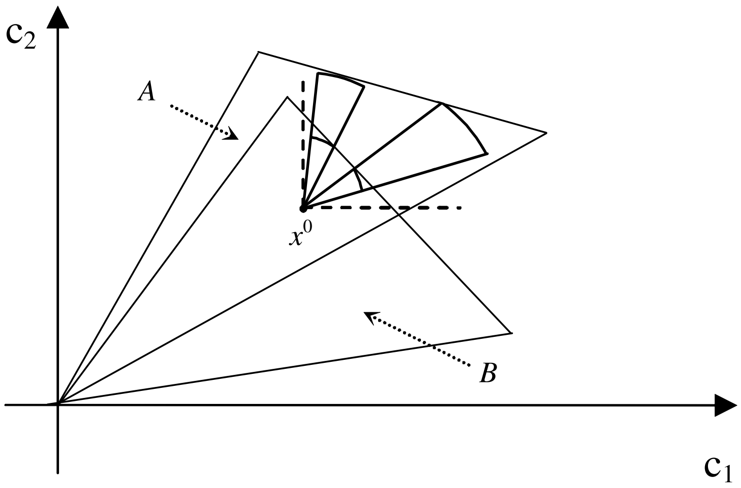

Figure 2

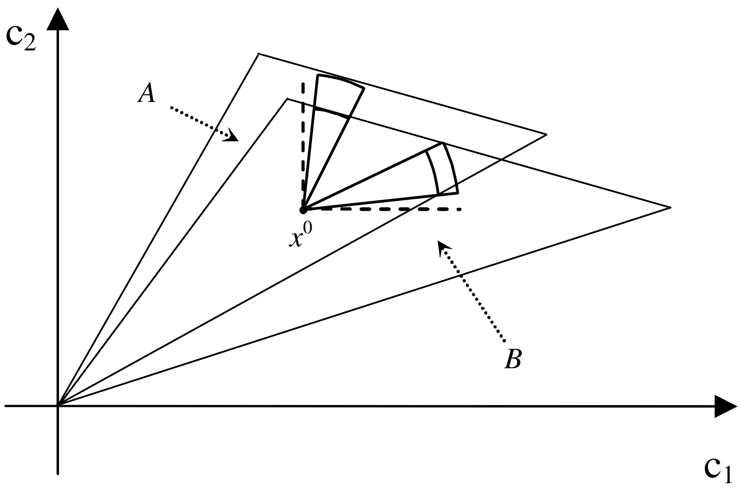

Figure 3

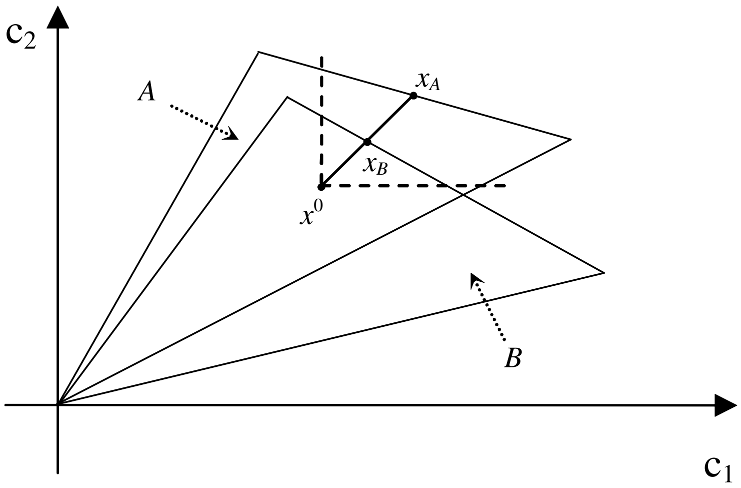

Figure 4

Let us now define, for all $A \in \mathcal{K}$ and all $d \in \mathcal{D}$,

$$r_\delta(A,d) = \begin{cases} -1 & \text{if } x^0 \notin A, \\ \max_t\{t \in \mathbb{R}_+ : \{x_\delta \in \mathbb{R}^n_+ : x_\delta \geq x^0, d(x_\delta, x^0) \leq t\} \subseteq A\} & \text{if } x^0 \in A. \end{cases}$$

The similarity to $r(A,d)$ in section 2 is obvious; $r_\delta(A,d)$ measures the maximal extension of sectors in north–east direction with $0 \leq \delta \leq 90$.

