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Jorgen Kratz

Department of Economics and Related Studies
University of York
Heslington
York, YO10 5DD

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Jörgen Kratz†

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Abstract

There is no conflict between maximizing the number of transplants and giving priority to, e.g., highly HLA-sensitized recipients in kidney exchange programs that only permit pairwise exchanges. In some programs that feature cyclic exchanges or chains, however, giving priority to some recipients may reduce the number of transplants that can be carried out. This paper identifies the conditions under which there is a trade-off between prioritization and transplant maximization objectives. The results show that kidney exchange programs can permit some cyclic exchanges and chains without introducing such trade-offs. Whether or not a kidney exchange program has conflicting objectives and regardless of how recipients are prioritized, it can ensure a Pareto efficient outcome by selecting from a new class of matchings. These generalize several classes of matchings used in practice and studied in the literature.

Keywords: Kidney exchange, desensitization, priority, matroid, simplicial complex.
JEL Classification: C78, D47.

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†University of York, Department of Economics and Related Studies. *E-mail address:* jorgen.kratz@york.ac.uk

1 Introduction

While most humans are born with two kidneys, it is possible to lead a normal life with only one. As a result, a person in need of a kidney transplant, called a *recipient*, might be able to receive a new kidney from a living friend or relative. Unfortunately, the recipient may have antibodies against the donor's blood group or tissue type, making transplantation infeasible. Recipients in this predicament may still be able to receive immunologically compatible kidneys by exchanging donors with one another. The purpose of a *Kidney exchange program* (KEP) is to facilitate exchanges of this kind by evaluating the compatibility of different recipients and donors and selecting a *matching*: a set of exchanges to be carried out.[1]

KEPs have different objectives and constraints and consequently follow different approaches when selecting matchings. For example, the primary objective of most European KEPs is to maximize the number of transplants. Many KEPs also have objectives designed to improve the number of transplants for disadvantaged groups of recipients. For instance, the Belgian, Italian, Dutch, Polish, Portuguese, Scandinavian, Spanish, and UK KEPs all assign higher priority to highly HLA-sensitized recipients. Highly HLA-sensitized recipients are recipients that are tissue type incompatible with a significant share of the population and therefore difficult to find a compatible donor for. However, giving priority to highly HLA-sensitized recipients may reduce the number of transplants that can be carried out.

This paper investigates the conditions under which the objective of maximizing the number of transplants is in conflict with the objective of prioritizing particular groups of recipients. For example, the French KEP only permits *pairwise exchanges* involving two recipients, while the Dutch KEP also permits *cyclic exchanges* involving three or four recipients (called 3-way and 4-way exchanges), and the UK program permits *chains* involving two recipient-donor pairs and one altruistic kidney donor (Biró et al., 2021a). KEPs that only permit pairwise exchanges are known to be immune to trade-offs between maximizing the number of transplants and prioritizing particular groups of recipients (Roth et al., 2005a). By contrast, Sönmez and Ünver (2014) observed that cyclic exchanges can give rise to trade-offs of this kind. However, the precise conditions under which these two objectives are in conflict have so far not been studied. KEPs that are immune to such conflicts are said to be *conflict-proof*.

This paper establishes that KEPs are conflict-proof if and only if they have a pure simplicial complex (PSC) structure. KEPs that only permit pairwise exchanges are shown to have a PSC structure, while KEPs that permit cyclic exchanges do not, even when exchanges are constrained to involve at most $k \geq 3$ recipients. These results confirm the observations by Roth et al. (2005a) and Sönmez and Ünver (2014). They also imply that none of the current KEPs that permit cyclic exchanges or chains are conflict-proof.

Sönmez and Ünver (2014) argued that, from a medical ethics perspective, the trade-offs introduced when cyclic exchanges are permitted give an edge to the priority mechanisms of Roth et al. (2005a), which are limited to 2-way exchanges. However, there are large welfare gains associated with 3-way exchanges (Saidman et al., 2006).[2] This paper shows that part of these welfare gains can be enjoyed without introducing conflicts between objectives, by only permitting *some* cyclic exchanges and chains. For example, 3-way exchanges with three *embedded* 2-way exchanges can safely be included in a conflict-proof KEP. 3-way exchanges with embedded 2-way exchanges are desirable from a risk mitigation perspective (Manlove & O'Malley, 2015; Smeulders et al., 2022) and are currently given priority in the UK KEP (NHS, 2017). Furthermore, it is

[1] The first KEP to make use of optimization techniques, the New England Program for Kidney Exchange, was based on the work by Roth et al. (2004, 2005a).

[2] While Roth et al. (2007) demonstrated that the benefits of increasing the permitted size of cyclic exchanges from three to four pairs are typically small, there may be large gains from long cyclic exchanges if many recipients are highly HLA-sensitized (Ashlagi et al., 2012).

possible to introduce short chains initiated by altruistic kidney donors without losing conflict-proofness.

In addition to evaluating the conditions under which there is a conflict between prioritization and transplant maximization, this paper also studies various ways in which this conflict can be managed for different approaches to prioritization. While most KEPs prioritize recipients based on a variety of criteria when selecting a matching, prioritization is often strictly subordinate to the primary goal of maximizing the number of transplants (Biró et al., 2021a). In such KEPs, priorities are only used to break ties when there are multiple matchings that result in the same number of transplants. There are exceptions, however. For example, CIAT is a set of KEPs that were recently piloted in the Netherlands, with the primary objective of maximizing the number of transplants for certain highly HLA-sensitized recipients and the secondary objective of maximizing the total number of transplants (de Klerk et al., 2023). Another related example is the Spanish KEP, which assigns highly HLA-sensitized recipients a higher score in a points-based mechanism (Biró et al., 2021a).

In CIAT, highly HLA-sensitized recipients are sorted into a higher *priority group*. The primary objective is to match as many recipients as possible in the higher priority group and the secondary objective is to match as many recipients as possible in the lower priority group. This paper generalizes the approaches above by allowing recipients to be sorted into an arbitrary number of priority groups and introduces a new class of matchings that match recipients and donors in line with priority groups. These *priority group matchings* are guaranteed to be Pareto efficient in a setting general enough to encompass most KEPs. Furthermore, they are shown to generalize several classes of matchings, both from the literature and current practice.

Priority group matchings primarily maximize the number of transplants for recipients in the first priority group, secondarily maximize the number of transplants for recipients in the second priority group, and so on. Finally, priority group matchings minimize the use of *desensitization*, a treatment through which some recipients may receive kidneys from blood group incompatible and sometimes even tissue type incompatible donors.[3] A similar sequential approach with tiered priority groups has been proposed in the context of blood allocation mechanisms (Han et al., 2022).

To understand why some KEPs would be willing to match fewer recipients in order to prioritize some highly HLA-sensitized recipients, it is important to note that "unmatched" recipients may still receive kidney transplants the next time a matching is selected. A typical KEP will select several matchings per year. For example, matching algorithms are used to find new matchings on a quarterly basis in the Czech Republic, France, the Netherlands, Portugal, the United Kingdom and Switzerland (Biró et al., 2019). Highly HLA-sensitized recipients are more difficult to match and consequently less likely to receive kidney transplants in the future if they remain unmatched. To combat the tendency of such recipients to accumulate in the pool of unmatched pairs, the KEP may wish to give them priority when medically feasible exchanges present themselves. Because of these dynamics, even though prioritizing difficult-to-match recipients can reduce the number of transplants in the short run, it may still improve the number of transplants or some other measure of welfare in the long run. In addition to HLA-sensitization, recipients can receive priority based on a number of different criteria. For example, children are given priority over adults in the Spanish and Italian programs and recipients on waitlists for kidney transplants from deceased donors are often prioritized based on how urgent their conditions are (Costa et al., 2006; Celebi et al., 2006).

The matching outcome will depend on a KEP's approach to prioritizing recipients. For example, a simple approach is to set a threshold[4] in terms of HLA-sensitization. Recipients above the threshold are

[3]The primary motivation for minimizing its use is to improve recipient welfare, not to save money. As long as the recipient is expected to survive a few years, blood group incompatible transplantation using desensitization is still cheaper than letting the recipient remain on dialysis (Thydén et al., 2012; Wennberg, 2010).

[4]Not to be confused with the threshold mechanism studied by Ünver (2010).

then prioritized in accordance with some predetermined rule. For example, a KEP selecting *simple threshold matchings* will first maximize the number of transplants for recipients above the threshold and then maximize the number of transplants for recipients below it. This is in line with the CIAT KEPs discussed above, where the HLA-sensitization threshold is a vPRA score[5] of at least 85% (de Klerk et al., 2023). In settings without desensitization, simple threshold matchings are closely related to the class of matchings studied by Dickerson and Sandholm (2014) and Dickerson et al. (2014).

In an alternate approach, recipients above the threshold are prioritized in accordance with the priority mechanism introduced in Roth et al. (2005a). In this case, priority group matchings reduce to a new class of matchings called *threshold matchings*. A higher threshold is shown to always result in a weakly higher number of transplants. Intuitively, threshold matchings then reduce to *maximum matchings* (maximizing the number of transplants) at sufficiently high thresholds. At sufficiently low thresholds, threshold matchings instead reduce to the *priority matchings* studied by Andersson and Kratz (2020) and Roth et al. (2005a) in settings with and without desensitization, respectively.

This suggests that the shift in focus from priority matchings (Roth et al., 2005a; Okumura, 2014) to maximum matchings in models that incorporate cyclic exchanges and chains (Roth et al., 2005b, 2007; Saidman et al., 2006) was not a transition between two unrelated classes of matchings. Threshold matchings were selected in both cases and when the KEPs studied were no longer conflict-proof due to the introduction of more advanced exchanges, this issue was sidestepped by raising the threshold high enough for no recipients to qualify for preferential treatment. Two exceptions to this are Dickerson et al. (2014) and McElfresh and Dickerson (2018) who studied the reduction in the number of transplants caused by prioritizing highly HLA-sensitized recipients using the *price of fairness* (Bertismas et al., 2011).

The model in this paper is sufficiently general to encompass many different KEPs with varying characteristics. With the exception of Aziz et al. (2021), it is the only model general enough to incorporate both cyclic exchanges with length constraints and desensitization. Andersson and Kratz (2020) showed that desensitization can improve the number of transplants significantly if implemented correctly. Heo et al. (2021) also allow for cyclic exchanges in a setting with desensitization, but their model does not permit restrictions on cycle lengths and assumes that all incompatibilities can be overcome using desensitization.[6]

While the recipient preferences considered in this paper generalize the preferences in Andersson and Kratz (2020), they still remain closer to the dichotomous preferences in Roth et al. (2005a) than other models with non-dichotomous preferences, such as Roth et al. (2004), Nicolò and Rodríguez-Álvarez (2012, 2017), and Biró et al. (2021b). Non-dichotomous preferences make it possible to incentivize compatible recipient-donor pairs to participate in a KEP by offering them a "better" kidney (Andersson & Kratz, 2020; Chipman et al., 2022). It is well established that the number of transplants can be increased by including compatible pairs in this manner (Roth et al., 2005b; Gentry et al., 2007). Participation incentives can compensate for the fact that recipients in compatible pairs are able receive kidneys from their own donors outside the KEP. Sönmez et al. (2020) proposed incentivizing participation by offering priority in the waitlist for kidneys from deceased donors as insurance against future kidney failure. Unlike other papers permitting the inclusion of compatible recipient-donor pairs, such as Sönmez and Ünver (2014), the model in this paper does not assume that all compatible pairs are available for exchanges. Instead, some recipients are willing to participate in exchanges as long as they are not made worse off, while others would only participate in exchanges that

[5](Virtual) Panel Reactive Antibody (PRA) scores give a measure of a recipient's degree of HLA-sensitization.

[6]Transplantation across the HLA (tissue type) barrier is associated with significantly lower graft and recipient survival rates (Haririan et al., 2009) and transplantation across the blood group barrier is only deemed safe if the recipient's titer values can be kept below some threshold for a period of time before transplantation (Sönmez et al., 2018).

make them strictly better off. The number of transplants can be improved further using *chains*, in which a single altruistic donation may result in a large number transplants (Roth et al., 2006; Ashlagi et al., 2012; Anderson et al., 2015).

The remainder of the paper is structured as follows. Section 2 introduces the model. Section 3 explores the conditions under which KEPs are conflict-proof. Section 4 focuses on priority group matchings and the subclasses they generalize. Section 5 provides some concluding remarks. Appendix A describes how the structure of Pareto efficient matchings changes in settings with desensitization. Finally, Appendix B contains all proofs.

2 Model

This section is divided into two parts focusing on compatibility between recipients and donors and on matchings, respectively.

2.1 Compatibility

Let $N = \{1, \dots, n\}$ be a set of $n \geq 2$ *recipients*, where each recipient $i \in N$ has a living *donor* d_i. Let D be a (possibly empty) set of *altruistic donors* with no associated recipient in N, where $D = \{n+1, \dots, n+|D|\}$ whenever $|D| \geq 1$. For ease of notation, let $d_j := j$ for all $j \in D$. Unlike the donors belonging to recipient-donor pairs, the donors in D are willing to donate a kidney without a guarantee that some particular recipient receive a kidney. This paper follows the compatibility structure in Andersson and Kratz (2020), in which each donor is either *compatible*, *half-compatible* or *incompatible* with a given recipient. A recipient can always receive a kidney from a compatible donor and never from an incompatible donor. A donor is half-compatible if transplantation is only possible using desensitization. Let C be an $n \times (n + |D|)$ *compatibility matrix*. The ijth entry of C is given by the following.

$$c_{i,j} = \begin{cases} 2 & \text{if recipient } i \text{ is compatible with donor } d_j, \\ 1 & \text{if recipient } i \text{ is half-compatible with donor } d_j, \\ 0 & \text{if recipient } i \text{ is incompatible with donor } d_j. \end{cases}$$

In settings without desensitization, recipients and donors are never half-compatible and the compatibility structure reduces to the standard structure in, e.g., Roth et al. (2005a). There are two types of recipients: regular and altruistic. *Regular* recipients are only willing[7] to participate in kidney exchanges that make them strictly better off in terms of compatibility. If a regular recipient is incompatible with her own donor, she would accept any exchange that assigns her a half-compatible or compatible donor. If she is half-compatible with her own donor, she would only accept exchanges that assign her a compatible donor (thereby helping her avoid desensitization). *Altruistic* recipients would, on the other hand, accept participating in any exchanges as long as they are not assigned a strictly less compatible donor. Unlike regular recipients, altruistic recipients with compatible donors would be willing to participate in KEPs, despite having no incentive beyond altruism to do so. Improved participation rates are known to increase the number of transplants (Roth et al., 2005b; Sönmez & Ünver, 2014). Regular recipients are gathered in the set I_R and altruistic recipients are gathered in the set I_A, where $I_R \cup I_A = N$ and $I_R \cap I_A = \emptyset$. Since no recipients can receive kidney transplants

[7]Recipient preferences are discussed further in Section 4 and defined formally in Appendix A.

from incompatible donors, all recipients with incompatible donors must be regular recipients. I_R contains no recipients with compatible donors, since such recipients would be unwilling to participate in the KEP. By allowing recipients to be either regular or altruistic, this model contains models where all recipients are regular (Roth et al., 2005a) and models where all recipients are altruistic (Sönmez & Ünver, 2014) as special cases.

Let $G = (N \cup D, E)$ be a directed *compatibility graph*, with vertex set $N \cup D$ and arc set E. There is an arc $ij \in E$ from some recipient i to some recipient or altruistic donor j if i would accept receiving a kidney from recipient j's donor d_j or from the altruistic donor j. The set of arcs from some recipient i to other recipients and altruistic donors is therefore determined by the compatibility matrix and whether i is a regular or altruistic recipient.

$$\text{For } i \in I_R, \quad ij \in E \iff \begin{cases} i = j \text{ and } c_{i,i} = 1, \text{or} \\ j \in (N \cup D) \setminus \{i\} \text{ and } c_{i,j} > c_{i,i}. \end{cases}$$
$$\text{For } i \in I_A, \quad ij \in E \iff c_{i,j} \geq c_{i,i}.$$
$$\text{For } i \in D, \quad ij \notin E \text{ for all } j \in N \cup D.$$

This means that there is a loop ii at each vertex i corresponding to a recipient who is half-compatible with her own donor d_i, or an altruistic recipient who is compatible with her own donor. There is an arc ij from a recipient i to another recipient j if i is strictly "more compatible" with donor d_j than with her own donor d_i. There is also an arc ij from an altruistic recipient i to another recipient j if i is equally compatible with d_j and her own donor d_i. Note that $c_{i,j} \geq c_{i,i}$ implies that i and d_j are at least half-compatible as no recipient in I_A is incompatible with her own donor. Finally, there are no arcs from altruistic donors since they are not in need of kidney transplants.

