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On the Use of the Lasso for Instrumental Variables Estimation with Some Invalid Instruments*

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Abstract

We investigate the behaviour of the Lasso for selecting invalid instruments in linear instrumental variables models for estimating causal effects of exposures on outcomes, as proposed recently by Kang, Zhang, Cai and Small (2016, Journal of the American Statistical Association). Invalid instruments are such that they fail the exclusion restriction and enter the model as explanatory variables. We show that for this setup, the Lasso may not consistently select the invalid instruments if these are relatively strong. We propose a median estimator that is consistent when less than 50% of the instruments are invalid, but its consistency does not depend on the relative strength of the instruments, or their correlation structure. We show that this estimator can be used for adaptive Lasso estimation, with the resulting estimator having oracle properties. The methods are applied to a Mendelian randomisation study to estimate the causal effect of BMI on diastolic blood pressure, using data on individuals from the UK Biobank, with 96 single nucleotide polymorphisms as potential instruments for BMI.

Key Words: causal inference, instrumental variables estimation, invalid instruments, Lasso, Mendelian randomisation

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1 Introduction

Instrumental variables estimation is a procedure for the identification and estimation of causal effects of exposures on outcomes where the observed relationships are confounded by non-random selection of exposure. This problem is likely to occur in observational studies, but also in randomised clinical trials if there is selective participant non-compliance. An instrumental variable (IV) can be used to solve the problem of non-ignorable selection. In order to do this, an IV needs to be associated with the exposure, but only associated with the outcome indirectly through its association with the exposure. The former condition is referred to as the 'relevance' and the latter as the 'exclusion' condition. Examples of instrumental variables are quarter-of-birth for educational achievement to determine its effect on wages, see Angrist and Krueger (1991), randomisation of patients to treatment as an instrument for actual treatment when there is non-compliance, see e.g. Greenland (2000), and Mendelian randomisation studies use IVs based on genetic information, see e.g. Lawlor et al. (2008). For recent reviews and further examples see e.g. Clarke and Windmeijer (2012), Imbens (2014), Burgess et al. (2015) and Kang et al. (2016).

Whether instruments are relevant can be tested from the observed association between exposure and instruments. The effects on the standard linear IV estimator of 'weak instruments', i.e. the case where instruments are only weakly associated with the exposure of interest, have been derived for the linear model using weak instrument asymptotics by Staiger and Stock (1997). This has led to the derivation of critical values for the simple F-test statistic for testing the null of weak instruments by Stock and Yogo (2005). Another strand of the literature focuses on instrument selection in potentially high-dimensional settings, see e.g. Belloni et al. (2012), Belloni et al. (2014), Chernozhukov et al. (2015) and Lin et al. (2015), where the focus is on identifying important covariate effects and selecting optimal instruments from a (large) set of a priori valid instruments, where optimality is with respect to the variance of the IV estimator.

In this paper we consider violations of the exclusion condition of the instruments, following closely the setup of Kang et al. (2016) for the linear IV model where some of the available instruments can be invalid in the sense that they can have a direct effect on the outcomes or are associated with unobserved confounders. Kang et al. (2016) propose

a Lasso type procedure to identify and select the set of invalid instruments. Liao (2013) and Cheng and Liao (2015) also considered shrinkage estimation for identification of invalid instruments, but in their setup there is a subset of instruments that is known to be valid and that contains sufficient information for identification and estimation of the causal effects. In contrast, Kang et al. (2016) do not assume any prior knowledge about which instruments are potentially valid or invalid. This is a similar setup as in Andrews (1999) who proposed a selection procedure using information criteria based on the so-called J-test of over-identifying restrictions, as developed by Sargan (1958) and Hansen (1982). The Andrews (1999) setup is more general than that of Kang et al. (2016) and requires a large number of model evaluations, which has a negative impact on the performance of the selection procedure.

This paper assesses the performance of the Kang et al. (2016) Lasso type selection and estimation procedure in their setting of a fixed number of potential instruments. If the set of invalid instruments were known, the oracle Two-Stage Least Squares (2SLS) estimator would be the estimator of choice in their setting. As the focus is estimation of and inference on the causal effect parameter, denoted by β, and as the standard Lasso approach does not have oracle properties, see e.g. Zou (2006), we show how the adaptive Lasso procedure of Zou (2006) can be used in order to obtain an estimator with oracle properties. In order to do so, we propose an initial consistent estimator of the parameters that is consistent also when the irrepresentable condition for consistent Lasso selection of Zhao and Yu (2006) and Zou (2006) fails.

Applying the irrepresentable condition to this IV setup, we derive conditions under which the Lasso method does not consistently select the invalid instruments. As is well known from Zhao and Yu (2006), Zou (2006), Meinshausen and Bühlmann (2006) and Wainwright (2009), certain correlation structures of the variables prevent consistent selection. New in our results are the conditions on the strength of the invalid instruments relative to that of the valid ones that result in violations of the irrepresentable condition, where the strength of an instrument is its standardised effect on the exposure. From this we can show that consistent selection of the invalid instruments may not be possible if these are relatively strong, even when less than 50% of the instruments are invalid, which is a sufficient condition for the identification of the parameters.

We show that under the condition that less than 50% of the instruments are invalid, a

simple median type estimator is a consistent estimator for the parameters in the model, independent of the strength of the invalid instruments relative to that of the valid instruments, or their correlation structure. It can therefore be considered for use in the adaptive Lasso procedure as proposed by Zou (2006). With n the sample size, we show that the median estimator converges at the $\sqrt{n}$ rate, but with an asymptotic bias, as the limiting distribution is that of an order statistic. It does, however, satisfy the conditions for the adaptive Lasso procedure to enjoy oracle properties.

Because of this oracle property, and as in practice instrument strength is very likely to vary by instruments and invalid instruments could be relatively strong, it will be important to consider our adaptive Lasso approach for assessing instrument validity and estimating causal effects. In Mendelian randomisation studies it is clear that genetic markers have differential impacts on exposures from examining the results from genome wide association studies and one cannot rule out ex-ante that invalid instruments with a direct effect are also stronger predictors for the exposure.[1]

The next section, Section 2, introduces the model and the Lasso estimator as proposed by Kang et al. (2016). In Section 3, we derive the irrepresentable condition for this particular Lasso selection problem and present the result on the relationship between the relative strengths of the instruments and consistent selection. Section 4 presents the median estimator, establishes its consistency and shows that its asymptotic properties are such that the adaptive Lasso estimator enjoys oracle properties. Section 5 presents some Monte Carlo simulation results. In Section 6, we link the Andrews (1999) method to the Lasso selection problem and show how the test of overidentifying restrictions can be used as a stopping rule. Section 7 investigates how close the behaviour of the adaptive Lasso estimator is to that of the oracle 2SLS estimator in the Monte Carlo simulations, by comparing the performances of the Wald tests on the causal parameter under the null for different sample sizes. It further discusses more generally the issue of information content for this estimation problem, including that of weak instruments. In Section 8, the methods are applied to a Mendelian randomisation study to estimate the causal effect of Body Mass Index (BMI) on diastolic blood pressure using data on individuals from

[1] Bowden et al. (2015) and Kolesar et al. (2015) allow for all instruments to be invalid and show that the causal effect can be consistently estimated if the number of instruments increases with the sample size under the assumption of uncorrelatedness of the instrument strength and their direct effects on the outcome variable.

the UK Biobank, with 96 single nucleotide polymorphisms as potential instruments for BMI. Section 9 concludes.

The following notation is used in the remainder of the paper. For a full column rank matrix $\mathbf{X}$ with n rows, $\mathbf{M}_X = \mathbf{I}_n - \mathbf{P}_X$, where $\mathbf{P}_X = \mathbf{X}\left(\mathbf{X}'\mathbf{X}\right)^{-1}\mathbf{X}'$ is the projection onto the column space of $\mathbf{X}$, and $\mathbf{I}_n$ is the n-dimensional identity matrix. A k-vector of ones is denoted $\boldsymbol{\iota}_k$. The l_p-norm is denoted by $\|.\|_p$, and the l_0-norm, $\|.\|_0$, denotes the number of non-zero components of a vector. We use $\|.\|_\infty$ to denote the maximal element of a vector.

2 Model and Lasso Estimator

We follow Kang, Zhang, Cai and Small (2016) (KZCS from now on), who considered the following potential outcomes model. For $i = 1, ..., n$, let $Y_i^{(d,\mathbf{z})}$, be the potential outcome if the individual i were to have exposure d and instrument values $\mathbf{z}$. The observed outcome for an individual i is denoted by the scalar Y_i, the treatment by the scalar D_i and the vector of L potential instruments by $\mathbf{Z}_{i.}$. The instruments may not all be valid and can have a direct or indirect effect. For two possible values of the exposure d^*, d and instruments $\mathbf{z}^*$, $\mathbf{z}$, assume the following potential outcomes model

$$Y_i^{(d^*,\mathbf{z}^*)} - Y_i^{(d,\mathbf{z})} = (\mathbf{z}^* - \mathbf{z})'\boldsymbol{\phi} + (d^* - d)\beta \quad (1)$$

$$E\left[Y_i^{(0,0)}|\mathbf{Z}_{i.}\right] = \mathbf{Z}_{i.}'\boldsymbol{\psi}, \quad (2)$$

where $\boldsymbol{\phi}$ measures the direct effect of $\mathbf{z}$ on Y, and $\boldsymbol{\psi}$ represents the presence of unmeasured confounders that affect both the instruments and the outcome.

We have a random sample $\{Y_i, D_i, \mathbf{Z}_{i.}'\}_{i=1}^n$. Combining (1) and (2), the observed data model for the random sample is given by

$$Y_i = D_i\beta + \mathbf{Z}_{i.}'\boldsymbol{\alpha} + \varepsilon_i, \quad (3)$$

where $\boldsymbol{\alpha} = \boldsymbol{\phi} + \boldsymbol{\psi}$;

$$\varepsilon_i = Y_i^{(0,0)} - E\left[Y_i^{(0,0)}|\mathbf{Z}_{i.}\right]$$

and hence $E\left[\varepsilon_i|\mathbf{Z}_{i.}\right] = 0$. For ease of exposition, we further assume that $E\left[\varepsilon_i^2|\mathbf{Z}_{i.}\right] = \sigma^2$.

The KZCS definition of a valid instrument is then linked to the exclusion restriction and given as follows: Instrument j, $j \in \{1, ..., L\}$, is valid if $\alpha_j = 0$ and it is invalid if

$\alpha_j \neq 0$. As in the KZCS setting, we are interested in the identification and estimation of the scalar treatment effect β in large samples with a fixed number L of potential instruments.

Let $\mathbf{y}$ and $\mathbf{d}$ be the n-vectors of n observations on $\{Y_i\}$ and $\{D_i\}$ respectively, and let $\mathbf{Z}$ be the $n \times L$ matrix of potential instruments. As an intercept is implicitly present in the model, $\mathbf{y}$, $\mathbf{d}$ and the columns of $\mathbf{Z}$ have all been taken in deviation from their sample means. Following the notation of Zou (2006), let $\mathbf{Z}_A$ be the set of invalid instruments, $A = \{j : \alpha_j \neq 0\}$ and $\boldsymbol{\alpha}_A$ the associated coefficient vector. The oracle Instrumental Variables, or Two-Stage Least Squares (2SLS) estimator is obtained when the set $\mathbf{Z}_A$ is known. Let $\mathbf{R}_A = \begin{bmatrix} \mathbf{d} & \mathbf{Z}_A \end{bmatrix}$, the oracle 2SLS estimator is then given by

$$\widehat{\boldsymbol{\theta}}_{or} = \begin{pmatrix} \widehat{\beta}_{or} \\ \widehat{\boldsymbol{\alpha}}_A \end{pmatrix} = \left(\mathbf{R}_A' \mathbf{P}_Z \mathbf{R}_A\right)^{-1} \mathbf{R}_A' \mathbf{P}_Z \mathbf{y}. \tag{4}$$

Let $\widehat{\mathbf{d}} = \mathbf{P}_Z \mathbf{d}$, with individual elements $\widehat{D}_i$, then $\widehat{\boldsymbol{\theta}}_{or}$ is the OLS estimator in the model

$$Y_i = \widehat{D}_i \beta + \mathbf{Z}_{A,i.}' \boldsymbol{\alpha}_A + \xi_i,$$

where ξ_i is defined implicitly, and hence

$$\begin{aligned} \widehat{\boldsymbol{\alpha}}_A &= \left(\mathbf{Z}_A' \mathbf{M}_{\widehat{d}} \mathbf{Z}_A\right)^{-1} \mathbf{Z}_A' \mathbf{M}_{\widehat{d}} \mathbf{y} \\ &= \left(\mathbf{Z}_A' \mathbf{M}_{\widehat{d}} \mathbf{Z}_A\right)^{-1} \mathbf{Z}_A' \mathbf{M}_{\widehat{d}} \mathbf{P}_Z \mathbf{y}. \end{aligned} \tag{5}$$

The oracle 2SLS estimator for β is given by

$$\widehat{\beta}_{or} = \left(\widehat{\mathbf{d}}' \mathbf{M}_{Z_A} \widehat{\mathbf{d}}\right)^{-1} \widehat{\mathbf{d}}' \mathbf{M}_{Z_A} \mathbf{y}.$$

Under standard assumptions, as defined below,

$$\sqrt{n}\left(\widehat{\beta}_{or} - \beta\right) \xrightarrow{d} N\left(0, \sigma^2_{\beta_{or}}\right), \tag{6}$$

where

$$\sigma^2_{\beta_{or}} = \sigma^2 \left(E\left[\mathbf{Z}_{i.} D_i\right]' E\left[\mathbf{Z}_{i.} \mathbf{Z}_{i.}'\right]^{-1} E\left[\mathbf{Z}_{i.} D_i\right] - E\left[\mathbf{Z}_{A,i.} D_i\right]' E\left[\mathbf{Z}_{A,i.} \mathbf{Z}_{A,i.}'\right]^{-1} E\left[\mathbf{Z}_{A,i.} D_i\right] \right)^{-1}. \tag{7}$$

The vector $\widehat{\mathbf{d}}$ is the linear projection of $\mathbf{d}$ on $\mathbf{Z}$. If we define $\widehat{\boldsymbol{\gamma}} = \left(\mathbf{Z}'\mathbf{Z}\right)^{-1} \mathbf{Z}'\mathbf{d}$, then $\widehat{\mathbf{d}} = \mathbf{Z}\widehat{\boldsymbol{\gamma}}$, or $\widehat{D}_i = \mathbf{Z}_{i.}' \widehat{\boldsymbol{\gamma}}$. We specify

$$D_i = \mathbf{Z}_{i.}' \boldsymbol{\gamma} + v_i, \tag{8}$$

where $\boldsymbol{\gamma} = E[\mathbf{Z}_{i.}\mathbf{Z}_{i.}']^{-1} E[\mathbf{Z}_{i.}D_i]$, and hence $E[\mathbf{Z}_{i.}v_i] = 0$. Further, as in KZCS, let $\boldsymbol{\Gamma} = E[\mathbf{Z}_{i.}\mathbf{Z}_{i.}']^{-1} E[\mathbf{Z}_{i.}Y_i] = \boldsymbol{\gamma}\beta + \boldsymbol{\alpha}$. Then define π_j as

$$\pi_j \equiv \frac{\Gamma_j}{\gamma_j} = \beta + \frac{\alpha_j}{\gamma_j}, \tag{9}$$

for $j = 1, ..., L$. Theorem 1 in KZCS states the conditions under which, given knowledge of $\boldsymbol{\gamma}$ and $\boldsymbol{\Gamma}$, a unique solution exists for values of β and α_j. A necessary and sufficient condition to identify β and the α_j is that the valid instruments form the largest group, where instruments form a group if they have the same value of π. Corollary 1 in KZCS then states a sufficient condition for identification. Let $s = ||\boldsymbol{\alpha}||_0$ be the number of invalid instruments. A sufficient condition is that $s < L/2$, as then clearly the largest group is formed by the valid instruments.