We now use the idea that the direction of desirable improvement is restricted and formulate the following modification of axiom 3.1.4:

(5.1.1) **Asymmetric Expansion from the Reference Point** iff, for all $A, B \in \mathcal{K}$, if, for each $d \in \mathcal{D}$, there exists $t_d > 0$ such that $X_\delta(x^0, t_d, d) \cap A = X_\delta(x^0, t_d, d)$, and $B \cap X_\delta(x^0, t_d, d) \subset X(x^0, t_d, d)$, then $A \succ B$.

If $\mathcal{D}$ consists of a single element d, then for all $A, B \in \mathcal{K}$,

$$A \overset{\succeq^r}{\underset{\delta}{}} B \leftrightarrow r_\delta(A,d) \geq r_\delta(B,d)\,.$$

We can now formulate Theorem 5.1 in analogy to Theorem 4.1.

Theorem 5.1. Suppose $\succeq$ over $\mathcal{K}$ is a quasi–ordering. Then, $\succeq$ satisfies Monotonicity, Betweenness, Dominance, and Asymmetric Expansion from the Reference Point if and only if $\succeq \in \mathcal{B}^r$.

Since the proof of this theorem is very close to that for Theorem 4.1, we omit it.

For the case that $\mathcal{D} = \{d\}$ and $\succeq$ is an ordering, we get, as in section 4, the following corollary.

Corollary 5.2. Suppose that $\succeq$ over $\mathcal{K}$ is an ordering and $D = \{d\}$. Then, $\succeq$ satisfies Monotonicity, Betweenness, Dominance, and Asymmetric Expansion from the Reference Point if and only if $\succeq = \underset{\delta}{\succeq^r}$.

6 A Generalization: A Reference Set

Up to this point, it has been assumed that the reference level of the individual is just a single point. What happens when this reference level extends to more than one point? In the sequel, we shall assume that the reference level becomes a line with a negative slope. The negativity of this line allows for trade–offs among characteristics. For example, as far as vacation resorts are concerned, a higher degree of tranquility combined with lower food quality may be considered equivalent (in terms of a reference level) to a lower degree of tranquility combined with higher food quality. In terms of nutrition, different combinations of vitamins, calcium and protein may serve as equivalent levels of reference. The length of the negatively sloped line may vary from individual to individual. We shall call this line $e(c)$ and define it by $e(c) := \{x \in e(c)$: there exists no $y \in e(c)$ such that $y > x\}$. In this section, we assume that $\mathcal{D}$ contains a single distance function given by the Euclidean distance function. For any characteristics vectors $x, y \in \mathbb{R}^n_+$, $||x - y||$ henceforth denotes the Euclidean distance between x and y.

Given the locus of reference points $e(c)$, let $x^i \in e(c)$. For any $t \geq 0$, define $X(e(c), x^i, t) := \{x \in \mathbb{R}^n_+ : x \geq x^i, ||x - x^i|| \leq t\}$. For any set $A \in \mathcal{K}$ and any $x^i \in e(c)$, let $t(A, x^i) = \max\{t : X(e(c), x^i, t) \subseteq A\}$. So $t(A, x^i)$ is the maximal extension north–east starting from $x^i \in e(c)$ such that the quarter–ball is still contained in A. For any set A and any $x^i \in e(c)$, let $A(x^i, t(A, x^i)) = X(e(c), x^i, t(A, x^i))$. Furthermore, let $t^*(A) = \underset{x^i \in e(c)}{\min} t(A, x^i)$ denote the minimum of all the maximal extensions along $e(c)$ for set A, and let $X(e(c), x^a, t^*(A))$ be the minimal maximal set extending north–east, emanating from $x^a \in e(c)$ and lying in A. So $x^a \in e(c)$ is the location on the reference locus from where the quarter–ball north–east has its minimal–maximal size.