2.2 Matchings

Assignments of kidneys to recipients are represented by cycles and chains. A *cycle* of length $q \geq 2$, or a *q-cycle*, is an ordered list of unique recipients $(i_1, i_2, \ldots, i_q)$ such that $i_j i_{j+1} \in E$ for all $j \in \{1, 2, \ldots, q\}$ taken modulo q, and $i_1 \leq i_j$ for all $j \in \{1, 2, \ldots, q\}$.[8] In other words, an ordered list of recipients is a cycle if there is an arc from each recipient to the next recipient in the list and an arc from the last recipient to the first recipient in the list. A 1-cycle is an ordered list (i) containing a single recipient i for which $ii \in E$. Note that a cycle can never involve an altruistic donor in D. A q-cycle represents a q-way exchange involving q recipient-donor pairs. For example, the 3-cycle $(1, 2, 3)$ would assign donor d_2 to recipient 1, donor d_3 to recipient 2, and donor d_1 to recipient 3.

An ordered list of unique recipients and an altruistic donor $(a_1, a_2, \ldots, a_q)$ is called a *chain* if $a_q \in D$ and $a_j a_{j+1} \in E$ for all $j \in \{1, \ldots, q-1\}$. Recipients that appear in a cycle or chain are said to *participate* or *be involved* in the cycle or chain. A chain of length $q \geq 1$, or a *q-chain*, is a chain that involves q recipients. The 2-chain $(1, 2, d)$ assigns altruistic donor d to recipient 2 and donor d_2 to recipient 1. Since $(1, 2, d)$ is not a cycle, donor d_1's kidney can be donated to some recipient $i \notin N$ on the waitlist for recipients who do not have a living donor d_i that they can participate in the KEP with. Interpreted in this way, a q-chain results in $q + 1$ kidney transplants.

[8] The requirement that $i_1 \leq i_j$ for all $j \in \{1, 2, \ldots, q\}$ prevents copies of cycles by inducing a unique indexing of the recipients in a cycle.

Let $\mathcal{K}$ contain all cycles and chains in G and let $K = \{k_1, \dots, k_{|K|}\}$ be a subset of $\mathcal{K}$. K can be interpreted as the set of all *permitted cycles and chains* and is determined by the constraints imposed on the KEP. For instance, there may be a constraint that forbids exchanges involving more than three recipients, in which case any cycles in $\mathcal{K}$ involving four or more recipients would be excluded from K. K is always assumed to be non-empty to avoid trivial cases where no transplants are possible. A KEP may also face other types of constraints. For example, altruistic participation of compatible pairs can be voluntary, but it can also be forbidden or enforced. Such constraints are modeled by letting all recipients be regular or altruistic, respectively. Additionally, KEPs that do not make use of desensitization are modeled by letting $c_{i,j} \neq 1$ for all $i, j \in N \cup D$.

A *matching* is defined as a subset $\mu \subseteq K$ such that each recipient and altruistic donor participates in at most one cycle or chain in μ. Let $\mathcal{M}$ denote the set of all matchings. Matchings specify which donor (if any) each recipient will receive a kidney transplant from. At a matching μ, a recipient i is assigned a donor d_j, where $i \neq j$, if i is the direct predecessor of j in some cycle or chain $k \in \mu$. That is, i is assigned d_j whenever μ contains a cycle or chain of the form $(\dots, i, j, \dots)$ or a cycle or the form $(j, \dots, i)$. If a 2-cycle is selected, this is referred to as a *pairwise exchange*. If a k-cycle involving $k \geq 3$ recipients is selected, this is referred to as a *cyclic exchange* or a *k-way exchange*. Furthermore, a recipient i is assigned her own donor d_i if μ contains the 1-cycle (i). The requirement that no recipient or altruistic donor participates in more than one cycle or chain ensures that no recipient receives more than one kidney and no donor donates more than one kidney.

Recipient i is said to be *matched* at μ if she participates in some cycle or chain in μ. Let $N^*(\mu) \subseteq N$ denote the set of recipients that are matched at μ, i.e., the set of recipients receiving transplants. If $i \notin N^*(\mu)$, then i is said to be *unmatched* at μ. Furthermore, let $B(\mu) \subseteq N^*(\mu)$ be the set of recipients that are assigned compatible donors at μ and thereby avoid desensitization.

3 Conflicting objectives

The following example illustrates a conflict between the objectives of maximizing the number of transplants and prioritizing certain recipients (e.g., highly HLA-sensitized recipients).

Example 1. Let $N = \{1, 2, 3, 4\}$ and $\mathcal{K} = K = \{(1, 2), (2, 3), (2, 3, 4)\}$.

In this example, there are three matchings, $\mu = \{(1, 2)\}$, $\mu' = \{(2, 3)\}$, and $\mu'' = \{(2, 3, 4)\}$. That is, there is a choice between a matching μ, at which recipients 1 and 2 receive transplants, a matching μ', at which recipients 2 and 3 receive transplants, and a matching μ'', at which recipients 2, 3 and 4 receive transplants. If the primary objective is to maximize the number of transplants, matching μ'' would be selected. Now suppose that recipient 1 is highly HLA-sensitized. Rather than selecting μ'', a KEP may then wish to give priority to recipient 1 and select matching μ instead. While selecting μ would reduce the number of transplants from three to two, giving priority to highly HLA-sensitized recipients could improve the number of transplants in the long run and mitigate the tendency of highly HLA-sensitized recipients to accumulate. As KEPs typically select matchings several times per year (Biró et al., 2019), the unmatched recipients may still have a chance to receive transplants at some point in the future. Highly HLA-sensitized recipients are particularly difficult to find compatible donors for and consequently unlikely to receive transplants in the future. It can then make sense to give priority to such recipients in the rare event that a medically feasible exchange is available, even if the selected matching involves a smaller number of transplants.

The conflict between the objective of maximizing the number of transplants and the objective of prioritizing highly HLA-sensitized recipients in Example 1 arose because all exchanges in $\mathcal{K}$ were permitted. Suppose instead that $K = \{(1,2),(2,3)\} \subset \mathcal{K}$, i.e., that the cyclic exchange $(2,3,4)$ is not permitted. In this case there is no conflict between the two objectives since the matching $\{(1,2)\}$ both maximizes the number of transplants given the set of permitted exchanges and ensures that the highly HLA-sensitized recipient 1 receives a transplant. This is in line with the more general observation by Roth et al. (2005a) that there is never a trade-off between maximizing the number of transplants and prioritizing recipients in KEPs that only permit pairwise exchanges. To understand and expand upon this observation, note that for any set of recipients N, a matching problem can be represented by a set-family $\mathcal{S}$ containing each subset of recipients in N that can be matched simultaneously[9]:

$$\mathcal{S} := \{I \subseteq N \mid I \subseteq N^*(\mu) \text{ for some } \mu \in \mathcal{M}\}$$

If $I \in \mathcal{S}$, then there exists a matching at which every agent in I is matched and I is said to be a *face* of $\mathcal{S}$. If I is not a face of $\mathcal{S}$, then there exists no such matching.

Whether or not there is a conflict between transplant maximization and prioritization objectives depends on the structure of $\mathcal{S}$, which is in turn determined by the constraints imposed on the KEP. For any KEP, and indeed any matching problem, $\mathcal{S}$ has the property that if $I \in \mathcal{S}$ and $J \subset I$, then $J \in \mathcal{S}$.[10] A set-family with this property is called an *abstract simplicial complex*. The absence of a conflict between the two objectives observed by Roth et al. (2005a) has been explained with reference to the fact that $\mathcal{S}$ is guaranteed to constitute a *matroid* when only pairwise exchanges are permitted. An abstract simplicial complex $\mathcal{S}$ is a matroid if it has the additional property (called *the exchange property*) that for any $I, J \in \mathcal{S}$ with $|J| < |I|$, there exists some $i \in I \setminus J$ such that $J \cup \{i\} \in \mathcal{S}$. Matroids were first studied by Whitney (1935).

To understand the exchange property, suppose there are two sets of recipients, I and J, such that all recipients in I can be matched simultaneously and all recipients in J can be matched simultaneously. Furthermore, suppose that I contains a strictly larger number of recipients than J. The exchange property then ensures that there exists some recipient i who is a member of I but not of J, and can be matched simultaneously with all recipients in J. An immediate implication of this property is that for any matching μ that does not maximize the number of transplants, there must exist some other matching μ' that results in a strictly higher number of transplants than μ while ensuring that all recipients matched at μ still receive kidney transplants.

As Example 1 demonstrated, whether some set of recipients can be matched simultaneously depends on the constraints imposed on the KEP. In practice, the constraints on the permitted set of exchanges are typically determined prior to observing the recipients and donors participating in the program (Biró et al., 2018). Let a *problem* be defined as a triple $P = \langle N, D, C \rangle$ consisting of a set of recipients, a set of altruistic donors and a compatibility matrix. Furthermore, let $\mathscr{P}$ denote the set of all *potential problems*. A KEP is said to have a *matroid structure* if $\mathcal{S}$ is a matroid for any potential problem $P \in \mathscr{P}$. The results in Roth et al. (2005a) imply that KEPs that only permit pairwise exchanges have a matroid structure. The authors used this finding to show that every Pareto efficient matching results in the same number of transplants in their setting.

As will be shown later, a matroid structure is sufficient to ensure the absence of conflicts in a KEP between the two objectives, but it is not a necessary condition. A necessary condition can be found by generalizing

[9] Note that the setting is still static, despite the connotations of the word "simultaneous".
[10] Any matching that matches all recipients in some set I will also match all recipients in any subset of I.

the notion of a matroid slightly by relaxing the exchange property. An abstract simplicial complex is said to satisfy the *relaxed exchange property* if for any $I, J \in \mathcal{S}$ with $|J| < |I|$, there exists some $i \in N \setminus J$ such that $J \cup \{i\} \in \mathcal{S}$. Since $I \subseteq N$ for all $I \in \mathcal{S}$, every matroid satisfies the relaxed exchange property. Lemma 1 below shows that an abstract simplicial complex satisfies the relaxed exchange property if and only if all of its *facets* have the same cardinality. A *facet* of an abstract simplicial complex $\mathcal{S}$ is a face of $\mathcal{S}$ that is not a subset of any other face of $\mathcal{S}$.

Lemma 1. An abstract simplicial complex satisfies the relaxed exchange property if and only if all of its facets have the same cardinality.

An abstract simplicial complex with the property that all of its facets have the same cardinality is known as a *pure simplicial complex*. In the context of kidney exchange, a facet of $\mathcal{S}$ corresponds to the set of recipients that receive transplants under some *maximal matching* in $\mathcal{M}$. A matching $\mu \in \mathcal{M}$ is a maximal matching if there exists no other matching $\nu \in \mathcal{M}$ such that $N^*(\mu) \subset N^*(\nu)$. That is, μ is maximal if it is impossible to match an additional recipient without preventing any recipient who is matched at μ from receiving a transplant. On the other hand, a matching is a *maximum matching* if it maximizes the total number of transplants over all matchings. A face of $\mathcal{S}$ corresponds to the set of recipients that receive transplants under some maximum matching in $\mathcal{M}$ if its cardinality is at least as high as the cardinality of any other face of $\mathcal{S}$. These observations imply the following lemma.

Lemma 2. For any problem $\langle N, D, C \rangle$, the set of maximal matchings equals the set of maximum matchings if and only if $\mathcal{S}$ is a pure simplicial complex.

Lemma 1 and Lemma 2 jointly imply that the relaxed exchange property is the weakest condition an abstract simplicial complex must satisfy for this equivalence between maximal and maximum matchings to hold.

In settings without desensitization, recipients are typically assumed to be indifferent between any compatible donors (preferences are dichotomous) and a matching is therefore Pareto efficient if and only if it is a maximal matching. The equivalence between maximal and maximum matchings then implies that all Pareto efficient matchings maximize the number of transplants in their setting and consequently result in the same number of transplants, as observed by Roth et al. (2005a) for the special case of matroids.

It is also this equivalence that ensures the absence of a trade-off between prioritizing recipients and maximizing the number of transplants. It implies that if there exists some matching at which some recipient i receives a kidney transplant, there must also exist some maximum matching at which i receives a kidney transplant.[11] As a result, there is no trade-off between matching i and maximizing the number of transplants, for any recipient $i \in N$. The results in Roth et al. (2005a) therefore imply that there is no conflict between recipient prioritization and transplant maximization objectives in KEPs that only permit pairwise exchanges.

While $\mathcal{S}$ being a pure simplicial complex is both a necessary and a sufficient condition for the equivalence between maximal and maximum matchings, $\mathcal{S}$ being a matroid is only a sufficient condition. This distinction is illustrated in the following example.

Example 2. Let $N = \{1, 2, 3, 4, 5\}$ and $\mathcal{K} = \{(1, 2, 3), (2, 3), (3, 4, 5)\}$. Furthermore, suppose that 2 is highly HLA-sensitized. Consider the following three cases.

(a) The set of permitted exchanges is given by $K = \{(2, 3), (3, 4, 5)\}$. In this case, the matching $\{(2, 3)\}$ matches 2 and 3 simultaneously, while $\{(3, 4, 5)\}$ matches 3, 4 and 5 simultaneously. Thus, both $\{2, 3\}$

[11] This observation explains why the greedy algorithm in Roth et al. (2005a) maximizes the number of transplants.

and $\{3,4,5\}$ belong to $\mathcal{S}$. Since $|\{2,3\}| = 2 < |\{3,4,5\}| = 3$, the relaxed exchange property requires that there exist some recipient $i \in N \setminus \{2,3\}$ such that 2, 3 and i can be matched simultaneously. However, since $(2,3)$ is the only exchange in K that involves 2, this is impossible. Since the relaxed exchange property is violated, $\mathcal{S}$ is neither a pure simplicial complex nor a matroid. Note that $\{(2,3)\}$ is a maximal matching but not a maximum matching and that there is a conflict between selecting $\{(2,3)\}$ to prioritize 2 and selecting $\{(3,4,5)\}$ to maximize the number of transplants.

(b) The set of permitted exchanges is given by $K = \{(1,2,3),(3,4,5)\}$. In this case, the exchange property is trivially satisfied and $\mathcal{S}$ is both a pure simplicial complex and a matroid. Note that both maximal matchings $\{(1,2,3)\}$ and $\{(3,4,5)\}$ are maximum matchings, and there is no conflict between the two objectives since $\{(1,2,3)\}$ both matches the highly HLA-sensitized 2 and maximizes the number of transplants.

(c) All exchanges in $\mathcal{K}$ are permitted: $K = \{(1,2,3),(2,3),(3,4,5)\}$. As in case (a), the matching $\{(2,3)\}$ matches 2 and 3 simultaneously, while $\{3,4,5\}$ matches 3, 4 and 5 simultaneously. Thus, $\{2,3\},\{3,4,5\} \in \mathcal{S}$. As $|\{2,3\}| = 2 < |\{3,4,5\}| = 3$, the exchange property is violated since it requires that there exist some recipient $i \in \{3,4,5\} \setminus \{2,3\}$ such that 2, 3 and i can be matched simultaneously. This means that $\mathcal{S}$ is not a matroid. However, the relaxed exchange property is satisfied since it only requires that there exist some recipient $i \in N \setminus \{2,3\}$ such that 2, 3 and i can be matched simultaneously. This condition is satisfied for $i = 1$ and $\mathcal{S}$ is consequently a pure simplicial complex, but not a matroid. Note that it is still the case that both maximal matchings $\{(1,2,3)\}$ and $\{(3,4,5)\}$ are maximum matchings and that there is no conflict between prioritizing 2 and maximizing the number of transplants.

In Example 2, $\mathcal{S}$ is a matroid when only the 3-cycles $(1,2,3)$ and $(3,4,5)$ are permitted. When $(1,2,3)$ also contains the "embedded" 2-cycle $(2,3)$, $\mathcal{S}$ is still a pure simplicial complex but no longer a matroid. Such embedded cycles have desirable properties that will be discussed later in this section. The example shows that, if the goal is to avoid introducing conflicts between these objectives, it is not necessary to restrict attention to KEPs in which $\mathcal{S}$ is guaranteed to be a matroid, such as KEPs that only permit pairwise exchanges. This hints at one of the conclusions that can be drawn from the results in this section. Namely that within the bounds of conflict-proof KEPs, there are potential welfare gains to be found by considering KEPs that lack a matroid structure.

A KEP is said to have a *pure simplicial complex (PSC) structure* if $\mathcal{S}$ is a pure simplicial complex for any potential problem $P \in \mathscr{P}$. By Lemma 2, a maximal matching is always guaranteed to maximize the number of transplants whenever the KEP has a PSC structure and vice versa.