In model (3), some elements of $\boldsymbol{\alpha}$ are assumed to be zero, but it is not known ex-ante which ones they are and the selection problem therefore consists of correctly identifying those instruments with non-zero α. KZCS propose to estimate the parameters $\boldsymbol{\alpha}$ and β by using l_1 penalisation on $\boldsymbol{\alpha}$ and to minimise

$$\left(\widehat{\boldsymbol{\alpha}}^{(n)}, \widehat{\beta}^{(n)}\right) = \arg\min_{\alpha,\beta} \frac{1}{2}\|\mathbf{P}_Z(\mathbf{y} - \mathbf{d}\beta - \mathbf{Z}\boldsymbol{\alpha})\|_2^2 + \lambda_n\|\boldsymbol{\alpha}\|_1, \tag{10}$$

where $\|\boldsymbol{\alpha}\|_1 = \sum_j |\alpha_j|$ and the squared l_2 norm is $(\mathbf{y} - \mathbf{d}\beta - \mathbf{Z}\boldsymbol{\alpha})' \mathbf{P}_Z (\mathbf{y} - \mathbf{d}\beta - \mathbf{Z}\boldsymbol{\alpha})$. This method is closely related to the Lasso, and the regularization parameter λ_n determines the sparsity of the vector $\widehat{\boldsymbol{\alpha}}^{(n)}$. From (5), a fast two-step algorithm is proposed as follows. For a given λ_n solve

$$\widehat{\boldsymbol{\alpha}}^{(n)} = \arg\min_{\alpha} \frac{1}{2}\|\mathbf{M}_{\widehat{d}}\mathbf{P}_Z\mathbf{y} - \mathbf{M}_{\widehat{d}}\mathbf{Z}\boldsymbol{\alpha}\|_2^2 + \lambda_n\|\boldsymbol{\alpha}\|_1 \tag{11}$$

and obtain $\widehat{\beta}^{(n)}$ by

$$\widehat{\beta}^{(n)} = \frac{\widehat{\mathbf{d}}'\left(\mathbf{y} - \mathbf{Z}\widehat{\boldsymbol{\alpha}}^{(n)}\right)}{\widehat{\mathbf{d}}'\widehat{\mathbf{d}}}. \tag{12}$$

In order to find $\widehat{\boldsymbol{\alpha}}^{(n)}$ in (11), the Lasso modification of the LARS algorithm of Efron, Hastie, Johnstone and Tibshirani (2004) can be used and KZCS have developed an R-routine for this purpose, called *sisVIVE* (some invalid and some valid IV estimator), where the regularisation parameter λ_n is obtained by cross-validation.

The standard Lasso estimator does not have oracle properties, see e.g. Zou (2006). In order to obtain an adaptive Lasso estimator with oracle properties, we propose an

initial consistent estimator that is well behaved also when the irrepresentable condition for consistent Lasso variable selection fails. In the next section we show under what conditions the Lasso selection of invalid instruments is inconsistent using the irrepresentable condition of Zhao and Yu (2006) and Zou (2006). We show that this does depend on the strength of the invalid instruments relative to that of the valid ones in combination with the number of invalid instruments, and the correlation structure of the instruments. KZCS did show analytically that the performance of the Lasso estimator is influenced by these factors, but did not relate them to consistent selection. In particular, we show how relatively strong invalid instruments may result in the Lasso method selecting the valid instruments as invalid in large samples.

Under the assumptions specified below, $\boldsymbol{\gamma}$ and $\boldsymbol{\Gamma}$ can be consistently estimated by $\widehat{\boldsymbol{\gamma}}$ and $\widehat{\boldsymbol{\Gamma}} = (\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'\mathbf{y}$ respectively. Let $\widehat{\pi}_j = \widehat{\Gamma}_j/\widehat{\gamma}_j$, for $j = 1, ..., L$. We show that the median of the $\widehat{\pi}_j$ is a consistent estimator for β when $s < L/2$, without any further restrictions on the relative strengths or correlations of the instruments, and that the asymptotic properties of this estimator are such that the resulting estimator for $\boldsymbol{\alpha}$ can be used for the adaptive Lasso of Zou (2006). We therefore obtain a desired estimator with the same limiting distribution as the oracle 2SLS estimator (6), also when the irrepresentable condition is violated, as long as $s < L/2$, which is the same condition as the sufficient condition of KZCS for identification.

For the random variables and i.i.d. sample $\{Y_i, D_i, \mathbf{Z}'_{i.}\}_{i=1}^n$, and model (3) and (8), we assume throughout that the following conditions hold:

Assumption 1 $E[\mathbf{Z}_{i.}\mathbf{Z}'_{i.}] = \mathbf{Q}$ *is full rank.*

Assumption 2 $\operatorname{plim}(n^{-1}\mathbf{Z}'\mathbf{Z}) = E[\mathbf{Z}_{i.}\mathbf{Z}'_{i.}]$; $\operatorname{plim}(n^{-1}\mathbf{Z}'\mathbf{d}) = E[\mathbf{Z}_{i.}D_i]$; $\operatorname{plim}(n^{-1}\mathbf{Z}'\boldsymbol{\varepsilon}) = E[\mathbf{Z}_{i.}\varepsilon_i] = 0$.

Assumption 3 $\gamma = (E[\mathbf{Z}_{i.}\mathbf{Z}'_{i.}])^{-1}E[\mathbf{Z}_{i.}D_i]$, $\gamma_j \neq 0$, $j = 1, ..., L$.

The setting is thus a relatively straightforward one with fixed parameters β, $\boldsymbol{\alpha}$ and $\boldsymbol{\gamma}$, and fixed number $L \ll n$ of potential instruments. This is the setting under which the oracle 2SLS estimator has the limiting distribution (6), and is a setting of interest in many applications. To identify in this simple setting an ex-ante unknown subset of

invalid instruments using the Lasso is challenging, as highlighted in the next section. In the Monte Carlo simulation section we present results confirming that the adaptive Lasso estimator we propose performs well in large samples, but may not perform well in small samples. A small sample is essentially a small information setting, which we discuss further in Section 7.

For the case of many weak instruments, even the oracle 2SLS estimator would not be the estimator of choice, due to its poor asymptotic performance, and the median estimator may not be consistent. Oracle estimators with better asymptotic properties in this setting are the Limited Information Maximum Likelihood (LIML) estimator, see Bekker (1994) and Hansen, Hausman and Newey (2008), or the Continuous Updating Estimator (CUE), see Newey and Windmeijer (2009). Selection of invalid instruments in this setting is outside the scope of this paper.

3 Irrepresentable Condition

As $\mathbf{Z}'\mathbf{M}_{\widehat{d}}\mathbf{M}_{\widehat{d}}\mathbf{P}_Z\mathbf{y} = \mathbf{Z}'\mathbf{M}_{\widehat{d}}\mathbf{P}_Z\mathbf{y} = \mathbf{Z}'\mathbf{M}_{\widehat{d}}\mathbf{y}$, it follows that

$$\begin{aligned} \|\mathbf{M}_{\widehat{d}}(\mathbf{P}_Z\mathbf{y} - \mathbf{Z}\boldsymbol{\alpha})\|_2^2 &= \mathbf{y}'\mathbf{P}_Z\mathbf{M}_{\widehat{d}}\mathbf{P}_Z\mathbf{y} - 2\mathbf{y}'\mathbf{M}_{\widehat{d}}\mathbf{Z}\boldsymbol{\alpha} + \boldsymbol{\alpha}'\mathbf{Z}'\mathbf{M}_{\widehat{d}}\mathbf{Z}\boldsymbol{\alpha} \\ &= \mathbf{y}'\mathbf{P}_Z\mathbf{M}_{\widehat{d}}\mathbf{P}_Z\mathbf{y} - 2\mathbf{y}'\widetilde{\mathbf{Z}}\boldsymbol{\alpha} + \boldsymbol{\alpha}'\widetilde{\mathbf{Z}}'\widetilde{\mathbf{Z}}\boldsymbol{\alpha}, \end{aligned}$$

where $\widetilde{\mathbf{Z}} = \mathbf{M}_{\widehat{d}}\mathbf{Z}$. As

$$\|\mathbf{y} - \widetilde{\mathbf{Z}}\boldsymbol{\alpha}\|_2^2 = \mathbf{y}'\mathbf{y} - 2\mathbf{y}'\widetilde{\mathbf{Z}}\boldsymbol{\alpha} + \boldsymbol{\alpha}'\widetilde{\mathbf{Z}}'\widetilde{\mathbf{Z}}\boldsymbol{\alpha},$$

it follows that the Lasso estimator $\widehat{\boldsymbol{\alpha}}^{(n)}$ as defined in (11) can equivalently be obtained as

$$\widehat{\boldsymbol{\alpha}}^{(n)} = \arg\min_{\alpha} \frac{1}{2}\|\mathbf{y} - \widetilde{\mathbf{Z}}\boldsymbol{\alpha}\|_2^2 + \lambda_n\|\boldsymbol{\alpha}\|_1. \tag{13}$$

This minimization problem looks very much like a standard Lasso approach with $\widetilde{\mathbf{Z}}$ as explanatory variables. However, an important difference is that $\widetilde{\mathbf{Z}}$ does not have full rank, but its rank is equal to $L-1$. This is related to the standard Lasso case where we have an overcomplete dictionary implying that the OLS solution is not feasible. Intuitively, we cannot set $\lambda_n = 0$ in (13) as we have to shrink at least one element of α to zero to identify the parameter β. All just-identified models with $L-1$ instruments included as invalid result in a residual correlation of 0, and hence $\lambda_n = 0$ does not lead to a

unique 2SLS estimator. Therefore, when using the LARS/Lasso algorithm, it has to start from a model without any instruments included in the model as invalid, and at the last LARS/Lasso step one instrument is excluded from the model, i.e. treated as valid. When $L-1$ instruments have been selected as invalid and included in the model, the resulting Lasso estimator is the (just identified) 2SLS estimator and this final model is the model for which $\lambda_n = 0$. Clearly, it can then be the case that the LARS/Lasso path is such that it does not include a model where all invalid instruments have been selected as such, which is the case when the final instrument selected as valid is in fact invalid.

We follow Zhao and Yu (2006) and Zou (2006) who developed the irrepresentable conditions for consistent Lasso variable selection. Let $\mathbf{C} = \text{plim}\left(n^{-1}\widetilde{\mathbf{Z}}'\widetilde{\mathbf{Z}}\right)$. As before, let $A = \{j : \alpha_j \neq 0\}$ and assume wlog that $A = \{1, 2, ..., s\}$.[2] Let

$$\mathbf{C} = \begin{bmatrix} \mathbf{C}_{11} & \mathbf{C}_{21}' \\ \mathbf{C}_{21} & \mathbf{C}_{22} \end{bmatrix}, \tag{14}$$

where $\mathbf{C}_{11}$ is an $s \times s$ matrix. Further, define $\widehat{A}_n = \left\{j : \widehat{\alpha}_j^{(n)} \neq 0\right\}$. Let $\mathbf{s}(\boldsymbol{\alpha}_1)$ denote the vector $\text{sgn}(\boldsymbol{\alpha}_1)$, where $\boldsymbol{\alpha}_1 = \boldsymbol{\alpha}_A = (\alpha_1, ..., \alpha_s)'$, $\text{sgn}(a) = 1$ if $a > 0$ and $\text{sgn}(a) = -1$ if $a < 0$. A sufficient condition for consistent Lasso variable selection, meaning that $\lim_{n\to\infty} P\left(\widehat{A}_n = A\right) = 1$, is the irrepresentable condition

$$\left\|\mathbf{C}_{21}\mathbf{C}_{11}^{-1}\mathbf{s}(\boldsymbol{\alpha}_1)\right\|_\infty < 1. \tag{15}$$

If $\left\|\mathbf{C}_{21}\mathbf{C}_{11}^{-1}\mathbf{s}(\boldsymbol{\alpha}_1)\right\|_\infty > 1$, then the lasso variable selection is inconsistent, see Zhao and Yu (2006) and Zou (2006).

Partition $\mathbf{Q} = \text{plim}\,(n^{-1}\mathbf{Z}'\mathbf{Z})$ and $\boldsymbol{\gamma}$ commensurate with the partitioning of $\mathbf{C}$ as

$$\mathbf{Q} = \begin{bmatrix} \mathbf{Q}_{11} & \mathbf{Q}_{21}' \\ \mathbf{Q}_{21} & \mathbf{Q}_{22} \end{bmatrix}, \ \boldsymbol{\gamma} = \begin{pmatrix} \boldsymbol{\gamma}_1 \\ \boldsymbol{\gamma}_2 \end{pmatrix}, \tag{16}$$

where the instruments have been standardised such the diagonal elements of $\mathbf{Q}$ are equal to 1. Then for the Lasso specification (13) we have the following result.

Proposition 1 *Consider the observational models (3) and (8) under Assumptions 1, 2, and 3. Let* $\mathbf{C} = \text{plim}\left(n^{-1}\widetilde{\mathbf{Z}}'\widetilde{\mathbf{Z}}\right)$*;* $\mathbf{Q} = \text{plim}\,(n^{-1}\mathbf{Z}'\mathbf{Z})$*; and* $\mathbf{C}_{11}$*,* $\mathbf{C}_{21}$*,* $\mathbf{Q}_{11}$*,* $\mathbf{Q}_{21}$*,* $\mathbf{Q}_{22}$*,* $\boldsymbol{\gamma}_1$

[2] We will use subscripts A and 1 interchangeably from here onwards, and subscript 2 for associations with the set $A^c = \{j : \alpha_j = 0\}$.

and $\boldsymbol{\gamma}_2$ *as specified in (14) and (16). Then* $\mathbf{C}_{21}\mathbf{C}_{11}^{-1}$ *is given by*

$$\mathbf{C}_{21}\mathbf{C}_{11}^{-1} = \mathbf{Q}_{21}\mathbf{Q}_{11}^{-1} - \widetilde{\mathbf{Q}}_{22}\boldsymbol{\gamma}_2 \frac{\boldsymbol{\gamma}_1' + \boldsymbol{\gamma}_2'\mathbf{Q}_{21}\mathbf{Q}_{11}^{-1}}{\boldsymbol{\gamma}_2'\widetilde{\mathbf{Q}}_{22}\boldsymbol{\gamma}_2}, \tag{17}$$

where

$$\widetilde{\mathbf{Q}}_{22} = \mathbf{Q}_{22} - \mathbf{Q}_{21}\mathbf{Q}_{11}^{-1}\mathbf{Q}_{21}' = \operatorname{plim}\left(n^{-1}\mathbf{Z}_2'\mathbf{M}_{Z_1}\mathbf{Z}_2\right).$$

Proof. See Appendix A.1. ■

Proposition 1 shows that consistent selection of the instruments is not only affected by the correlation structure of the instruments, but also by the values of $\boldsymbol{\gamma}_1$ and $\boldsymbol{\gamma}_2$. The next Proposition derives conditions on $\boldsymbol{\gamma}_1$ and $\boldsymbol{\gamma}_2$ under which consistent selection is not possible.

Proposition 2 *Under the assumptions of Proposition 1, the Lasso variable selection is not consistent if*

$$\left|\boldsymbol{\gamma}_1'\mathbf{s}\left(\boldsymbol{\alpha}_1\right)\right| > \left\|\boldsymbol{\gamma}_2\right\|_1.$$

Proof. It follows from (17) that

$$\left|\boldsymbol{\gamma}_2'\mathbf{C}_{21}\mathbf{C}_{11}^{-1}\mathbf{s}\left(\boldsymbol{\alpha}_1\right)\right| = \left|\boldsymbol{\gamma}_1'\mathbf{s}\left(\boldsymbol{\alpha}_1\right)\right|.$$

Therefore,

$$\begin{aligned} \left\|\boldsymbol{\gamma}_2\right\|_1 \left\|\mathbf{C}_{21}\mathbf{C}_{11}^{-1}\mathbf{s}\left(\boldsymbol{\alpha}_1\right)\right\|_\infty &\geq \left|\boldsymbol{\gamma}_1'\mathbf{s}\left(\boldsymbol{\alpha}_1\right)\right| \\ \left\|\mathbf{C}_{21}\mathbf{C}_{11}^{-1}\mathbf{s}\left(\boldsymbol{\alpha}_1\right)\right\|_\infty &\geq \frac{\left|\boldsymbol{\gamma}_1'\mathbf{s}\left(\boldsymbol{\alpha}_1\right)\right|}{\left\|\boldsymbol{\gamma}_2\right\|_1}. \end{aligned}$$

Hence, $\left\|\mathbf{C}_{21}\mathbf{C}_{11}^{-1}\mathbf{s}\left(\boldsymbol{\alpha}_1\right)\right\|_\infty > 1$ if $\left|\boldsymbol{\gamma}_1'\mathbf{s}\left(\boldsymbol{\alpha}_1\right)\right| > \left\|\boldsymbol{\gamma}_2\right\|_1$. ■

Remark 1 *If* $\mathbf{s}\left(\boldsymbol{\alpha}_1\right) = \mathbf{s}\left(\boldsymbol{\gamma}_1\right)$*, then* $\left|\boldsymbol{\gamma}_1'\mathbf{s}\left(\boldsymbol{\alpha}_1\right)\right| = \left\|\boldsymbol{\gamma}_1\right\|_1$*, its maximum. Regardless of the correlation structure of the instruments, the Lasso variable selection is therefore not consistent in that case if* $\left\|\boldsymbol{\gamma}_1\right\|_1 > \left\|\boldsymbol{\gamma}_2\right\|_1$*, i.e. when the invalid instruments are stronger (in* l_1*-norm) than the valid ones.*

From Proposition 1 we can investigate consistent selection for various cases of interest. Related to the Monte Carlo simulations in KZCS and below in Section 5, Corollary 1 considers the case with $\boldsymbol{\gamma}_1 = \widetilde{\gamma}_1\boldsymbol{\iota}_s$ and $\boldsymbol{\gamma}_2 = \widetilde{\gamma}_2\boldsymbol{\iota}_{L-s}$.

