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Abstract: This paper proposes an ascending auction for selling multiple heterogeneous indivisible items to several potential bidders. Every bidder demands at most one item and faces a budget constraint. His valuations and budget are private information. Budget constraints may lead to the failure of competitive equilibrium. Bidders are not assumed to behave as price-takers and may therefore act strategically. We prove that the auction always induces bidders to bid truthfully and finds a strongly Pareto efficient core allocation when bidders are budget constrained, otherwise a Walrasian equilibrium with the minimum equilibrium price vector.

Keywords: Ascending auction, core, equilibrium, budget constraint, incentive, assignment market.

JEL classification: D44.

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1 Introduction

This paper addresses the problem of how to efficiently allocate multiple heterogeneous inherently indivisible items or goods among a group of bidders who can be financially constrained. To be more precise, an auctioneer (or seller) wants to sell n indivisible items to m potential bidders. Every bidder acquires at most one item and views his n valuations over those items and his budget as his private information. Every bidder is initially endowed with a budget but his budget is limited and may not be able to pay up to his valuation. In this setting, it is not possible to follow the traditional approach of using market-clearing prices as an effective means to allocate goods, as market-clearing prices are not guaranteed to exist due to budget constraints. We aim to develop a new dynamic auction mechanism that can not only overcome the nonexistence problem of market-clearing prices but also give bidder right incentives to achieve an efficient market outcome.

Auctions have been long used for the sale of a variety of items since a few thousand years ago when they were applied by the Babylonians. Nowadays auctions can be conducted online and off-line. They are powerful market mechanisms and have been widely explored by both private and public sectors to carry out a broad range of and a huge volume of economic activities. They are used by governments to sell radio spectrum licenses, treasury bills, timber rights, off-shore oil leases, mineral rights and pollution permits, and to procure public projects including goods and services, and to privatize state companies (in the former Soviet Unions and other eastern European socialist states), and by private sectors to sell all kinds of commodities and services ranging from antiques, art works, flowers and fish, to airline routes, takeoff and landing slots, and keywords; see e.g., Klemperer (2004), Milgrom (2004), and Krishna (2010).

A key assumption in auction theory has been that all potential bidders are not subject to any budget constraints so that they can pay up to their valuations on the goods for sale. It is well-known that financial or budget constraints pose a serious obstacle to the efficient allocation of resources; see Che and Gale (1998, 2000), Laffont and Robert (1996), Maskin (2000) and Krishna (2010) among others. A longstanding guiding economic principle is that efficient allocation of goods can be achieved through market-clearing or Walrasian equilibrium prices. In the presence of budget constraints, this principle can no longer be applied, because market-clearing prices are not guaranteed to exist.

To overcome the absence of market-clearing prices caused by financial constraints, we have to adopt a more general approach—the notion of core—to the current challenging allocation problem. The concept of core is a generalization of Edgeworth's contract curve and is one of the most fundamental solution concepts in game theory and general equilibrium theory; see Gillies (1953), Debreu and Scarf (1963), Scarf (1967), Shapley and Shubik (1971), Shapley (1973), Shapley and Scarf (1974), Quinzii (1984), and Predtetchinski and Herings (2004) among others. A core allocation consists of an assignment of items and its supporting price system and is Pareto efficient. It specifies a feasible distribution of items and incomes among all market participants that is stable against every possible deviation from any coalition. Because of budget constraints, agents will not be able to transfer part of their utilities to others. In spite of budget constraints and non-transferable utilities we prove that there exists at least one strongly Pareto-efficient core allocation in the market and thus a strongly Pareto efficient allocation can be achieved. Our major contribution goes further by designing a dynamic auction for actually locating a strongly Pareto efficient core allocation and inducing bidders to bid truthfully.