This has implications for the existence of trade-offs between prioritizing recipients and maximizing the number of transplants. Let $\mathscr{N}$ denote the set of all *potential recipients* (i.e., the set of all recipients that exist in at least one problem in $\mathscr{P}$). A *priority structure* $\hat{\pi} : \mathscr{N} \to \mathbb{R}$ is a function that assigns some priority $\hat{\pi}(i)$ to each potential recipient $i \in \mathscr{N}$, where a lower number indicates higher priority. The priority structure can reflect any criterion or combination of criteria used to prioritize recipients, such as urgency, age, or PRA scores.

Definition 1. A kidney exchange program is *conflict-proof* if there is no conflict between matching high priority recipients and maximizing the number of transplants in each problem $P \in \mathscr{P}$ and for each priority structure $\hat{\pi}$.

Conflict-proof KEPs are guaranteed to never face trade-offs between transplant maximization and prioritizing, e.g., highly HLA-sensitized recipients. This means that there is no conflict between these objectives and no need to evaluate their relative importance. By contrast, there will always exist scenarios in which both objectives cannot be met in KEPs that are not conflict-proof. Without prior knowledge of the medical details of future participants (recipients and donors), establishing precisely how the two objectives are weighed against each other is therefore an unavoidable necessity in programs that are not conflict-proof. The following result shows that a PSC structure is a necessary and sufficient condition for conflict-proofness.

Proposition 1. A kidney exchange program is conflict-proof if and only if it has a PSC structure.

As a characterization of conflict-proof KEPs, Proposition 1 is useful for evaluating in which types of programs the objectives of maximizing the number of transplants and prioritizing recipients are in conflict. While it has already been established in the literature that these objectives are not in conflict in KEPs that only permit pairwise exchanges and that KEPs that only permit pairwise exchanges have a matroid structure, the connection between these results has not been explored in the past. Proposition 1 makes the relationship between KEPs that only permit pairwise exchanges and conflict-proofness explicit. It also highlights the limitations of focusing on matroids when studying conflict-proof mechanisms.

For example, Sönmez and Ünver (2014) observed that trade-offs between prioritization and transplant maximization can arise in KEPs that permit cyclic exchanges. This observation can not be explained by the observation that such KEPs lack a matroid structure, as a matroid structure is a sufficient condition for conflict-proofness, but not a necessary condition. However their observation can be confirmed using the concept of a pure simplicial complex, since Proposition 1 implies that a conflict between transplant maximization and prioritization can arise in a KEP if and only if it lacks a PSC structure. With the help of Proposition 1, it is possible to provide some new insights into what types of KEPs are and are not conflict-proof.

Proposition 2. For any $q \geq 3$, a kidney exchange program is not conflict-proof if it permits all q-cycles.

Proposition 2 is proven by demonstrating that such KEPs do not have a PSC structure. The majority of KEPs permit any available 3-way exchanges. By Proposition 2, these programs are not conflict-proof and must consequently address the trade-off between prioritization and transplant maximization. Note that Proposition 2 does not assume that exchanges involving fewer recipients than q are permitted. Example 1 showcased a conflict between transplant maximization and prioritization that arose because there was a choice between a 2-cycle involving a highly HLA-sensitized recipient and a 3-cycle. Proposition 2 implies that these trade-offs do not arise because the KEP involves exchanges of different sizes: A KEP that permits all 3-cycles and forbids all 2-cycles would still not be conflict-proof. Similarly, introducing chains can cause a KEP to lose its PSC structure.

Proposition 3. For any $q \geq 2$, a kidney exchange program is not conflict-proof if it permits all chains involving at most q recipients.

A natural question is then whether KEPs that only permit pairwise exchanges and chains involving one altruistic donor $d \in D$ and one recipient $i \in N$ are conflict-proof. Such chains are called *short chains*. In addition to helping i receive a kidney transplant from d, the kidney of donor d_i can also be assigned to a recipient $j \notin N$ not participating in the KEP, such as a recipient on the waitlist. The UK KEP initially only permitted short chains and introduced longer chains involving two recipient-donor pairs in 2015 (NHS, 2017).

Proposition 4 shows that short chains can be introduced into a KEP that only permits pairwise exchanges without losing its matroid structure.

Proposition 4. Kidney exchange programs that only permit pairwise exchanges and short chains have a matroid structure.

Proposition 4 generalizes Proposition 1 in Roth et al. (2005a) and Proposition 11 in Andersson and Kratz (2020) by including short chains. It implies that some KEPs that are known to be conflict-proof can safely incorporate short chains and remain conflict-proof. The remainder of this section explores other types of exchanges with potential for welfare gains that can be incorporated in a KEP without introducing any trade-offs between maximizing the number of transplants and prioritizing recipients.

For this purpose, it is important to note that Proposition 2 does not imply that conflict-proof KEPs are limited to pairwise exchanges (and short chains). It would, for example, be possible to make the permissibility of a cyclic exchange conditional on $\mathcal{S}$ remaining a pure simplicial complex after its inclusion in K. This would allow a KEP to enjoy the welfare gains associated with more advanced cyclic exchanges to a limited extent without introducing trade-offs between conflicting objectives, as demonstrated in Example 2.

A more practical example of a conflict-proof KEP that permits cyclic exchanges is a program that permits 3-cycles with sufficiently many *embedded 2-cycles*. A 3-cycle $(1, 2, 3) \in K$ is said to have an embedded 2-cycle if there exists a 2-cycle $(i, i') \in K$ such that $i, i' \in \{1, 2, 3\}$ and $i \neq i'$. A 3-cycle can have between zero and three embedded 2-cycles. Since 2010, the UK program has given priority to 3-cycles with embedded 2-cycles (NHS, 2017). Embedded 2-cycles are desirable as they mitigate some of the risk associated with the 3-cycles they are embedded in (Manlove & O'Malley, 2015; Smeulders et al., 2022). For example, suppose a 3-cycle $(1, 2, 3)$ containing the embedded 2-cycle $(1, 2)$ is selected. Before the corresponding transplantations are carried out, $(1, 2, 3)$ may be deemed unfeasible due to previously undiscovered medical incompatibilities involving recipient 3. Alternatively, 3 may simply change her mind before the transplantations take place. In both cases, even though the cyclic exchange $(1, 2, 3)$ can no longer be carried out, 1 and 2 can still receive kidney transplants by carrying out the embedded pairwise exchange $(1, 2)$.

The following result shows that requiring 3-cycles to contain embedded 2-cycles is insufficient for avoiding the trade-off between the conflicting objectives discussed above. However, if each 3-cycle is required to contain three embedded 2-cycles, then the KEP is conflict-proof.

Proposition 5. A kidney exchange program that only permits pairwise exchanges and 3-cycles with at least q embedded 2-cycles is conflict-proof if and only if $q = 3$.

Proposition 5 is proven by demonstrating that such KEPs have a matroid structure when $q = 3$, but lack a PSC structure when $q \neq 3$.

Sönmez and Ünver (2014) wrote that *"[A] program never matches a high-priority patient at the expense of multiple patients under the Pareto-efficient pairwise priority mechanisms offered by [Roth et al. (2005a)]. This result does not hold [...] for mechanisms that allow larger exchanges than pairwise. Hence, from a medical ethics perspective, it gives pairwise priority mechanisms an edge. However, this advantage comes at a high cost to aggregate patient welfare."*

The results in this section have shown that while this desirable property of pairwise kidney exchange is lost when 3-way exchanges are introduced *generally*, it is still possible to introduce *some* larger exchanges without compromising this feature. Proposition 5 gives an example of this, showing that is possible to improve the number of transplants by including certain 3-cycles with embedded 2-cycles, without introducing conflicts between transplant maximization and prioritizing, e.g., highly HLA-sensitized recipients. However, making full use of cyclic exchanges of a limited size or chains involving two or more recipient-donor pairs requires weighing competing objectives against each other.

4 Priority group matchings

Having established the conditions under which transplant maximization and prioritizing, e.g., highly HLA-sensitized recipients are conflicting objectives, this section focuses on how the resulting trade-off can be managed. As discussed in the introduction, the portion of the kidney exchange literature that emphasizes priorities has traditionally avoided this conflict by restricting attention to conflict-proof KEPs, such as programs that only permit pairwise exchanges. Similarly, the portion of the literature focusing on KEPs that are not conflict-proof, such as programs that permit any cyclic exchanges below some maximum length, has also avoided this conflict by using priorities to at most break ties between matchings that maximize the number of transplants. However, there are exceptions in the literature and in practice. For example, the CIAT KEPs use a mechanism that prioritizes highly HLA-sensitized recipients even if this would reduce the number of transplants.

This section generalizes all of these approaches and demonstrates that KEPs that make use of desensitization can guarantee Pareto efficient outcomes regardless of how this conflict is managed or how recipients are prioritized. A new class of Pareto efficient matchings is introduced, which enables priorities to be taken into account even in KEPs that are not conflict-proof. They are shown to generalize several classes of matchings from the literature and real world practice, corresponding to different approaches to prioritizing recipients and managing the conflict discussed in the previous section.

In KEPs with desensitization, it is reasonable to assume that recipients have a preference for avoiding the delays and additional treatments associated with desensitization. Such preferences constitute a departure from the dichotomous preferences in Roth et al. (2005a), which makes it possible to provide incentives for recipients with half-compatible donors to participate in KEPs by ensuring that any kidney exchanges they are involved in help them avoid desensitization (Andersson & Kratz, 2020). However, such non-dichotomous preferences complicate matters for KEPs as they must take desensitization into account in order to find Pareto efficient matchings.[12] See Appendix A for a characterization of Pareto efficient matchings and a discussion on the structure of Pareto efficient matchings in KEPs with desensitization. The appendix also contains the formal definitions of recipient preferences and Pareto efficiency.

Each recipient's preferences over matchings are determined by the donor she is assigned at each matching. All recipients prefer compatible donors to half-compatible donors and half-compatible donors to being unmatched. Regular recipients prefer their own donors to other equally compatible donors, while altruistic recipients are indifferent between their own donors and other equally compatible donors. A matching $\mu \in \mathcal{M}$ is *Pareto efficient* if there is no other matching that is weakly preferred to μ by all recipients and strictly preferred by at least one recipient.

All recipients are sorted into *priority groups* in accordance with the KEP's approach to prioritizing recipients. Formally, N is partitioned into $m \in \{1, \ldots, n\}$ priority groups $N_1, \ldots, N_m$, where each recipient belongs to exactly one priority group. N_1 is called the first priority group, N_2 is called the second priority group and so on. For example, the recipients in the first and mth priority groups could be the recipients with the highest and lowest PRA scores, respectively. A higher PRA score indicates a more HLA-sensitized recipient. Let $N_t^*(\mu) \coloneqq N^*(\mu) \cap N_t$ be the set of recipients in N_t that are matched at μ and let $N_0 \coloneqq \emptyset$ for

[12] By contrast, when preferences are dichotomous, any maximal matching is Pareto efficient.

notational convenience.

Let the objectives of a KEP be represented by a binary *preference relation* R. For any $\mu, \mu' \in \mathcal{M}$, $\mu R \mu'$ indicates that μ is at least as good as μ'. *Priority group preferences* are a class of preferences that align with the KEP's approach to prioritization. A program with priority group preferences is primarily concerned with matching as many recipients in the first priority group as possible. The secondary concern is to match as many recipients in the second priority group as possible, and so on. If the same number of recipients receive transplants in each priority group at two different matchings, then the matching that requires fewer recipients to undergo desensitization treatments is preferred. Priority group preferences are denoted by $\succsim$ and defined formally below, where $t \in \{1, \ldots, m\}$ and $\tau \in \{0, \ldots, m-1\}$.[13]

$$\mu \succ \mu' \iff \left(\begin{array}{l} |N_t^*(\mu)| > |N_t^*(\mu')| \text{ for some } t \leq m \text{ and } |N_\tau^*(\mu)| = |N_\tau^*(\mu')| \text{ for all } \tau < t, \text{ or} \\ |N_t^*(\mu)| = |N_t^*(\mu')| \text{ for all } t \leq m \text{ and } |B(\mu)| > |B(\mu')|, \end{array} \right)$$

$$\mu \sim \mu' \iff |N_t^*(\mu)| = |N_t^*(\mu')| \text{ for all } t \leq m \text{ and } |B(\mu)| = |B(\mu')|.$$

A matching $\mu \in \mathcal{M}$ is a *priority group matching* whenever $\mu \succsim \mu'$ for all $\mu' \in \mathcal{M}$. That is, priority group matchings satisfy the objectives of a KEP with priority group preferences at least as well as any other matching. Priority group matchings are gathered in the set $\mathcal{M}^*$. Computationally, priority group matchings can be found using sequential optimization techniques (Delorme et al., 2023). Proposition 6 shows that priority group matchings are Pareto efficient for any approach to prioritization and any constraints imposed on the KEP.

Proposition 6. Priority group matchings are Pareto efficient.

Later in this section, many classes of matchings studied in the literature and selected in practice are shown to be special cases of priority group matchings. Proposition 6 thus implies that all of these matchings are Pareto efficient even when extending their corresponding models to the typically more general setting considered in this paper. Roth et al. (2005a) demonstrated that all Pareto efficient matchings result in the same number of transplants. While this observation does not carry over to KEPs without a PSC structure, the corresponding statement for priority group matchings does. Specifically, for a given approach to prioritization, every priority group matching will match the same number of recipients belonging to each priority group.

Lemma 3. $|N_t^*(\mu)| = |N_t^*(\mu')|$ for all $\mu, \mu' \in \mathcal{M}^*$ and all $t \leq m$.

Of course, the manner in which recipients are prioritized may affect the number of transplants, as seen in Example 1.

In a small KEP with only 25 recipients, there are over four quintillion ways to sort the recipients into priority groups. Rather than evaluating each sorting individually before selecting a matching, KEPs would normally follow some simple approach to prioritization that can be applied in any scenario. One option is to order all recipients by, e.g., PRA scores and to set some *threshold* in terms of that measure. The HLA-sensitization of recipients above the threshold, called the *prioritized recipients*, is deemed severe enough to warrant preferential treatment. These recipients are prioritized in accordance with some predetermined rule, while recipients below the threshold have less severe conditions and are not prioritized in any way.

Let π be a priority structure in which no two recipients receive the same priority. Such a priority structure is called a *priority order* and can be modeled as a bijective function $\pi : N \to \{1, \ldots, n\}$, where a lower value

[13] Strict preference and indifference are denoted by $\succ$ and $\sim$, respectively.

indicates higher priority. For example, if $\pi(i) = 1$, then i is the recipient with highest priority. It should be noted that a higher priority need not represent a more HLA-sensitized recipient as recipients could, in principle, be prioritized in accordance with any criteria. Depending on how the threshold is set and how recipients above the threshold are prioritized, matchings will be selected from a variety of different classes of matchings.

4.1 Threshold matchings

First, suppose recipients above the threshold are prioritized in accordance with an adaptation of the *priority mechanism* introduced by Roth et al. (2005a). Let m be the number of recipients above the threshold plus one. In other words, there are $m-1$ recipients above the threshold. Let $\mathcal{E}_0^m := \mathcal{M}$ and let the set of *threshold matchings*, $\mathcal{M}_T^m$, be recursively defined by the following.

$$\text{For } t \in (0,m),\ \ \mathcal{E}_t^m = \begin{cases} \{\mu \in \mathcal{E}_{t-1}^m \mid \pi^{-1}(t) \in N^*(\mu)\} \text{ if } \pi^{-1}(t) \in N^*(\mu) \text{ for some } \mu \in \mathcal{E}_{t-1}^m, \\ \mathcal{E}_{t-1}^m \text{ otherwise.} \end{cases}$$

$$\mathcal{E}_m^m = \operatorname*{arg\,max}_{\mu \in \mathcal{E}_{m-1}^m} |N^*(\mu)|$$

$$\mathcal{M}_T^m = \operatorname*{arg\,max}_{\mu \in \mathcal{E}_m^m} |B(\mu)|$$

The first condition corresponds to the priority mechanism in Roth et al. (2005a). The second condition ensures that the number of recipients below the threshold receiving kidney transplants is maximized subject to the constraint that the recipients above the threshold are prioritized in accordance with the first condition. That is, first the recipients above the threshold are prioritized as prescribed by the priority mechanism, then as many recipients below the threshold as possible are matched. The third condition simply states that if there are multiple matchings satisfying the first and second conditions, the number of recipients assigned compatible donors should be maximized in order to minimize the use of desensitization.

It should be noted that the priority mechanism in Roth et al. (2005a) was introduced in a setting with only pairwise exchanges. When only pairwise exchanges are allowed, the KEP is conflict-proof. As a result, the only role of the priority mechanism in their model is to break ties between matchings that maximize the number of transplants. In the present setting with cyclic exchanges and chains, on the other hand, the KEP is not conflict-proof and the priority mechanism may reduce the number of transplants by prioritizing highly HLA-sensitized recipients.