Corollary 1 *If* $\boldsymbol{\gamma}_1 = \widetilde{\gamma}_1 \boldsymbol{\iota}_s$ *and* $\boldsymbol{\gamma}_2 = \widetilde{\gamma}_2 \boldsymbol{\iota}_{L-s}$*, then* $|\boldsymbol{\gamma}_1' \mathbf{s}(\boldsymbol{\alpha}_1)| > \|\boldsymbol{\gamma}_2\|_1$ *if* $\left|\frac{\widetilde{\gamma}_1}{\widetilde{\gamma}_2}\right| |\boldsymbol{\iota}_s' \mathbf{s}(\boldsymbol{\alpha}_1)| > L - s$*. Let* $g = |\boldsymbol{\iota}_s' \mathbf{s}(\boldsymbol{\alpha}_1)|$*, then it follows that selection is inconsistent if* $\left|\frac{\widetilde{\gamma}_1}{\widetilde{\gamma}_2}\right| g > L - s$*. Hence if* $g = s$*,* $\left|\left|\mathbf{C}_{21}\mathbf{C}_{11}^{-1}\mathbf{s}(\boldsymbol{\alpha}_1)\right|\right|_\infty > 1$ *if* $s > L/\left(1 + \left|\frac{\widetilde{\gamma}_1}{\widetilde{\gamma}_2}\right|\right)$*.*

When instruments are uncorrelated, such that $\mathbf{Q} = \mathbf{I}_L$*, it follows that* $\left\|\mathbf{C}_{21}\mathbf{C}_{11}^{-1}\mathbf{s}(\boldsymbol{\alpha}_1)\right\|_\infty < 1$ *if* $s < L - \left|\frac{\widetilde{\gamma}_1}{\widetilde{\gamma}_2}\right| g$*. Hence if* $g = s$*,* $\lim_{n\to\infty} P\left(\widehat{A}_n = A\right) = 1$ *if* $s < L/\left(1 + \left|\frac{\widetilde{\gamma}_1}{\widetilde{\gamma}_2}\right|\right)$*.*

Remark 2 *For equal strength instruments,* $\widetilde{\gamma}_1 = \widetilde{\gamma}_2$*, the result of Corollary 1 shows that the Lasso approach will not select the invalid instruments consistently for all possible configurations of* $\boldsymbol{\alpha}_1$ *if* $s > L/2$*. For uncorrelated equal strength instruments, the selection is consistent for all possible configurations of* $\boldsymbol{\alpha}_1$ *if* $s < L/2$*.*

Remark 3 *The result of Corollary 1 shows that for the uncorrelated instruments design, the condition* $s < L/2$*, whilst sufficient for the formal identification result of Theorem 1 in KZCS, is not a sufficient condition for consistent selection of the invalid instruments, as the number of invalid instruments permitted for consistent selection shrinks with an increasing relative strength of the invalid instruments,* $|\widetilde{\gamma}_1/\widetilde{\gamma}_2|$*.*

Remark 4 *It follows from Corollary 4 in Zhao and Yu (2006), that if the columns of* $\widetilde{\mathbf{Z}}$ *are standardised such that the diagonal elements of* $\mathbf{C}$ *are equal to 1, with the correlations bounded from 1, then the irrepresentable condition holds and the selection is consistent when* $s = 1$*, irrespective of the values of* $\boldsymbol{\gamma}$*.*

4 A Consistent Estimator when $s < L/2$ and Adaptive Lasso

As the results above highlight, the LARS/Lasso path may not include the correct model, leading to an inconsistent estimator of β. This is the case even if less than 50% of the instruments are invalid because of differential instrument strength and/or correlation patterns of the instruments. In this section we present an estimation method that consistently selects the invalid instruments when less than 50% of the potential instruments are invalid. This is the same condition as that for the LARS/Lasso selection to be guaranteed to be consistent for equal strength uncorrelated instruments, but the proposed

estimator below is consistent when the instruments have differential strength and/or have a general correlation structure.

We consider the adaptive Lasso approach of Zou (2006) using an initial consistent estimator of the parameters. In the standard linear case, the OLS estimator in the model with all explanatory variables included is consistent. As explained in Section 3, in the instrumental variables model this option is not available. We build on the result of Han (2008), who shows that the median of the L IV estimates of β using one instrument at the time is a consistent estimator of β in a model with invalid instruments, but where the instruments cannot have direct effects on the outcome, unless the instruments are uncorrelated.

As before, let $\widehat{\mathbf{\Gamma}} = (\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'\mathbf{y}$; $\widehat{\boldsymbol{\gamma}} = (\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'\mathbf{d}$, and let $\widehat{\boldsymbol{\pi}}$ be the L-vector with j-th element

$$\widehat{\pi}_j = \frac{\widehat{\Gamma}_j}{\widehat{\gamma}_j}. \tag{18}$$

Under the standard assumptions, Theorem 1 below shows that the median of the $\widehat{\pi}_j$, denoted $\widehat{\beta}_m$, is a consistent estimator for β when $s < L/2$, without any further restrictions on the relative strengths or correlations of the instruments. Theorem 1 also shows that $\sqrt{n}\left(\widehat{\beta}_m - \beta\right)$ converges in distribution to that of an order statistic. From these results it follows that the consistent estimator $\widehat{\boldsymbol{\alpha}}_m = \widehat{\mathbf{\Gamma}} - \widehat{\boldsymbol{\gamma}}\widehat{\beta}_m$ can be used for the adaptive Lasso approach of Zou (2006), resulting in oracle properties of the resulting estimator of β.

Theorem 1 *Under model specifications (3) and (8) with Assumptions 1 - 3, let $\widehat{\boldsymbol{\pi}}$ be the L-vector with elements as defined in (18). If $s < L/2$, then the estimator $\widehat{\beta}_m$ defined as*

$$\widehat{\beta}_m = \text{median}\left(\widehat{\boldsymbol{\pi}}\right)$$

is a consistent estimator for β,

$$\text{plim}\left(\widehat{\beta}_m\right) = \beta.$$

Let $\widehat{\boldsymbol{\pi}}_2$ be the $L-s$ vector with elements $\widehat{\pi}_j$, $j = s+1, ..., L$. The limiting distribution of $\widehat{\beta}_m$ is given by

$$\sqrt{n}\left(\widehat{\beta}_m - \beta\right) \xrightarrow{d} q_{[l],L-s},$$

where for L odd, $q_{[l],L-s}$ is the l-th order-statistic of the limiting normal distribution of $\sqrt{n}\left(\widehat{\boldsymbol{\pi}}_2 - \beta\boldsymbol{\iota}_{L-s}\right)$, where l is determined by L, s and the signs of $\delta_j = \frac{\alpha_j}{\gamma_j}$, $j = 1, ..., s$.

For L even, $q_{[l],L-s}$ is defined as the average of either the $[l]$ and $[l-1]$ order statistics, or the $[l]$ and $[l+1]$ order statistics.

Proof. Under the stated assumptions,

$$\begin{aligned} \operatorname{plim}\left(\widehat{\boldsymbol{\Gamma}}\right) &= \boldsymbol{\gamma}\beta + \boldsymbol{\alpha}; \\ \operatorname{plim}\left(\widehat{\boldsymbol{\gamma}}\right) &= \boldsymbol{\gamma}. \end{aligned}$$

Hence

$$\operatorname{plim}\left(\widehat{\pi}_j\right) = \frac{\gamma_j\beta + \alpha_j}{\gamma_j} = \beta + \frac{\alpha_j}{\gamma_j},$$

for $j = 1, ..., L$. As $s < L/2$, more than 50% of the αs are equal to zero and hence it follows that more than 50% of the elements of $\operatorname{plim}\left(\widehat{\boldsymbol{\pi}}\right)$ are equal to β. Using a continuity theorem, it then follows that

$$\operatorname{plim}\left(\widehat{\beta}_m\right) = \operatorname{median}\left\{\operatorname{plim}\left(\widehat{\boldsymbol{\pi}}\right)\right\} = \beta.$$

For the limiting distribution, let $\boldsymbol{\delta}_1$ be the s-vector with elements

$$\delta_j = \frac{\alpha_j}{\gamma_j},$$

for $j = 1, ..., s$. Let $\boldsymbol{\delta} = \begin{pmatrix} \boldsymbol{\delta}_1' & \mathbf{0}_{L-s}' \end{pmatrix}'$. Partition $\widehat{\boldsymbol{\pi}}$ accordingly as $\widehat{\boldsymbol{\pi}} = \begin{pmatrix} \widehat{\boldsymbol{\pi}}_1' & \widehat{\boldsymbol{\pi}}_2' \end{pmatrix}'$. Under the standard conditions, the limiting distribution of $\widehat{\boldsymbol{\pi}}$ is given by

$$\sqrt{n}\left(\widehat{\boldsymbol{\pi}} - \left(\beta\boldsymbol{\iota}_L + \boldsymbol{\delta}\right)\right) \xrightarrow{d} N\left(0, \boldsymbol{\Sigma}_\pi\right).$$

As $\widehat{\beta}_m = \operatorname{median}\left(\widehat{\boldsymbol{\pi}}\right)$,

$$\begin{aligned} \sqrt{n}\left(\widehat{\beta}_m - \beta\right) &= \sqrt{n}\left(\operatorname{median}\left(\widehat{\boldsymbol{\pi}}\right) - \beta\right) \\ &= \operatorname{median}\left(\sqrt{n}\left(\widehat{\boldsymbol{\pi}} - \beta\boldsymbol{\iota}_L\right)\right). \end{aligned}$$

As

$$\sqrt{n}\left(\widehat{\boldsymbol{\pi}} - \beta\boldsymbol{\iota}_L\right) = \begin{pmatrix} \sqrt{n}\left(\widehat{\boldsymbol{\pi}}_1 - \left(\beta\boldsymbol{\iota}_s + \boldsymbol{\delta}_1\right)\right) + \sqrt{n}\boldsymbol{\delta}_1 \\ \sqrt{n}\left(\widehat{\boldsymbol{\pi}}_2 - \beta\boldsymbol{\iota}_{L-s}\right) \end{pmatrix},$$

it follows that

$$\sqrt{n}\left(\widehat{\beta}_m - \beta\right) = \operatorname{median}\left(\sqrt{n}\left(\widehat{\boldsymbol{\pi}} - \beta\boldsymbol{\iota}_L\right)\right) \xrightarrow{d} q_{[l],L-s}.$$

■

Given the consistent estimator $\widehat{\beta}_m$, we obtain a consistent estimator for $\boldsymbol{\alpha}$ as

$$\widehat{\boldsymbol{\alpha}}_m = (\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'\left(\mathbf{y} - \mathbf{d}\widehat{\beta}_m\right) = \widehat{\boldsymbol{\Gamma}} - \widehat{\boldsymbol{\gamma}}\widehat{\beta}_m,$$

which can then be used for the adaptive Lasso specification of (13) as proposed by Zou (2006). The adaptive Lasso estimator for $\boldsymbol{\alpha}$ is defined as

$$\widehat{\boldsymbol{\alpha}}_{ad}^{(n)} = \arg\min_{\alpha} \frac{1}{2}\|\mathbf{y} - \widetilde{\mathbf{Z}}\boldsymbol{\alpha}\|_2^2 + \lambda_n \sum_{l=1}^{L} \frac{|\alpha_l|}{|\widehat{\alpha}_{m,l}|^{\upsilon}}, \tag{19}$$

and, for given values of υ can be estimated straightforwardly using the LARS algorithm, see Zou (2006). The resulting adaptive Lasso estimator for β is obtained as

$$\widehat{\beta}_{ad}^{(n)} = \frac{\widehat{\mathbf{d}}'\left(\mathbf{y} - \mathbf{Z}\widehat{\boldsymbol{\alpha}}_{ad}^{(n)}\right)}{\widehat{\mathbf{d}}'\widehat{\mathbf{d}}}.$$

As the result for the limiting distribution of the median estimator shows, $\widehat{\beta}_m$, although converging at the $\sqrt{n}$ rate, has an asymptotic bias. This clearly also results in an asymptotic bias of $\widehat{\boldsymbol{\alpha}}_m$. As $\sqrt{n}\left(\widehat{\boldsymbol{\alpha}}_m - \boldsymbol{\alpha}\right) = O_p(1)$, Theorem 2 together with Remark 1 in Zou (2006) states the following properties of the adaptive Lasso estimator $\widehat{\boldsymbol{\alpha}}_{ad}^{(n)}$, where $\widehat{A}_{ad,n} = \left\{j : \widehat{\alpha}_{ad,j}^{(n)} \neq 0\right\}$.

Proposition 3 *Suppose that $\lambda_n = o(\sqrt{n})$ and $(\sqrt{n})^{\nu-1}\lambda_n \to \infty$, then the adaptive Lasso estimator $\widehat{\boldsymbol{\alpha}}_{ad}^{(n)}$ satisfies*

1. *Consistency in variable selection:* $\lim_{n\to\infty} P\left(\widehat{A}_{ad,n} = A\right) = 1$.
2. *Asymptotic normality:* $\sqrt{n}\left(\widehat{\boldsymbol{\alpha}}_{ad,A}^{(n)} - \boldsymbol{\alpha}_A\right) \xrightarrow{d} N\left(0, \sigma^2 C_{11}^{-1}\right)$.

Proof. See Zou (2006), Theorem 2 and Remark 1. ■

From the results of Proposition 3, it follows that the limiting distribution of $\widehat{\beta}_{ad}^{(n)}$ is that of the oracle 2SLS estimator, as stated in the next Corollary.

Corollary 2 *Under the conditions of Proposition 3, the limiting distribution of the adaptive Lasso estimator $\widehat{\beta}_{ad}^{(n)}$ is given by*

$$\sqrt{n}\left(\widehat{\beta}_{ad}^{(n)} - \beta\right) \xrightarrow{d} N\left(0, \sigma_{\beta_{or}}^2\right), \tag{20}$$

with $\sigma_{\beta_{or}}^2$ as defined in (7).

5 Some Simulation Results

5.1 Equal strength instruments

We start with presenting some estimation results from a Monte Carlo exercise which is similar to that in KZCS. The data are generated from

$$\begin{aligned} Y_i &= D_i\beta + \mathbf{Z}_{i.}'\boldsymbol{\alpha} + \varepsilon_i \\ D_i &= \mathbf{Z}_{i.}'\boldsymbol{\gamma} + v_i, \end{aligned}$$

where

$$\begin{pmatrix} \varepsilon_i \\ v_i \end{pmatrix} \sim N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}\right);$$
$$\mathbf{Z}_{i.} \sim N\left(0, \mathbf{I}_L\right);$$

and we set $\beta = 0$; $L = 10$; $\rho = 0.25$; $s = 3$, and the first s elements of α are equal to $a = 0.2$. Further, $\boldsymbol{\gamma}_1 = \widetilde{\gamma}_1\boldsymbol{\iota}_s$ and $\boldsymbol{\gamma}_2 = \widetilde{\gamma}_2\boldsymbol{\iota}_{L-s}$ with $\widetilde{\gamma}_1 = \widetilde{\gamma}_2 = 0.2$. Note that none of the estimation results presented here and below depend on the value of β. Table 1 presents estimation results for estimators of β in terms of bias, standard deviation, root mean squared error (rmse) and median absolute deviation (mad) for 1000 replications for sample sizes of $n = 500$, $n = 2000$ and $n = 10,000$.

The information content for IV estimation can be summarised by the concentration parameter, see Rothenberg (1984). For the oracle estimation of β by 2SLS, the concentration parameter is given by $\mu_n^2 = \boldsymbol{\gamma}_2'\mathbf{Z}_2'\mathbf{M}_{Z_1}\mathbf{Z}_2\boldsymbol{\gamma}_2/\sigma_v^2$. For this data generating process with independent instruments, the concentration parameter is therefore approximately $n\left(L-s\right)\left(0.2^2\right)$ and hence equal to 140, 560 and 2800 for the three sample sizes. The corresponding population F-statistics are equal to $n\left(0.2^2\right)$, or 20, 80 and 400 for the sample sizes 500, 2000 and $10,000$ respectively. The F-statistic is a test for $H_0 : \boldsymbol{\gamma}_2 = 0$.

We will discuss information content for the Lasso selection in more detail in Section 7, but the (squared) Signal to Noise Ratio (SNR), denoted by η^2, is defined as

$$\eta^2 = \frac{\boldsymbol{\alpha}_1'\mathbf{C}_{11}\boldsymbol{\alpha}_1}{\sigma_\varepsilon^2},$$

and is here equal to 0.084.

