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Using copulas to measure association between ordinal measures of health and income

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Abstract

This paper introduces a new approach to measuring the association between health and socioeconomic status. Measuring inequalities in health is difficult when health is measured qualitatively, specifically on an ordinal scale. This paper demonstrates a rank-based dependence measure - the copula - that is invariant to both the scale and any monotonic transformations of its dimensions. Accordingly, the copula measure of association between health and income is robust under different cardinal scales for health as well as different income distributions, and can be used for ordering countries. The copula is also used to generate contingency tables of joint probability, which illustrate how this ordering can be due to polarity in the distributions of health and income, as well as stronger association between the distributions of health and income.

1 Introduction

In this paper I consider income-related inequalities in health across 4 countries within the European Community Household Panel (ECHP) survey: the United Kingdom, Germany, Greece and Portugal. Income is measured, in all 4 countries, in various forms, leading ultimately to a measure of post-tax household equivalised income, adjusted for Purchasing Power Parity (PPP) across the countries. For most comparisons (e.g. considering income inequality) this is not problematic. What *is* problematic is the measurement of Self-Assessed Health (SAH) on an ordinal scale - typically "Very Good", "Good", "Fair", "Poor" and "Very Poor". As is demonstrated elsewhere (see for example Allison and Foster 2004, and Zheng 2006), the distribution of the responses on this scale - referring to their subsequent cardinal values - becomes different depending upon the cardinal scale to which it is transformed. In particular, mean-based measures of inequality (or, in multiple dimensions, covariance) are not robust under some such transformations.

This analysis is, by design, straightforward, relative to the broader consideration of health inequalities. Inequalities in health are not considered explicitly, while income-related inequalities in health are defined according to, for example, Wagstaff and van Doorslaer (2000). In Wagstaff, *et al.*'s (1991) paper, the authors asked the question, "To what extent are there inequalities in health that are systematically related to socioeconomic status?". Similarly, the International Society for Equity in Health (ISEqH, 2001) defined this as equity, specifically: "...the absence of potentially remediable, systematic differences in one or more aspects of health across socially, economically, demographically or geographically defined population groups or subgroups."

Bommier and Stecklov (2002) consider an extra-welfarist definition, stating that an ideal equitable society was one in which *access* to health, rather than health itself, was not determined by socioeconomic status or income, according to Rawlsian principles of justice. The analysis here follows the former definitions. This assumes that, for empirical purposes, health outcomes (SAH specifically) can be used to consider the distribution of health, and how it is related to the distribution of income.[1]

According to such definitions, health inequality of a socioeconomic nature can be defined by its absence: a perfectly 'fair' distribution of health contains an absence of association between socioeconomic status and health, or between the distribution of income and the distribution of health. This is the one considered in the construction of the concentration curve and index (Wagstaff, *et al.* 1991; Kakwani, *et al.* 1997). In their measure of inequality, the covariance between health status and the rank of individual income in the income distribution is used. The meaning is the same: higher values of the concentration index result from stronger covariance, i.e. stronger association between health and income. Returning to Bommier and Stecklov (2002), however, the optimal measure of socioeconomic inequalities in health is one that can measure the association between the distributions of health and income without influence from the distributions themselves (i.e. invariant to the cardinal scale given to SAH, and invariant to any inequality in the distribution only of income).

The method used in this paper - the copula - is a rank-based measure of association between monotonic transformations of random variables rather than the variables themselves: i.e. a bivariate distribution of univariate distribution functions. Specifically, this paper takes advantage of the functional relationship between the Archimedean class of copulas and measures of rank correlation to demonstrate that the health and income distributions of different countries can be ordered according to principles of stochastic dominance. This ordering is robust under both the discrete nature of SAH and transformations of the cardinal scale(s) applied to SAH. Following work by Contoyannis and Wildman (2006), the copula for each country shows that, for example, Greece exhibits inequality in more of the bivariate distribution of health and income, however Portugal exhibits greater polarity between low health and high income, resulting in Greece dominating Portugal in terms of association between the distributions of health and income.

2 Measuring ordinal health and health inequality

Ordinal measurement of SAH is known to be problematic for the analysis of health. As an ordinal measure of health, SAH also suffers by comparison to continuous measures of welfare. As a categorical measure, SAH is not necessarily Lorenz consistent; according to the Pigou-Dalton transfer principle that progressive transfers reduce inequality and/or make society better-off (Chateauneuf and Moyes 2005; Zheng 2006). First and foremost is the practical consideration: 'health' itself is not transferable. Secondly, a progressive transfer in ordinal scale does not necessarily mean a transfer across cut-points. A transfer of health from a person with *very* Very Good health to a person who only *just* has Very Good health will not alter any measure of health inequality, irrespective of the cardinal scale to which SAH is adapted.

In terms of the first consideration, some authors recommend measuring inequality still within the framework of progressive transfers, but such that non-transferable dimensions of the social welfare function - progressive or otherwise - be left out of the measure (Bosmans, *et al.* 2006). Bommier and Stecklov (2002) discuss this also with respect to Rawlsian principles of justice: health itself should not be considered a basic freedom, deserving of equal distribution, but access to health or health care should, similar to Sen's (1979) equality of opportunity (Rosa Dias and Jones 2007).

Retaining health in the measurement of welfare and inequality means addressing the nature of common health measures, as well as defining how inequality in health is to be considered: for example access to health care versus its utilisation versus its outcomes (Braveman 2006). One such measure is the concentration index, which considers the association an individual's health and the rank of their income.

2.1 Concentration curves and indices with discrete SAH

The concentration curve was defined in Kakwani (1977) as representing the relationship between the distribution function $F(y)$ - where y is given to be income, for example, and some other monotonic function $F_1(y)$, or in the case of Kakwani's (1977) demonstration, $F_1(g(y))$. Kakwani (1977) in particular identified $g(y)$ and $F_1(g(y))$ such that the Lorenz curve is seen as a special form of the concentration curve - what he called a relative concentration curve.[2]

The concentration curve, as it is known to health economists, is one representing the relationship between functions of two related variables, rather than a single variable. I.e., individuals are ranked according to income, for example, while we are interested in their cumulative share of health (rather than ranking them according to income and being interested in each rank's cumulative share of income, also). Wagstaff, *et al.* (1991) present their *generalised* concentration curve relating the cumulative amount of health to the rank of individuals according to socioeconomic status, their interest being in the socioeconomic dimension of inequalities in health.[3] The concentration curve was similarly employed with mortality, previously (Preston, *et al.* 1981; Leclerc, *et al.* 1990).