Suppose that there are $m \in \{1, \ldots, n\}$ priority groups, that $|N_m| = n - (m-1)$ and that whenever $m \geq 2$, $N_t = \{\pi^{-1}(t)\}$ for all $t \in \{1, \ldots, m-1\}$. That is, each recipient i above the threshold belongs to her own priority group $N_{\pi(i)}$, while each recipient below the threshold is not prioritized and sorted into the last priority group N_m. Proposition 7 states that the set of priority group matchings reduces to the set of threshold matchings whenever recipients are sorted into priority groups in this way.

Proposition 7. Consider some priority order π and some $m \leq n$. Let $|N_m| = n - (m-1)$ and if $m \geq 2$, let $N_t = \{\pi^{-1}(t)\}$ for all $t < m$. Then $\mathcal{M}^* = \mathcal{M}_T^m$.

Threshold matchings can therefore be thought of as a subclass of priority group matchings corresponding to a particular approach to prioritization. There is a clear relationship between the choice of threshold and the number of recipients receiving transplants. Proposition 8 shows that the higher the threshold is, the (weakly) higher the total number of transplants will be.

Proposition 8. Consider some priority order π. For any $m \leq n$, any $m' \leq m$, any $\mu \in \mathcal{M}_T^m$ and any $\mu' \in \mathcal{M}_T^{m'}$, $|N^*(\mu')| \geq |N^*(\mu)|$.

This result is intuitive since a lower threshold implies that a higher number of recipients qualify for preferential treatment. A donor is a scarce resource that may be used more or less efficiently. A prioritized recipient may be assigned a donor that could otherwise have participated in a large cyclic exchange or chain. The lower the threshold is, the more likely it is that some prioritized recipients prevent certain cyclic exchanges and chains from being selected and thereby reduce the number of transplants.

These considerations have clear similarities with the use of triage in the event of mass casualty incidents. Treatments may be more costly and time consuming for certain recipients. If a doctor's goal is to treat as many patients as possible, they would focus on patients that can be treated quickly and easily. On the other hand, they may wish to treat patients with very severe conditions first, even if their treatment would consume a lot of time and resources and thereby reduce the total number of treated patients in any given time period. The doctor's problem of determining how severe conditions should be to justify giving patients preferential treatment corresponds to the problem facing a KEP when setting the threshold. The lower the requirements, the larger the number of patients that can potentially reduce the number of treated patients by consuming the doctor's time and resources.

Dickerson et al. (2012) discussed donors as resources that can be used more or less efficiently from a different perspective. They pointed out that some altruistic donors have higher potential usefulness than others (such as donors with blood group O) and suggested that, in dynamic settings, it may sometimes be beneficial to save more valuable altruistic donors until they can be used to form a long chain rather than using them right away in some exchange involving a smaller number of recipients.

4.2 Maximum matchings and priority matchings

For the two most prevalent approaches in the literature to managing the conflict between prioritization and transplant maximization, threshold matchings coincide with two classes of matchings that are commonly selected in KEPs that are and are not conflict-proof, respectively. Let $\mathcal{M}_{\max} \coloneqq \arg\max_{\mu \in \mathcal{M}} |N^*(\mu)|$ be the set of maximum matchings and let $\mathcal{M}_{\max}^{\varepsilon} \coloneqq \arg\max_{\mu \in \mathcal{M}_{\max}} |B(\mu)|$ be the set of *desensitization-minimizing maximum matchings*. Maximum matchings maximize the number of transplants, while desensitization-minimizing maximum matchings are the subset of maximum matchings that also minimize the use of desensitization. Currently, most European KEPs are not conflict-proof and select maximum matchings (Biró et al., 2021a). This means that they avoid the trade-off between prioritization and transplant maximization by making the former objective strictly subordinate to the latter objective.

The *priority matchings* introduced by Roth et al. (2005a) provide an alternative to maximum matchings by putting emphasis on prioritizing, e.g., highly HLA-sensitized recipients. Priority matchings are normally only proposed for conflict-proof KEPs and can be found using the priority mechanism discussed in Section 4.1. This mechanism first selects the set of matchings at which the highest priority recipient is matched (if non-empty), then it selects the subset of that set at which the second highest priority recipient is matched (if non-empty), and so on. The set of matchings that remains when the process terminates is the set of priority matchings. Formally, let $\mathcal{E}_0 \coloneqq \mathcal{M}$ and for each $t \in \{1, \dots, n\}$, let $\mathcal{E}_t$ be recursively defined by

$$\mathcal{E}_t = \begin{cases} \{\mu \in \mathcal{E}_{t-1} \mid \pi^{-1}(t) \in N^*(\mu)\} \text{ if } \pi^{-1}(t) \in N^*(\mu) \text{ for some } \mu \in \mathcal{E}_{t-1}, \\ \mathcal{E}_{t-1} \text{ otherwise.} \end{cases}$$

$\mathcal{E}_n$ is the set of priority matchings.[14] Roth et al. (2005a) introduced priority matchings in a setting without desensitization. This means that the distinction between half-compatible and compatible donors is not taken into account in the definition of priority matchings. As a consequence, priority matchings are no longer Pareto efficient when desensitization is introduced (Andersson & Kratz, 2020). The relationship between desensitization, preferences, and Pareto efficiency is explored further in Appendix A.

Andersson and Kratz (2020) showed that Pareto efficiency can be preserved by restricting attention to the subset of priority matchings that also minimize the use of desensitization, called *half-compatibility priority matchings*. Formally, let $\mathcal{E}_0 := \mathcal{M}$ and let the set of half-compatibility priority matchings $\mathcal{M}_B$ be defined recursively as follows.

$$\text{For } t \in \{1, \ldots, n\},\ \ \mathcal{E}_t = \begin{cases} \{\mu \in \mathcal{E}_{t-1} \mid \pi^{-1}(t) \in N^*(\mu)\} \text{ if } \pi^{-1}(t) \in N^*(\mu) \text{ for some } \mu \in \mathcal{E}_{t-1}, \\ \mathcal{E}_{t-1} \text{ otherwise.} \end{cases}$$

$$\mathcal{M}_B = \underset{\mu \in \mathcal{E}_n}{\arg\max} |B(\mu)|$$

So far, priority matchings have only been studied in conflict-proof KEPs that only permit pairwise exchanges. This implies that even though priority matchings put heavy emphasis on priorities, this never comes at the expense of the total number of transplants. Sönmez and Ünver (2014) point out that priority matchings may fail to maximize the number of transplants in settings with cyclic exchanges and therefore restrict attention to pairwise exchanges. However, there are KEPs (such as the CIAT and Spanish programs) that are not conflict-proof and actively balance prioritization and transplant maximization objectives. It is therefore meaningful to evaluate the properties of priority matchings in a more general setting with larger exchanges and chains.

The following result shows that both half-compatibility priority matchings and desensitization-minimizing maximum matchings are special cases of threshold matchings. Specifically, Proposition 9 states that threshold matchings reduce to desensitization-minimizing maximum matchings when the threshold is sufficiently high and to half-compatibility priority matchings when the threshold is sufficiently low.

Proposition 9. For any priority order π:

$$\mathcal{M}_T^m = \begin{cases} \mathcal{M}_{\max}^{\varepsilon} & \text{at } \ m = 1, \\ \mathcal{M}_B & \text{at } \ m \in \{n, n+1\}. \end{cases}$$

At $m = 1$, the threshold is set so high that no recipient qualifies for preferential treatment. In this case, all recipients are sorted into the same priority group and the primary objective of the planner is simply to maximize the number of transplants without prioritizing highly HLA-sensitized recipients. A sufficiently low threshold therefore corresponds to the common approach to managing the trade-off between prioritization and transplant-maximization in KEPs that are not conflict proof discussed above: the primary goals is to maximize the number of transplants and then, at most, use priorities to break ties. Conversely, at $m = n + 1$ and $m = n$, either all recipients or all recipients except one are prioritized. In both cases, each recipient is sorted into her own priority group. Recipients are then prioritized lexicographically in accordance with, e.g., their PRA scores, following the priority mechanism in Roth et al. (2005a). Since threshold matchings are priority group matchings, Proposition 9 implies half-compatibility priority matchings and desensitization-

[14]Priority matchings can also be defined in terms of a class of preference relations as in Okumura (2014). Andersson and Kratz (2020) show that these definitions are equivalent.

minimizing maximum matchings are Pareto efficient. This shows that Proposition 3 in Andersson and Kratz (2020), which states that half-compatibility priority matchings are Pareto efficient,[15] remains valid for KEPs that permit cyclic exchanges and chains and are consequently not conflict-proof. It is not obvious that the Pareto efficiency of half-compatibility priority matchings would carry over to more general settings, as they no longer maximize the number of transplants when cyclic exchanges and chains are introduced. Half-compatibility priority matchings are still maximal matchings, but there is no longer an equivalence between maximality and Pareto efficiency in settings with desensitization.

In settings without desensitization, desensitization-minimizing maximum matchings coincide with maximum matchings and half-compatibility priority matchings coincide with priority matchings. Consequently, the following result is a corollary of Proposition 9.

Corollary 1. Suppose that desensitization is not possible. Then, for any priority order π:

$$\mathcal{M}_T^m = \begin{cases} \mathcal{M}_{\max} & \text{at} \quad m = 1, \\ \mathcal{E}_n & \text{at} \quad m \in \{n, n+1\}. \end{cases}$$

This implies that, in settings without desensitization, threshold matchings correspond to maximum matchings at sufficiently high thresholds and priority matchings at sufficiently low thresholds. Priority matchings and maximum matchings are consequently Pareto efficient when desensitization is not permitted and preferences are dichotomous, confirming that the corresponding result in Roth et al. (2005a) carries over into more general settings. This result is not surprising since maximal matchings and Pareto efficient matchings coincide when preferences are dichotomous. While priority matchings are no longer maximum matchings in KEPs that are not conflict-proof, they are still maximal matchings and therefore Pareto efficient. However, Proposition 12 in Appendix A confirms that priority matchings are no longer Pareto efficient when desensitization is permitted.

Corollary 1 and Proposition 9 tell a similar story about maximum matchings. Proposition 6, Proposition 7 and Corollary 1 jointly imply that maximum matchings are Pareto efficient when recipient preferences are dichotomous. However, in settings with non-dichotomous preferences motivated by the use of desensitization, Proposition 12 implies that maximum matchings are no longer Pareto efficient. That is, priority matchings and maximum matchings both lose Pareto efficiency in settings with desensitization, and Pareto efficiency can be preserved in both cases by simply considering the subsets of the two classes of matchings that minimize the use of desensitization.

There is one more interesting special case to consider. Some authors, such as Roth et al. (2005b) and Sönmez and Ünver (2014), have advocated the inclusion of compatible recipient-donor pairs in KEPs where only pairwise exchanges are permitted. There is no desensitization in their settings, which means that a recipient is either incompatible or compatible with her own donor. Then a recipient with a compatible donor has no incentive beyond altruism to participate in a KEP, since she could receive a kidney from her own donor without participating or using desensitization.[16] Nevertheless, Roth et al. (2005b) and Sönmez and Ünver (2014) show that the number of transplants can be increased by including compatible recipient-donor pairs in a KEP selecting priority matchings.

KEPs that include all compatible recipient-donor pairs can be studied in the present model by letting

[15] The proposition also states that half-compatibility priority matchings are maximum matchings in their setting, which follows immediately from the fact that all maximal matchings are maximum matchings in settings where only pairwise exchanges are allowed. Formally, their Proposition 3 follows immediately from Propositions 1, 5, 7, 8, and 10.

[16] Unless given some external incentive, as in Sönmez et al. (2020).

all recipients with compatible or half-compatible donors be altruistic recipients. The matchings considered in Sönmez and Ünver (2014) are called *altruistically unbalanced priority matchings* and are gathered in the set $\mathcal{M}_U$. In their setting, a kidney exchange is only feasible if it involves at least one incompatible recipient-donor pair.[17] The model in this paper is general enough to contain the problem studied in Sönmez and Ünver (2014) as a special case. By designing the KEP as in their paper and by sorting each recipient into a unique priority group, it is straightforward to see that the set of threshold matchings reduces to the set of altruistically unbalanced priority matchings.

Corollary 2. Suppose that desensitization is not possible and that for each $i \in N$, $c_{i,i} = 2$ if and only if $i \in I_A$. Let $m \in \{n, n+1\}$, let $D = \emptyset$ and let K be the set of all 2-cycles involving at least one incompatible recipient-donor pair. Then $\mathcal{M}_T^m = \mathcal{M}_U$.

Specifically, suppose that desensitization is not possible, all recipients with incompatible donors are regular, all recipients with compatible donors are altruistic, there are no altruistic donors and only pairwise exchanges involving at least one incompatible pair are permitted. Then the set of threshold matchings and the set of altruistically unbalanced priority matchings coincide whenever the threshold is sufficiently low. In other words, altruistically unbalanced priority matchings are, just like maximum matchings and priority matchings, a special case of priority group matchings corresponding to a particular set of constraints imposed on the KEP and a particular approach to prioritizing recipients.

4.3 Simple threshold matchings

The recipients above the threshold do not necessarily need to be prioritized in accordance with the priority mechanism in Roth et al. (2005a). A simple alternative is to prioritize the recipients above the threshold equally, without regard for their relative positions in the priority order. This is the approach taken in the CIAT KEPs discussed in the introduction. First, the number of recipients above the threshold that receive transplants is maximized. Second, the number of recipients below the threshold that receive transplants is maximized. Finally, the use of desensitization is minimized. Formally, let $N_\Gamma \subseteq N$ be the set of recipients above the threshold. The set of *simple threshold matchings* $\mathcal{M}_S$ is then defined by the following.

$$\Gamma = \arg\max_{\mu \in \mathcal{M}} |N^*_\Gamma(\mu)|$$
$$\mathcal{M}_\Gamma = \arg\max_{\mu \in \Gamma} |N^*(\mu)|$$
$$\mathcal{M}_S = \arg\max_{\mu \in \mathcal{M}_\Gamma} |B(\mu)|$$

That is, simple threshold matchings are found by first maximizing the number of matched recipients above the threshold, then maximizing the number of matched recipients below the threshold and finally minimizing the use of desensitization. An interesting question is whether the reduction in the number of transplants caused by prioritizing some recipients would be lower if all recipients above the threshold were assigned equal weight in this way, rather than being weighted in accordance with the priority mechanism. While this could improve the number of transplants, the answer is, in general, no. Note that whenever $N_\Gamma = \{i \in N \mid \pi(i) < m\}$ for some $m \in \{1, \ldots, n\}$, the set of prioritized recipients is the same whether threshold matchings or simple threshold matchings are selected. The only difference is that the prioritized recipients are given equal

[17] Exchanges involving two compatible recipient-donor pairs are wasteful in settings with dichotomous preferences, since they are likely to introduce both delays and logistic complications.

priority rather than being ranked internally. The following example demonstrates that this approach could even reduce the total number of transplants.

Example 3. Let $N = \{1, 2, 3, 4, 5\}$, $N_\Gamma = \{1, 2, 3\}$, $K = \{(1, 4, 5), (2, 3)\}$ and $\pi(i) = i$ for all $i \in N$. Note that matchings in $\mathcal{M}_S$ and $\mathcal{M}_T^5$ prioritize the same set of recipients: N_Γ. In this case $\{(2, 3)\}$ is the unique matching in $\mathcal{M}_S$, resulting in a lower number of transplants than the unique matching in $\mathcal{M}_T^3$, $\{(1, 4, 5)\}$.

All else equal, simple threshold matchings will always match a weakly higher number of recipients *above the threshold* than threshold matchings. However, as the example above demonstrated, the total number of transplants may be lower. Whenever the objective is to help as many prioritized recipients as possible to receive transplants, simple threshold matchings are preferable to threshold matchings. Furthermore, while it has been shown in Proposition 8 that a higher threshold is associated with a higher number of transplants when considering threshold matching, the same conclusion does not hold for simple threshold matchings. A higher threshold could either result in a higher or a lower number of transplants in a KEP that selects simple threshold matchings. To see this, consider the following examples.

Example 4. Let $N = \{1, 2, 3, 4\}$, $N_\Gamma = \{i \in N \mid \pi(i) < m\}$, $K = \{(1, 3), (2, 3, 4)\}$, and $\pi(i) = i$ for all $i \in N$. At a low threshold with $m = 3$, $\{(2, 3, 4)\}$ is the unique matching in $\mathcal{M}_S$. At a higher threshold with $m = 2$, $\{(1, 3)\}$ is the unique matching in $\mathcal{M}_S$. Thus, a higher threshold results in a lower number of transplants.