The "2SLS" results are for the naive 2SLS estimator of β that treats all instruments as valid. The probability limit of this estimator is given by

$$\text{plim}\left(\widehat{\beta}_{naive}\right) = \beta + \frac{\boldsymbol{\gamma}'\mathbf{Q}\boldsymbol{\alpha}}{\boldsymbol{\gamma}'\mathbf{Q}\boldsymbol{\gamma}} = \beta + \frac{\boldsymbol{\gamma}_1'\mathbf{Q}_{11}\boldsymbol{\alpha}_1 + \boldsymbol{\gamma}_2'\mathbf{Q}_{21}\boldsymbol{\alpha}_1}{\boldsymbol{\gamma}_1'\mathbf{Q}_{11}\boldsymbol{\gamma}_1 + 2\boldsymbol{\gamma}_2'\mathbf{Q}_{21}\boldsymbol{\gamma}_1 + \boldsymbol{\gamma}_2'\mathbf{Q}_{22}\boldsymbol{\gamma}_2}. \tag{21}$$

Therefore, in the design specified here, we have plim $\left(\widehat{\beta}_{naive}\right) = s/L = 0.3$.

The "2SLS or" is the oracle 2SLS estimator that correctly includes the three invalid instruments in the model as explanatory variables. For the Lasso estimates, the value for λ_n has been obtained by 10-fold cross-validation, using the one-standard error rule, as in KZCS. This estimator is denoted "Lasso$_{cvse}$" and is the one produced by the *sisVIVE* routine. We also present results for the cross-validated estimator that does not use the one-standard error rule, denoted "Lasso$_{cv}$". For the Lasso estimation procedure, we standardise throughout such that the diagonal elements of $\widetilde{\mathbf{Z}}'\widetilde{\mathbf{Z}}/n$ are equal to 1.

We further present results for the so-called post-Lasso estimator, see e.g. Belloni et al. (2012), which is called the LARS-OLS hybrid by Efron et al. (2004). In this case this is simply the 2SLS estimator in the model that includes $\mathbf{Z}_{\widehat{A}_n}$, the set of instruments with non-zero estimated Lasso coefficients. Clearly, when $\widehat{A}_n = A$, the post-Lasso 2SLS estimator is equal to the oracle 2SLS estimator. The post-Lasso 2SLS estimator is expected to have a smaller bias as it avoids the bias in the Lasso estimate of β due to the shrinkage of the Lasso estimate of $\boldsymbol{\alpha}$ towards $\mathbf{0}$, see also Hastie et al. (2009, p. 91). This shrinkage bias effect on $\widehat{\beta}^{(n)}$ for models where $A \subseteq \widehat{A}_n$ is in the direction of the bias of $\widehat{\beta}_{naive}$, where $\boldsymbol{\alpha}$ is assumed to be $\mathbf{0}$.[3]

Further entries in Table 1 are the average number of instruments selected as invalid, i.e. the average number of instruments in $\widehat{A}_n = \left\{j : \widehat{\alpha}_j^{(n)} \neq 0\right\}$, together with the minimum and maximum number of selected instruments, and the proportion of times the instruments selected as invalid include all 3 invalid instruments.

The results in Table 1 reveal some interesting patterns. First of all, the Lasso$_{cv}$ estimator outperforms the Lasso$_{cvse}$ estimator in terms of bias, rmse and mad for all sample sizes, but this is reversed for the post-Lasso estimators, i.e. the post-Lasso$_{cvse}$ outperforms the post-Lasso$_{cv}$. The Lasso$_{cv}$ estimator selects on average around 6.5 instruments as invalid, which is virtually independent of the sample size. The Lasso$_{cvse}$ estimator selects on average around 3.8 instruments as invalid for $n = 2000$ and $n = 10,000$, but fewer, 3.16 for $n = 500$. Although the 3 invalid instruments are always jointly selected as invalid for the larger sample sizes, the Lasso$_{cvse}$ is substantially biased, the biases being

[3] In an OLS setting, Belloni and Chernozhukov (2013) show that the post-Lasso estimator can perform at least as well as Lasso in terms of rate of convergence, but is less biased even if the Lasso-based model selection misses some components of the true model.

larger than twice the standard deviations. The post-Lasso$_{cvse}$ estimator performs best, but is still outperformed by the oracle 2SLS estimator at $n = 10,000$. Although the post-Lasso$_{cvse}$ estimator has a larger standard deviation than the Lasso$_{cvse}$ estimator, it has a smaller bias, rmse and mad for all sample sizes.

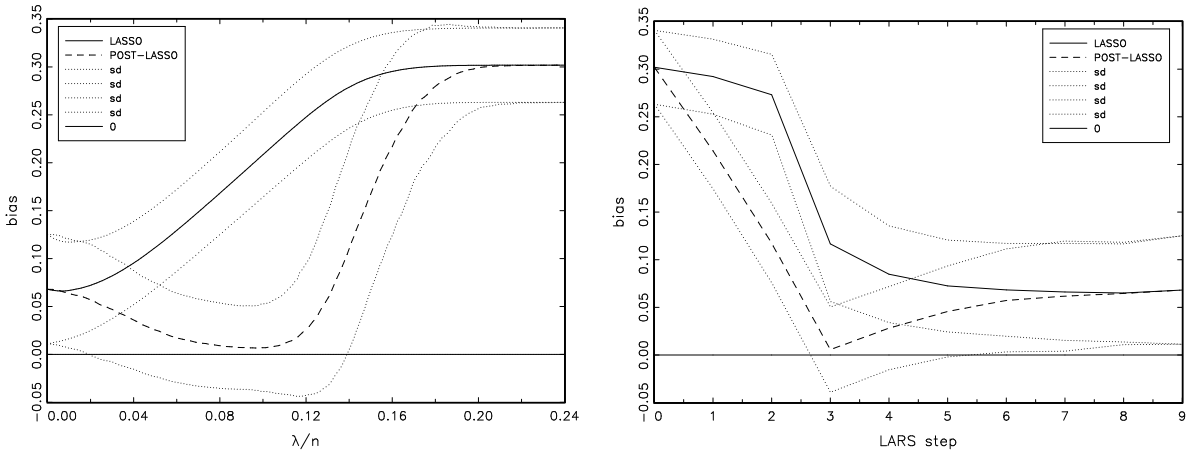
Table 1. Estimation results for 2SLS and Lasso estimators for β; $L = 10$, $s = 3$, $\widetilde{\gamma}_1 = \widetilde{\gamma}_2$

β	bias	std dev	rmse	mad	av. # instr selected as invalid [min, max]	freq. all invalid instr selected
$n = 500$						
2SLS	0.2966	0.0808	0.3074	0.2944	0	0
2SLS or	0.0063	0.0843	0.0845	0.0570	3	1
Lasso$_{cv}$	0.1384	0.0965	0.1687	0.1352	6.41 [2,9]	0.990
Post-Lasso$_{cv}$	0.1169	0.1136	0.1630	0.1143		
Lasso$_{cvse}$	0.2206	0.0847	0.2363	0.2174	3.16 [0,8]	0.664
Post-Lasso$_{cvse}$	0.0905	0.1243	0.1537	0.0994		
$n = 2000$						
2SLS	0.3019	0.0387	0.3044	0.3007	0	0
2SLS or	0.0047	0.0422	0.0424	0.0285	3	1
Lasso$_{cv}$	0.0721	0.0509	0.0882	0.0705	6.64 [3,9]	1
Post-Lasso$_{cv}$	0.0617	0.0577	0.0845	0.0644		
Lasso$_{cvse}$	0.1140	0.0430	0.1218	0.1165	3.76 [3,8]	1
Post-Lasso$_{cvse}$	0.0277	0.0521	0.0590	0.0387		
$n = 10,000$						
2SLS	0.2996	0.0177	0.3002	0.2992	0	0
2SLS or	0.0006	0.0182	0.0182	0.0126	3	1
Lasso$_{cv}$	0.0317	0.0236	0.0395	0.0311	6.44 [3,9]	1
Post-Lasso$_{cv}$	0.0272	0.0267	0.0380	0.0282		
Lasso$_{cvse}$	0.0479	0.0187	0.0514	0.0489	3.81 [3,9]	1
Post-Lasso$_{cvse}$	0.0118	0.0238	0.0265	0.0176		

Notes: Results from 1000 MC replications; $\beta = 0$; $\rho = 0.25$; $a = 0.2$; $\widetilde{\gamma}_2 = 0.2$

Figures 1a and 1b illustrate the different behaviours of the Lasso and post-Lasso estimators. Figure 1a shows the bias and standard deviations of the two estimators for different values of λ/n, for the design above with $n = 2000$, again from 1000 replications. It is clear that the Lasso estimator exhibits a positive bias for all values of λ, declining from that of the naive 2SLS estimator to the minimum bias of 0.0664 at $\lambda/n = 0.0060$. In contrast, the post-Lasso estimator is (much) less biased, obtaining its minimum bias

of 0.0068 at the value of λ/n of 0.0965. Figure 1b displays the same information but now as a function of the LARS steps, where additional variables enter the model (we have omitted 3 replications where there were Lasso steps). At step 3, the correct 3 invalid instruments have been selected 991 times out of the 997 replications, and the post-Lasso estimator has a bias there of 0.0058, only fractionally larger than that of the oracle 2SLS estimator. In contrast, the Lasso estimator for β still has a substantial upward bias at step 3. Its bias decreases from 0.116 at step 3 to a minimum of 0.0650 at step 8. The bias of the post-Lasso estimator increases again after step 3, reaching the same bias as the Lasso estimator at the last step, as there $\lambda_n = 0$ and the Lasso and post-Lasso estimators are equal.



Figures 1a and 1b. Bias and standard deviations of Lasso and post-Lasso estimators as functions of λ/n, and LARS steps. Same design as in Table 1, $n = 2000$. 3 replications out of 1000 omitted in 1b due to Lasso steps.

As the results in Table 1 show, the difference in bias between the Lasso and post-Lasso estimators is larger for the *cvse* procedure compared to the *cv* procedure, as the *cvse* procedure selects a larger value for λ_n and hence shrinks the α coefficients more towards 0. This results in a larger bias in the estimator for β, as depicted in Figure 1a.

Table 2 presents results for the median and Adaptive Lasso estimators. The estimation results for the adaptive Lasso are based on setting $\upsilon = 1$. The resulting estimators are denoted "ALasso". As L is even here, the median is defined as $\widehat{\beta}_m = \left(\widehat{\pi}_{[5]} + \widehat{\pi}_{[6]}\right)/2$,

where $\widehat{\pi}_{[j]}$ is the j-th order statistic. The results show the oracle properties of the adaptive Lasso procedure, especially for the post-ALasso$_{cvse}$ estimator, with its estimation results very close to that of the oracle 2SLS estimator for $n = 2000$ and $n = 10,000$.

Table 2. Estimation results for Adaptive Lasso estimators for β; $L = 10$, $s = 3$, $\widetilde{\gamma}_1 = \widetilde{\gamma}_2$

β	bias	std dev	rmse	mad	av. # instr selected as invalid [min, max]	freq. all invalid instr selected
$n = 500$						
$\widehat{\beta}_m$	0.1197	0.1029	0.1578	0.1159		
ALasso$_{cv}$	0.0952	0.0981	0.1366	0.0934	4.62 [2,9]	0.974
Post-ALasso$_{cv}$	0.0732	0.1182	0.1390	0.0894		
ALasso$_{cvse}$	0.1984	0.0847	0.2157	0.1974	2.63 [0,6]	0.583
Post-ALasso$_{cvse}$	0.0703	0.1156	0.1353	0.0859		
$n = 2000$						
$\widehat{\beta}_m$	0.0634	0.0502	0.0809	0.0648		
ALasso$_{cv}$	0.0350	0.0496	0.0607	0.0403	4.17 [3,9]	1
Post-ALasso$_{cv}$	0.0281	0.0573	0.0638	0.0386		
ALasso$_{cvse}$	0.0960	0.0434	0.1053	0.0977	3.03 [3,5]	1
Post-ALasso$_{cvse}$	0.0059	0.0433	0.0437	0.0287		
$n = 10,000$						
$\widehat{\beta}_m$	0.0277	0.0225	0.0357	0.0278	3	1
ALasso$_{cv}$	0.0111	0.0224	0.0250	0.0169	3.85 [3,9]	1
Post-ALasso$_{cv}$	0.0081	0.0250	0.0263	0.0169		
ALasso$_{cvse}$	0.0382	0.0191	0.0428	0.0391	3.01 [3,4]	1
Post-ALasso$_{cvse}$	0.0008	0.0184	0.0184	0.0127		

Notes: Results from 1000 MC replications; $\beta = 0$; $\rho = 0.25$; $a = 0.2$; $\widetilde{\gamma}_2$= 0.2

The design in Tables 1 and 2 has $L = 10$, $s = 3$ and $\delta_1 = \delta_2 = \delta_3 > 0$. As the median is defined as $\widehat{\beta}_m = \left(\widehat{\pi}_{[5]} + \widehat{\pi}_{[6]}\right)/2$, it follows that

$$\sqrt{n}\left(\widehat{\beta}_m - \beta\right) = \text{median}\left(\sqrt{n}\left(\widehat{\boldsymbol{\pi}} - \beta\boldsymbol{\iota}_{10}\right)\right) \xrightarrow{d} q_{[5,6],7}$$

where $q_{[5,6],7}$ is the average of the fifth and sixth order statistic of the limiting distribution

$$\sqrt{n}\left(\begin{pmatrix} \widehat{\pi}_4 \\ \vdots \\ \widehat{\pi}_{10} \end{pmatrix} - \beta\boldsymbol{\iota}_7\right) \xrightarrow{d} N\left(0, \boldsymbol{\Sigma}_{\pi}^{*}\right).$$

For the design in Table 2, $\mathbf{\Sigma}_{\pi}^{*} = 25\mathbf{I}_7$, as $\sigma_{\varepsilon}^2 = 1$ and $1/\gamma_j^2 = 25$ for $j = 4, ..., 10$. From a simple simulation, drawing repeatedly from the $N(0, 25\mathbf{I}_7)$ distribution, we find that $E\left[q_{[5,6],7}\right] = 2.78$. Therefore $E\left[q_{[5,6],7}\right]/\sqrt{n} = 0.0278$ for $n = 10,000$, almost exactly the result found for the bias of $\widehat{\beta}_m$ in Table 2. For this design, the asymptotic bias of the median estimator is affected by the number of invalid instruments in the following way. For $n = 10,000$ we get for $s = 4$, 2, 1, 0 respectively $E\left[q_{[5,6],6}\right]/\sqrt{n} = 0.0477$; $E\left[q_{[5,6],8}\right]/\sqrt{n} = 0.0156$; $E\left[q_{[5,6],9}\right]/\sqrt{n} = 0.0069$; and $E\left[q_{[5,6],10}\right]/\sqrt{n} = 0$.

Having all elements of $\boldsymbol{\delta}_1$ with the same sign is clearly the worst case scenario for the asymptotic bias of the median estimator. The best case scenario is for even s, if half the elements in $\boldsymbol{\delta}_1$ are positive and half negative, as we then have that $\sqrt{n}\left(\widehat{\beta}_m - \beta\right)$ converges to the median of the limiting distribution of $\sqrt{n}\left(\widehat{\boldsymbol{\pi}}_2 - \beta\boldsymbol{\iota}_{L-s}\right)$, and therefore has no asymptotic bias.

For the results in Table 2, for $n = 2000$, the means of the estimates $\widehat{\alpha}_{m,j}$ for the positive $\alpha_j = 0.2$, $j = 1, ..3$, are approximately 0.187, whereas the means of the estimates for the $\alpha_j = 0$, $j = 4, ..., 10$, are approximately 0.0186. For $n = 10,000$, these are approximately 0.194 and 0.0085. The ratios of the biases for $n = 10,000$, relative to those of $n = 2000$ are approximately 0.45 which is equal to $\sqrt{2000}/\sqrt{10,000}$, confirming that the bias in $\widehat{\boldsymbol{\alpha}}_m$ decreases at the $\sqrt{n}$ rate.

5.2 Strong invalid instruments

Table 3 presents estimation results for the same Monte Carlo design as in Tables 1 and 2, but now with stronger invalid than valid instruments, with $\widetilde{\gamma}_2 = 0.2$ and $\widetilde{\gamma}_1 = 3\widetilde{\gamma}_2$. At these relative values, the irrepresentable condition (15) is not satisfied and the Lasso selection will here select the valid instruments as invalid. Note that the behaviour of the oracle 2SLS estimator is the same as in Table 1. In this case $\beta + a/\widetilde{\gamma}_2 = 0 + 0.2/0.6 = 0.33$, which is the parameter value estimated by the invalid instruments. The SNR is smaller here, with $\eta^2 = 0.0247$.

The results in Table 3 confirm that, for large sample sizes, the Lasso selects the valid instruments as invalid because of the relative strength of the invalid instruments. The post-ALasso$_{cvse}$ estimator does not perform well for $n = 500$, but does for the sample sizes of $n = 2000$, and $n = 10,000$, with results for the latter very similar to the oracle

2SLS results. The Post-ALasso$_{cv}$ estimator performs better at $n = 500$, as it selects more instruments as invalid with a larger proportion correctly selecting all invalid instruments, although it is outperformed there by the simple median estimator $\widehat{\beta}_m$.