Interpretation of the concentration curve is the same as for the Lorenz curve: the $45°$ line represents perfect (socioeconomic) equality in health. A curve observed above this diagonal at all points represents pro-poor inequality, while one below represents pro-rich inequality. For the purposes of comparison the same principle applies to the concentration curve. For two random variables Y_1 and Y_2 representing the incomes in country 1 and country 2, respectively, and with the same ranking function $F(.)$, their concentration curves C_1 and C_2 will be the cumulative share of health in each country, $G(.)$. Thus C_1 will *Lorenz* dominate (i.e. be at all points above) C_2 iff $C_1 \geq C_2 \; \forall \; F(y_1)$ or $F(y_2)$. That is, for all ranks of income, the concentration curve of country 1 must be strictly no less than that of country 2. Dominance does not occur when the lines cross. Since this is more commonly observed, the concentration index is employed instead as the measure of relative (in)equality (see Hernández Quevedo, *et al.* 2006 for their comparison of all ECHP countries).

The concentration index is used as a numerical measure of distance between the observed concentration curve and the line of equality - twice the area, like the Gini coefficient. It can therefore be used to compare two or more overlapping concentration curves, establishing generalised Lorenz dominance of one country over another.

There are several representations of the concentration index; the most useful for this analysis is the 'convenient covariance' representation, in which each individual's health h_i (from a distribution with mean μ_h) is indexed against their rank R_i in the distribution of income Y (Kakwani 1980; Wagstaff, *et al.* 2003). Thus

$$\begin{aligned} CI &= \frac{2}{\mu_h (n-1)} \sum_{i=1}^{n} (h_i - \mu_h) \left(R_i - \frac{1}{2} \right) \\ &= \frac{2}{\mu_h} \mathrm{cov}(h_i, R_i) \end{aligned} \tag{1}$$

for a sample size n. The concentration index is the scaled covariance between the health of the individual and their rank in the income distribution. Thus income-related inequality in health is this measure of covariance such that $CI = 0$ when there is no inequality - i.e. there is no observed association between socioeconomic status and health - and $-1 \leq CI \leq 1$ due to the $\frac{2}{\mu_h}$ term.[4] This association, or lack thereof, forms the basis of the measurement of socioeconomic-related inequalities in health, according to the criteria of Wagstaff, *et al.* (1991) and Bommier and Stecklov (2002), among others.

As a measure of covariance, the concentration index relies upon the means of the distributions of health and income

$$
\begin{aligned}
CI &= \frac{2}{\mu_h}\operatorname{cov}(h, R) \\
&= \frac{2}{\mu_h}\operatorname{cov}(h, F(y)) \\
&= \frac{2}{\mu_h}\left[E(h, F(y)) - E(h)\, E(F(y))\right]
\end{aligned}
\tag{2}
$$

Reliance on the mean is problematic in the case of the distribution of health, which is typically measured on an ordinal scale, rather than a cardinal one. Allison and Foster (2004), for example, demonstrate that the mean of a cardinal scale is not a robust measure of ordinal-scale health. They employ different cardinal scales, from linear to highly concave, for the ordinal SAH scale $c = [1, 2, 3, 4, 5]$. They show that inequality rankings can be reversed by using different cardinal scales - i.e. different relative values of health. This is also shown to be the case with first-order dominance, when the mean is used to normalise the measure of inequality, as it is in Equations (1) and (2). In the case of the concentration index, using covariance has a similar effect: as long as the function R_i is used to rank income, non-linear transformations of income will have no effect (e.g. a change in the tax schedule), but those of health will. A linear cardinal scale will provide a different measure of linear covariance than a highly concave one, for example. The implication of this for concentration index-based ordering of countries is demonstrated in Tables 4.1 and 4.2.

2.2 Transforming SAH so that it is continuous

SAH has been rendered continuous in the past by inverting a covariate-dependent distribution function $F(h^*)$. Procedurally, this involves using, for example, an ordered probit model of latent health $h^* = X'\beta + \varepsilon$. The regression link between latent health h^* and assessed health h is given by the ordered probit, such that $h = j$ if $\mu_{j-1} < X'\beta \leq \mu_j$ for $j = 1, .., 5$ (van Doorslaer and Jones 2003), where unobserved

$\varepsilon \sim N(0,1)$. For each individual, $h_i^* = X_i'\beta$ is their predicted latent health, and can be used as a proxy for actual health.

This method is constrained by the fact that regression is imprecise: observed association between $X'\beta$ and R will not be the same as between h^* or h and R. Moreover, this imprecision may be structural, due to omitted variables or some other source of heterogeneity (Vanness and Mullahy 2005). In particular, one faces the problem of what to do with income. That it affects health status can be taken as given, however including it as a regressor obscures the association between the two. Moreover, it can introduce into the distribution of health the distribution of income, such that inequalities in the distribution of income become entangled with income-related inequalities in health. Omitting income from the explanation of health, on the other hand, worsens the accuracy of subsequent predictions.

The optimal measure of income-related inequalities in health would be one that accepts the discrete, even ordinal nature of SAH without penalty, and can relate it to the income distribution in a manner that is robust against the scale and transformation issues mentioned above. Following, for example, Wagstaff, *et al.*'s (1991) consideration of the structural relationship between socioeconomic status and health, a form of bivariate distribution called the copula can do this, by measuring the association between the distributions of health and income, rather than health and income directly.

2.3 Copulas as measures of dependence between SAH and income

Consider the random variables health h, income y, and their marginal distribution functions $F(h)$ and $G(y)$, respectively. Then, by a theorem due to Sklar (1959) the joint distribution of health and income can be written in the form of a copula, C, where

$$H(h,y) = C(F(h), G(y)) \tag{3}$$

i.e. the copula is a multivariate distribution function not of random variables, but the distribution functions of those random variables: a multivariate distribution with strictly uniform margins. The copula is a function that parameterises the dependence between univariate marginal distributions (in this case $F(h)$ and $G(y)$) and binds them to form the joint distribution function (given by $C(F(h), G(y))$).

Copulas use measures of association that are invariant to monotonic - but not necessarily strict or linear - transformations of random variables. Ergo unlike, for example, the bivariate normal distribution, the association between h and y is the same as between $F(h)$ and $G(y)$. The rank correlations of Kendall and Spearman are familiar examples of such measures of association. Like the bivariate normal, any bivariate copula is an approximation to the true bivariate distribution H; except with the advantage that the marginal distributions are, by construction, tractable: they are free of every other marginal distribution in the joint distribution, and separated also from the measure of association.