We consider a basic auction market in which there are a finite number of heterogenous indivisible items like houses for sale and a finite number of potential bidders. Every bidder wants to consume at most one item but faces a budget constraint and may not be able to pay up to his valuations on those items. When bidders are not budget constrained, this market model becomes the classic assignment market models as studied by Koopmans and Beckmann (1957), Shapley and Shubik (1971), Crawford and Knoer (1981), Leonard (1983), Demange et al. (1986), Mishra and Talman (2010), Andersson et al. (2013), Andersson and Erlanson (2013), and Herings and Zhou (2022). In the current model, both valuations and budgets are bidders' private information and bidders are not assumed to be price-takers and may therefore act strategically as long as it serves their interests. In the auction literature, private information concerns typically every bidder's valuations on goods; see e.g., Ausubel (2004), Perry and Reny (2005), Ausubel (2006), Mishra and Parkes (2007), and Sun and Yang (2014). The current model has an additional dimension of private information concerning also budgets and makes the design more challenging, as it becomes a multi-dimensional dynamic auction design problem; see Armstrong and Rochet (1999) for a survey on the multi-dimensional static contract design. We propose an ascending auction in which bidders determine their own bids and pay as they bid. We will show that the proposed auction always induce bidders to bid truthfully and finds a strongly Pareto efficient core allocation when bidders are budget constrained, otherwise a Walrasian equilibrium with the minimum equilibrium prices. So when bidders are not budget constrained, the proposed auction can recover the well-known results of Leonard (1983) and Demange et al. (1986).

The proposed auction works roughly as follows. Every bidder reports his initial bids to the auctioneer. The auctioneer then selects a provisional assignment based on the reported bids to maximize her revenues. If a bidder gets no item from the provisional assignment and can make new bids or withdraw some of his previous bids, the auctioneer chooses again a provisional assignment based on reported renewed bids. The auction stops when no bidder is willing to make any new bid. This ascending auction shares several common features with other ascending auctions. Compared with the sealed-bid auctions such as the VCG mechanism, our ascending auction has the advantage of demanding less information from bidders, allowing them to learn and adjust, being detail-free, and being independent of any probability distribution. This feature is very important and attractive for auction design; see Wilson (1987), Rothkopf et al. (1990), Ausubel (2004), Perry and Reny (2005), Ausubel (2006), Bergemann and Morris (2007), Milgrom (2007), and Rothkopf (2007) among others.

1.1 A Brief Literature Review

This article relates to the early literature on auctions of selling one or two items with budget constrained bidders. Che and Gale (1998, 2000), Laffont and Robert (1996), Maskin (2000), Krishna (2010), and Zheng (2001) have examined the cases of selling a single item when bidders face budget constraints. Hafalir et al. (2012) have studied a sealed-bid Vickrey auction for selling one divisible good to budget constrained bidders. Benoit and Krishna (2001), Brusco and Lopomo (2008), and Pitchik (2009) have analyzed auctions for selling two items under budget constraints.

Our article further connects with a number of recent studies on models with multiple items. Ausubel and Milgrom (2002) have briefly discussed a stylized model with budget constraints in their Section 8. In their model, bidders' budgets and utility functions are not given instead they require that every bidder has a strict preference relation over a finite set of choices. They propose a procedure for finding a core allocation. Ashlagi et al. (2010) have investigated a position auction model with budget constrained bidders. They obtain incentive compatibility and Pareto efficiency results under the assumption that every bidder has a different private budget and one private valuation. Dobzinski et al. (2012) have examined an auction model in which several identical items are sold to budget constrained bidders. They have shown that there does not exist a deterministic mechanism which satisfies individual rationality, incentive compatibility and no positive transfers.

Our current study is very closely related to the recent works by Talman and Yang (2015), van der Laan and Yang (2016), Rong et al. (2019), and Herings and Zhou (2022) on the assignment markets with budget constrained bidders. The first paper proposes a dynamic auction that finds a core allocation. The second one introduces an ascending auction that locates a constrained equilibrium, which possesses some desirable properties but is not necessarily a core allocation. The third proposes a novel criterion for mechanism design that exhibits various appealing properties. Herings and Zhou (2022) introduce a new notion of quantity-constrained competitive equilibrium. At this equilibrium, bidders form expectations about possible trades and may foresee that a trade will not take place if the corresponding budget constraint is binding. They establish the existence of their equilibrium through a dynamic process and the equivalence between equilibrium outcomes and stable outcomes. These papers, however, do not discuss the incentive issue. Their algorithms are considerably different from ours. Our current model deals with an incomplete information environment in which every bidder is assumed to have private valuations over multiple items and a private budget. We achieve both efficiency and strategy-proof results through a new dynamic auction design. Our model can accommodate all kinds of budget constraints. For instance, budget constraints can be soft so that a Walrasian equilibrium exists. Budget constraints can be also hard so that there is no Walrasian equilibrium at all.