Example 5. Let $N = \{1, 2, 3, 4\}$, $N_\Gamma = \{i \in N \mid \pi(i) < m\}$, $K = \{(1, 2), (1, 3, 4)\}$, and $\pi(i) = i$ for all $i \in N$. At a low threshold with $m = 3$, $\{(1, 2)\}$ is the unique matching in $\mathcal{M}_S$. At a higher threshold with $m = 2$, $\{(1, 3, 4)\}$ is the unique matching in $\mathcal{M}_S$. Thus, a higher threshold results in a higher number of transplants.

Proposition 10 shows that simple threshold matchings are a special case of priority group matchings corresponding to a particular approach to prioritization. In particular, if recipients are sorted into two priority groups, then the set of priority group matchings reduces to the set of simple threshold matchings.

Proposition 10. Consider some priority order π, let $m = 2$ and set $N_\Gamma = N_1$. Then $\mathcal{M}^* = \mathcal{M}_S$.

This implies that simple threshold matchings are Pareto efficient. It should be noted that this result relies on the assumption that the KEP can make full use of available desensitization opportunities. For example, the simple threshold matchings selected by CIAT may not be Pareto efficient since desensitization opportunities are awarded as a privilege for a subset of recipients. Proposition 6 and Proposition 10 jointly suggest that this could be rectified by giving equal access to desensitization treatments, which would guarantee Pareto efficiency while implementing the same mechanism as before.

Finally, a related class of matchings has been studied by Dickerson and Sandholm (2014) and Dickerson et al. (2014). They set up a kidney exchange problem as an integer programming problem, starting out with an objective function assigning equal weight to all recipients. This objective function is maximized whenever the number of transplants is maximized. The objective function is then amended by scaling up the weight assigned to certain "marginalized" recipients (e.g., children or highly HLA-sensitized recipients) by a factor of $(1 + \beta)$. Solutions to the resulting integer programming problem are called MaxCard-Fair solutions. Proposition 11 shows that for sufficiently large values of β, the set of MaxCard-Fair solutions coincides with $\mathcal{M}_\Gamma$.

Proposition 11. Sort all "marginalized" recipients into N_Γ and let $\beta > n - |N_\Gamma| - 1$. Then $\mathcal{M}_\Gamma$ is the set of *MaxCard-Fair* solutions.

This means that, in settings without desensitization, the set of simple threshold matchings and the *MaxCard-Fair* solutions coincide whenever β is large enough.

5 Concluding remarks

This paper identifies the conditions under which prioritizing certain recipients and maximizing the number of transplants are conflicting objectives, expanding upon several observations in the literature. For example, KEPs that only permit pairwise exchanges are conflict-proof, while KEPs that permit 3-way exchanges are not. These observations could be used to justify restricting attention to pairwise exchanges. However, the results in this paper indicate that it is possible for a KEP to remain conflict-proof while still permitting *some* cyclic exchanges and chains, as long as its PSC structure is preserved. For example, it is possible for a KEP to permit short chains and 3-way exchanges with sufficiently many embedded 2-way exchanges and still remain immune to trade-offs between prioritization and transplant maximization. There are potential welfare gains associated with such exchanges that could be evaluated experimentally in future research.

While KEPs that only permit pairwise exchanges are conflict-proof, many KEPs permit chains and longer cyclic exchanges. However, most programs that are not conflict-proof have prioritization objectives that are strictly subordinate to transplant maximization objectives, thereby avoiding situations where a matching involving fewer recipients has to be selected in order to prioritize, e.g., highly HLA-sensitized recipients. There are, however, recent examples of KEPs that permit trade-offs of this kind, such as the CIAT and the Spanish KEPs. If these innovations are deemed successful, other KEPs may follow suit and begin experimenting with matching algorithms that prioritize various groups of recipients in similar ways. It is therefore important to have tools capable of keeping up with such developments. The priority group matchings introduced in this paper are guaranteed to be Pareto efficient regardless of whether the KEP is conflict-proof and regardless of how different recipients are prioritized. This was demonstrated in a model general enough to encompass a wide variety of KEPs facing different constraints. In the case of the CIAT KEP, the results in this paper imply that their current implementation is not Pareto efficient. However, this could be rectified by expanding the use of desensitization while still implementing the same mechanism.

This paper not only sheds some light on the shift in the literature from priority matchings to maximum matchings when cyclic exchanges and chains were introduced, similar developments have also been observed in practice. The Swedish KEP (2016–2019) only focused on pairwise exchanges[18] and used a mechanism (introduced in Andersson and Kratz (2020)) that selected half-compatibility priority matchings (Biró et al., 2018). The Swedish program has since been replaced by the international Scandiatransplant program (STEP). In line with the developments in the literature described above, STEP abandoned the mechanism selecting half-compatibility priority matchings when introducing cyclic exchanges involving three or more recipient-donor pairs (Scandiatransplant, 2019). Similarly, the primary objective of most European KEPs that permit cyclic exchanges or chains is to maximize the number of transplants (Biró et al., 2021a).

While not yet a widespread practice, kidneys from deceased donors can be used to initiate chains. In fact, Italy has had a deceased-kidney paired donation program that allows the use of deceased donor kidneys for this purpose since 2019 (Furian et al., 2019, 2020). Furthermore, Combe et al. (2019) identified welfare gains associated with the possibility of introducing deceased donor-initiated chains in the French KEP. However, kidneys from deceased donors exhibit lower graft survival and function (Bos et al., 2007), which raises the question to what extent recipients would be willing to exchange a living donor for a deceased donor. The model introduced in this paper is particularly well suited for taking such considerations into account by letting the selection of chains initiated with a deceased donor be conditional on the priority group membership of the first recipient in the chain. Receiving a kidney from a brain dead donor would presumably be more

[18] Though an ad hoc cyclic exchange involving three pairs was carried out in 2018 (Njurförbundet, 2018).

acceptable to highly HLA-sensitized recipients or recipients with particularly severe or urgent conditions.

The preference domain is an expansion of the dichotomous preferences in Roth et al. (2005a). It follows the approach in Andersson and Kratz (2020), assuming that recipients have a preference for avoiding desensitization. However, it is important to stress that there are other reasonable ways to model non-dichotomous recipient preferences. While most of the results in this paper are independent of recipient preferences, how the results relating to Pareto efficiency would translate into other preference domains, such as the preference domains of Nicolò and Rodríguez-Álvarez (2012) and Biró et al. (2021b), is still an open question. Transplant quality is affected by many factors, such as the age of the donor (Nicolò & Rodríguez-Álvarez, 2017), and it is not clear how recipient preferences should weigh, e.g., a younger half-compatible donor against an older fully compatible donor.

Finally, it should be noted that while prioritization in KEPs that are not conflict-proof can be motivated using dynamic arguments, this paper still models the kidney exchange problem as a static problem. A static model is sufficient for the purposes of this paper, since any knowledge that may be available on the likely distribution of future arrivals can be taken into account through the KEP's approach to prioritization. Precisely how knowledge of future recipient-donor pair distributions ought to be taken into account is left to be answered in future research.

Evaluating the welfare impact of different prioritization approaches in KEPs that are not conflict-proof would not only require answers to a number of ethical and normative questions, but also a dynamic model. The optimal prioritization approach could either be studied as an optimal control problem as in Ünver (2010) and Akbarpour et al. (2020) or by means of simulations in a dynamic setting as in Dickerson et al. (2012). An adaptation of the integer linear program introduced in Aziz et al. (2021), which both distinguishes compatible and half-compatible recipients and allows for cycle length constraints could serve as the basis for such simulations.

A Desensitization and Pareto efficiency

In settings with desensitization, it is reasonable to assume that recipients not only care about whether they receive a kidney transplant, but also from whom they receive a transplant as this will determine the need for desensitization. This expansion of the preference domain affects the structure of Pareto efficient matchings. Denote the recipient or altruistic donor that recipient i is matched to at matching $\mu \in \mathcal{M}$ by $\mu(i)$. For notational convenience, let $\mu(i) = \emptyset$ whenever i is unmatched at μ and define $c_{i,\emptyset} = 0$ for all $i \in N$. Consider two arbitrary matchings $\mu, \nu \in \mathcal{M}$ and let each recipient $i \in N$ have preferences $\succsim_i$ over matchings, defined by the following.

$$\begin{aligned}
\text{For } i \in I_R, \quad & \mu \succ_i \nu \iff \begin{cases} \mu(i) \neq i \text{ and } c_{i,\mu(i)} > c_{i,\nu(i)}, \\ \mu(i) = i \neq \nu(i) \text{ and } c_{i,\mu(i)} \geq c_{i,\nu(i)}, \end{cases} \\
& \mu \sim_i \nu \iff \begin{cases} \mu(i) = \nu(i), \\ \mu(i) \neq i, \nu(i) \neq i \text{ and } c_{i,\mu(i)} = c_{i,\nu(i)}. \end{cases} \\
\text{For } i \in I_A, \quad & \mu \succ_i \nu \iff c_{i,\mu(i)} > c_{i,\nu(i)}, \\
& \mu \sim_i \nu \iff c_{i,\mu(i)} = c_{i,\nu(i)}.
\end{aligned}$$

Their preferences over matchings can be thought to reflect their underlying preferences over donors. That is, all recipients prefer compatible donors to half-compatible donors and half-compatible donors to incompatible donors. Recipients in I_R differ from recipients in I_A by, all else equal, having a preference for their own donors. A matching $\mu \in \mathcal{M}$ is *Pareto efficient* if there exists no matching $\nu \in \mathcal{M}$ such that $\nu \succsim_i \mu$ for all $i \in N$ and $\nu \succ_i \mu$ for some $i \in N$.

If desensitization is not possible, recipients will never receive transplants from half-compatible donors and all transplants will involve recipients and donors that are both blood group and tissue type compatible. In that case, the non-dichotomous preference domain in this paper reduces to the standard dichotomous preference domain introduced in Roth et al. (2005a), in which the only concern of recipients is whether or not they receive transplants. They can only receive compatible kidneys through kidney exchange and they are indifferent between all compatible donors.[19] In other words, whenever desensitization is not a possibility, all recipients will have dichotomous preferences over all donors they could feasibly be assigned within the KEP.

In settings with dichotomous preferences, a matching is Pareto efficient if and only if it is a maximal matching. However, with the introduction of desensitization, preferences are no longer dichotomous, maximal matchings are no longer guaranteed to be Pareto efficient and Pareto efficient matchings are no longer guaranteed to be maximal. The following characterization of Pareto efficient matchings in settings with desensitization highlights this change.

Proposition 12. A matching $\mu \in \mathcal{M}$ is Pareto efficient if and only if the following two conditions hold for all $\nu \in \mathcal{M}$:

(a) $N^*(\mu) \subset N^*(\nu) \implies B(\mu) \not\subseteq B(\nu)$

[19] This statement does not apply to regular recipients with compatible donors, but such recipients would never be included in the recipient-donor pool since they would receive kidney transplants from their own donors outside the KEP.

(b) $B(\mu) \subset B(\nu) \implies N^*(\mu) \nsubseteq N^*(\nu)$

A Pareto efficient matching will always exist as long as there is at least one matching. Proposition 12 implies that a Pareto efficient matching μ need not be maximal in terms of $N^*(\mu)$ or $B(\mu)$. Furthermore, a matching μ that is maximal in terms of $N^*(\mu)$ or $B(\mu)$ need not be Pareto efficient. A matching μ that is maximal in terms of both $N^*(\mu)$ and $B(\mu)$ is Pareto efficient, but such a matching may not exist. The proposition shows that, at a Pareto efficient matching μ, it is impossible to match an additional recipient or reduce the use of desensitization without either preventing some other recipient (who is matched at μ) from receiving a transplant or forcing some recipient (with a fully compatible donor at μ) to undergo desensitization treatments. Note that in settings without desensitization, $N^*(\mu) = B(\mu)$ for all $\mu \in \mathcal{M}$. This means that a matching μ is maximal in terms of $N^*(\mu)$ if and only if it is maximal in terms of $B(\mu)$. When desensitization is not possible, Proposition 12 thereby reduces to the statement that a matching is Pareto efficient if and only if it is a maximal matching. In other words, Proposition 12 generalizes the observation in Roth et al. (2005a) that maximal matchings and Pareto efficient matchings coincide in settings with dichotomous preferences. It explains why various subclasses of maximal matchings such as the priority matchings studied by Roth et al. (2005a) fail to satisfy Pareto efficiency in settings with desensitization, and why Pareto efficiency can be attained by first maximizing the number of transplants and then minimizing the use of desensitization, as in Andersson and Kratz (2020).

B Proofs

This appendix contains all proofs. The propositions are restated for convenience.

Lemma 1. An abstract simplicial complex satisfies the relaxed exchange property if and only if all of its facets have the same cardinality.

Proof. Consider an abstract simplicial complex $\mathcal{S}$ containing subsets of some set N. Let $\mathcal{S}_F := \{I \in \mathcal{S} \mid \forall J \in \mathcal{S}, I \not\subset J\}$ denote its facets and let $\mathcal{S}_{MC} := \{I \in \mathcal{S} \mid \forall J \in \mathcal{S}, |I| \geq |J|\}$ denote the subset of its faces that have maximum cardinality.

($\Rightarrow$) First, assume that $\mathcal{S}$ satisfies the relaxed exchange property and that there exist $I, J \in \mathcal{S}_F$ such that $|J| < |I|$ to reach a contradiction. By the relaxed exchange property, there exists some $i \in N \setminus J$ such that $J \cup \{i\} \in \mathcal{S}$. This contradicts the assumption that $J \in \mathcal{S}_F$.

($\Leftarrow$) Next, assume that all facets have the same cardinality and that $\mathcal{S}$ does not satisfy the relaxed exchange property to reach a contradiction. Note that if all facets have the same cardinality, then $\mathcal{S}_F = \mathcal{S}_{MC}$. Since the relaxed echange priority is violated, there exists some $I, J \in \mathcal{S}$ such that $|J| < |I|$ while there is no $i \in N \setminus J$ such that $J \cup \{i\} \in \mathcal{S}$. This implies that $J \in \mathcal{S}_F$. By definition, there must exist some $I' \in \mathcal{S}$ such that $I \subseteq I'$ and $I' \in \mathcal{S}_F$. Since, $|J| < |I'|$, this contradicts the assumption that all facets of $\mathcal{S}$ have the same cardinality. $\square$

Lemma 2. For any problem $\langle N, D, C \rangle$, the set of maximal matchings equals the set of maximum matchings if and only if $\mathcal{S}$ is a pure simplicial complex.

Proof. Consider some problem $\langle N, D, C \rangle$, let $\mathcal{M}_{\text{mal}} := \{\mu \in \mathcal{M} \mid N^*(\mu) \not\subset N^*(\nu) \text{ for all } \nu \in \mathcal{M}\}$ denote the set of maximal matchings and let $\mathcal{M}_{\max} := \arg\max_{\mu \in \mathcal{M}} |N^*(\mu)|$ denote the set of maximum matchings.

(⇒) First, assume that $\mathcal{M}_{\max} = \mathcal{M}_{\text{mal}}$ and that $\mathcal{S}$ is not a pure simplicial complex to reach a contradiction. Then there exist two facets of $\mathcal{S}$, I and J, such that $|J| < |I|$. By the definition of $\mathcal{S}$, $I \subseteq N^*(\mu)$ for some $\mu \in \mathcal{M}$ and $J \subseteq N^*(\nu)$ for some $\nu \in \mathcal{M}$. Suppose that $J \subset N^*(\nu)$. Then there exists some $i \in N^*(\nu) \setminus J$, implying that $J \cup \{i\}$ is also a face of $\mathcal{S}$. Since $|J| < |J \cup \{i\}|$, this contradicts the assumption that J is a facet of $\mathcal{S}$. Thus, $J = N^*(\nu)$. Since J is a facet of $\mathcal{S}$, there exists no face J' of $\mathcal{S}$ such that $J \subset J'$. This implies that ν is a maximal matching. However, since $|J| < |I|$, $|N^*(\nu)| < |N^*(\mu)|$ and $\nu \in \mathcal{M}_{\text{mal}} \setminus \mathcal{M}_{\max}$. This contradicts $\mathcal{M}_{\max} = \mathcal{M}_{\text{mal}}$.

(⇐) Next, assume that $\mathcal{S}$ is a pure simplicial complex and that $\mathcal{M}_{\max} \neq \mathcal{M}_{\text{mal}}$ to reach a contradiction. Since every maximum matching is a maximal matching, it must be the case that there exists some $\mu \in \mathcal{M}_{\text{mal}} \setminus \mathcal{M}_{\max}$. Let $\nu \in \mathcal{M}_{\max}$ and note that there is some facet $J \in \mathcal{S}$ such that $J = N^*(\nu)$ and some facet $I \in \mathcal{S}$ such that $I = N^*(\mu)$. Since $\mu \in \mathcal{M}_{\max}$ and $\nu \notin \mathcal{M}_{\max}$, $|J| < |I|$. Thus, not all facets of $\mathcal{S}$ have the same cardinality, contradicting the assumption that $\mathcal{S}$ is a pure simplicial complex. □

Lemma 3. $|N_t^*(\mu)| = |N_t^*(\mu')|$ for all $\mu, \mu' \in \mathcal{M}^*$ and all $t \leq m$.