There are many families of copulas (see Joe 1997; Nelsen 2006). Here I use a specific class known as the Archimedean copula. This class is distinguishable by the fact that its measure of association, θ, is functionally related to rank correlation. Specifically, for any copula C with continuous univariate margins $u|_{u=F(h)}$ and $v|_{v=G(y)}$, the copula $C(u,v;\theta)$ is such that (Nelsen 2006)

$$\text{Kendall's } \tau = 4 \int\int_{I^2} C(u,v)\, dC(u,v) - 1 \tag{4}$$

and

$$\text{Spearman's } \rho = 12 \int\int_{I^2} C(u,v)\, du dv - 3 \tag{5}$$

where I^2 refers to the bivariate uniform $(0,1)^2$ space. Archimedean copulas in particular are constructed by so-called *generator* functions such that

$$C(u,v) = \varphi^{-1}(\varphi(u) + \varphi(v)) \tag{6}$$

where the generator $\varphi(.)$ is unique to each copula (see Nelsen 2006).

The differences between Spearman's ρ and Kendall's τ are discussed in Nelsen (2006) and Fredericks and Nelsen (2007). For absolutely continuous distributions u and v the use of either is equivalent. No general guideline exists, suggesting which circumstances are preferred for one method or another.[5] For applied research purposes the use of Kendall's τ can be more convenient as the functional form of the relationship with the copula parameter θ is available (Genest and Rivest 1993; Nelsen 2006). As discussed below, the use of SAH makes Kendall's τ a preferred reference for the Genest-Rivest solutions for estimating θ.

For each Archimedean copula, Kendall's τ can be given by

$$\begin{aligned} \tau &= 4\int\int_{\mathbf{I}^2} C(u,v)dC(u,v) - 1 \\ &= 1 + 4\int_0^1 \frac{\varphi(t)}{\varphi'(t)}dt \end{aligned} \tag{7}$$

For some marginal distribution t. Then in sample space for health and income (h,y)

$$\hat{\tau} = \binom{n}{2}^{-1} \sum_{i<j} sign\left[(h_i - h_j)(y_i - y_j)\right] \tag{8}$$

Solving Equations (7, 8) for $\hat{\tau} = \tau$ and a given Archimedean generator function $\varphi(t)$ will provide a sample estimate of θ (Genest and Rivest 1993).

The complexity of Equations (4 - 8) belies the simplicity of this approach in practice, and particularly when analysing SAH, due to its distribution being discrete, i.e.

not a strictly-increasing function of the random variable SAH. For the representations in Equations (4) and (5) to hold strictly, τ must be an increasing function of the association parameter θ.

That SAH is distributed discretely while income is continuous is not problematic: the margins of a copula can be mixed, so that non-parametric analysis can used; the histogram of SAH can be used with the continuous empirical or kernel distribution of income. Because of the functional relationship between θ and τ, Kendall's so-called tau-b (hereafter denoted τ_b) can be used to estimate θ, following Vandenhende and Lambert (2000, 2003), which can be combined to calculate joint probabilities.Kendall's τ_b is one of two generalisation for ordinal data with ties. For a square contingency table with C concordant pairs, D discordant pairs and non-tied pairs on the vectors of health or income ($\bar{O}_h$ and $\bar{O}_y$ respectively), τ_b is given by

$$\tau_b = \frac{(C-D)}{\sqrt{\left(C+D-\bar{O}_h\right)\left(C+D-\bar{O}_y\right)}} \tag{9}$$

Vandenhende and Lambert (2000) analyse the extent of this problem; the degree to which the copula τ in Equation (4) will not correspond to τ_b. I.e. the degree to which τ_b may not be a strictly increasing function of the copula's association parameter θ. They found that the relationship was preserved in the case of the Frank copula, which is positively ordered; they also observed monotonocity for non-ordered copulas when dependence was not weak.[6] The relationship between τ_b and θ also proved to be stronger as the number of categories in the margins increased. They observed the margins to behave more like continuous margins, rather than discrete as they increased categories from 4 to 10.[7]

Association θ can be estimated directly for each copula: Full-Information Maximum Likelihood (FIML), for example, was used by Cameron, *et al.* (2003; Kolesárová and Mordelová 2006 also discuss this issue). However the discrete/continuous mixture makes this more difficult to estimate directly. Cameron, et al. (2004) in particular discuss the relative merits of this approach compared to FIML, using binary data. Following Vandenhende and Lambert (2003), and considering that SAH is usually measured in 5 categories, the indirect approach is preferable here.

The tractability of copulas means the copula of one country is directly comparable to that of another country, allowing for countries to be ordered according to stochastic dominance, similar though not entirely analogous to lorenz dominance. Finally, either measure of association θ or τ can be used for *rank* dominance.

2.3.1 The Frank copula

The Frank copula is given by (Frank, 1979)

$$C(u,v;\theta) = -\frac{1}{\theta}\ln\left(1 + \frac{\left(e^{-\theta u}-1\right)\left(e^{-\theta v}-1\right)}{e^{-\theta}-1}\right) \tag{10}$$

where $\theta \in (-\infty,\infty)\setminus\{0\}$. The Frank copula is constructed using the generator function

$$\varphi_\theta(t) = -\ln\left(\frac{e^{-\theta t}-1}{e^{-\theta}-1}\right) \tag{11}$$

This is a comprehensive family, such that association θ corresponds to $\tau \in [-1,1]\setminus\{0\}$.[8] The tau link function for the Frank generator is given by

$$\tau = 1 - \frac{4}{\theta}\left[\frac{1}{\theta}\int_0^\theta \frac{t}{e^t-1}dt + \frac{\theta-2}{2}\right] \tag{12}$$

2.3.2 The Gumbel copula

The Gumbel copula is a single-parameter distribution function, like the Frank.[9] However, the Gumbel copula is skewed such that dependence is measured more precisely in the upper tail than the lower (i.e. "high-high" combinations: see Figure 4.1; Trivedi and Zimmer 2006). It is therefore suited in particularly not only to random variables that are positively correlated, but to those in which high values of each are more strongly correlated than low values.