We now want to consider the case that for another set $B \in \mathcal{K}$, one has $x^b \neq x^a$ so that $X(e(c), x^b, t^*(B))$ is neither contained nor does contain $X(e(c), x^a, t^*(A))$.

How can we compare these two sets? We consider a shift from x^a to x^b along $e(c)$, the locus of equivalent reference points and a shift from x^b to x^a along $e(c)$. We denote $X(e(c), x^a, t^*(A))$ shifted to x^b by $\widehat{X}(e(c), x^a \to x^b, t^*(A))$ and $X(e(c), x^b, t^*(B))$ shifted to x^a by $\widehat{X}(e(c), x^b \to x^a, t^*(B))$.

We now define $\underset{e}{\succeq}$ over $\mathcal{K}$ as follows: for all $A, B \in \mathcal{K}$,

(i) if $e(c) \cap B = e(c) \cap A = \emptyset$, then $A \underset{e}{\sim} B$,

(ii) if $e(c) \cap B = \emptyset, e(c) \cap A \neq \emptyset$, then $A \underset{e}{\succ} B$,

(iii) if $e(c) \cap B \neq \emptyset, e(c) \cap A \neq \emptyset$, then
$A \underset{e}{\succeq} B$, if $B(x^i, t(B, x^i)) \subseteq A(x^i, t(A, x^i))$ for all $x^i \in e(c)$,
$A \underset{e}{\succeq} B$, if there exist $x^a, x^b \in e(c)$ with $x^a = x^b$ such that $X(e(c), x^b, t^*(B)) \subseteq X(e(c), x^a, t^*(A))$,
$A \underset{e}{\succeq} B$, if $\widehat{X}(e(c), x^b \to x^a, t^*(B)) \subseteq X(e(c), x^a, t^*(A))$,
$B \underset{e}{\succ} A$, iff $\widehat{X}(e(c), x^a \to x^b, t^*(A)) \subset X(e(c), x^b, t^*(B))$.

Figures 5 and 6 try to depict clause (iii) which is the interesting case. In Figure 5, one can see that at point $x^a, X(e(c), x^a, t^*(A))$ is contained in the maximal quarter–ball north–east with respect to set B, and at x^b, the opposite is the case, i.e., $X(e(c), x^b, t^*(B))$ is contained in the maximally extending quarter–ball north–east with respect to A. The figure shows that $B \underset{e}{\succ} A$, since $\widehat{X}(e(c), x^a \to x^b, t^*(A)) \subset X(e(c), x^b, t^*(B))$. A simple subcase within (iii) is depicted in Figure 6, where the quarter–ball extensions north–east with respect to B are contained in the quarter–balls with respect to A for all $x^i \in e(c)$. In such a case, set B can completely lie inside set A which would be a case of set dominance, but not necessarily, as Figure 6 depicts. Another simpler case is the one where sets A and B are such that points x^a and x^b collapse. In this case, the path $x^b \to x^a$ and the path $x^a \to x^b$ have length zero.