The rest of the paper is organized as follows. Section 2 presents the model and basic concepts. Section 3 introduces and analyzes the auction and present the main results. Section 4 concludes.

2 The Model

An auctioneer (seller) wishes to sell a set of n heterogeneous indivisible goods (items) $N = \{1, 2, \ldots, n\}$ to a group of m potential bidders $M = \{1, 2, \ldots, m\}$. Let 0 represent the seller (she) and let $M_0 = M \cup \{0\}$ stand for all agents in the market. We also use 0 to denote a harmless null item which has no value and let $N_0 = N \cup \{0\}$. Every bidder $i \in M$ attaches a monetary value (units of money) to each item, namely, each bidder has a valuation function $v^i : N_0 \mapsto \mathbb{Z}_+$ with $v^i(0) = 0$. Every bidder i is endowed with an amount $m^i \in \mathbb{Z}_+$ of money. We say that bidder i is *budget or financial constrained* if $m^i < \max_{a \in N} v^i(a)$, that is, the valuation of bidder i for some bundle exceeds what he can afford. Otherwise, bidder i is *not budget constrained.* We use $((v^i, m^i), i \in M, N)$ to represent this model. Without loss of generality we have assumed that the seller's reserve price for every item is zero.

The following mild assumptions are imposed upon the model:

(A1) *Private Information on Values and Budget*: Every bidder $i \in M$ knows privately his own valuation function v^i and budget m^i.

(A2) *Quasilinear Utility*: For any bidder $i \in M$, if he pays $p(a)$ in exchange for item $a \in N$, he gets utility of $v^i(a) + m^i - p(a)$ for $p(a) \leq m^i$ and utility of $-\infty$ for $p(a) > m^i$.

When no bidder is financially constrained, the model reduces to the celebrated assignment market models as studied by Koopmans and Beckmann (1957), Shapley and Shubik (1971), Crawford and Knoer (1981), Leonard (1983), Demange et al. (1986), Mishra and Talman (2010), Andersson et al. (2013), Andersson and Erlanson (2013), and Herings and Zhou (2022).

An *assignment* $\pi = (\pi(0), \pi(1), \ldots, \pi(m))$ assigns every bidder $i \in M$ exactly one item $\pi(i) \in N_0$ such that no real item $a \in N$ is assigned to more than one bidder and any item which is not assigned to a bidder is retained by the seller 0. So an assignment may assign the null item to several bidders. At π, a real item $a \in N$ is *unassigned* if it is not assigned to any bidder. So $\pi(0)$ contains all unassigned items. Let $\mathcal{A}$ denote the family of all assignments. An assignment π is *fully efficient* if for every assignment ρ, we have

$$\sum_{i \in M} v^i(\pi(i)) \geq \sum_{i \in M} v^i(\rho(i)). \tag{1}$$

A vector $r = (r^0, r^1, \ldots, r^m)$ is a feasible income distribution if $r^i \geq 0$ for all $i \in M_0$ and $\sum_{i \in M_0} r^i = \sum_{i \in M_0} m^i$. A pair (π, r) of an assignment π and a feasible income distribution r is called an allocation. At (π, r), agent $i \in M$ receives item $\pi(i)$ and holds r^i a total amount of income. Then the utility that the bidders and the seller achieve are given by

$$u^i(\pi, r) = v^i(\pi(i)) + r^i, \forall i \in M, \text{ and } u^0(\pi, r) = r^0 = \sum_{i \in M} (m^i - r^i),$$

respectively.