The Gumbel copula is given by

$$C(u,v;\theta) = \exp\left[-\left[(-\ln u)^{\theta} + (-\ln v)^{\theta}\right]^{\frac{1}{\theta}}\right] \tag{13}$$

where $\theta \in [1,\infty)$. The Gumbel copula is constructed using the generator function

$$\varphi_{\theta}(t) = (-\ln t)^{\theta} \tag{14}$$

For which the tau link function is given by

$$\tau = 1 - \theta^{-1} \tag{15}$$

This corresponds to positive association only $\theta \to \tau \in [0,1]$.

2.3.3 The AMH copula

The AMH copula is skewed in its dependence structure, similar to the Gumbel, however it estimates dependence more precisely in the lower tail (i.e. "low-low" combinations, when dependence is positive: see Figure 4.1). It is suited to joint distributions where positive correlation (in this case) is strongest between low values, relative to high values, of the random variables.

The AMH copula is given by (Ali, *et al.* 1978)

$$C(u,v;\theta) = \frac{uv}{1-\theta(1-u)(1-v)} \tag{16}$$

where $\theta \in [-1,1)$ corresponds to $\tau \in \left[-0.181726, \frac{1}{3}\right]$. The AMH copula is constructed using the generator function

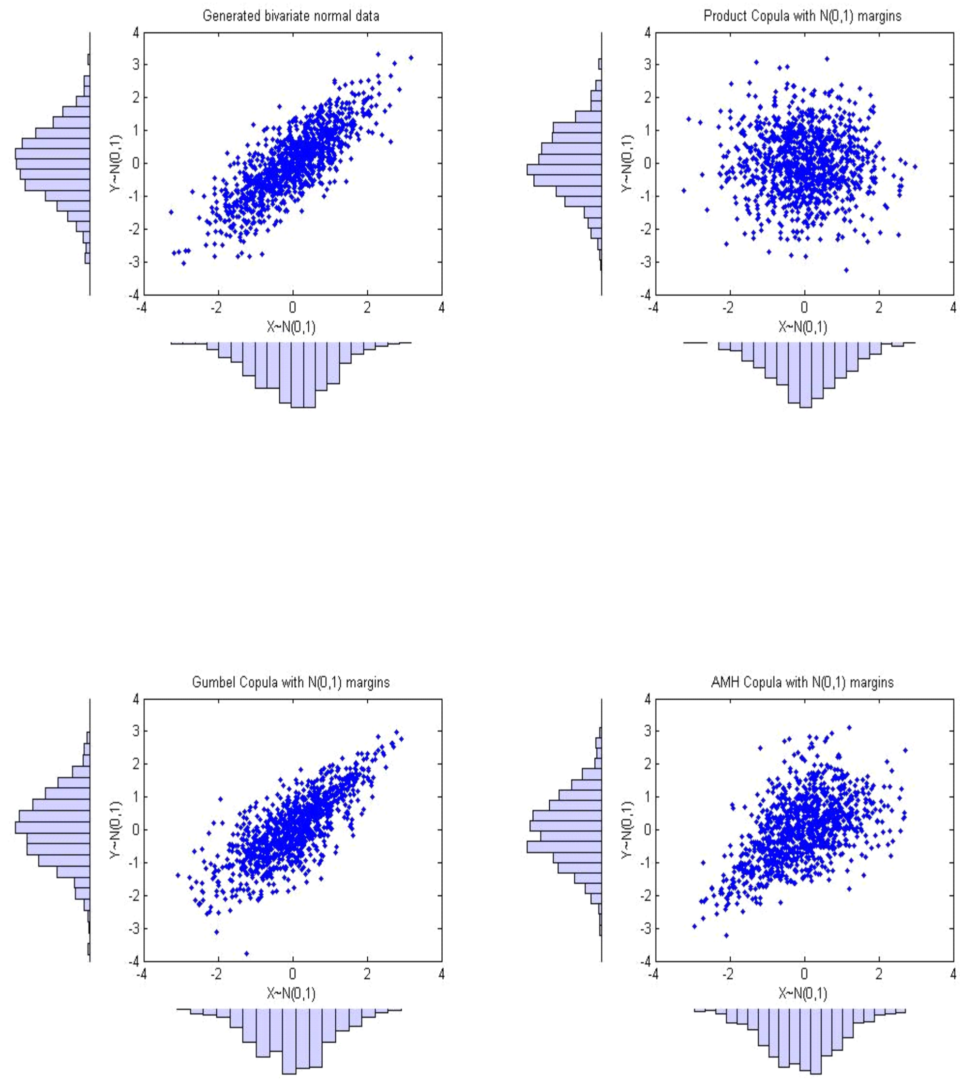
$$\varphi_\theta(t) = \ln\left(\frac{1-\theta(1-t)}{t}\right) \tag{17}$$

For which the tau link function is given by

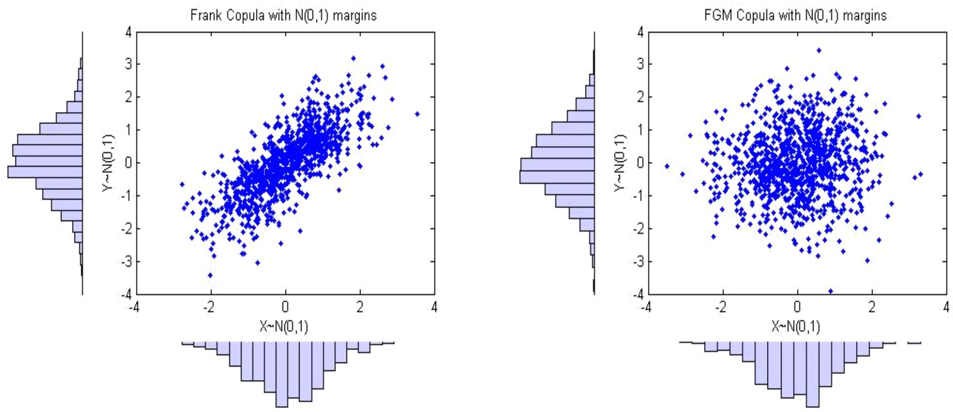
$$\tau = 1 + 2\left[\frac{\frac{-1}{6\theta} - \left[(\theta-1)^2 \ln(1-\theta)\right]}{3\theta^2}\right] \tag{18}$$

Figure 4.1 contains an illustration of the distribution of simulated bivariate normal data ($\rho = 0.8$). The only difference between the distributions is the copula used to simulate the data: the margins are all standard-normally distributed, and they are equally dependent, but the structure of that dependence is different for each. Greater density of simulated data in the right tail of the Gumbel copula and the left tail of the AMH copula reflect properties discussed above.