Table 3. Estimation results for β; $L = 10$, $s = 3, \widetilde{\gamma}_1 = 3\widetilde{\gamma}_2$

β	bias	std dev	rmse	mad	av. # instr selected as invalid [min, max]	freq. all s invalid instr selected
$n = 500$						
Post-Lasso$_{cv}$	0.2696	0.0583	0.2759	0.2718	5.06 [0,9]	0.03
Post-Lasso$_{cvse}$	0.2658	0.0429	0.2692	0.2651	0.45 [0,8]	0
$\widehat{\beta}_m$	0.1128	0.0936	0.1466	0.1129		
ALasso$_{cv}$	0.1735	0.0952	0.1979	0.1830	3.73 [0,9]	0.48
Post-ALasso$_{cv}$	0.1324	0.1321	0.1870	0.1591		
ALasso$_{cvse}$	0.2586	0.0420	0.2620	0.2568	0.46 [0,6]	0.04
Post-ALasso$_{cvse}$	0.2428	0.0787	0.2552	0.2568		
$n = 2000$						
Post-Lasso$_{cv}$	0.3004	0.0308	0.3020	0.3023	8.89 [3,9]	0.01
Post-Lasso$_{cvse}$	0.2910	0.0352	0.2931	0.2932	6.58 [0,9]	0.00
$\widehat{\beta}_m$	0.0634	0.0500	0.0808	0.0649		
ALasso$_{cv}$	0.0600	0.0527	0.0798	0.0596	4.42 [3,9]	0.998
Post-ALasso$_{cv}$	0.0360	0.0626	0.0722	0.0442		
ALasso$_{cvse}$	0.1656	0.0489	0.1726	0.1668	3.07 [0,6]	0.89
Post-ALasso$_{cvse}$	0.0281	0.0774	0.0823	0.0348		
$n = 10,000$						
Post-Lasso$_{cv}$	0.3197	0.0120	0.3199	0.3202	8.97 [8,9]	0
Post-Lasso$_{cvse}$	0.3202	0.0122	0.3204	0.3204	8.70 [7,9]	0
$\widehat{\beta}_m$	0.0278	0.0226	0.0358	0.0284		
ALasso$_{cv}$	0.0153	0.0222	0.0270	0.0190	3.92 [3,9]	1
Post-ALasso$_{cv}$	0.0092	0.0253	0.0269	0.0177		
ALasso$_{cvse}$	0.0661	0.0212	0.0694	0.0668	3.02 [3,6]	1
Post-ALasso$_{cvse}$	0.0010	0.0186	0.0187	0.0129		

Notes: Results from 1000 MC replications; $a = 0.2$; $\beta = 0$; $\widetilde{\gamma}_2$= 0.2 $\rho = 0.25$

6 Alternative Stopping Rule

The results for the Lasso estimator in Table 1 show that the 10-fold cross-validation method tends to select too many valid instruments as invalid over and above the in-

valid ones, and that the ad-hoc one-standard error rule does improve the selection. The fact that the cross-validation method selects too many variables is well known, see e.g. Bühlmann and Van de Geer (2011), who argue that use of the cross-validation method is appropriate for prediction purposes, but that the penalty parameter needs to be larger for variable selection, as achieved by the one-standard error rule. Selecting valid instruments as invalid in addition to correctly selecting the invalid instruments clearly does not lead to an asymptotic bias, but results in a less efficient estimator as compared to the oracle estimator. Table 1 shows better results for the Lasso$_{cv}$ estimator as compared to the Lasso$_{cvse}$ estimator in terms of bias, rmse and mad, whereas the Lasso$_{cvse}$ estimator has a smaller standard deviation. However, the more favourable results for the Lasso$_{cv}$ estimator are largely due to the smaller shrinkage bias resulting from selecting a smaller λ_n, and results are reversed when comparing the post-Lasso estimators, with the estimation results of the post-Lasso$_{cvse}$ best overall, even at $n = 500$, where the one-standard error rule includes the full set of invalid instruments much less frequently than the 10-fold cross-validation method. There clearly is a trade-off between selecting too few instruments such that the full set of invalid instruments have not been included, which leads to bias in the estimator, and selecting too many instruments as invalid, which does not result in an asymptotic bias as long as all invalid instruments have been selected, but a loss of efficiency, although Figure 1 shows that it can lead to a larger finite sample bias.

We propose a stopping rule for the LARS/Lasso algorithm based on the approach of Andrews (1999) for moment selection, which is particularly well-suited for the IV selection problem. We can use this approach because the number of instruments $L \ll n$. This stopping rule is less computationally expensive than cross validation. Consider again the oracle model

$$\begin{aligned} \mathbf{y} &= \mathbf{d}\beta + \mathbf{Z}_A\boldsymbol{\alpha}_A + \boldsymbol{\varepsilon} \\ &= \mathbf{R}_A\boldsymbol{\theta}_A + \boldsymbol{\varepsilon}. \end{aligned} \tag{22}$$

Let $\mathbf{g}_n(\boldsymbol{\theta}_A) = n^{-1}\mathbf{Z}'(\mathbf{y} - \mathbf{R}_A\boldsymbol{\theta}_A)$, and $\mathbf{W}_n$ a $k_z \times k_z$ weight matrix, then the oracle Generalised Method of Moments (GMM) estimator is defined as

$$\widehat{\boldsymbol{\theta}}_{A,gmm} = \arg\min_{\theta_A} \mathbf{g}_n(\boldsymbol{\theta}_A)' \mathbf{W}_n^{-1} \mathbf{g}_n(\boldsymbol{\theta}_A),$$

see Hansen (1982). 2SLS is a one-step GMM estimator, setting $\mathbf{W}_n = n^{-1}\mathbf{Z}'\mathbf{Z}$. Given the

moment conditions $E(\mathbf{Z}_{i.}\varepsilon_i) = 0$, 2SLS is efficient under conditional homoskedasticity, $E(\varepsilon_i^2|\mathbf{Z}_{i.}) = \sigma^2$. Under general forms of conditional heteroskedasticity, an efficient two-step oracle GMM estimator is obtained by setting

$$\mathbf{W}_n = \mathbf{W}_n\left(\widehat{\boldsymbol{\theta}}_{A,1}\right) = n^{-1}\sum_{i=1}^{n}\left(\left(y_i - \mathbf{R}'_{A,i.}\widehat{\boldsymbol{\theta}}_{A,1}\right)^2 \mathbf{Z}_{i.}\mathbf{Z}'_{i.}\right)$$

where $\widehat{\boldsymbol{\theta}}_{A,1}$ is an initial consistent estimator, with a natural choice the 2SLS estimator. Then, under the null that the moment conditions are correct, $E(\mathbf{Z}_{i.}\varepsilon_i) = 0$, the Hansen (1982) J-test statistic and its limiting distribution are given by

$$J_n\left(\widehat{\boldsymbol{\theta}}_{A,gmm}\right) = n\mathbf{g}_n\left(\widehat{\boldsymbol{\theta}}_{A,gmm}\right)'\mathbf{W}_n^{-1}\left(\widehat{\boldsymbol{\theta}}_{A,1}\right)\mathbf{g}_n\left(\widehat{\boldsymbol{\theta}}_{A,gmm}\right) \xrightarrow{d} \chi^2_{(L-\dim(\mathbf{R}_A))}.$$

For any set A^+, such that $A \subset A^+$, we have that

$$J_n\left(\widehat{\boldsymbol{\theta}}_{A^+,gmm}\right) \xrightarrow{d} \chi^2_{\left(L-\dim\left(\mathbf{R}_{A^+}\right)\right)},$$

whereas for any set A^-, such that $A \not\subset A^-$, $J_n\left(\widehat{\boldsymbol{\theta}}_{A^-,gmm}\right) = O_p(n)$.

Note that the J-test is a robust score, or Lagrange Multiplier, test for testing H_0 : $\boldsymbol{\alpha}_C = 0$ in the just identified specification

$$\mathbf{y} = \mathbf{d}\beta + \mathbf{Z}_B\boldsymbol{\alpha}_B + \mathbf{Z}_C\boldsymbol{\alpha}_C + \boldsymbol{\varepsilon},$$

where $\mathbf{Z}_B$ is a k_B set of instruments included in the model and $\mathbf{Z}_C$ is any selection of $L - k_B - 1$ instruments from the $L - k_B$ set of instruments not in $\mathbf{Z}_B$, see e.g. Davidson and MacKinnon (1993, p. 235). This makes clear the link between the J-test and testing for additional invalid instruments of the form as specified in model (3).

We can now combine the LARS/Lasso algorithm with the Hansen J-test, which is a directed downward testing procedure in the terminology of Andrews (1999). Compute at every LARS/Lasso step as described above $J_n\left(\widehat{\theta}_{\widehat{A}_n^{[j]}}\right)$, for $j = 0, 1, 2, ...$, where $\widehat{A}_n^{[0]} = \varnothing$ and $\left\|\widehat{A}_n^{[1]}\right\|_0 = 1$, compare it to a corresponding critical value $\zeta_{n,L-k}$ of the $\chi^2_{(L-k)}$ distribution, where $k = \dim\left(\mathbf{R}_{\widehat{A}_n^{[j]}}\right)$. We then select the model with the largest degrees of freedom $L - k$, for which $J_n\left(\widehat{\theta}_{\widehat{A}_n^{[j]}}\right)$ is smaller than the critical value. If two models of the same dimension pass the test, which can happen with a Lasso step, the model with the smallest value of the J-test gets selected.[4] Clearly, this approach is a post-Lasso

[4]If there is no empirical evidence at all for any invalid instruments, i.e., if $J_n\left(\widehat{\theta}_{\widehat{A}_n^{[0]}}\right)$ is smaller than its corresponding critical value, then the model with all instruments as valid gets selected.

approach, where the LARS/Lasso algorithm is used purely for selection of the invalid instruments. For consistent model selection, the critical values $\zeta_{n,L-k}$ need to satisfy

$$\zeta_{n,L-k} \to \infty \text{ for } n \to \infty, \text{ and } \zeta_{n,L-k} = o(n), \tag{23}$$

see Andrews (1999). Let $\zeta_{n,L-k} = \chi^2_{L-k}(p_n)$ be the $1-p_n$ quantile of the χ^2_{L-k} distribution. Here, p_n is the p-value of the test. This combination of the Andrews/Hansen method with the LARS/Lasso steps therefore results in having to choose a p-value p_n instead of a penalty parameter λ_n. Keeping n fixed, choosing a large value for p_n leads to selecting a larger set as invalid instruments as compared to choosing a smaller value for p_n.

Table 4 presents the estimation results using this stopping rule as a selection device for the Lasso estimator for the design with equal instrument strength as in Table 1. The resulting 2SLS estimator we denote "post-Lasso$_{ah}$", the subscript ah standing for Andrews/Hansen. The p-values here are chosen as $p_n = 0.1/\ln(n)$, following Belloni et al. (2012), and are equal to 0.0161, 0.0132 and 0.0109 for n equal to 500, 2000 and 10,000 respectively. The ah approach selects too few invalid instruments for $n = 500$, resulting in an upward bias, with bias, std dev, rmse and mad very similar to those of the post-Lasso$_{cvse}$ estimator in Table 1. For $n = 2000$ and $n = 10,000$, this post-Lasso procedure performs well with properties very similar to that of the oracle 2SLS estimator, and with smaller bias, rmse and mad than the post-Lasso$_{cvse}$ method.

Table 4. Results for post-Lasso$_{ah}$ 2SLS estimator for β; $L = 10$, $s = 3$, $\widetilde{\gamma}_1 = \widetilde{\gamma}_2$

n	bias	std dev	rmse	mad	av. # instr selected as invalid [min, max]	freq. all invalid instr selected
500	0.0896	0.1252	0.1539	0.1007	2.56 [0,5]	0.391
2000	0.0055	0.0430	0.0434	0.0286	3.02 [3,5]	1
10,000	0.0009	0.0186	0.0186	0.0129	3.02 [3,5]	1

Notes: Results from 1000 MC replications; $\beta = 0$; $a = 0.2$; $\widetilde{\gamma}_2 = 0.2$; $\rho = 0.25$

Table 5 presents the results for the 2SLS post-ALasso$_{ah}$ estimator for the design as in Table 3, with strong invalid instruments. For $n = 10,000$ the results are virtually identical to those of the oracle and post-ALasso$_{cvse}$ estimators, whereas the post-ALasso$_{ah}$ estimator performs better in terms of bias, std dev, rmse and mad than the post-ALasso$_{cvse}$

estimator when $n = 2000$. Again, when $n = 500$, the method does not select the invalid instruments.

Table 5. Results for post-ALasso$_{ah}$ 2SLS estimator for β; $L = 10$, $s = 3$, $\widetilde{\gamma}_1 = 3\widetilde{\gamma}_2$

n	bias	std dev	rmse	mad	av. # instr selected as invalid [min, max]	freq. all invalid instr selected
500	0.2172	0.1091	0.2431	0.2471	0.86 [0,5]	0.07
2000	0.0173	0.0677	0.0699	0.0303	3.05 [1,5]	0.93
10,000	0.0008	0.0186	0.0186	0.0129	3.01 [3,5]	1

Notes: Results from 1000 MC replications; $\beta = 0$; $a = 0.2$; $\widetilde{\gamma}_2$= 0.2; $\rho = 0.25$

7 Inference and Information

7.1 Inference

From the limiting distribution result (20), a simple approach to estimating the asymptotic variance of the post-ALasso 2SLS estimator for β is by calculating the standard 2SLS variance estimator. The post-ALasso 2SLS estimator is given by

$$\widehat{\beta}^{(n)}_{ad,post} = \left(\widehat{\mathbf{d}}'\mathbf{M}_{Z_{\widehat{A}_{ad,n}}}\widehat{\mathbf{d}}\right)^{-1}\widehat{\mathbf{d}}'\mathbf{M}_{Z_{\widehat{A}_{ad,n}}}\mathbf{y}$$

and its estimated variance given by

$$\widehat{Var}\left(\widehat{\beta}^{(n)}_{ad,post}\right) = \widehat{\sigma}^2_{\varepsilon}\left(\widehat{\mathbf{d}}'\mathbf{M}_{Z_{\widehat{A}_{ad,n}}}\widehat{\mathbf{d}}\right)^{-1}, \tag{24}$$

where

$$\begin{aligned}\widehat{\sigma}^2_{\varepsilon} &= \widehat{\boldsymbol{\varepsilon}}'\widehat{\boldsymbol{\varepsilon}}/n \\ \widehat{\boldsymbol{\varepsilon}} &= \mathbf{y} - \mathbf{d}\widehat{\beta}^{(n)}_{ad,post} - \mathbf{Z}_{\widehat{A}_{ad,n}}\widehat{\alpha}^{(n)}_{\widehat{A}_{ad,n},post}.\end{aligned}$$

Under the conditions of Proposition 3, the standard assumptions and conditional homoskedasticity, $n\widehat{Var}\left(\widehat{\beta}^{(n)}_{ad,post}\right) \xrightarrow{p} \sigma^2_{\beta_{or}}$. A standard robust version, robust to general forms of heteroskedasticity, is given by

$$\widehat{Var}_r\left(\widehat{\beta}^{(n)}_{ad,post}\right) = \left(\widehat{\mathbf{d}}'\mathbf{M}_{Z_{\widehat{A}_{ad,n}}}\widehat{\mathbf{d}}\right)^{-1}\widehat{\mathbf{d}}'\mathbf{M}_{Z_{\widehat{A}_{ad,n}}}\widehat{\mathbf{H}}\mathbf{M}_{Z_{\widehat{A}_{ad,n}}}\widehat{\mathbf{d}}\left(\widehat{\mathbf{d}}'\mathbf{M}_{Z_{\widehat{A}_{ad,n}}}\widehat{\mathbf{d}}\right)^{-1},$$

where $\widehat{\mathbf{H}}$ is a $n \times n$ diagonal matrix with diagonal elements $\widehat{\mathbf{H}}_{ii} = \widehat{\varepsilon}_i^2$, for $i = 1, .., n$. The robust Wald test for the null $H_0 : \beta = \beta_0$ is then given by