When bidders face no budget constraints, the Walrasian equilibrium has been the most widely used solution for auction and equilibrium models and market-clearing prices are used in auction design. Given a price vector $p = \big(p(a)\big)_{a \in N_0}$ which specifies a price for each item with $p(0) = 0$, the demand set of bidder i is defined by

$$D_p(i|v^i, m^i) = \Big\{a \in N_0 \mid p(a) \leq m^i \text{ and } v^i(a) - p(a) \geq v^i(b) - p(b) \\ \text{for any } b \in N_0 \text{ and } p(b) \leq m^i\Big\}.$$

We always omit v^i and m^i when there is no confusion. The set $D_p(i)$ contains all optimal affordable items of the bidder at prices p.

Definition 1. *A* Walrasian equilibrium *is a pair (π, p) of assignment π and prices p such that $\pi(i) \in D_p(i)$ for every $i \in M$ and $p(a) = 0$ for every unassigned item $a \in \pi(0)$.*

At equilibrium, every bidder gets his best item at the prices within his budget and the price of every unsold item is equal to zero.

If (π, p) is a Walrasian equilibrium, then p is called an equilibrium or market-clearing price vector and π a Walrasian equilibrium assignment. It is well-known from Koopmans and Beckmann (1957) and Shapley and Shubik (1971) that there will be at least one Walrasian equilibrium and the set of Walrasian equilibrium price vectors forms a lattice when no agent is budget constrained. It is known when bidders are not budget constrained, any Walrasian assignment must be fully efficient. However, if bidders are budget constrained, a Walrasian assignment need not be fully efficient.

The following example shows that when buyers are budget constrained and even if their budgets are different, the Walrasian equilibrium still cannot be guaranteed to exist.

Example 1. *A seller has two items $\{a, b\}$ for sale. There are three bidders 1, 2 and 3. Each bidder demands no more than one item and has valuation and budget as given in Table 1.Observe that each bidder has a different budget and both bidders 2 and 3 are financially constrained.*

Table 1: Valuation and budget

Bidder	$v^i(0)$	$v^i(a)$	$v^i(b)$	Budget m^i
1	0	8	6	9
2	0	7	0	5
3	0	0	6	3

We will prove that there exists no Walrasian equilibrium due to budget constraints. Suppose there would be a Walrasian equilibrium price vector $p = \big(p(a), p(b)\big)$. It is easy to see that both items must be sold. This means that it is necessary to have $p(a) \leq 8$ and $p(b) \leq 6$. We need to consider the following cases in which the two inequalities hold.

Case 1. When $p(a) = p(b) + 2$, we have $D_p(1) = \{a, b\}$. If $p(a) \leq m^2 = 5$, then we have $D_p(2) = \{a\}$ and $D_p(3) = \{b\}$ and the set $\{a, b\}$ is over-demanded. If $p(a) > m^2 = 5$, then $D_p(2) = D_p(3) = 0$ and the set $\{a, b\}$ is under-demanded. In either case, there is no equilibrium.

Case 2. When $p(a) < p(b) + 2$, we have $D_p(1) = \{a\}$. In order to have an equilibrium we must have $p(b) \leq m^3 = 3$, which implies $p(a) < 5 = m^2$. Then we have $D^2(p) = \{a\}$. So item a is over-demanded and we cannot have an equilibrium.

Case 3. When $p(a) > p(b) + 2$, we have $D_p(1) = \{b\}$. In order to have an equilibrium we must have $p(a) \leq m^2 = 5$, which implies $p(b) < 3 = m^3$. Then we have $D_p(3) = \{b\}$. So item b is over-demanded and we cannot have an equilibrium.

Observe that because goods are indivisible and bidders are budget constrained, some utilities cannot be transferred from one agent to another. This example motivates us to make use of a more general solution: the core. The notion of core has been widely used in general equilibrium theory and cooperative game theory; see e.g., Debreu and Scarf (1963), Scarf (1967), Shapley (1973), Shapley and Scarf (1974), Quinzii (1984), and Predtetchinski and Herings (2004). We now introduce this concept of core for nontransferable utility environments.