Figure 1: MARGINAL HISTOGRAMS AND BIVARIATE SCATTERPLOTS FOR X N(0,1), Y N(0,1), SIMULATED VIA INVERTING CONDITIONAL COPULAS (rho = 0.8)



MARGINAL HISTOGRAMS AND BIVARIATE SCATTERPLOTS FOR X N(0,1), Y N(0,1), SIMULATED VIA INVERTING CONDITIONAL COPULAS (rho = 0.8)



The value of this is demonstrated later, when goodness-of-fit testing is employed to determine which copula appears to be the better fit to the data. As well as functionality as the bivariate distribution, this selection provides information about the structure or shape of the dependence between random variables.

Solving Equations (12), (15) and (18) is possible using software such as Maple or Mathematica, or with an implementable Matlab package (see Perkins and Lane 2003). Solutions for the estimates of Kendall's τ_b can be seen in Table 4.3.

Estimates of θ are necessary for each copula, but not for rank-ordering the countries according to their association between health and income. Association θ is, by design, a monotonic function of τ: the rank-order of the countries will not change because a different copula is used. What may change is the joint distribution as a whole, due to the different shapes of copulas (see for example Joe, 1997; Bouyé, *et al.* 2000).

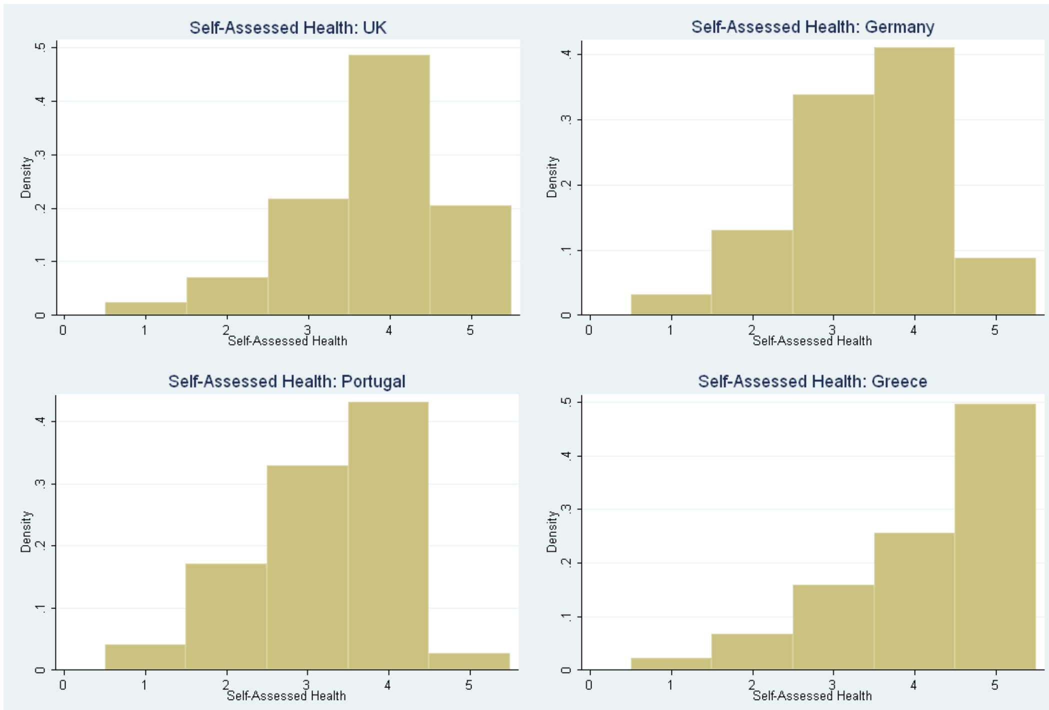
3 Application: income-related inequalities in health in the ECHP

3.1 Data

Data on SAH and income are drawn from the 7th wave (2000) of the European Community Household Panel Survey (ECHP: see Peracchi 2002 and Hernández Quevedo, *et al.* 2006 for descriptions of the ECHP Users' Database). Income in this data is equivalised household income, adjusted for Purchasing Power Parity between the countries.[10] SAH is taken from responses to the question "How is your health in general?" and contains 5 categorical responses: "Very Good", "Good", "Fair", "Poor" and "Very Poor".[11] Sample sizes ranged from 8,573 (the UK) to 11,035 (Portugal).

The 4 countries used were chosen for purposes of comparison with the UK. Looking at inequalities in both health and income, Portugal has the greatest of both, Greece has proximate inequalities in health (relative to the UK) but greater inequalities in income, while Germany is among the countries with the lowest inequalities in each - it sits on an interior frontier, along with Austria and Denmark (see for example Jones and Rice 2004). Their respective distributions of SAH and income are shown in Figures 4.2 and 4.3.

Figure 2: HISTOGRAMS FOR SELF-ASSESSED HEALTH IN THE UK, GERMANY, PORTUGAL AND GREECE



In distributional terms, SAH is roughly similar for all of the countries except Greece, whose elevated self-assessment has been documented elsewhere (Cantarero and Pascual 2005). The skewness in Greece's SAH also generates a slightly higher average health status than the UK, who would otherwise have the highest average of the three less-skewed distributions. The distributions of equivalised household income are also similar, although the UK and Germany have, predictably, higher levels of household income. Portugal enjoys much less household income, relative to the other three countries.

Figure 3: HISTOGRAMS FOR INCOME IN THE UK, GERMANY, PORTUGAL AND GREECE

Table 1: CONCENTRATION INDICES AND MEAN HEALTH USING DIFFERENT CARDINAL SCALES FOR HEALTH

3.2 Estimation

Estimation is undertaken according to the procedures described above. Methods for estimating concentration curves and indices can be found in, for example, Wagstaff, *et al.* (1991) and Wagstaff (2000). For the copulas, the procedure following Genest and Rivest (1993) and Vandenhende and Lambert (2003) was employed: sample estimates of Kendall's τ_c, shown in Table 4.1, were used to solve for θ in Matlab (see Perkins and Lane, 2003). The copulas themselves were calculated in Stata, however they too could have been calculated in Matlab.[12]

3.3 Results: concentration indices and curves

Following the example of Allison and Foster (2004), consider the concentration indices CI of the UK, Portugal, Germany and Greece using CI_{UK}, $CI_{Portugal}$, $CI_{Germany}$ and CI_{Greece} and with health scales $S_1 = [1,2,3,4,5]$, $S_2 = [1,2,3,4,10]$, $S_3 = [1,2,3,4,15]$ and $S_4 = [1,4,9,16,25]$. The resulting indices from Equation (1) are shown in Table 4.1, followed by the changes in index-based orderings of countries shown in Table 4.2.