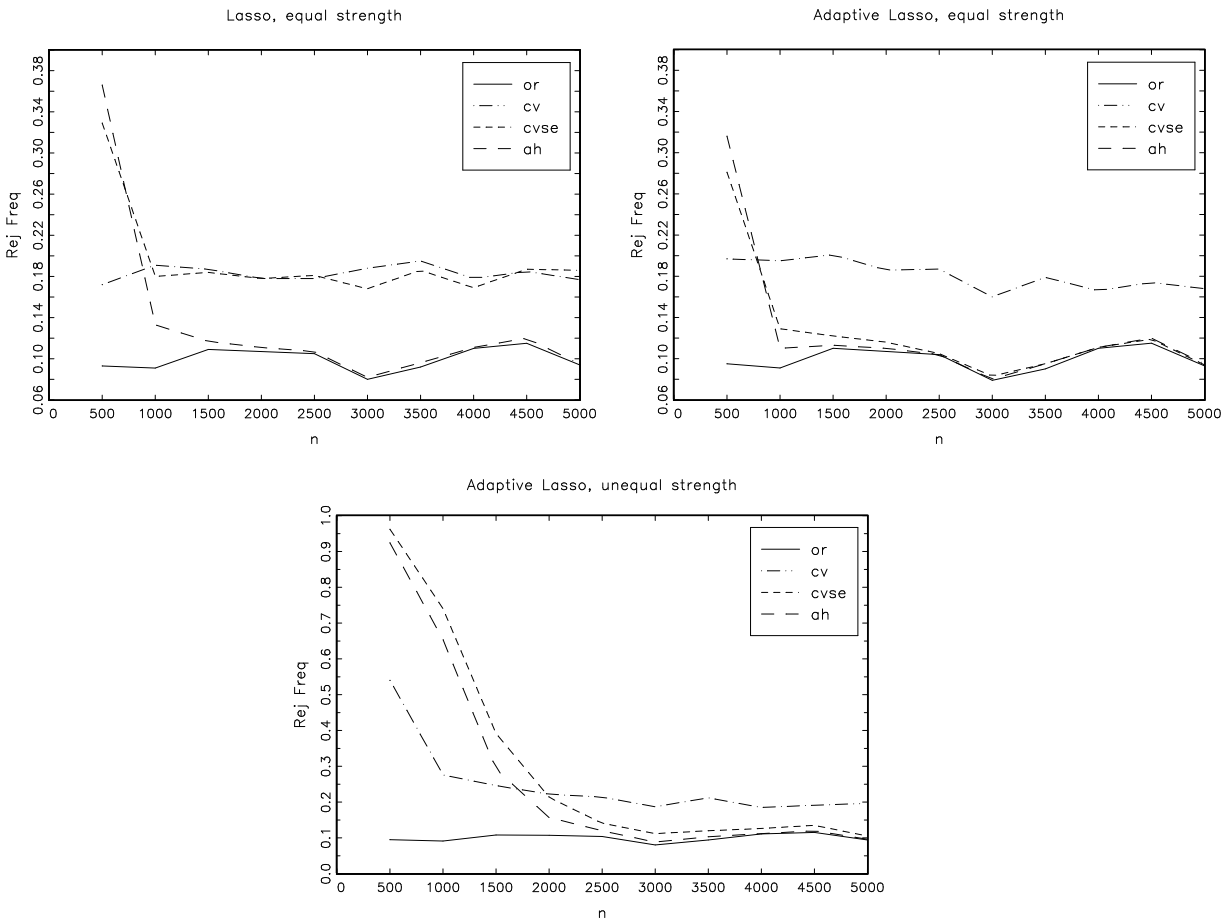
$$W_{\beta,r} = \frac{\left(\widehat{\beta}_{ad,post}^{(n)} - \beta_0\right)^2}{\widehat{Var}_r\left(\widehat{\beta}_{ad,post}^{(n)}\right)}.$$

From the results for the post-ALasso$_{cvse}$ and post-ALasso$_{ah}$ estimators for the unequal strength instruments design as presented in Tables 3 and 5 respectively, one would expect this approach to work well for the large sample case, $n = 10,000$, as there the estimation results are very close to those of the oracle 2SLS estimator. The robust Wald test for the null $H_0 : \beta = 0$, the true value of β, at the 10% level for $n = 10,000$ has a rejection frequency of 9.3% and 9.2% for the post-ALasso$_{cvse}$ and post-ALasso$_{ah}$ estimators respectively, very close to that of the robust Wald test based on the oracle 2SLS estimator, which has a rejection frequency of 9.0%.

For the equal strength instruments design, we perform the same analysis for the post-Lasso estimators. Figures 2a-2c show the performance of the robust Wald test $W_{\beta,r}$, its rejection frequency at the 10% level, as a function of the sample size in steps of 500, $n = 500, 1000, ..., 5000$. Figures 2a and 2b show the results for the post-Lasso and post-ALasso estimators for the equal strength instruments design. Figure 2c shows the results for the post-ALasso estimators for the unequal strength instruments design.

Figure 2a clearly shows that the Lasso$_{cv}$ and Lasso$_{cvse}$ procedures do not result in consistent selection and the resulting post-Lasso estimators do not have oracle properties. The Wald tests rejection frequencies remain constant for increasing sample size and larger than those of the oracle estimator. In contrast, the post-Lasso$_{ah}$ estimator behaves very similar to the oracle estimator in this design from $n = 1500$ onwards. Figure 2b shows that both the post-ALasso$_{cvse}$ and post-ALasso$_{ah}$ behave like the oracle estimator, again from $n = 1500$ onwards in this design.

The results in Figure 2c show that for the unequal instruments strength design considered here, the performances of the post-adaptive Lasso estimators are far from that of the oracle estimator in small samples, as expected from the results in Tables 3 and 5. The post-ALasso$_{ah}$ behaves like the oracle estimator here from $n = 4000$ onwards, with the post-ALasso$_{cvse}$ estimator behaving similarly, but having a larger rejection frequency for all sample sizes considered here that are less than $n = 5000$.



Figures 2a, 2b and 2c. Rejection frequencies of robust Wald tests for H_0: $\beta = 0$ at 10% level as a function of sample size, in steps of 500. Equal strength instruments design, Post-Lasso in Figure 2a, Post-ALasso in Figure 2b. Unequal strength instruments design, Post-ALasso in Figure 2c. Based on 1000 MC replications for each sample size.

The results in Tables 1-5 and Figures 2a-2c show clearly that the information content in the data, given the parameter values chosen here, is insufficient at $n = 500$ for the (adaptive) Lasso procedures to correctly select the invalid instruments and hence the resulting estimators have poor properties, far removed from those of the oracle estimator. At these levels of information the ALasso$_{cv}$ estimator is actually the preferred estimator as it counteracts the selection of too few invalid instruments of the ALasso$_{cvse}$ and ALasso$_{ah}$ estimators. We next explore how the performance of the adaptive Lasso estimators depends on the information contained in the data.

7.2 Information content

We distinguish two different measures of information in the IV Lasso selection problem. First, as mentioned in Section 5, the information content for the estimation of β in the oracle instrumental variables model is characterised by the concentration parameter μ_n^2, Rothenberg (1984), which is here given by

$$\mu_n^2 = \frac{\boldsymbol{\gamma}_2' \mathbf{Z}_2' \mathbf{M}_{Z_1} \mathbf{Z}_2 \boldsymbol{\gamma}_2}{\sigma_v^2}.$$

μ_n^2 is approximately equal to $n\,(L-s)\,\widetilde{\gamma}_2^2 = 140$ for the $n = 500$ cases above, which is equivalent to a first-stage population F-test value for the null $H_0 : \boldsymbol{\gamma}_2 = 0$ of $n\widetilde{\gamma}_2^2 = 20$. This is a reasonably large value of the F-test, see Staiger and Stock (1997) and Stock and Yogo (2005), which is reflected in the good properties of the oracle 2SLS estimator. Clearly, though, the Lasso procedures do not perform well for the $n = 500$ case, especially when $\widetilde{\gamma}_1 = 3\widetilde{\gamma}_2$.

Note that the Wald test for $H_0 : \boldsymbol{\gamma}_2 = 0$ is given by $W_{\boldsymbol{\gamma}_2} = \frac{\widehat{\boldsymbol{\gamma}}_2' \mathbf{Z}_2' \mathbf{M}_{Z_1} \mathbf{Z}_2 \widehat{\boldsymbol{\gamma}}_2}{\widehat{\sigma}_v^2}$ and so μ_n^2 is the population counterpart of the Wald test. The corresponding F-test is $W_{\boldsymbol{\gamma}_2} / (L-s)$. The weak instrument asymptotics of Staiger and Stock (1997) is obtained by considering $\boldsymbol{\gamma}_2$ in a neighbourhood of 0, as $\boldsymbol{\gamma}_2 = \mathbf{c}_{\gamma_2}/\sqrt{n}$, with $\mathbf{c}_{\gamma_2}$ a vector of constants. The information content μ_n^2 does then not increase with the sample size and converges to $\mu_c^2 = \mathbf{c}_{\gamma_2}' \widetilde{Q}_{22} \mathbf{c}_{\gamma_2} / \sigma_v^2$ as $n \to \infty$. The oracle 2SLS estimator is in that case not consistent and converges to a random variable with expected value different from β, with the difference larger for smaller values of μ_c^2.

Second, a measure of information for the Lasso selection is the (squared) Signal to Noise Ratio (SNR), see e.g. Bühlmann and Van de Geer (2011, p. 25), defined as

$$\eta^2 = \frac{\boldsymbol{\alpha}' \mathbf{C} \boldsymbol{\alpha}}{\sigma_\varepsilon^2} = \frac{\boldsymbol{\alpha}_1' \mathbf{C}_{11} \boldsymbol{\alpha}_1}{\sigma_\varepsilon^2},$$

where, as before, $\mathbf{C} = \text{plim}\left(\frac{1}{n}\widetilde{\mathbf{Z}}'\widetilde{\mathbf{Z}}\right)$ and $\mathbf{C}_{11} = \text{plim}\left(\frac{1}{n}\widetilde{\mathbf{Z}}_1'\widetilde{\mathbf{Z}}_1\right)$. From the proof of Proposition 1 in the Appendix we obtain

$$\boldsymbol{\alpha}_1' \mathbf{C}_{11} \boldsymbol{\alpha}_1 = \boldsymbol{\alpha}_1' \mathbf{Q}_{11} \boldsymbol{\alpha}_1 - \frac{\left(\boldsymbol{\gamma}_1' \mathbf{Q}_{11} \boldsymbol{\alpha}_1 + \boldsymbol{\gamma}_2' \mathbf{Q}_{21} \boldsymbol{\alpha}_1\right)^2}{\boldsymbol{\gamma}_1' \mathbf{Q}_{11} \boldsymbol{\gamma}_1 + 2\boldsymbol{\gamma}_2' \mathbf{Q}_{21} \boldsymbol{\gamma}_1 + \boldsymbol{\gamma}_2' \mathbf{Q}_{22} \boldsymbol{\gamma}_2}. \tag{25}$$

It follows from (25) that if we multiply $\boldsymbol{\alpha}_1$ by a factor m, η^2 gets multiplied by m^2 whereas multiplying $\boldsymbol{\gamma}$ by a factor m does not affect the value of η^2.

For the Monte Carlo design above, with $\sigma_\varepsilon^2 = 1$, it follows that

$$\begin{aligned}\eta^2 &= sa^2 - \frac{\left(sa\widetilde{\gamma}_1\right)^2}{s\widetilde{\gamma}_1^2 + (L-s)\,\widetilde{\gamma}_2^2} \\ &= \frac{(L-s)\,a^2}{\left(\frac{\widetilde{\gamma}_1}{\widetilde{\gamma}_2}\right)^2 + \frac{L-s}{s}},\end{aligned}$$

and so the SNR is here directly influenced by the relative value $|\widetilde{\gamma}_1/\widetilde{\gamma}_2|$, with an increase in this value decreasing the value of η^2. For the unequal strength design above, $\eta^2 = 0.0247$, and for the equal strength design it is $\eta^2 = 0.0840$. As Zou (2006) indicated, the smaller value of the SNR in the unequal design explains the poorer performance of the adaptive Lasso estimators for a given sample size, as depicted in Figures 2b and 2c.

The concentration parameter equivalent of η^2 is

$$\eta_n^2 = \frac{\boldsymbol{\alpha}_1' \widetilde{\mathbf{Z}}_1' \widetilde{\mathbf{Z}}_1 \boldsymbol{\alpha}_1}{\sigma_\varepsilon^2},$$

which is the population counterpart of the Wald test for $H_0 : \boldsymbol{\alpha}_1 = 0$ based on the oracle 2SLS estimator

$$W_{\boldsymbol{\alpha}_1} = \frac{\widehat{\boldsymbol{\alpha}}_{1,or}' \widetilde{\mathbf{Z}}_1' \widetilde{\mathbf{Z}}_1 \widehat{\boldsymbol{\alpha}}_{1,or}}{\widehat{\sigma}_{\varepsilon,or}^2}.$$

In order to illustrate the performance of the adaptive Lasso estimator in relation to the value of η_n^2, Table 6 shows the rejection frequencies at the 10% level of the robust Wald tests $W_{\beta,r}$ based on the oracle and post-ALasso$_{ah}$ estimators, for the $n = 500$ unequal strength case as in Section 5.2, but increasing $\boldsymbol{\alpha}$ by a multiplicative factor $m = \sqrt{1}, \sqrt{3}, ..., \sqrt{9}$, and hence increasing η^2 and η_n^2 (approximately) by a multiplicative factor m^2. At $m^2 = 9$, the post-ALasso$_{ah}$ estimator again behaves like the oracle estimator. The mean of η_n^2 is 116 there, with the associated population F-test statistic equal to $\eta_n^2/3 = 38.7$. Increasing the value of a, whilst keeping $\boldsymbol{\gamma}$ constant, increases of course the bias of the naive 2SLS estimator as is clear from (21).

Increasing $\widetilde{\gamma}_1$ and $\widetilde{\gamma}_2$ by the same multiplicative factor whilst keeping a and the sample size constant does not alter η^2 and η_n^2, and does not lead to an improvement of the performance of the Wald test $W_{\beta,r}$, as confirmed by the results in Table 6. The bias of the naive 2SLS estimator decreases here with increasing values of m.

Table 6. Rejection frequencies of $W_{\beta,r}$ at 10% level, varying $\boldsymbol{\alpha}$ or $\boldsymbol{\gamma}$

			m^2		
$\boldsymbol{\alpha} \times m$	1	3	5	7	9
post-ALasso$_{ah}$	0.9300	0.3130	0.1350	0.1100	0.1070
Oracle	0.1020	0.1030	0.1020	0.1010	0.1040
η_n^2 (mean)	12.90	38.69	64.51	90.31	116.13
$\boldsymbol{\gamma} \times m$					
post-ALasso$_{ah}$	0.9300	0.9270	0.9340	0.9360	0.9360
Oracle	0.1020	0.0960	0.0980	0.1010	0.0980
η_n^2 (mean)	12.90	12.51	12.43	12.39	12.37

Notes: $n = 500$, same design as in Tables 3 and 5 when $m = 1$, 1000 MC replications

Multiplying both $\boldsymbol{\gamma}$ and $\boldsymbol{\alpha}$ by a factor m increases both μ_n^2 and η_n^2 by a factor m^2, which is then similar to an increase in the sample size n by a factor m^2. For example, Table 7 displays estimation results for the unequal strength design for $n = 500$, multiplying a, $\widetilde{\gamma}_1$ and $\widetilde{\gamma}_2$ by $m = \sqrt{20}$. The estimation results are very similar to those for $n = 10,000$ as reported in Tables 3 and 5.

Table 7. Estimation results for β; $n = 500$, $\widetilde{\gamma}_1 = 3\widetilde{\gamma}_2$, $a = \widetilde{\gamma}_2 = 0.8944$

β	bias	std dev	rmse	mad	av. # instr selected as invalid [min, max]	freq. all s invalid instr selected
2SLS	0.2650	0.0106	0.2652	0.2651	0	0
2SLS or	-0.0002	0.0191	0.0191	0.0128	3	1
$\widehat{\beta}_m$	0.0272	0.0229	0.0356	0.0267		
Post-ALasso$_{cvse}$	0.0003	0.0194	0.0194	0.0132	3.02 [3,5]	1
Post-ALasso$_{ah}$	0.0000	0.0192	0.0192	0.0130	3.02 [3,5]	1

Notes: Results from 1000 MC replications; $L = 10$, $s = 3$, $\beta = 0$, $\rho = 0.25$

Whilst η_n^2 does seem to provide information content for the Lasso selection, its value itself does not convey the same information about the performance of the adaptive Lasso estimators as μ_n^2 does for the oracle 2SLS estimator. We first illustrate this with an example where the coefficients in $\boldsymbol{\alpha}_1$ take different values. In this case, the performance of the Lasso estimator is driven by how well it does in selecting the variable with the smallest value $|\alpha_j|$, $j = 1, ..., s$. If we for example change $\boldsymbol{\alpha}_1$ to $\boldsymbol{\alpha}_1 = \begin{pmatrix} 0.1 & 0.2 & 0.2063 \end{pmatrix}'$, we get the same value of $\eta^2 = 0.0247$ in the unequal strength design, but as Figure 3 shows,

a much larger sample size is needed for the inference based on the post-ALasso$_{cvse}$ and post-ALasso$_{ah}$ estimators to be similar to that of the oracle estimator, due to the presence of the smaller coefficient $\alpha_1 = 0.1$.