Proof. Consider some $\mu, \mu' \in \mathcal{M}^*$ assume that $|N_t^*(\mu)| \neq |N_t^*(\mu')|$ for some $t \leq m$ to reach a contradiction. Let $t' \leq m$ be the lowest integer at which $|N_{t'}^*(\mu)| \neq |N_{t'}^*(\mu')|$. Without loss of generality, assume that $|N_{t'}^*(\mu)| > |N_{t'}^*(\mu')|$. Then $|N_{t'}^*(\mu)| > |N_{t'}^*(\mu')|$ for some $t \in \{1, \ldots, m\}$ and $|N_\tau^*(\mu)| = |N_\tau^*(\mu')|$ for all $\tau \in \{0, \ldots, t-1\}$. Thus, $\mu \succ \mu'$ by definition. This contradicts $\mu' \in \mathcal{M}^*$. □

Proposition 1. A kidney exchange program is conflict-proof if and only if it has a PSC structure.

Proof. (⇒) Suppose a KEP does not have a PSC structure. Then there exists some problem $\langle N, D, C\rangle \in \mathscr{P}$ such that $\mathcal{S}$ is not a pure simplicial complex. Let $\mathcal{M}_{\text{mal}} \coloneqq \{\mu \in \mathcal{M} \mid N^*(\mu) \not\subset N^*(\nu) \text{ for all } \nu \in \mathcal{M}\}$ denote the set of maximal matchings and let $\mathcal{M}_{\max} \coloneqq \arg\max_{\mu \in \mathcal{M}} |N^*(\mu)|$ denote the set of maximum matchings. Since $\mathcal{S}$ is not a pure simplicial complex, Lemma 2 implies that $\mathcal{M}_{\text{mal}} \neq \mathcal{M}_{\max}$. Furthermore, since every maximum matching is a maximal matching, $\mathcal{M}_{\text{mal}} \neq \mathcal{M}_{\max}$ implies that there exists some $\mu \in \mathcal{M}_{\text{mal}} \setminus \mathcal{M}_{\max}$. Consider a priority structure $\hat{\pi}$ such that $\hat{\pi}(i) < \hat{\pi}(j)$ for all $i \in N^*(\mu)$ and all $j \in \mathscr{N} \setminus N^*(\mu)$. Since $\mu \in \mathcal{M}_{\text{mal}}$, there is no $\nu \in \mathcal{M}_{\max}$ such that $N^*(\mu) \subseteq N^*(\nu)$. Thus, the KEP is not conflict-proof.

(⇐) Next, suppose a KEP is not conflict-proof. Then there exists some problem $\langle N, D, C\rangle \in \mathscr{P}$ at which there is a trade-off between matching some set of high priority recipients $I \subset N$ and maximizing the number of transplants. Furthermore, assume that the KEP has a PSC structure to reach a contradiction. First note that it must be possible to match all recipients in I, i.e., $I \in \mathcal{S}$. Then there must exist some matching $\mu \in \mathcal{M}$ such that $I \subseteq N^*(\mu)$. By definition, there also exists some maximal matching ν such that $I \subseteq N^*(\nu)$. Since $\mathcal{S}$ is a pure simplicial complex, Lemma 2 implies that ν is a maximum matching. Hence, it is possible to match all recipients in I and maximize the number of transplants. This contradicts the observation that there is a trade-off between matching the recipients in I and maximizing the number of transplants.

Thus, every conflict-proof KEP has a PSC structure (⇒) and every KEP that has a PSC structure is conflict-proof (⇐). □

Proposition 2. For any $q \geq 3$, a kidney exchange program is not conflict-proof if it permits all q-cycles.

Proof. Consider a problem $\langle N, D, C\rangle$ in which $N = \{1, 2, \ldots, 3q-2\}$ and $K = \{k_1, k_2, k_3\}$. Let $k_1 = (1, 2, \ldots, q)$, $k_2 = (q+1, \ldots, 2q-1, 1)$, and $k_3 = (2q, \ldots, 3q-2, 2)$. The notation $(x, \ldots, y, z)$ is used to indicate that the cycle involves recipient i for every integer between x and y. Each $k \in K$ represents a q-cycle. Note that there are four matchings, $\mu_1 = \{k_1\}$, $\mu_2 = \{k_2\}$, $\mu_3 = \{k_3\}$, and $\mu_4 = \{k_2, k_3\}$. Assume that the

KEP is conflict-proof to reach a contradiction. By Proposition 1, the KEP has a PSC structure and $\mathcal{S}$ is a pure simplicial complex. By definition, $N^*(\mu_1), N^*(\mu_4) \in \mathcal{S}$. Furthermore, $|N^*(\mu_1)| = q < |N^*(\mu_4)| = 2q$. By Lemma 1, $\mathcal{S}$ satisfies the relaxed exchange property and there must exist some $i' \in N \setminus N^*(\mu_1)$ such that $N^*(\mu_1) \cup \{i'\} \in \mathcal{S}$. Some matching $\nu \in \mathcal{M}$ such that $N^*(\mu_1) \cup \{i'\} \subseteq N^*(\nu)$ must therefore exist. First note that $\nu \neq \mu_1$. Since $\mathcal{M} = \{\mu_1, \mu_2, \mu_3, \mu_4\}$, $\mu \in \{\mu_2, \mu_3, \mu_4\}$. However, since $q \in N^*(\mu_1) \setminus N^*(\mu)$ for all $\mu \in \{\mu_2, \mu_3, \mu_4\}$ there exists no $\mu \in \{\mu_2, \mu_3, \mu_4\}$ such that $N^*(\mu_1) \cup \{i'\} \subseteq N^*(\nu)$. This contradicts $\nu \in \mathcal{M}$. □

Proposition 3. For any $q \geq 2$, a kidney exchange program is not conflict-proof if it permits all chains involving at most q recipients.

Proof. Consider a problem $\langle N, D, C\rangle$ in which $N = \{1, 2, 3\}$ and $D = \{d\}$, and $K = \{(3, d), (1, 2, d)\}$. Note that there are two matchings, $\mu_1 = \{(3, q)\}$ and $\mu_2 = \{(1, 2, d)\}$. Furthermore, note that $\mu_1, \mu_2 \in \mathcal{M}$ for any KEP that permits all chains involving at most q recipients, where $q \geq 2$. Assume that the KEP is conflict-proof to reach a contradiction. By Proposition 1, the KEP has a PSC structure and $\mathcal{S}$ is a pure simplicial complex. By definition, $N^*(\mu_1), N^*(\mu_2) \in \mathcal{S}$. Furthermore, $|N^*(\mu_1)| = 1 < |N^*(\mu_4)| = 2$. By Lemma 1, $\mathcal{S}$ satisfies the relaxed exchange property. Thus, there exists some $i' \in N \setminus N^*(\mu_1)$ such that $N^*(\mu_1) \cup \{i'\} \in \mathcal{S}$. Some matching $\nu \in \mathcal{M}$ such that $N^*(\mu_1) \cup \{i'\} \subseteq N^*(\nu)$ must therefore exist. This is a contradiction since $3 \in N^*(\mu_1) \setminus N^*(\mu_2)$ and $\{\mu_1\} \cup \{\mu_2\} = \mathcal{M}$. □

Proposition 4. Kidney exchange programs that only permit pairwise exchanges and short chains have a matroid structure.

Proof. Consider a KEP that does not have a matroid structure to reach a contradiction. Then there must exist some problem $\langle N, D, C\rangle$ in which $\mathcal{S}$ is not a matroid. Let μ and μ' be two matchings such that $|N^*(\mu)| < |N^*(\mu')|$. The exchange property must be violated since $\mathcal{S}$ is not a matroid. Thus, there exists no recipient $i \in N^*(\mu') \setminus N^*(\mu)$ such that $N^*(\mu) \cup \{i\} \in \mathcal{S}$. Since $\mathcal{S}$ is an abstract simplicial complex, it follows that $N^*(\mu) \nsubseteq N^*(\mu')$. Furthermore, since $|N^*(\mu)| < |N^*(\mu')|$, this implies that $|N^*(\mu) \setminus N^*(\mu')| < |N^*(\mu') \setminus N^*(\mu)|$. Hence, $N^*(\mu') \setminus N^*(\mu) \neq \emptyset$.

Consider some recipient $i_1 \in N^*(\mu') \setminus N^*(\mu)$. Then the exchange $k' \in \mu'$ that i_1 is involved in must also involve some $i_2 \in N^*(\mu)$. Otherwise, $\mu \cup \{k'\} \in \mathcal{M}$, which would contradict the non-existence of a recipient $i \in N^*(\mu') \setminus N^*(\mu)$ such that $N^*(\mu) \cup \{i\} \in \mathcal{S}$. Furthermore, since i_2 is matched at μ, there must exist some exchange $k = (i_2, i_3) \in \mu$ for some recipient $i_3 \in N^*(\mu) \setminus \{i_2\}$. To see why, note that if not, then $k = (i_2)$ or $k = (i_2, d)$ for some $d \in D$. In both cases, $(\mu \setminus \{k\}) \cup \{k'\} \in \mathcal{M}$, which contradicts the non-existence of a recipient $i \in N^*(\mu') \setminus N^*(\mu)$ such that $N^*(\mu) \cup \{i\} \in \mathcal{S}$.

Any exchange $k'_2 \in \mu'$ that involves i_3 must involve another recipient $i_4 \in N^*(\mu)$. If $k'_2 = (i_3)$ or $k'_2 = (i_3, d)$ for some $d \in D$, then $(\mu \setminus \{(i_2, i_3)\}) \cup \{(i_1, i_2), k'_2\} \in \mathcal{M}$. Furthermore, if $k'_2 = (i_3, i_4)$ for some recipient $i_4 \in N^*(\mu') \setminus N^*(\mu)$, then $(\mu \setminus \{(i_2, i_3)\}) \cup \{(i_1, i_2), (i_3, i_4)\} \in \mathcal{M}$. All these cases lead to the same contradiction as above.

Thus, either (a) $i_3 \in N^*(\mu) \setminus N^*(\mu')$ or (b) there is an exchange $k'_2 = (i_3, i_4) \in \mu'$ for some $i_4 \in N^*(\mu)$. In case (b), the argument above can be repeated as follows. Since $i_4 \in N^*(\mu)$, there is some exchange $k_3 \in \mu$ that involves i_4. If $k_3 = (i_4)$ or $k_3 = (i_4, d)$, then $(\mu \setminus \{k, k_3\}) \cup \{(i_1, i_2), (i_3, i_4)\} \in \mathcal{M}$, leading to the same contradiction as above. Thus, there exists an exchange $k_3 = (i_4, i_5) \in \mu$ for some recipient $i_5 \in N^*(\mu) \setminus \{i_4\}$. Continuing the same argument, any exchange $k'_3 \in \mu'$ that involves i_5 must involve another recipient $i_6 \in N^*(\mu)$. If $k'_3 = (i_5)$ or $k'_3 = (i_5, d)$ for some $d \in D$, then $(\mu \setminus \{(i_2, i_3), (i_4, i_5)\}) \cup$

$\{(i_1, i_2), (i_3, i_4), k'_3\} \in \mathcal{M}$. Furthermore, if $k'_3 = (i_5, i_6)$ for some recipient $i_6 \in N^*(\mu') \setminus N^*(\mu)$, then $(\mu \setminus \{(i_2, i_3), (i_4, i_5)\}) \cup \{(i_1, i_2), (i_3, i_4), (i_5, i_6)\} \in \mathcal{M}$. All these cases lead to the same contradiction as above.

Again, either (a') $i_5 \in N^*(\mu) \setminus N^*(\mu')$ or (b') there is an exchange $k'_3 = (i_5, i_6) \in \mu'$ for some $i_6 \in N^*(\mu)$. Note repeating this argument produces a sequence of recipients in $N^*(\mu) \cup N^*(\mu')$ that continues until reaching some recipient $i \in N^*(\mu) \setminus N^*(\mu')$. Since i_1 is an arbitrary recipient in $N^*(\mu') \setminus N^*(\mu)$, this implies that each recipient in $i \in N^*(\mu') \setminus N^*(\mu)$ has a unique corresponding recipient in $i^* \in N^*(\mu) \setminus N^*(\mu')$. Thus, $|N^*(\mu) \setminus N^*(\mu')| > |N^*(\mu') \setminus N^*(\mu)|$, which contradicts the observation above that $|N^*(\mu) \setminus N^*(\mu')| < |N^*(\mu') \setminus N^*(\mu)|$. This implies that the exchange property holds. In conclusion, there exists no problem $\langle N, D, C\rangle$ in which $\mathcal{S}$ is a not matroid. □

Proposition 5. A kidney exchange program that only permits pairwise exchanges and 3-cycles with at least q embedded 2-cycles is conflict-proof if and only if $q = 3$.

Proof. ($\Leftarrow$) First, let $q = 3$. Consider a KEP that does not have a matroid structure to reach a contradiction. Then there must exist some problem $\langle N, D, C\rangle$ in which $\mathcal{S}$ is not a matroid. Let μ and μ' be two matchings such that $|N^*(\mu)| < |N^*(\mu')|$. Since $\mathcal{S}$ is not a matroid, the exchange property must be violated. Thus, there exists no recipient $i \in N^*(\mu') \setminus N^*(\mu)$ such that $N^*(\mu) \cup \{i\} \in \mathcal{S}$. Since $\mathcal{S}$ is an abstract simplicial complex, it follows that $N^*(\mu) \not\subseteq N^*(\mu')$. Furthermore, since $|N^*(\mu)| < |N^*(\mu')|$, this implies that $|N^*(\mu) \setminus N^*(\mu')| < |N^*(\mu') \setminus N^*(\mu)|$. Hence, $N^*(\mu') \setminus N^*(\mu) \neq \emptyset$.

Consider some recipient $i_1 \in N^*(\mu') \setminus N^*(\mu)$. Then the exchange $k' \in \mu'$ that i_1 is involved in must also involve some $i_2 \in N^*(\mu)$. Otherwise, $\mu \cup \{k'\} \in \mathcal{M}$, which would contradict the non-existence of a recipient $i \in N^*(\mu') \setminus N^*(\mu)$ such that $N^*(\mu) \cup \{i\} \in \mathcal{S}$. Furthermore, the exchange $k \in \mu$ that i_2 is involved in must also involve at least one recipient $i_3 \in N^*(\mu) \setminus \{i_2\}$. To see why, first note that if not, then $k = (i_2)$. If k' is a 2-cycle, then $(\mu \setminus \{k\}) \cup \{k'\} \in \mathcal{M}$. If k' is a 3-cycle, then it contains the embedded 2-cycle (i_1, i_2) and $(\mu \setminus \{k\}) \cup \{(i_1, i_2)\} \in \mathcal{M}$. Both cases contradict the non-existence of a recipient $i \in N^*(\mu') \setminus N^*(\mu)$ such that $N^*(\mu) \cup \{i\} \in \mathcal{S}$.

Now suppose k is a 3-cycle, involving recipients i_2, i_3, and i_4. Then k contains the embedded 2-cycle (i_3, i_4) and $(\mu \setminus \{k\}) \cup \{(i_1, i_2), (i_3, i_4)\} \in \mathcal{M}$. This again contradicts the non-existence of a recipient $i \in N^*(\mu') \setminus N^*(\mu)$ such that $N^*(\mu) \cup \{i\} \in \mathcal{S}$. Thus, $k = (i_2, i_3)$.

Any exchange $k'_2 \in \mu'$ that involves i_3 must involve some recipient $i_4 \in N^*(\mu)$ and no recipient in $N^*(\mu') \setminus N^*(\mu)$. If $k'_2 = (i_3)$, then $(\mu \setminus \{(i_2, i_3)\}) \cup \{(i_1, i_2), (i_3)\} \in \mathcal{M}$, resulting in the same contradiction as above. If k'_2 involves some recipient $i'_4 \in N^*(\mu') \setminus N^*(\mu)$, then either k'_2 equals (i_3, i'_4) or contains the embedded 2-cycle (i_3, i'_4). In both cases, $(\mu \setminus \{(i_2, i_3)\}) \cup \{(i_1, i_2), (i_3, i'_4)\} \in \mathcal{M}$, leading to the same contradiction as above.