Any two (or more) non-degenerate distributions of discretely-identified health status can be ordered differently according to different inequality indices or cardinal scale (Zheng 2006; Allison and Foster 2004, respectively): any concentration index similarly applied can be ordered differently depending upon the cardinal scale used, as Tables 4.1 and 4.2 illustrate.

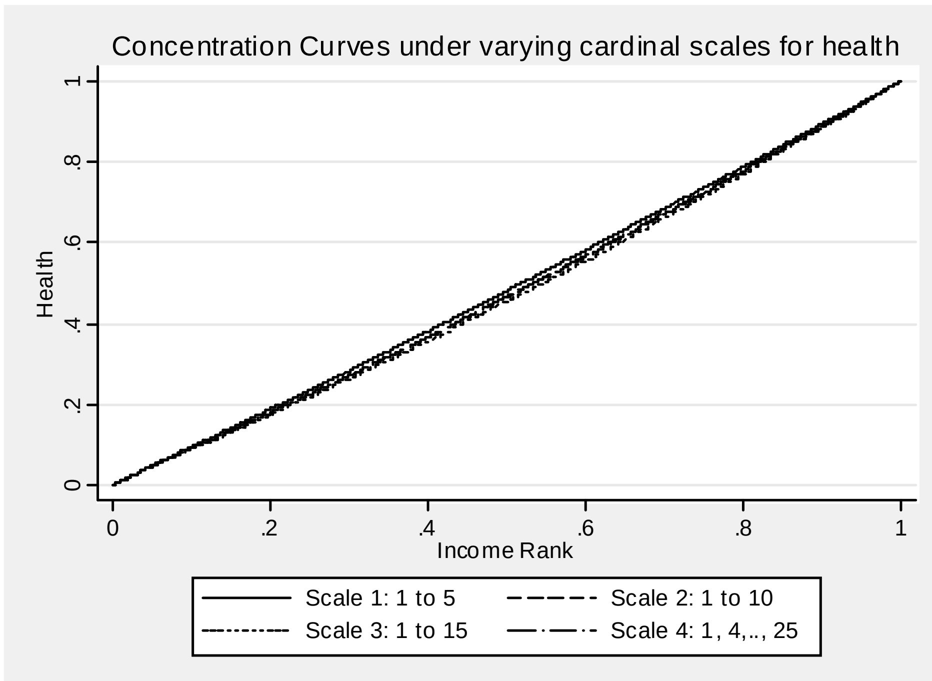
In Table 4.2, increasing the value or weight of only the highest health status re-orders Portugal and Greece. Another increase in the value of the highest health status subsequently re-orders Portugal and the UK, and so forth. Having the lower

Table 2: RANK-ORDERING USING DIFFERENT CARDINAL SCALES FOR HEALTH

Health Scale	Ordering
$S_1 \in [1, 2, 3, 4, 5]$	$CI_{\text{Germany}} \geq CI_{\text{UK}} \geq CI_{\text{Greece}} \geq CI_{\text{Portugal}}$
$S_2 \in [1, 2, 3, 4, 10]$	$CI_{\text{Germany}} \geq CI_{\text{UK}} \geq CI_{\text{Portugal}} \geq CI_{\text{Greece}}$
$S_3 \in [1, 2, 3, 4, 15]$	$CI_{\text{Germany}} \geq CI_{\text{Portugal}} \geq CI_{\text{UK}} \geq CI_{\text{Greece}}$
$S_4 \in [1, 4, 9, 16, 25]$	$CI_{\text{Germany}} \geq CI_{\text{UK}} \geq CI_{\text{Greece}} \geq CI_{\text{Portugal}}$

mean health, Portugal slowly moves up the order. Squaring the values relative to the linear scale changes the values for the concentration indices, but not the order. This re-ordering occurs despite relatively small absolute changes in the concentration indices. The concentration curves under each cardinal scale of the UK, for example, are nearly indistinguishable visually (see Figure 4.3).

Figure 5: CONCENTRATION CURVES FOR THE UK, GERMANY, PORTUGAL AND GREECE, USING DIFFERENT CARDINAL SCALES FOR HEALTH



3.4 Results from copulas: rank-ordering and stochastic dominance

Table 4.3 contains the estimates of the dependence parameter in each copula, following the Genest-Rivest solutions shown above.

Having estimated rank-correlation $\hat{\tau}_b$, using the work by Vandenhende and Lambert (2000), changing the cardinal scale of health will not alter this estimate, or cause any change in the estimates of the copula parameters, or in their ranking according to association between the distributions of SAH and income. Testing is necessary,

Table 3: CONCENTRATION INDEX-ORDERS USING DIFFERENT CARDINAL SCALES FOR HEALTH

Country	τ_b	Frank	Clayton	AMH
UK	0.1318	1.2032	0.3036	0.5100
Portugal	0.1783	1.6475	0.4340	0.6529
Germany	0.0454	0.4093	0.0951	0.1941
Greece	0.1571	1.4429	0.3728	0.5900

however, to determine which copula is preferred.

3.4.1 Goodness of fit

Copulas can be compared according to goodness of fit in order to select the most appropriate dependence structure, given the data. Following Fermanian (2005), a fairly straightforward Chi-squared test-statistic comparing each copula (as a bivariate distribution function) to the bivariate standard uniform distribution can be calculated. For well-fitting copulas, the hypothesis that the copula is *not* uniformly-distributed should always be rejected, however the copula that fits the best should be rejected the most strongly. Hence the highest of the Chi-squared test statistics will reveal the best-fitting copula.

In this case the Gumbel copula is the best fit, for all countries, followed very closely by the Frank copula. In fact the Frank copula's Chi-squared statistic is not significantly greater, in the statistical sense, than that of the Gumbel copula. According to the discussion preceding Figure 4.1, this indicates that the structure of the dependence between health and income is generally symmetric about the centres of the distributions, with slightly stronger dependence between high values of health and income than low values.

3.4.2 Rank-ordering countries

The Gumbel copula has been retained hereon for comparison of the results.[13] The bivariate distribution function $C\left(\hat{F}(h_i), \hat{G}(y_i); \hat{\theta}\right)$ for SAH status h_i and income y_i can be calculated by using the estimates of dependence $\hat{\theta}$ (based on $\hat{\tau}_b$), as well as histograms for SAH, $\hat{F}(h)$ and empirical CDFs for income, $\hat{G}(y)$. For the purposes of determining stochastic dominance of one country over another, consider the joint distributions of health and income for 2 countries, $C_1(u_1, v_1)$ and $C_2(u_2, v_2)$.