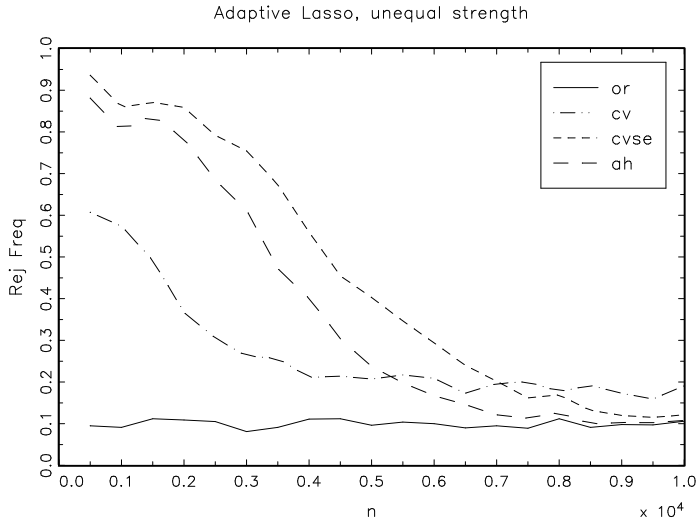


Figure 3. Rejection frequencies of robust Wald tests for H_0: $\beta = 0$ at 10% level as a function of sample size, in steps of 500. Unequal strength instruments design, Post-ALasso, $\alpha_1 = \begin{pmatrix} 0.1 & 0.2 & 0.2063 \end{pmatrix}$. Based on 1000 MC replications for each sample size.

In order to convey this information, instead of η_n^2 related to the Wald test on $\boldsymbol{\alpha}_1$, we compute the $\eta_{n,j}^2$ related to the individual Wald tests for $H_0 : \alpha_j = 0$ in the oracle model, resulting in

$$\eta_{n,j}^2 = \frac{\alpha_j^2 \left(\widetilde{\mathbf{Z}}_{1,.j}' \mathbf{M}_{\widetilde{\mathbf{Z}}_{1[j]}} \widetilde{\mathbf{Z}}_{1,.j} \right)}{\sigma_\varepsilon^2},$$

for $j = 1, ..s$, where $\widetilde{\mathbf{Z}}_{1,.j}$ is the j-th column of $\widetilde{\mathbf{Z}}_1$ and $\widetilde{\mathbf{Z}}_{1[j]}$ are the other $s-1$ columns of $\widetilde{\mathbf{Z}}_1$. With $\eta_j^2 = \text{plim}\left(\frac{1}{n}\eta_{n,j}^2\right)$, it follows for the specific design above with $\alpha_1 = \alpha_2/2$ that $\eta_1^2 = \frac{1}{4}\eta_2^2$, and $\eta_{n,1}^2 \approx \frac{1}{4}\eta_{n,2}^2$, conveying the smaller amount of information. As shown in Figure 3, at, and from, $n = 8500$ the behaviour of the post-ALasso$_{ah}$ estimator is close to that of the oracle estimator with the rejection frequencies of $W_{\beta,r}$ at the 10% level equal to 0.1000 for the post-ALasso$_{ah}$ estimator and 0.0920 for the oracle estimator. The mean value of η_n^2 at $n = 8500$ is equal to 220, whereas those of $\eta_{n,1}^2$, $\eta_{n,2}^2$ and $\eta_{n,3}^2$ are equal to 37, 149, and 159 respectively. In comparison, for the unequal strength design results as

depicted in Figure 2c, the mean value of the population F-test statistic, $\eta_n^2/3$ is equal to 33 at $n = 4000$. For the equal strength design test results as depicted in Figure 2b, the population F-test mean value is 42 at $n = 1500$ and it is 39 in the example of Table 6 when multiplying $\boldsymbol{\alpha}$ by $m = 3$.

For the oracle properties of the adaptive Lasso estimator, we need that $|\boldsymbol{\alpha}_1|_{\min} \gg O\left(n^{-1/2}\right)$. Similar to μ_n^2, η_n^2 does not increase with the sample size if $\boldsymbol{\alpha}_1 = \mathbf{c}_{\alpha_1}/\sqrt{n}$. Although the naive 2SLS estimator is then consistent in this finite s setting, provided $\boldsymbol{\gamma}_2 \gg O\left(n^{-1/2}\right)$, the adaptive Lasso estimator will not have oracle properties in this case, and the properties of the Wald test $W_{\beta,r}$ do not improve with an increasing sample size, as the information does not increase and $\widehat{\beta}_{ad}^{(n)}$ has an asymptotic bias. For example, setting $\boldsymbol{\alpha}_1 = a_n \boldsymbol{\iota}_s$ in the equal strength instruments design, with $a_n = 0.2 * \sqrt{500}/\sqrt{n}$, gives rejection frequencies of the robust Wald test based on the post-ALasso$_{ah}$ estimator at the 10% level of 35%, 33% and 34% for $n = 500$, $n = 2000$ and $n = 10,000$ respectively. For the unequal strength instruments example above, setting $\boldsymbol{\alpha}_1 = \begin{pmatrix} a_n & 0.2 & 0.2063 \end{pmatrix}$ with $a_n = 0.1 * \sqrt{500}/\sqrt{n}$, we get rejection frequencies of the post-ALasso$_{ah}$ based Wald test of 56%, 53% and 54% respectively at $n = 2000$, $n = 10,000$ and $n = 30,000$. In both examples, the adaptive Lasso procedure does not select the full set of invalid instruments as invalid in large samples. This leads to an asymptotic bias, as the random variable $\sqrt{n}\left(\widehat{\beta}_{ad}^{(n)} - \beta\right)$ then does not have a mean of zero when $n \to \infty$, as $a_n = O\left(n^{-1/2}\right)$. Of course, in the finite dimension, fixed parameters case we consider here, we have the limiting distribution of the adaptive Lasso estimator as in (20), but, as shown in Figure 3, the presence of small coefficients in $\boldsymbol{\alpha}_1$ may affect the behaviour of the estimator in any given application.

Whilst the $\eta_{n,j}^2$ provide information on the performance of the Lasso estimation procedure, they are of limited value in practice, as they can only meaningfully be estimated if the set of invalid instruments is known. The Sargan test based on the naive 2SLS estimator is related to the SNR, as it is a score test for $H_0 : \boldsymbol{\alpha}_C = 0$ in the model

$$\mathbf{y} = \mathbf{d}\beta + \mathbf{Z}_C\boldsymbol{\alpha}_C + \boldsymbol{\varepsilon},$$

where $\mathbf{Z}_C$ is any $(L-1)$ subset of instruments from $\mathbf{Z}$. The SNR η^2 is not affected when $\boldsymbol{\gamma}$ is multiplied by a factor m and therefore is not affected by weak instruments, where the value of $\boldsymbol{\gamma}$ is such that the concentration parameter μ_n^2 is small. However, it

is well known that weak instruments decreases the power of the Sargan/Hansen test, see Staiger and Stock (1997) and Kitamura (2006), which would then affect the behaviour of the post-ALasso$_{ah}$ estimator, as the decrease in power will result in less instruments selected as invalid. This is illustrated in Table 8, which presents results for the unequal instrument, large a design as in Table 6, for $n = 500$, $a = 0.6$, $\widetilde{\gamma}_1 = 0.6$, $\widetilde{\gamma}_2 = 0.2$, and $\boldsymbol{\gamma}$ multiplied by a factor $1/m$ for $m^2 = 1, 3, ..., 9$. We present results here for the median bias, as the variability of the estimators increases substantially for larger values of m. For increasing value of m, the number of selected invalid instruments decreases for the ALasso$_{ah}$ estimator, whilst that of the ALasso$_{cvse}$ actually increases, and the median bias of the ALasso$_{ah}$ estimator increases relative to that of the ALasso$_{cvse}$ estimator. Increasing the p-value for the Hansen test decreases the bias of the ALasso$_{ah}$ estimator. For example, for $m = 3$, setting the p-value to 0.2 instead of 0.016 increases the average number of selected instruments as invalid to 3.13 and reduces the median bias of the estimator to 0.1022. The post-ALasso$_{ah}$ estimator is, however, a much noisier estimator here with a standard deviation of 0.66 compared to 0.35 for the post-ALasso$_{cvse}$ estimator and 0.21 for the oracle 2SLS estimator.

Table 8. Median bias for 2SLS estimators of β; $L = 10$, $s = 3$

	m^2				
$\boldsymbol{\gamma} \times (1/m)$	1	3	5	7	9
2SLS	0.7883	1.3311	1.6778	1.9464	2.1570
2SLS or	0.0087	0.0344	0.0504	0.0666	0.0740
post-ALasso$_{cvse}$	0.0138	0.0410	0.0632	0.0899	0.0992
# Inv Inst	3.07	3.10	3.17	3.26	3.31
post-ALasso$_{ah}$	0.0103	0.0348	0.0556	0.0920	0.1257
# Inv Inst	3.01	3.01	2.88	2.61	2.37
μ_n^2	678.51	225.66	136.40	96.86	75.14
η_n^2	116.13	127.44	135.88	146.86	157.21

Notes: $n = 500$, $a = 0.6$, $\widetilde{\gamma}_1 = 3\widetilde{\gamma}_2$, $\widetilde{\gamma}_2 = 0.2/m$, 1000 MC replications

Even when all the coefficients in $\boldsymbol{\alpha}_1$ are the same, the value of η_n^2 itself is not sufficient as a guide to the performance of the adaptive Lasso estimator. As illustrated above, within the same design, a larger value for η_n^2 was associated with a better performance of the adaptive Lasso estimator. However, across designs this may not be the case. For example, we show in Appendix A.2 that introducing correlation to the instruments,

whilst keeping $\mu^2 = \text{plim}\,(\mu_n^2/n)$ and η^2 constant, requires larger samples and hence larger values of η_n^2 than in the uncorrelated instruments designs, for the performance of the robust Wald test $W_{\beta,r}$ based on the post-ALasso estimators to be close to that based on the oracle estimator.

Assumption 3 states that the instruments are relevant in the sense that $\gamma_j \neq 0$, for $j = 1, ..., L$. If this assumption is relaxed and some of the γ_j are equal to 0, then the instruments are potentially invalid for two reasons as the relevance and/or exclusion condition may fail. If $\gamma_j = 0$, then $|\pi_j| = \left|\left(\beta\gamma_j + \alpha_j\right)/\gamma_j\right|$ is either ∞ for instruments with $\alpha_j \neq 0$, or ill defined as $\beta + 0/0$. The median and adaptive Lasso estimators still have the properties as stated in Section 4 if we make the assumption that more than 50% of the instruments are valid *and* relevant. Another remedy to a relevance problem would be to do an initial selection of the instruments by for example a Lasso selection in the model $\mathbf{d} = \mathbf{Z}\boldsymbol{\gamma} + \mathbf{v}$, with the identification condition then that less than 50% of the selected (i.e. sufficiently strong) instruments are invalid. This is the approach taken by Guo et al. (2016), who then proceed to select the invalid instruments by a pairwise comparison of the $\widehat{\pi}_j$.

8 The Effect of BMI on Diastolic Blood Pressure Using Genetic Markers as Instruments

We use data on 105, 276 individuals from the UK Biobank and investigate the effect of BMI on diastolic blood pressure (DBP). See Sudlow et al. (2015) for further information on the UK Biobank. We use 96 single nucleotide polymorphisms (SNPs) as instruments for BMI as identified in independent GWAS studies, see Locke et al. (2015).

With Mendelian randomisation studies the SNPs used as potential instruments can be invalid for various reasons, such as linkage disequilibrium, population stratification and horizontal pleiotropy, see e.g. von Hinke et al. (2016) or Davey Smith and Hemani (2014). For example, a SNP has pleiotropic effects if it not only affects the exposure but also has a direct effect on the outcome. Whilst we guard against population stratification by considering only white European origin individuals in our data, the use of the Lasso methods can be extremely useful here to identify the SNPs with direct effects on the outcome and to estimate the causal effect of BMI on diastolic blood pressure taking

account of this.

Because of skewness, we log-transformed both BMI and DBP. The linear model specification includes age, age^2 and sex, together with 15 principal components of the genetic relatedness matrix as additional explanatory variables. Table 9 presents the estimation results for the causal effect parameter, which is here the percentage change in DBP due to a 1% change in BMI. As p-value for the Hansen test based procedures we take again $0.1/\ln(n) = 0.0086$.

The OLS estimate of the causal parameter is equal to 0.206 (s.e. 0.003), whereas the 2SLS estimate treating all 96 instruments as valid is much smaller at 0.087 (s.e. 0.016), with a 95% confidence interval of [0.056,0.118]. The J-test, however, rejects the null that all the instruments are valid. The Lasso$_{cv}$ estimator identifies a large number of 56 instruments as invalid and the Lasso$_{cv}$ estimate is equal to 0.126, the post-Lasso$_{cv}$ estimate equal to 0.145. The Lasso$_{cvse}$ procedure identifies 20 instruments as invalid and the Lasso$_{cvse}$ estimate is equal to 0.111. The post-Lasso$_{cvse}$ estimate is larger and equal to 0.142, which is in line with our findings above that the Lasso estimator is biased towards the 2SLS estimator that treats all instruments as valid due to shrinkage. The post-Lasso$_{ah}$ procedure selects a subset of 12 instruments as invalid, and the post-Lasso$_{ah}$ parameter estimate is equal to 0.122.

The median estimate $\widehat{\beta}_m$ is equal to 0.148. Using this estimate for the adaptive Lasso results in the cv method selecting 54 instruments as invalid and the $cvse$ method selecting 17 instruments as invalid. The adaptive Lasso$_{ah}$ method selects a subset of 11 instruments as invalid. The post-ALasso$_{cv}$, post-ALasso$_{cvse}$ and post-ALasso$_{ah}$ estimates are equal to 0.161, 0.151 and 0.163 respectively, with the 95% confidence intervals of the post-ALasso$_{cvse}$ and post-ALasso$_{ah}$ estimators given by [0.113,0.189] and [0.127,0.198] respectively. These results indicate that the OLS estimator is less confounded than suggested by the 2SLS estimation results using all 96 instruments as valid instruments.

The strongest potential instrument is the FTO SNP. For all Lasso estimators in Table 9 it is selected as an invalid instrument. The value for $\widehat{\pi}_{FTO} = -0.009$, i.e. negative, which is contrary to the direction of the found causal effect.

Table 9. Estimation results, the effect of $\ln(BMI)$ on $\ln(DBP)$

	estimate	rob st err	# instr selected as invalid	p-value J-test
OLS	0.206	0.003		
2SLS	0.087	0.016	0	0.0000
Lasso_{cv}	0.126		56	
Post-Lasso_{cv}	0.145	0.033		1.0000
Lasso_{cvse}	0.111		20	
Post-Lasso_{cvse}	0.142	0.020		0.6435
Post-Lasso_{ah}	0.122	0.018	12	0.0123
median, $\widehat{\beta}_m$	0.148			
ALasso_{cv}	0.158		54	
Post-ALasso_{cv}	0.161	0.029		1.0000
ALasso_{cvse}	0.131		17	
$\text{Post-ALasso}_{cvse}$	0.151	0.019		0.4091
Post-ALasso_{ah}	0.163	0.018	11	0.0102

Notes: sample size $n = 105,276$; $L = 96$

The F-test statistic for $H_0 : \boldsymbol{\gamma}_2 = 0$ for the model resulting from the ALasso_{ah} procedure is equal to 18.21 with the associated estimate of the concentration parameter equal to 1547.81. The F-test result indicates that the 2SLS estimator may have some many weak instruments bias, see Stock and Yogo (2005). However, the LIML (Limited Information Maximum Likelihood) estimator in this model is very similar to the 2SLS estimator and equal to 0.159 (s.e. 0.019), indicating that there is not a many weak instruments problem here, see Davies et al. (2015).

9 Conclusions

Instrumental variables estimation is a well established procedure for the identification and estimation of causal effects of exposures on outcomes where the observed relationships are confounded by non-random selection of exposure. The main identifying assumption is that the instruments satisfy the exclusion restriction, i.e. they only affect the outcomes through their relationship with the exposure. In an important contribution, Kang et al. (2016) show that the Lasso method for variable selection can be used to select invalid

instruments in linear IV models, even though there is no prior knowledge about which instruments are valid.

We have shown here that, even under the sufficient condition for identification that less than 50% of the instruments are invalid, the Lasso selection may select the valid instruments as invalid if the invalid instruments are relatively strong, i.e. the case where an invalid instrument explains more of the exposure variance than a valid instrument. Consistent selection of invalid instruments also depends on the correlation structure of the instruments.