Thus, either (a) $i_3 \in N^*(\mu) \setminus N^*(\mu')$ or (b) i_3 is involved in some exchange $k'_2 \in \mu'$ that only involves recipients in $N^*(\mu)$. In case (b), repeat the argument above for each recipient i_5 involved in k'_2 to show the following. Let k_3 be a cycle in μ that involves i_5. Then for any $i_6 \neq i_5$ involved in k_3, either (a') $i_6 \in N^*(\mu) \setminus N^*(\mu')$ or (b') i_6 is involved in some exchange $k'_3 \in \mu'$ that only involves recipients in $N^*(\mu)$. To see this, first note that if k_3 is a 3-cycle involving i_5, i_6, and some recipient i'_5, then $(\mu \setminus \{(i_2, i_3), k_3\}) \cup \{(i_1, i_2), (i_3, i_5), (i'_5, i_6)\} \in \mathcal{M}$ resulting in the same contradiction as above. Thus, $k_3 = (i_5, i_6)$. If $k'_3 = (i_6)$, then $(\mu \setminus \{(i_2, i_3), (i_5, i_6)\}) \cup \{(i_1, i_2), (i_3, i_5), (i_6)\} \in \mathcal{M}$ again leading to the same contradiction as above. If k'_3 involves some recipient $i'_6 \in N^*(\mu') \setminus N^*(\mu)$, then either k'_3 equals (i_6, i'_6) or contains the embedded

2-cycle (i_6, i_6'). In both cases, $(\mu \setminus \{(i_2, i_3), (i_5, i_6)\}) \cup \{(i_1, i_2), (i_3, i_5), (i_6, i_6'\} \in \mathcal{M}$, again resulting in a contradiction.

Thus, either (a) $i_6 \in N^*(\mu) \setminus N^*(\mu')$ or (b) i_3 is involved in some exchange $k_2' \in \mu'$ that only involves recipients in $N^*(\mu)$. Note that repeating the argument above induces a process that always leads to a recipient $i \in N^*(\mu) \setminus N^*(\mu')$ and never involves any recipient in $N^*(\mu') \setminus N^*(\mu)$ except for i_1. Since i_1 is an arbitrary recipient in $N^*(\mu') \setminus N^*(\mu)$, this implies that each recipient in $i \in N^*(\mu') \setminus N^*(\mu)$ has some corresponding recipient in $i^* \in N^*(\mu) \setminus N^*(\mu')$. Furthermore, since i_2 is the only recipient in the process above involved in an exchange with a recipient in $N^*(\mu') \setminus N^*(\mu)$ at μ', i^* is unique for each $i \in\in N^*(\mu') \setminus N^*(\mu)$. This implies that $|N^*(\mu) \setminus N^*(\mu')| > |N^*(\mu') \setminus N^*(\mu)|$, which contradicts the observation above that $|N^*(\mu) \setminus N^*(\mu')| < |N^*(\mu') \setminus N^*(\mu)|$. This concludes the proof by contradiction, demonstrating that the exchange property is satisfied and that $\mathcal{S}$ is a matroid. Thus, the KEP has a matroid structure. By Proposition 1, the program is conflict-proof when $q = 3$.

($\Rightarrow$) Next, let $q \neq 3$. Since a 3-cycle can involve at most three embedded 2-cycles, $q \leq 2$. Consider a problem $\langle N, D, C\rangle$ in which $N = \{1, 2, 3, 4\}$, $D = \emptyset$, and $K = \{(1, 2), (2, 3), (2, 4), (2, 3, 4)\}$. Note that $(2, 3, 4)$ has two embedded 2-cycles, $(2, 3)$ and $(2, 4)$. There are four matchings, $\mu_1 = \{(1, 2)\}$, $\mu_2 = \{(2, 3)\}$, $\mu_3 = \{(3, 4)\}$, and $\mu_4 = \{(2, 3, 4)\}$. Assume that the KEP has a PSC structure to reach a contradiction. Then $\mathcal{S}$ is a pure simplicial complex. By definition, $N^*(\mu_1), N^*(\mu_4) \in \mathcal{S}$. Furthermore, $|N^*(\mu_1)| = 2 < |N^*(\mu_4)| = 3$. By Lemma 1, $\mathcal{S}$ satisfies the relaxed exchange property. Thus, there exists some $i' \in N \setminus N^*(\mu_1)$ such that $N(\mu_1) \cup \{i'\} \in \mathcal{S}$. Some matching $\nu \in \mathcal{M}$ such that $N^*(\mu_1) \cup \{i'\} \subseteq N^*(\nu)$ must therefore exist. This is a contradiction since $1 \notin N^*(\mu)$ for all $\mu \in \mathcal{M}$ except for μ_1. Consequently, $\mathcal{S}$ is not a pure simplicial complex and the KEP does not have a PST structure. By Proposition 1, the program is not conflict-proof when $q \neq 3$. □

Proposition 6. Priority group matchings are Pareto efficient.

Proof. Consider any partitioning of N into priority groups and let π be any priority order[20] such that if $i \in N_t$, $j \in N_u$ and $t < u$, then $\pi(i) < \pi(j)$. Let μ be a priority group matching that is not Pareto efficient to reach a contradiction. Then there exists some $\nu \in \mathcal{M}$ such that $\nu \succsim_i \mu$ for all $i \in N$ and $\nu \succ_i \mu$ for some $i \in N$. Since $\nu \succsim_i \mu$ for all $i \in N$, it follows that $N^*(\mu) \subseteq N^*(\nu)$. Then, either $N^*(\mu) \subset N^*(\nu)$ or $N^*(\mu) = N^*(\nu)$.

First, suppose that $N^*(\mu) \subset N^*(\nu)$. Let j be the highest priority recipient in $N^*(\nu) \setminus N^*(\mu)$ and let N_t be the priority group for which $j \in N_t$. Since there is no $j' \in N^*(\nu) \setminus N^*(\mu)$ for whom $\pi(j') < \pi(j)$, it follows that $|N^*_\tau(\mu)| = |N^*_\tau(\nu)|$ for all $\tau \in \{0, \ldots, t-1\}$. Furthermore, since $N^*(\mu) \subset N^*(\nu)$ and $j \in N^*(\nu) \setminus N^*(\mu)$ it must be the case that $|N^*_t(\nu)| > |N^*_t(\mu)|$. Then, by the definition of $\succsim$, $\nu \succ \mu$. However, since $\mu \in \mathcal{M}^*$ and $\nu \in \mathcal{M}$, $\mu \succsim \nu$. This is a contradiction.

It follows that $N^*(\mu) = N^*(\nu)$. Since $\nu \succsim_i \mu$ for all $i \in N$, $c_{i,\nu(i)} \geq c_{i,\mu(i)}$ for all $i \in N$. Hence, $|B(\nu)| \geq |B(\mu)|$. Consider some recipient j for whom $\nu \succ_j \mu$. Note that j is either a regular recipient or an altruistic recipient. That is, either $j \in I_A$ or $j \in I_R$.

Suppose that $j \in I_A$. Whereas a regular recipient would prefer her own donor to a different donor with the same degree of compatibility, an altruistic recipient would only prefer one donor over another if the degree of compatibility is higher. That is, $\nu \succ_j \mu$ implies that $c_{j,\nu(j)} > c_{j,\mu(j)}$ by the definition of the preference relation $\succsim_j$. Since both μ and ν are matchings, $c_{j,\mu(j)} \geq 1$ and $c_{j,\nu(j)} \geq 1$. It therefore follows from $c_{j,\nu(j)} > c_{j,\mu(j)}$ that $c_{j,\mu(j)} = 1$ and $c_{j,\nu(j)} = 2$. In conjunction with the observation that $c_{i,\nu(i)} \geq c_{i,\mu(i)}$

[20] Defined on Page 14.

for all $i \in N$, this implies that $|B(\nu)| > |B(\mu)|$. However, since $\mu \succsim \nu$ and $N^*(\mu) = N^*(\nu)$ by assumption, it must be the case that $|B(\mu)| \geq |B(\nu)|$. This is a contradiction.

It must therefore be the case that $j \in I_R$. First suppose that $\nu(j) \neq j$. This means that recipient j's preference for ν over μ is not due to j being a regular recipient with a preference for her own donor. That is, since $\nu(j) \neq j$ and $\nu \succ_j \mu$ it must be the case that $c_{j,\nu(j)} > c_{j,\mu(j)}$ by the definition of the preference relation $\succsim_j$. Then the situation is identical to the case when $j \in I_A$, resulting in the same contradiction.

Thus, $j \in I_R$ and $\nu(j) = j$. Since $c_{j,\mu(j)} \geq c_{j,\nu(j)}$, either $c_{j,\mu(j)} = c_{j,\nu(j)}$ or $c_{j,\nu(j)} > c_{j,\mu(j)}$. First consider the case when $c_{j,\mu(j)} = c_{j,\nu(j)}$. Since $\nu \succ_j \mu$ it must be the case that $\mu(j) \neq \nu(j)$. However, since $j \in I_R$, $\mu(j) \in (N \cup D) \setminus \{j\}$ and $c_{j,\mu(j)} \not> c_{j,j} = c_{j,\nu(j)}$ it follows from the definition of the arc set E that the arc $j\mu(j) \notin E$. This violates the assumption that μ is a matching. In other words, since j is a regular recipient, she may only be assigned her own donor and donors with a strictly higher degree of compatibility. This contradicts the assumption that she is assigned a different donor with the same degree of compatibility as her own at μ.

Therefore, $N^*(\mu) = N^*(\nu)$, $j \in I_R$, $\nu(j) = j$ and $c_{j,\nu(j)} > c_{j,\mu(j)}$. In conjunction with the observation that $c_{i,\nu(i)} \geq c_{i,\mu(i)}$ for all $i \in N$, this, again, implies that $|B(\nu)| > |B(\mu)|$. However, since $\mu \succsim \nu$ and $N^*(\mu) = N^*(\nu)$ by assumption, it must be the case that $|B(\mu)| \geq |B(\nu)|$. This is a contradiction.

It has now been shown that whenever $\mu \in \mathcal{M}^*$, the existence of some $\nu \in \mathcal{M}$ such that $\nu \succsim_i \mu$ for all $i \in N$ and $\nu \succ_i \mu$ for some $i \in N$ will always result in a contradiction. This means that every priority group matching is Pareto efficient. □

Proposition 7. Consider some priority order π and some $m \leq n$. Let $|N_m| = n - (m-1)$ and if $m \geq 2$, let $N_t = \{\pi^{-1}(t)\}$ for all $t < m$. Then $\mathcal{M}^* = \mathcal{M}_T^m$.

Proof. The idea of the proof is to show that both $\mathcal{M}_T^m \subseteq \mathcal{M}^*$ and $\mathcal{M}^* \subseteq \mathcal{M}_T^m$, which is only true whenever $\mathcal{M}^* = \mathcal{M}_T^m$.

It will first be shown that $\mathcal{M}_T^m \subseteq \mathcal{M}^*$. Consider some $\mu \in \mathcal{M}$ and assume that $\mu \in \mathcal{M}_T^m \setminus \mathcal{M}^*$ to reach a contradiction. Since $\mu \notin \mathcal{M}^*$, there must exist some $\mu' \in \mathcal{M}$ such that $\mu' \succ \mu$. Recall the definition of priority group preferences on page 14. As $\mu' \succ \mu$, it follows that either $|N_t^*(\mu)| = |N_t^*(\mu')|$ for all $t \in [1, m]$ and $|B(\mu')| > |B(\mu)|$, or $|N_t^*(\mu')| > |N_t^*(\mu)|$ for some $t \in \{1, \ldots, m\}$ and $|N_\tau^*(\mu)| = |N_\tau^*(\mu')|$ for all $\tau \in \{0, \ldots, t-1\}$.

First, suppose that $|N_t^*(\mu)| = |N_t^*(\mu')|$ for all $t \in \{1, \ldots, m\}$ and $|B(\mu')| > |B(\mu)|$. Recall the definition of threshold matchings on page 15. If $m = 1$, then $\mathcal{E}_{m-1}^m = \mathcal{M}$ and $\mu' \in \mathcal{E}_{m-1}^m$. If $m \in \{2, \ldots, n\}$, then since $N_t = \{\pi^{-1}(t)\}$ for all $t \in \{1, \ldots, m-1\}$, $|N_t^*(\mu)| = |N_t^*(\mu')|$ for all $t \in \{1, \ldots, m\}$ implies that $\mu' \in \mathcal{E}_{m-1}^m$. That is, $\mu' \in \mathcal{E}_{m-1}^m$ for all $m \in \{1, \ldots, n\}$. Furthermore, $|N_m^*(\mu)| = |N_m^*(\mu')|$ implies that $\mu' \in \mathcal{E}_m^m$. Finally, $\mu' \in \mathcal{E}_m^m$ and $|B(\mu')| > |B(\mu)|$ contradict the assumption that $\mu \in \mathcal{M}_T^m$, since it requires that $\mu \in \arg\max_{\nu \in \mathcal{E}_m^m} |B(\nu)|$.

Hence, $|N_t^*(\mu')| > |N_t^*(\mu)|$ for some $t \in \{1, \ldots, m\}$ and $|N_\tau^*(\mu)| = |N_\tau^*(\mu')|$ for all $\tau \in \{0, \ldots, t-1\}$. Since $t \in \{1, \ldots, m\}$ and thus $N_\tau = \{\pi^{-1}(\tau)\}$ for all $\tau \in \{0, t-1\}$, $|N_\tau^*(\mu)| = |N_\tau^*(\mu')|$ for all $\tau \in \{0, t-1\}$ implies that $\mu' \in \mathcal{E}_{t-1}^m$.

Suppose that $t \in \{1, \ldots, m-1\}$. Then $N_t = \{\pi^{-1}(t)\}$. In that case, $|N_t^*(\mu')| > |N_t^*(\mu)|$ implies that $\pi^{-1}(t) \in N^*(\mu') \setminus N^*(\mu)$. That is, if $t \in \{1, \ldots, m-1\}$, then priority group N_t contains only the unique recipient with priority t. Since the number of recipients in priority group N_t is higher at μ' than at μ, it must therefore be the case that the only recipient in N_t, $\pi^{-1}(t)$, is matched at μ' but not at μ. Thus, since $\mu' \in \mathcal{E}^m_{t-1}$, $\mu \notin \mathcal{E}^m_t$ by the definition of $\mathcal{E}^m_t$. This contradicts the assumption that $\mu \in \mathcal{M}^m_T$, because $\mathcal{M}^m_T \subseteq \mathcal{E}^m_t$.

It must therefore be the case that $t = m$. Then $\mu' \in \mathcal{E}^m_{m-1}$. Since m is the number of priority groups and $|N^*_\tau(\mu)| = |N^*_\tau(\mu')|$ for all $\tau \in \{0, \dots, m-1\}$, $|N^*_t(\mu')| > |N^*_t(\mu)|$ if and only if $|N^*(\mu')| > |N^*(\mu)|$. This contradicts the assumption that $\mu \in \mathcal{M}^m_T \subseteq \arg\max_{\nu \in \mathcal{E}^m_{m-1}} |N^*(\nu)|$. Hence, $\mathcal{M}^m_T \subseteq \mathcal{M}^*$.

Next, it will be shown that $\mathcal{M}^* \subseteq \mathcal{M}^m_T$. Consider some $\mu \in \mathcal{M}$ and assume that $\mu \in \mathcal{M}^* \setminus \mathcal{M}^m_T$ to reach a contradiction. Since $\mathcal{M}$ is assumed to be non-empty, $\mathcal{M}^m_T$ is guaranteed to be non-empty. Consider some $\mu' \in \mathcal{M}^m_T$. Note that $\mu \in \mathcal{E}^m_0 \setminus \mathcal{M}^m_T$, as $\mathcal{E}^m_0$ is the set of all matchings. Since $\mathcal{E}^m_t \subseteq \mathcal{E}^m_{t-1}$ for all $t \in \{1, \dots, m\}$, it must therefore either be the case that $\mu \in \mathcal{E}^m_m \setminus \mathcal{M}^m_T$ or $\mu \in \mathcal{E}^m_{t'-1} \setminus \mathcal{E}^m_{t'}$ for some $t' \in \{1, \dots, m\}$.

First, suppose that $\mu \in \mathcal{E}^m_m \setminus \mathcal{M}^m_T$. In this case, $|N^*_t(\mu)| = |N^*_t(\mu')|$ for all $t \in \{1, \dots, m\}$. Hence, $\mu' \in \mathcal{M}^m_T$ implies that $|B(\mu')| > |B(\mu)|$. This, in turn, implies that $\mu' \succ \mu$, contradicting the assumption that $\mu \in \mathcal{M}^*$.