Following Dardanoni and Lambert (2001), the distribution of health and income in country 1 is known according to $C_1(u_1, v_1)$. It is to be compared to the distribution of health and income in country 1 *had it had the association of country 2.* The comparison will use the *copula* of country 2, containing the *margins* of country 1. Then $Country\ 1 \succeq_I Country\ 2 \Longleftrightarrow C_1(u_1, v_1) \geq C_2(u_1, v_1)$, i.e. if country 1 has a stronger association between the distributions of health and income it has, according to the definition here, more income-related inequality in health.

The counter-factual distribution can be manufactured with an exchange of copulas: for each country we can observe and estimate $C_1\left(u_1, v_1; \hat{\theta}_1\right)$ and $C_2\left(u_2, v_2; \hat{\theta}_2\right)$ respectively, but then calculate $C_2\left(u_1, v_1; \hat{\theta}_2\right)$ numerically. In fact the comparison is more straightforward: $C_1(u_1, v_1) \geq C_2(u_1, v_1)\ iff\ C_1(u_1, v_1) \geq C_2(u_2, v_2)$ as a result of the transformations at the margins (Dardanoni and Lambert 2001). Only the copula for each country is needed for comparison. If the association between the health and income is greater for one country than another, its copula will lie above that of the other country and, as above, it has more income-related inequality in health.

The results from use of the Gumbel copula for each country are shown in Table 4.4.

Each cell in Table 4.4 shows the highest joint probability at that intersection of the contingency table for $C\left(\hat{F}(h), \hat{G}(y); \hat{\theta}\right)$, according to the margins $\hat{F}(h)$ and $\hat{G}(y)$.[14] Table 4.4 shows the bivariate distribution of health and income; comparable to the *expected* contingency table had there been no income-related inequality (i.e. when $C\left(\hat{F}(h), \hat{G}(y); 0\right) = \hat{F}(h) \times \hat{G}(y)$). The distribution assuming no association between health and income is given in Table 4.5.

Table 4: BIVARIATE CUMULATIVE PROBABILITIES IN HEALTH-INCOME SPACE USING THE GUMBEL COPULA

UK

		Income				
		1	2	3	4	5
Health	1	0.0067	0.0122	0.0163	0.0203	0.0229
	2	0.0263	0.0476	0.0664	0.0820	0.0940
	3	0.0788	0.1485	0.2118	0.2670	0.3103
	4	0.1729	0.3411	0.5051	0.6622	0.7955
	5	0.1997	0.4001	0.6003	0.7989	0.9997

PORTUGAL

		Income				
		1	2	3	4	5
Health	1	0.0135	0.0230	0.0306	0.0367	0.0406
	2	0.0613	0.1114	0.1537	0.1880	0.2123
	3	0.1343	0.2574	0.3699	0.4692	0.5415
	4	0.1983	0.3960	0.5929	0.7890	0.9729
	5	0.1885	0.3940	0.5992	0.7980	0.9975

GERMANY

		Income				
		1	2	3	4	5
Health	1	0.0071	0.0138	0.0201	0.0261	0.0315
	2	0.0358	0.0701	0.1026	0.1339	0.1621
	3	0.1065	0.2107	0.3122	0.4101	0.5014
	4	0.1855	0.3695	0.5538	0.7362	0.9125
	5	0.2002	0.3998	0.5992	0.7997	0.9993

GREECE

		Income				
		1	2	3	4	5
Health	1	0.0071	0.0120	0.0164	0.0196	0.0220
	2	0.0265	0.0475	0.0649	0.0792	0.0890
	3	0.0677	0.1249	0.1752	0.2175	0.2484
	4	0.1229	0.2372	0.3415	0.4340	0.5042
	5	0.2000	0.4002	0.6005	0.8001	1.0000

Finally, Table 4.6 helps illustrate which copulas were furthest from independence. The table shows an indicator for which country's observed cell count was furthest from or nearest to the expected cell count, using relative differences, akin to the chi-squared testing procedure.

Table 5: CONTINGENCY TABLES OF HEALTH AND INCOME, IN HEALTH-INCOME SPACE, ASSUMING NO INCOME-RELATED INEQUALITY IN HEALTH

		UK - No Inequality				
		Income				
		1	2	3	4	5
Health	1	0.0046	0.0092	0.0139	0.0185	0.0231
	2	0.0188	0.0376	0.0564	0.0752	0.0940
	3	0.0621	0.1241	0.1862	0.2482	0.3103
	4	0.1593	0.3183	0.4776	0.6366	0.7956
	5	0.2002	0.4001	0.6003	0.8001	1.0000

		PORTUGAL - No Inequality				
		Income				
		1	2	3	4	5
Health	1	0.0081	0.0163	0.0244	0.0325	0.0407
	2	0.0424	0.0848	0.1272	0.1696	0.2120
	3	0.1083	0.2165	0.3246	0.4330	0.5410
	4	0.1947	0.3893	0.5837	0.7786	0.9728
	5	0.2001	0.4002	0.6000	0.8003	1.0000

		GERMANY - No Inequality				
		Income				
		1	2	3	4	5
Health	1	0.0064	0.0127	0.0191	0.0254	0.0318
	2	0.0326	0.0652	0.0978	0.1303	0.1629
	3	0.1006	0.2011	0.3017	0.4022	0.5028
	4	0.1827	0.3651	0.5477	0.7301	0.9127
	5	0.2002	0.4000	0.6001	0.8000	1.0000

		GREECE - No Inequality				
		Income				
		1	2	3	4	5
Health	1	0.0044	0.0088	0.0133	0.0177	0.0221
	2	0.0179	0.0358	0.0537	0.0715	0.0894
	3	0.0497	0.0995	0.1493	0.1989	0.2486
	4	0.1008	0.2017	0.3027	0.4034	0.5042
	5	0.2000	0.4002	0.6005	0.8001	1.0000

4 Discussion

Following Contoyannis and Wildman (2006), the contingency tables are able to show relative cell frequencies at each intersection of SAH and income quintile. Doing so enables further analysis or comparison of countries, in more detail than either Concentration Indices or copula ranking allows. Table 4.6 is used to highlight which country is nearest to, or farthest from, the relative cell frequency that would match independence between SAH and income. Table 4.6 shows that Germany has almost uniformly less departure from its perfect-equality distribution, corresponding to no association between health and income: this is not surprising, given how low its measures of association were, relative to the other countries used.