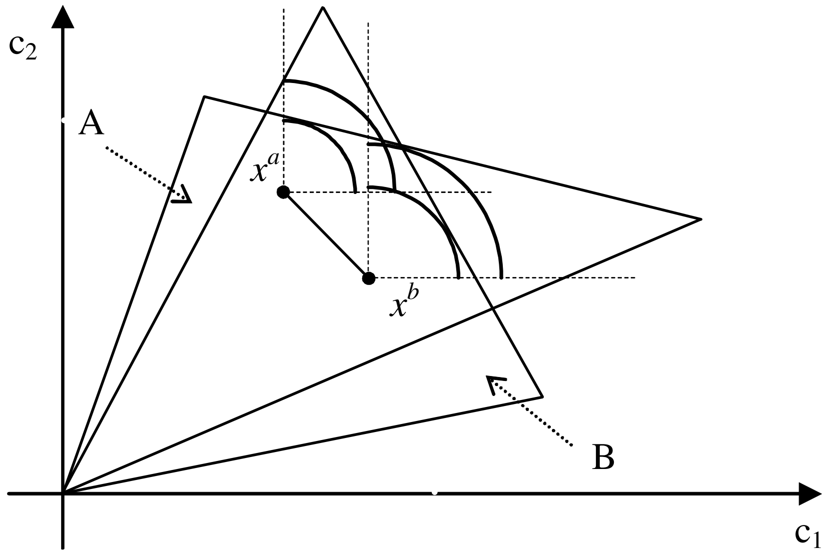

Figure 5

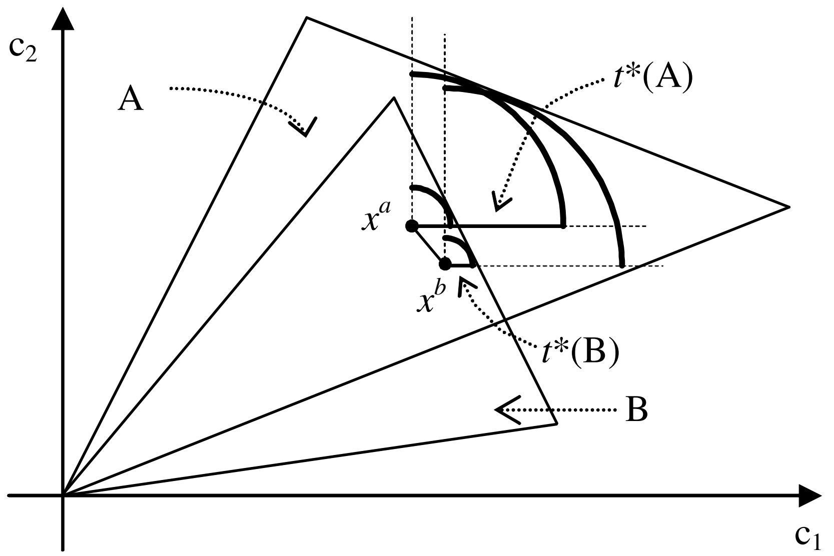

Figure 6

Definition 6.1. $\succeq$ over $\mathcal{K}$ satisfies

(6.1.1) **Reference Level Superiority** iff, for all $A, B \in \mathcal{K}$,
if $e(c) \cap B = \emptyset$, then $A \succeq B$,
if $e(c) \cap B = \emptyset$ and $e(c) \cap A \neq \emptyset$, then $A \succ B$;

(6.1.2) **Domination along the Reference Level** iff, for all $A, B \in \mathcal{K}$,
if for all $t \geq 0$ and all $x^i \in e(c)$, $X(e(c), x^i, t) \subseteq B \rightarrow X(e(c), x^i, t) \subseteq A$, then $A \succeq B$,

if for some $t \geq 0$ and some $x^i \in e(c)$, $X(e(c), x^i, t) \subseteq A$ but $X(e(c), x^i, t)$ is not a subset of B, then $A \succeq B \rightarrow A \succ B$;

(6.1.3) **Minimax Extension** iff, for all $A, B \in \mathcal{K}$,
if, given $t^*(A), t^*(B)$ and $x^a \neq x^b$, $\widehat{X}(e(c), x^b \rightarrow x^a, t^*(B)) \subseteq X(e(c), x^a, t^*(A))$,
then $A \succeq B$;

if, given $t^*(A), t^*(B), x^a = x^b$ and $X(e(c), x^b, t^*(B)) \subseteq X(e(c), x^a, t^*(A))$, then $A \succeq B$;

given $t^*(A), t^*(B), B \succ A$ iff $\widehat{X}(e(c), x^a \rightarrow x^b, t^*(A)) \subset X(e(c), x^b, t^*(B))$.

Theorem 6.1 Suppose that $\succeq$ over $\mathcal{K}$ is reflexive, complete and transitive. Then $\succeq$ satisfies Reference Level Superiority, Domination along the Reference Level and Minimax Extension if and only if $\succeq = \underset{e}{\succeq}$.
Proof. It can be checked that $\underset{e}{\succeq}$ is reflexive, complete and transitive and satisfies Reference Level Superiority, Domination along the Reference Level and Minimax Extension. We now show that if $\succeq$ over $\mathcal{K}$ satisfies Reference Level Superiority, Domination along the Reference Level and Minimax Extension, then $\succeq = \underset{e}{\succeq}$.