This means thas $\mu \in \mathcal{E}^m_{t'-1} \setminus \mathcal{E}^m_{t'}$ for some $t' \in \{1, \dots, m\}$. First suppose that $t' = m$ so that $\mu \in \mathcal{E}^m_{m-1} \setminus \mathcal{E}^m_m$. Then $|N^*_t(\mu)| = |N^*_t(\mu')|$ for all $t \in \{0, \dots, m-1\}$. (Recall that $N_0 = \emptyset$ by definition.) Since m is the number of priority groups, $\mu \notin \mathcal{E}^m_m$ and $\mu' \in \mathcal{E}^m_m$ then imply that $|N^*_m(\mu')| > |N^*_m(\mu)|$ and $|N^*(\mu')| > |N^*(\mu)|$. Hence, $\mu' \succ \mu$, contradicting the assumption that $\mu \in \mathcal{M}^*$.

Thus, $t' \in \{1, \dots, m-1\}$. Note that $m \geq 2$, since $m = 1$ requires that $t' = m$, which has been shown to result in a contradiction. Since $t' < m$, $N_\tau = \{\pi^{-1}(\tau)\}$ for all $\tau \in \{1, \dots, t\}$. Then, by the definition of $\mathcal{E}^m_{t'}$ and the fact that $\mu, \mu' \in \mathcal{E}^m_{t'-1}$, $|N^*_\tau(\mu)| = |N^*_\tau(\mu')|$ for all $\tau \in \{0, \dots, t'-1\}$. Since $\mu \in \mathcal{E}^m_{t'-1} \setminus \mathcal{E}^m_{t'}$ and $\mu' \in \mathcal{E}^m_{t'-1}$, $\pi^{-1}(t') \in N^*(\mu') \setminus N^*(\mu)$. Thus, $|N^*_{t'}(\mu')| > |N^*_{t'}(\mu)|$ as $|N_{t'}| = 1$. $|N^*_\tau(\mu)| = |N^*_\tau(\mu')|$ for all $\tau \in \{0, \dots, t'-1\}$ and $|N^*_{t'}(\mu')| > |N^*_{t'}(\mu)|$ together imply that $\mu' \succ \mu$. This contradicts $\mu \in \mathcal{M}^*$.

Hence, there exists no $\mu \in \mathcal{M}^* \setminus \mathcal{M}^m_T$, implying that $\mathcal{M}^* \subseteq \mathcal{M}^m_T$ since $\mathcal{M}^m_T$ is non-empty. In conclusion, as both $\mathcal{M}^m_T \subseteq \mathcal{M}^*$ and $\mathcal{M}^* \subseteq \mathcal{M}^m_T$ are true, it must be the case that $\mathcal{M}^* = \mathcal{M}^m_T$. □

Proposition 8. Consider some priority order π. For any $m \leq n$, any $m' \leq m$, any $\mu \in \mathcal{M}^m_T$ and any $\mu' \in \mathcal{M}^{m'}_T$, $|N^*(\mu')| \geq |N^*(\mu)|$.

Proof. First, let $m \in \{2, \dots, n\}$ and $m' \in \{1, \dots, m-1\}$. Consider some $\mu \in \mathcal{M}^m_T$ and some $\mu' \in \mathcal{M}^{m'}_T$. First note that $\mathcal{E}^m_t = \mathcal{E}^{m'}_t$ for all $t \in \{1, \dots, m'-1\}$. Consequently, $\mathcal{E}^m_{m'-1} = \mathcal{E}^{m'}_{m'-1}$. By definition, $|N^*(\mu')| \geq |N^*(\nu)|$ for all $\nu \in \mathcal{E}^{m'}_{m'-1} = \mathcal{E}^m_{m'-1}$. Furthermore, since $\mathcal{E}^m_\tau \subseteq \mathcal{E}^m_{m'-1}$ for all $\tau \in \{m'-1, \dots, m\}$ and $\mathcal{M}^m_T \subseteq \mathcal{E}^m_m$, it follows that $|N^*(\mu')| \geq |N^*(\mu)|$.

Next, let $m \in \{1, \dots, n\}$ and $m' = m$. Consider some $\mu \in \mathcal{M}^m_T$ and some $\mu' \in \mathcal{M}^{m'}_T$. Since $m = m'$, $\mathcal{M}^m_T = \mathcal{M}^{m'}_T$. By Proposition 7, there is a partitioning of N such that $\mathcal{M}^m_T = \mathcal{M}^*$. By Lemma 3, all priority group matchings match the same number of recipients. Then $|N^*(\mu)| = |N^*(\mu')|$, as $\mu, \mu' \in \mathcal{M}^*$.

Hence, for any $\mu \in \mathcal{M}^m_T$ and any $\mu' \in \mathcal{M}^{m'}_T$, $|N^*(\mu')| \geq |N^*(\mu)|$ both in the case when $m \in \{2, \dots, n\}$, $m' \in \{1, \dots, m-1\}$ and in the case when $m \in \{1, \dots, n\}$ and $m' = m$. □

Proposition 9. For any priority order π:

$$\mathcal{M}^m_T = \begin{cases} \mathcal{M}^\varepsilon_{\max} & \text{at} \quad m = 1 \\ \mathcal{M}_B & \text{at} \quad m \in \{n, n+1\} \end{cases}$$

Proof. First note that $\mathcal{M}^1_T = \arg\max_{\mu \in \mathcal{E}^1_m} |B(\mu)|$. Furthermore, $\mathcal{E}^1_m = \arg\max_{\mu \in \mathcal{E}^1_0} |N^*(\mu)|$. Since $\mathcal{E}^1_0 = \mathcal{M}$, $\mathcal{E}^1_m = \arg\max_{\mu \in \mathcal{M}} |N^*(\mu)| = \mathcal{M}_{\max}$. It follows from the definition of $\mathcal{M}^\varepsilon_{\max}$ that $\mathcal{M}^1_T = \arg\max_{\mu \in \mathcal{M}_{\max}} |B(\mu)| =$

$\mathcal{M}^{\varepsilon}_{\max}$. Next, recall that $\mathcal{M}^m_T = \arg\max_{\mu \in \mathcal{E}^m_m} |B(\mu)|$. $\mathcal{E}^m_m$ is defined by:

$$\text{For } t \in (0,m), \quad \mathcal{E}^m_t = \begin{cases} \{\mu \in \mathcal{E}^m_{t-1} \mid \pi^{-1}(t) \in N^*(\mu)\} \text{ if } \pi^{-1}(t) \in N^*(\mu) \text{ for some } \mu \in \mathcal{E}^m_{t-1} \\ \mathcal{E}^m_{t-1} \text{ otherwise} \end{cases}$$

$$\mathcal{E}^m_m = \arg\max_{\mu \in \mathcal{E}^m_{m-1}} |N^*(\mu)|$$

At $m = n+1$, this reduces to:

$$\text{For } t \in (0,n], \quad \mathcal{E}^n_t = \begin{cases} \{\mu \in \mathcal{E}^n_{t-1} \mid \pi^{-1}(t) \in N^*(\mu)\} \text{ if } \pi^{-1}(t) \in N^*(\mu) \text{ for some } \mu \in \mathcal{E}^n_{t-1} \\ \mathcal{E}^n_{t-1} \text{ otherwise} \end{cases}$$

At $m = n$, the definition is instead equivalent to:

$$\text{For } t \in (0,n), \quad \mathcal{E}^n_t = \begin{cases} \{\mu \in \mathcal{E}^n_{t-1} \mid \pi^{-1}(t) \in N^*(\mu)\} \text{ if } \pi^{-1}(t) \in N^*(\mu) \text{ for some } \mu \in \mathcal{E}^n_{t-1} \\ \mathcal{E}^n_{t-1} \text{ otherwise} \end{cases}$$

$$\mathcal{E}^n_n = \arg\max_{\mu \in \mathcal{E}^n_{n-1}} |N^*(\mu)|$$

In both cases, it is straightforward to confirm that $\mathcal{E}^n_n = \mathcal{E}^{n+1}_n = \mathcal{E}_n$. Thus, $\mathcal{M}^n_T = \mathcal{M}^{n+1}_T = \arg\max_{\mu \in \mathcal{E}_n} |B(\mu)| = \mathcal{M}_B$ by definition. □

Proposition 10. Consider some priority order π, let $m = 2$ and set $N_\Gamma = N_1$. Then $\mathcal{M}^* = \mathcal{M}_S$.

Proof. It follows immediately from the definition of priority group preferences on page 14 that matchings in $\mathcal{M}^{\varepsilon}_{\Gamma}$ are preferred to all other matchings in this setting. □

Proposition 11. Sort all "marginalized" recipients into N_Γ and let $\beta > n - |N_\Gamma| - 1$. Then $\mathcal{M}_\Gamma$ is the set of *MaxCard-Fair* solutions.

Proof. Dickerson and Sandholm (2014) assign each arc $e \in E$ some weight $w_e \in \mathbb{R}_{++}$. A cycle or chain c is defined in terms of the relevant arcs rather than (as in this paper) in terms of the participating recipients. A set $\mu \subseteq K$ is a matching if the cycles and chains it contains are disjoint. Consider an objective function u where for any $\mu \in \mathcal{M}$, $u(\mu) = \sum_{c \in \mu} \sum_{e \in c} w_e$. If each arc has the same weight, $w \in \mathbb{R}_{++}$, the solutions to the corresponding maximization problem will simply maximize the number of matched recipients. Dickerson and Sandholm (2014) call this objective function MaxCard. Since N_Γ is the set of marginalized recipients, the MaxCard-Fair objective function can be obtained from the MaxCard objective function by assigning each arc ij the same weight as before (w) whenever $i \in N \setminus N_\Gamma$ and increasing the weight to $(1+\beta)w$ whenever $i \in N_\Gamma$, where $\beta > 0$. Let $\beta > n - |N_\Gamma| - 1$. Then $(1+\beta)w > (n - |N_\Gamma|)w$. Since $(n - |N_\Gamma|)w$ is an upper bound on the sum of weights assigned to arcs ending in recipients belonging to $N \setminus N_\Gamma$, MaxCard-Fair's primary goal is to match as many recipients in N_Γ as possible and its secondary goal is to match as many recipients in $N \setminus N_\Gamma$ as possible. In other words, the set of MaxCard-Fair solutions is given by $\mathcal{M}_\Gamma$.

If the planner disallows desensitization, then $c_{i,j} \neq 1$ for all $i, j \in N$ and $|B(\mu)| = |N^*(\mu)|$ for all $\mu \in \mathcal{M}$. Since $|N^*(\mu)| = |N^*(\mu')|$ for all $\mu, \mu' \in \mathcal{M}_\Gamma$, $\mathcal{M}_S = \mathcal{M}_\Gamma$ in this case. Consequently, the set of simple threshold matchings is the set of MaxCard-Fair solutions. □

Proposition 12. A matching $\mu \in \mathcal{M}$ is Pareto efficient if and only if the following two conditions hold for all $\nu \in \mathcal{M}$:

(a) $N^*(\mu) \subset N^*(\nu) \implies B(\mu) \nsubseteq B(\nu)$

(b) $B(\mu) \subset B(\nu) \implies N^*(\mu) \nsubseteq N^*(\nu)$

Proof. It will first be shown that all matchings satisfying (a) and (b) are Pareto efficient. To reach a contradiction, suppose there exists some matching $\mu \in \mathcal{M}$ that satisfies (a) and (b), but is not Pareto efficient. Since μ is not Pareto efficient, there exists some $\nu \in \mathcal{M}$ such that $\nu \succsim_i \mu$ for all $i \in N$ and $\nu \succ_i \mu$ for some $i \in N$. The statement that $\nu \succsim_i \mu$ for all $i \in N$ implies that $c_{i,\nu(i)} \geq c_{i,\mu(i)}$ for all $i \in N$. Consequently, $N^*(\mu) \subseteq N^*(\nu)$ and $B(\mu) \subseteq B(\nu)$. Consider some $i \in N$ such that $\nu \succ_i \mu$. If $i \in I_A$, then $c_{i,\nu(i)} > c_{i,\mu(i)}$. If $i \in I_R$, then either $c_{i,\nu(i)} > c_{i,\mu(i)}$ or $\nu(i) = i \neq \mu(i)$ and $c_{i,\nu(i)} \geq c_{i,\mu(i)}$. Suppose that $\nu(i) = i \neq \mu(i)$ and $c_{i,\nu(i)} = c_{i,\mu(i)}$. By construction of E, this implies that $i\mu(i)$ is not an arc in the compatibility graph. Consequently, μ is not a matching, which contradicts $\mu \in \mathcal{M}$. It follows that $c_{i,\nu(i)} > c_{i,\mu(i)}$ in all possible cases. $N^*(\mu) \subseteq N^*(\nu)$, $B(\mu) \subseteq B(\nu)$ and $c_{i,\nu(i)} > c_{i,\mu(i)}$ imply that either $N^*(\mu) \subset N^*(\nu)$, $B(\mu) \subset B(\nu)$ or both. Thus, either (a), (b) or both (a) and (b) are violated, contradicting the assumption that μ satisfies both (a) and (b). Hence, every matching $\mu \in \mathcal{M}$ satisfying (a) and (b) is Pareto efficient.

Next, it will be shown that every Pareto efficient matching satisfies (a) and (b). To reach a contradiction, suppose there exists some Pareto efficient matching $\mu \in \mathcal{M}$ that fails to satisfy (a) or (b). Then there exists some $\nu \in \mathcal{M}$ such that one of the following two cases holds.

(a) $N^*(\mu) \subset N^*(\nu)$ and $B(\mu) \subseteq B(\nu)$. $N^*(\mu) \subset N^*(\nu)$ implies that $c_{i,\nu(i)} \geq 1$ for all $i \in N^*(\mu)$. This means that, $c_{i,\nu(i)} \geq c_{i,\mu(i)}$ for all $i \in N^*(\mu) \setminus B(\mu)$. Furthermore, $B(\mu) \subseteq B(\nu)$ implies that $c_{i,\nu(i)} = 2$ for all $i \in B(\mu)$. Hence, $c_{i,\nu(i)} \geq c_{i,\mu(i)}$ for all $i \in N^*(\mu)$. Consider some $i \in N^*(\mu)$. Then $\nu \succsim_i \mu$ unless $i \in I_R$, $\mu(i) = i \neq \nu(i)$ and $c_{i,\nu(i)} = c_{i,\mu(i)}$. However, in such a case, the arc $i\nu(i)$ would not be in the arc set E and ν would not be a matching. This would contradict $\nu \in \mathcal{M}$. Thus, $\nu \succsim_i \mu$ for all $i \in N^*(\mu)$. For each recipient $i \in N \setminus (N^*(\mu) \cup N^*(\nu))$, $c_{i,\nu(i)} = c_{i,\mu(i)} = 0$. Thus, $\nu \succsim_i \mu$ for all $i \in N \setminus N^*(\nu)$. Finally, $c_{i,\nu(i)} \geq 1 > c_{i,\mu(i)} = 0$ for all $i \in N^*(\nu) \setminus N^*(\mu)$. Hence, $\nu \succ_i \mu$ for all $i \in N^*(\nu) \setminus N^*(\mu)$ and $\nu \succsim_i \mu$ for all $i \in N$. This contradicts the assumption that μ is Pareto efficient.

(b) $N^*(\mu) \subseteq N^*(\nu)$ and $B(\mu) \subset B(\nu)$. $N^*(\mu) \subseteq N^*(\nu)$ implies that $c_{i,\nu(i)} \geq 1$ for all $i \in N^*(\mu)$. This means that, $c_{i,\nu(i)} \geq c_{i,\mu(i)}$ for all $i \in N^*(\mu) \setminus B(\mu)$. Furthermore, $B(\mu) \subset B(\nu)$ implies that $c_{i,\nu(i)} = 2$ for all $i \in B(\mu)$. Hence, $c_{i,\nu(i)} \geq c_{i,\mu(i)}$ for all $i \in N^*(\mu)$. Since $c_{i,\mu(i)} = 0$ for all $N \setminus N^*(\mu)$, $c_{i,\nu(i)} \geq c_{i,\mu(i)}$ for all $i \in N$. Consider any $i \in N$ and note that $\nu \succsim_i \mu$ unless $i \in I_R$, $\mu(i) = i \neq \nu(i)$ and $c_{i,\nu(i)} = c_{i,\mu(i)}$. However, in such a case, the arc $i\nu(i)$ would not be in the arc set E and ν would not be a matching. This would contradict $\nu \in \mathcal{M}$. Thus, $\nu \succsim_i \mu$ for all $i \in N$. Finally, $c_{i,\nu(i)} > c_{i,\mu(i)}$ for all $i \in B(\nu) \setminus B(\mu)$. Hence, $\nu \succ_i \mu$ for all $i \in B(\nu) \setminus B(\mu)$ and $\nu \succsim_i \mu$ for all $i \in N$. This contradicts the assumption that μ is Pareto efficient.

Both cases result in contradictions, implying that all Pareto efficient matchings satisfy both (a) and (b). □

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