Portugal and Greece, however, now make for a more interesting comparison. Both exhibit more income-related inequality than the UK, but not absolutely more than one another - as evidenced by their switching with relatively minor changes in the cardinal scale of SAH in Table 4.2. Specifically, Portugal has a higher proportion (than predicted for perfect equality) of people along the dimension of Very Poor health, as well as along the highest income quintile. It also had the closest-to-expected proportion of people with Very Good health, although the differences are far smaller. Greece, on the other hand, exhibits more income-related inequalities elsewhere in the health-income space of the contingency table. It is rank-ordered above Portugal, though, because Portugal's greater polarity at the lowest health and highest incomes generates stronger rank correlation than Greece's more wide-spread, but weaker, departures from perfect equality.

5 Conclusion

This paper has demonstrated the utilisation of copulas as measures of association between health and income, a means by which socioeconomic inequalities in health

Table 6: MIN AND MAX DISTANCE FROM ZERO ASSOCIATION BETWEEN HEALTH AND INCOME IN THE BIVARIATE CDF DUE TO THE GUMBEL COPULA

UK - Max distance		Income				
		1	2	3	4	5
Health	1	0	0	0	0	0
	2	0	0	0	0	0
	3	0	0	0	0	0
	4	0	0	0	0	0
	5	0	0	0	0	0

UK - Min distance		Income				
		1	2	3	4	5
Health	1	0	0	0	0	1
	2	0	0	0	0	0
	3	0	0	0	0	0
	4	0	0	0	0	0
	5	0	0	0	0	0

PORTUGAL - Max distance		Income				
		1	2	3	4	5
Health	1	1	1	1	1	1
	2	0	0	0	1	1
	3	0	0	0	0	1
	4	0	0	0	0	1
	5	0	0	0	0	0

PORTUGAL - Min distance		Income				
		1	2	3	4	5
Health	1	0	0	0	0	0
	2	0	0	0	0	0
	3	0	0	0	0	0
	4	0	0	0	0	0
	5	1	1	0	1	1

GERMANY - Max distance		Income				
		1	2	3	4	5
Health	1	0	0	0	0	0
	2	0	0	0	0	0
	3	0	0	0	0	0
	4	0	0	0	0	0
	5	1	0	0	0	0

GERMANY - Min distance		Income				
		1	2	3	4	5
Health	1	1	1	1	1	0
	2	1	1	1	1	1
	3	1	1	1	1	1
	4	1	1	1	1	1
	5	0	0	1	0	0

GREECE - Max distance		Income				
		1	2	3	4	5
Health	1	0	0	0	0	0
	2	1	1	1	0	0
	3	1	1	1	1	0
	4	1	1	1	1	0
	5	0	1	1	1	1

GREECE - Min distance		Income				
		1	2	3	4	5
Health	1	0	0	0	0	0
	2	0	0	0	0	0
	3	0	0	0	0	0
	4	0	0	0	0	0
	5	0	0	0	0	0

can be compared. The concentration curve and concentration index are considered the standard approach to measuring and comparing inequalities in health, controlling for individual rank in the income distribution. However, because Self-Assessed Health is commonly measured on an ordinal scale, the concentration curve and Index are dependent upon the cardinal scale subsequently given to health. Here I showed, using 4 countries from the European Community Household Panel, that changing the cardinal scale of health can re-order countries in an international comparison.

The copula method is based upon measures of rank correlation. When estimated in a manner that accommodates the discrete scale of health, the copula measure of association between health and income can be used comparatively as a measure of income-related inequality in health. The copula proved to be invariant to changes in the cardinal scales given to health. Moreover, contingency tables were shown, illustrating the copula in bivariate health-income space. Using this approach, departures from perfect equality can be traced across the 2-dimensional plane of heatlh-income space, identifying, for example, one country's inequality being greater along the income dimension, while another country's might be greater along the health dimension. Portugal ranks below Greece (i.e. has stronger association between health and income, and therefore greater income-related inequality in health), because of polarisation at the lower end of its health distribution. Greece exhibits more widely-spread, but smaller levels, of association elsewhere in the joint distribution of health and income.

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Notes

[1]That is to say, this analysis does not explicitly make any determinations of 'best' or 'worst' from any welfarist perspectives, only statistical perspectives.

[2]For income y, with rank function (or other CDF) $F(y)$, the Lorenz curve is given by $F_1(y)$, where F_1 is the income (or the proportion of all income) earned by all individuals with income less than or equal to y.

[3]Generalised concentration curves are those that multiply a concentration curve by total health: rather than the cumulative *share* of health, one sees the cumulative *amount* of health. Although the horizontal axis will be the cumulative proportion of a ranked population, the vertical axis will be the cumulative amount of health that ranks enjoys, up to the total health in the population.

[4]This relates to the previous note: division of the covariance by the mean of health is needed to force the limits of the generalised concentration index to match the Gini coefficient - in that the cumulative share of health extends from 0 to the total health health in the population - and facilitate more straightforward interpretation.

[5]For some families of copulas, closed-form solutions for either Spearman's ρ or Kendall's τ may - or may not - be available. Packages such as Mathematica or Maple, however, can be employed to find numerical solutions.

[6]In their study, 'not weak' was when $|\tau_b| > 0.02$. Observed dependence in this data not considered to be sufficiently weak for the discrete SAH to be a problem.

[7]Kendall's τ_b is often applied to 2×2 tables and/or binary data. The results in Vandenhende and Lambert (2000) suggest that, with 5 categories in SAH, τ_b can be considered an increasing function of θ.

[8] Algebraically, the Frank copula does not nest independence because of the term $\frac{1}{\theta}$. Nelsen (1998), however, demonstrates that $\lim_{\theta \to 0} C_{Frank} = uv$, i.e. the Product Copula.

[9] It is also referred to, in Nelsen (1999, 2006) and Hutchinson and Lai (1990) as the Gumbel-Hougaard copula.

[10] This was done by Cristina Hernández Quevedo, a colleague at the University of York, for other research - for which I am very grateful.

[11] Although the question and the number of categories is different in different countries. For this analysis the countries are comparable in this regard.

[12] Details of the procedure, as well as Matlab and Stata codes, are available from the author, as well as online here www.york.ac.uk/res/herd/hedg_stata.html.

[13] Results from the other copulas are available from the author.

[14] This is comparable to a bivariate cut-point.

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