Let $A, B, \in \mathcal{K}$. We consider the following cases:

(i) $e(c) \cap A = e(c) \cap B = \emptyset$. Since $e(c) \cap A = \emptyset$, by Reference Level Superiority, $B \succeq A$. Similarly, since $e(c) \cap B = \emptyset$, by the same requirement, $A \succeq B$. Therefore, $A \sim B$.

(ii) $e(c) \cap A \neq \emptyset$ and $e(c) \cap B = \emptyset$. By Reference Level Superiority, it follows that $A \succ B$.

(iii) $e(c) \cap A = \emptyset$ and $e(c) \cap B \neq \emptyset$. In this case, $B \succ A$ follows from an analogous argument.

(iv) $e(c) \cap A \neq \emptyset$ and $e(c) \cap B \neq \emptyset$

Consider first that $[B(x^i, t(B, x^i)) \subseteq A(x^i, t(A, x^i))$ for all $x^i \in e(c)]$ and $[A(x^i, t(A, x^i)) \subseteq B(x^i, t(B, x^i))$ for all $x^i \in e(c)]$. By Domination along the Reference Level, we obtain $A \succeq B$ and $B \succeq A$ and, therefore, $A \sim B$.

Next consider $[B(x^i, t(B, x^i)) \subseteq A(x^i, t(A, x^i))$ for all $x^i \in e(c)]$ and [for some $t \geq 0$ and some $x^j \in e(c)$, $X(e(c), x^j, t) \subseteq A$ but $X(e(c), x^j, t)$ is not a subset of B]. Note that in this case, by the first part of Domination along the Reference Level, $A \succeq B$. Then, by the second part of this requirement, $A \succ B$ follows immediately.

Similarly, when $[A(x^i, t(A, x^i)) \subseteq B(x^i, t(B, x^i))$ for all $x^i \in e(c)]$ and [for some $t \geq 0$ and some $x^j \in e(c)$, $X(e(c), x^j, t) \subseteq B$ but $X(e(c), x^j, t)$ is not a subset of A, $B \succ A$ follows directly.

Next, assume that for some $x^a = x^b$, $X(e(c), x^b, t^*(B)) \subseteq X(e(c), x^a, t^*(A))$. Then, due to Minimax Extension, $A \succeq B$.

Finally, consider $\widehat{X}(e(c), x^b \to x^a, t^*(B)) \subseteq X(e(c), x^a, t^*(A))$ for $x^a \neq x^b$. Assume that $B \succ A$. Then, due to Minimax Extension, $\widehat{X}(e(c), x^a \to x^b, t^*(A)) \subset X(e(c), x^b, t^*(B))$ which is in contradiction to our supposition that the minimal maximal quarterball with respect to B, shifted to x^a, is contained in $X(e(c), x^a, t^*(A))$. Thus, $B \succ A$ cannot hold. Therefore, due to completeness of $\succeq$, $A \succeq B$ holds.

7 Concluding Remarks

In this paper we used Lancaster's characteristics approach in order to rank sets of alternative characteristics combinations. We deviated from the standard approach which considers convex sets of characteristics combinations and then imposes a utility function that is to be maximized in the space of characteristics.

We introduced a reference vector as our point of orientation in order to measure the richness of choice and then established a ranking over different sets of characteristics (see also Gaertner and Xu (2006) for a measure in the space of functionings). When doing this, we considered several modifications. The first was that for a particular person, not all north–east directions may be equally desirable. The second was that the reference level may be more than one point. We assumed that it would be a negatively sloped line. In such a situation, an extension north–east can be measured in different ways. In the present version, we focused on the minimal–maximal extension north–east. Other types are, of course, possible.

We have only briefly mentioned the phenomenon of satiation. It may very well be the case that with higher income and a higher standard of living, some of the characteristics become less important for the consumer so that at some point, a satiation level is reached for these characteristics. In this case, the cone of desirable expansion will "lose" some dimensions. With respect to the other dimensions where there is no satiation, one would again proceed as much as possible in the still desirable directions. This aspect deserves further elaboration.

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