We show that a median estimator is consistent when less than 50% of the instruments are invalid, and its consistency does not depend on the relative strength of the instruments or their correlation structure. This initial consistent estimator can be used for the adaptive Lasso estimator of Zou (2006) and we show that it performs well for larger sample sizes/information settings in our simulations. This adaptive Lasso estimator has the same limiting distribution as the oracle 2SLS estimator, and solves the inconsistency problem of the Lasso method when the relative strength of the invalid instruments is such that the Lasso method selects the valid instruments as invalid.

Whilst less than 50% invalid instruments is a sufficient condition for identification, in principle the parameters are identified if the valid instruments form the largest group. Instruments form a group if they have the same estimate for the causal effect. Future research will therefore focus on how to obtain consistent results when more than 50% of the instruments are invalid, but the parameters are such that they are asymptotically identified.

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A Appendix

A.1 Proof of Proposition 1

Lemma 1 *For nonsingular $r \times r$ matrix $\mathbf{H}$ and r-vector $\mathbf{h}$, if $\mathbf{h}'\mathbf{H}^{-1}\mathbf{h} \neq 1$, then*

$$(\mathbf{H} - \mathbf{h}\mathbf{h}')^{-1} = \mathbf{H}^{-1} + \frac{1}{1 - \mathbf{h}'\mathbf{H}^{-1}\mathbf{h}}\mathbf{H}^{-1}\mathbf{h}\mathbf{h}'\mathbf{H}^{-1} \tag{26}$$

and so

$$\begin{aligned} \mathbf{h}'(\mathbf{H} - \mathbf{h}\mathbf{h}')^{-1} &= \left(1 + \frac{\mathbf{h}'\mathbf{H}^{-1}\mathbf{h}}{1 - \mathbf{h}'\mathbf{H}^{-1}\mathbf{h}}\right)\mathbf{h}'\mathbf{H}^{-1} \\ &= \frac{1}{1 - \mathbf{h}'\mathbf{H}^{-1}\mathbf{h}}\mathbf{h}'\mathbf{H}^{-1}, \end{aligned} \tag{27}$$

Proof. See Abadir and Magnus (2005, page 87). ■

From the definitions in Section 3, we have

$$\begin{aligned} \operatorname{plim}\left(\frac{1}{n}\widetilde{\mathbf{Z}}_1'\widetilde{\mathbf{Z}}_1\right) &= \mathbf{Q}_{11} - (\mathbf{Q}_{11}\boldsymbol{\gamma}_1 + \mathbf{Q}_{21}'\boldsymbol{\gamma}_2)(\boldsymbol{\gamma}'\mathbf{Q}\boldsymbol{\gamma})^{-1}(\boldsymbol{\gamma}_1'\mathbf{Q}_{11} + \boldsymbol{\gamma}_2'\mathbf{Q}_{21}) \\ \operatorname{plim}\left(\frac{1}{n}\widetilde{\mathbf{Z}}_2'\widetilde{\mathbf{Z}}_1\right) &= \mathbf{Q}_{21} - (\mathbf{Q}_{21}\boldsymbol{\gamma}_1 + \mathbf{Q}_{22}\boldsymbol{\gamma}_2)(\boldsymbol{\gamma}'\mathbf{Q}\boldsymbol{\gamma})^{-1}(\boldsymbol{\gamma}_1'\mathbf{Q}_{11} + \boldsymbol{\gamma}_2'\mathbf{Q}_{21}). \end{aligned}$$

Hence

$$\begin{aligned} \mathbf{C}_{21}\mathbf{C}_{11}^{-1} &= \operatorname{plim}\left(\frac{1}{n}\widetilde{\mathbf{Z}}_2'\widetilde{\mathbf{Z}}_1\right)\left(\operatorname{plim}\left(\frac{1}{n}\widetilde{\mathbf{Z}}_1'\widetilde{\mathbf{Z}}_1\right)\right)^{-1} \\ &= ((\boldsymbol{\gamma}'\mathbf{Q}\boldsymbol{\gamma})\mathbf{Q}_{21} - (\mathbf{Q}_{21}\boldsymbol{\gamma}_1 + \mathbf{Q}_{22}\boldsymbol{\gamma}_2)(\boldsymbol{\gamma}_1'\mathbf{Q}_{11} + \boldsymbol{\gamma}_2'\mathbf{Q}_{21})) \\ &\quad ((\boldsymbol{\gamma}'\mathbf{Q}\boldsymbol{\gamma})\mathbf{Q}_{11} - (\mathbf{Q}_{11}\boldsymbol{\gamma}_1 + \mathbf{Q}_{21}'\boldsymbol{\gamma}_2)(\boldsymbol{\gamma}_1'\mathbf{Q}_{11} + \boldsymbol{\gamma}_2'\mathbf{Q}_{21}))^{-1}. \end{aligned} \tag{28}$$

Let $\boldsymbol{\eta} = (\mathbf{Q}_{11}\boldsymbol{\gamma}_1 + \mathbf{Q}_{21}'\boldsymbol{\gamma}_2)$. From (27), we get

$$\begin{aligned} \boldsymbol{\eta}'((\boldsymbol{\gamma}'\mathbf{Q}\boldsymbol{\gamma})\mathbf{Q}_{11} - \boldsymbol{\eta}\boldsymbol{\eta}')^{-1} &= \frac{1}{1 - \boldsymbol{\eta}'\mathbf{Q}_{11}^{-1}\boldsymbol{\eta}/\boldsymbol{\gamma}'\mathbf{Q}\boldsymbol{\gamma}}\boldsymbol{\eta}'\mathbf{Q}_{11}^{-1}/\boldsymbol{\gamma}'\mathbf{Q}\boldsymbol{\gamma} \\ &= \frac{1}{\boldsymbol{\gamma}'\mathbf{Q}\boldsymbol{\gamma} - \boldsymbol{\eta}'\mathbf{Q}_{11}^{-1}\boldsymbol{\eta}}\boldsymbol{\eta}'\mathbf{Q}_{11}^{-1}. \end{aligned} \tag{29}$$

Using (26),

$$\begin{aligned} &(\boldsymbol{\gamma}'\mathbf{Q}\boldsymbol{\gamma})\mathbf{Q}_{21}((\boldsymbol{\gamma}'\mathbf{Q}\boldsymbol{\gamma})\mathbf{Q}_{11} - \boldsymbol{\eta}\boldsymbol{\eta}')^{-1} \\ &= \mathbf{Q}_{21}\mathbf{Q}_{11}^{-1} + (\boldsymbol{\gamma}'\mathbf{Q}\boldsymbol{\gamma})\mathbf{Q}_{21}\frac{1}{1 - \boldsymbol{\eta}'\mathbf{Q}_{11}^{-1}\boldsymbol{\eta}/\boldsymbol{\gamma}'\mathbf{Q}\boldsymbol{\gamma}}\mathbf{Q}_{11}^{-1}\boldsymbol{\eta}\boldsymbol{\eta}'\mathbf{Q}_{11}^{-1}/(\boldsymbol{\gamma}'\mathbf{Q}\boldsymbol{\gamma})^2 \\ &= \mathbf{Q}_{21}\mathbf{Q}_{11}^{-1} + \mathbf{Q}_{21}\mathbf{Q}_{11}^{-1}\boldsymbol{\eta}\frac{1}{\boldsymbol{\gamma}'\mathbf{Q}\boldsymbol{\gamma} - \boldsymbol{\eta}'\mathbf{Q}_{11}^{-1}\boldsymbol{\eta}}\boldsymbol{\eta}'\mathbf{Q}_{11}^{-1}. \end{aligned} \tag{30}$$

Hence, combining (28), (29) and (30),

$$\mathbf{C}_{21}\mathbf{C}_{11}^{-1} = \mathbf{Q}_{21}\mathbf{Q}_{11}^{-1} + \left(\mathbf{Q}_{21}\mathbf{Q}_{11}^{-1}\boldsymbol{\eta} - (\mathbf{Q}_{21}\boldsymbol{\gamma}_1 + \mathbf{Q}_{22}\boldsymbol{\gamma}_2)\right)\frac{1}{\boldsymbol{\gamma}'\mathbf{Q}\boldsymbol{\gamma} - \boldsymbol{\eta}'\mathbf{Q}_{11}^{-1}\boldsymbol{\eta}}\boldsymbol{\eta}'\mathbf{Q}_{11}^{-1}. \quad (31)$$

As

$$\begin{aligned}\boldsymbol{\gamma}'\mathbf{Q}\boldsymbol{\gamma} &= \boldsymbol{\gamma}_1'\mathbf{Q}_{11}\boldsymbol{\gamma}_1 + 2\boldsymbol{\gamma}_1'\mathbf{Q}_{21}'\boldsymbol{\gamma}_2 + \boldsymbol{\gamma}_2'\mathbf{Q}_{22}\boldsymbol{\gamma}_2 \\ \boldsymbol{\eta}'\mathbf{Q}_{11}^{-1}\boldsymbol{\eta} &= \boldsymbol{\gamma}_1'\mathbf{Q}_{11}\boldsymbol{\gamma}_1 + 2\boldsymbol{\gamma}_1'\mathbf{Q}_{21}'\boldsymbol{\gamma}_2 + \boldsymbol{\gamma}_2'\mathbf{Q}_{21}\mathbf{Q}_{11}^{-1}\mathbf{Q}_{21}'\boldsymbol{\gamma}_2,\end{aligned}$$

it follows that

$$\boldsymbol{\gamma}'\mathbf{Q}\boldsymbol{\gamma} - \boldsymbol{\eta}'\mathbf{Q}_{11}^{-1}\boldsymbol{\eta} = \boldsymbol{\gamma}_2'\left(\mathbf{Q}_{22} - \mathbf{Q}_{21}\mathbf{Q}_{11}^{-1}\mathbf{Q}_{21}'\right)\boldsymbol{\gamma}_2. \quad (32)$$

Further,

$$\mathbf{Q}_{11}^{-1}\boldsymbol{\eta} = \boldsymbol{\gamma}_1 + \mathbf{Q}_{11}^{-1}\mathbf{Q}_{21}'\boldsymbol{\gamma}_2 \quad (33)$$

$$\mathbf{Q}_{21}\mathbf{Q}_{11}^{-1}\boldsymbol{\eta} - (\mathbf{Q}_{21}\boldsymbol{\gamma}_1 + \mathbf{Q}_{22}\boldsymbol{\gamma}_2) = \left(\mathbf{Q}_{21}\mathbf{Q}_{11}^{-1}\mathbf{Q}_{21}' - \mathbf{Q}_{22}\right)\boldsymbol{\gamma}_2. \quad (34)$$

Combining (31), (32), (33) and (34), the result follows that

$$\mathbf{C}_{21}\mathbf{C}_{11}^{-1} = \mathbf{Q}_{21}\mathbf{Q}_{11}^{-1} - \left(\mathbf{Q}_{22} - \mathbf{Q}_{21}\mathbf{Q}_{11}^{-1}\mathbf{Q}_{21}'\right)\boldsymbol{\gamma}_2\frac{\boldsymbol{\gamma}_1' + \boldsymbol{\gamma}_2'\mathbf{Q}_{21}\mathbf{Q}_{11}^{-1}}{\boldsymbol{\gamma}_2'\left(\mathbf{Q}_{22} - \mathbf{Q}_{21}\mathbf{Q}_{11}^{-1}\mathbf{Q}_{21}'\right)\boldsymbol{\gamma}_2}.$$

A.2 Monte Carlo results for correlated instruments

Figures 4a-4c below show the rejection frequencies of the robust Wald test $W_{\beta,r}$ as in Figures 2a-2c for equal and unequal strength instruments, but where the instruments are correlated and distributed as follows

$$\mathbf{Z}_{i.} \sim N\left(0, \boldsymbol{\Sigma}\right),$$

with the elements of $\boldsymbol{\Sigma}$ given by

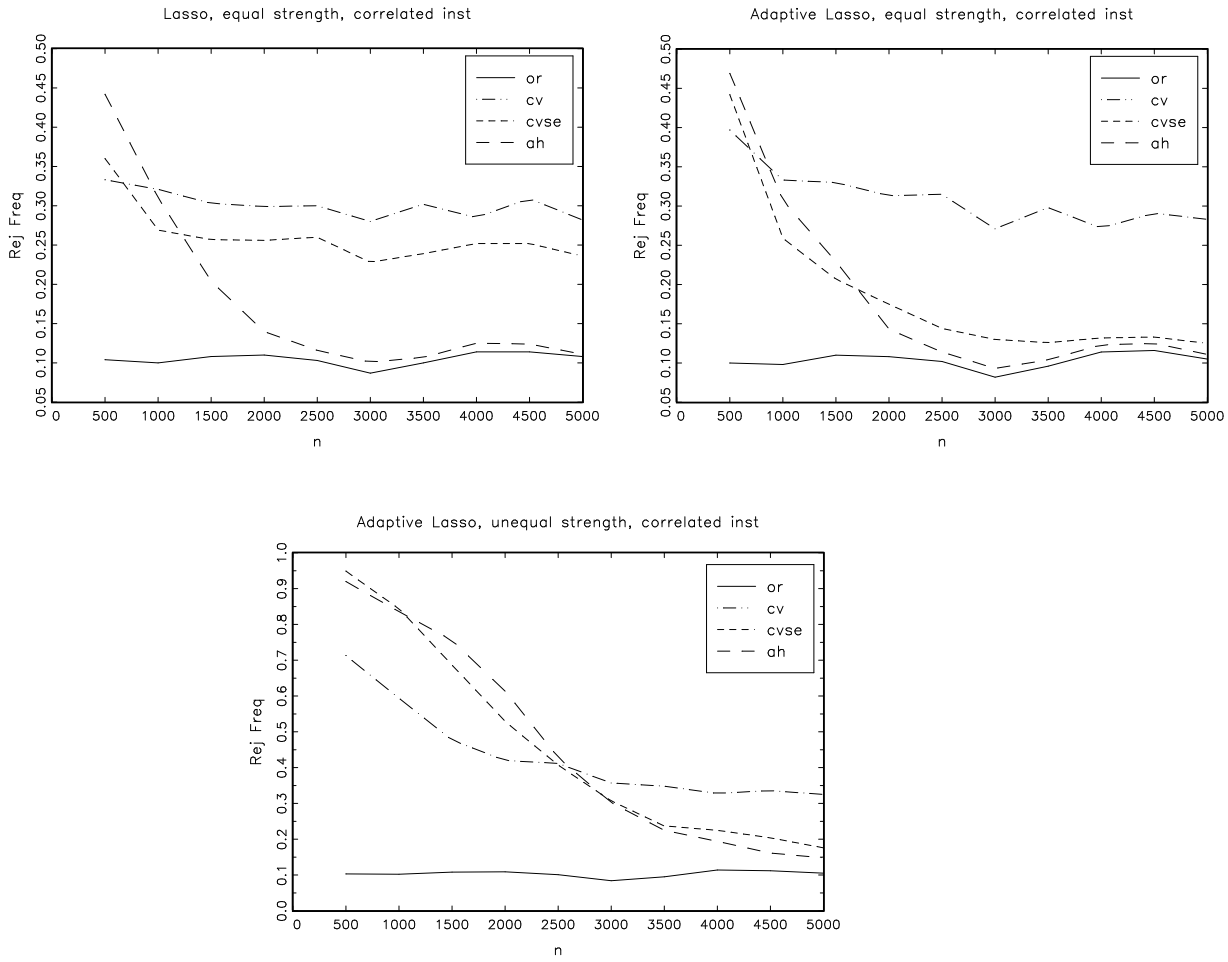
$$\boldsymbol{\Sigma}_{r,k} = \omega^{|r-k|}$$

and $\omega = 0.5$.

As in the designs of Figures 2a-2c, $\boldsymbol{\alpha}_1 = a\boldsymbol{\iota}_s$, $\boldsymbol{\gamma}_1 = \widetilde{\gamma}_1\boldsymbol{\iota}_s$, $\boldsymbol{\gamma}_2 = \widetilde{\gamma}_2\boldsymbol{\iota}_{L-s}$, $L = 10$, $s = 3$ and for the unequal strength design $\widetilde{\gamma}_1 = 3\widetilde{\gamma}_2$. We set the parameters a and $\widetilde{\gamma}_2$ such that $\mu^2 = \text{plim}\left(\mu_n^2/n\right)$ and η^2 are equal to those of the designs for Figures 2a-2c. This

results in $\widetilde{\gamma}_2 = 0.1321$ for both designs, $a = 0.1552$ in the equal instruments design, and $a = 0.1448$ in the unequal instruments design.

As the figures show, a larger sample size n is needed here for a performance of the Wald test $W_{\beta,r}$ similar to that of the uncorrelated instruments designs.



Figures 4a, 4b and 4c. Rejection frequencies of robust Wald tests for H_0: $\beta = 0$ at 10% level as a function of sample size, in steps of 500. Correlated instruments, equal strength instruments design, Post-Lasso in Figure 4a, Post-ALasso in Figure 4b. Unequal strength instruments design, Post-ALasso in Figure 4c. Based on 1000 MC replications for each sample size.