An allocation (π, r) is *individually rational* if every agent $i \in M_0$ achieves no less utility than they stand alone, i.e., $u^i(\pi, r) \geq m^i$ for every $i \in M$ and $u^0(\pi, r) \geq 0$ for the seller. An allocation (π, r) is *Pareto efficient* if there does not exist another allocation (ρ, τ) such that $u^i(\rho, \tau) > u^i(\pi, r)$ for all $i \in M_0$; otherwise, we say that (π, r) is strongly Pareto dominated by (ρ, τ). An allocation (π, r) is *strongly Pareto efficient* if there does not exist another allocation (ρ, τ) such that $u^i(\rho, \tau) \geq u^i(\pi, r)$ for all $i \in M_0$ with at least one strict inequality; otherwise, we say that (π, r) is Pareto dominated by (ρ, τ). A nonempty subset $S \subseteq M_0$ is called a viable coalition if S consists of either the seller with any number of bidders or a single bidder. Given a viable coalition S including the seller, an allocation (ρ^S, τ) is feasible for S, if $\tau = (\tau^i)_{i \in M_0}$ is an income distribution such that $\sum_{i \in S} \tau^i = \sum_{i \in S} m^i$, and $\rho^S(i) = \emptyset$ and $\tau^i = m^i$ for every bidder $i \in M_0 \setminus S$. An allocation (π, r) is blocked by a single bidder $i \in M$ if it is not individually rational for i such that $m^i > u^i(\pi, r)$. An allocation (π, r) is blocked by a viable coalition $S \ni 0$ if there exists a feasible allocation (ρ, τ) such that $u^i(\rho, \tau) > u^i(\pi, r)$ for all $i \in S$; the allocation (π, r) is weakly blocked by a viable coalition $S \ni 0$ if there exists a feasible allocation (ρ, τ) such that $u^i(\rho, \tau) \geq u^i(\pi, r)$ for all $i \in S$ and with at least one strict inequality.

Definition 2. *An allocation (π, r) is in the* core *and is called a core allocation if it is not blocked by any coalition. It is in the* strong core *and is called a strong core allocation if it cannot be weakly blocked by any coalition.*

Clearly, every core allocation or element is Pareto efficient and every strong core allocation is strongly Pareto efficient. It can be shown that if no bidder is budget constrained, then every strongly Pareto efficient allocation is fully efficient. However, when bidders face budget constraints, a strongly Pareto efficient need not be fully efficient.

Let us return to Example 1 which has no Walrasian equilibrium due to budget constraints. However, it is easy to verify that this example has the following core allocations $(\pi, r) = \big((0, a, 0, b), (9, 3, 5, 0)\big)$ and $(\pi', r') = \big((0, a, b, 0), (9, 5, 0, 3)\big)$. These are not in the strong core as they can be weakly blocked by a coalition.[1]

[1] The strong core of this problem is not empty. For example, allocation $(\pi'', r'') = \big((0, a, 0, b), (10, 2, 5, 0)\big)$ is a strong core allocation.

3 Main Results

In this section we present the main results of the paper including the new dynamic auction in Section 3.1, an illustrative example in Section 3.2, a strategic result in Section 3.3, and several results on the core and equilibrium in Section 3.4.

3.1 The Design of Dynamic Auction

We introduce an ascending auction which is a variant of the deferred acceptance algorithm (Gale and Shapley, 1962) and is also related to Crawford and Knoer (1981), Leonard (1983), Demange et al. (1986), Bernheim and Whinston (1986), Ausubel and Milgrom (2002), Andersson et al. (2013), Andersson and Erlanson (2013), and Herings and Zhou (2022) among others which were designed to deal with the situation without budget constraints. Differing from these existing auctions, this new auction can accommodate all kinds of budget constraints and induce bidders to act truthfully. The basic idea of the auction can be roughly described as follows. At the first round, each bidder makes some bids or no bid to the seller, and the seller chooses a set of bids yielding the highest (artificial) revenue to her and asks every provisionally losing bidder to make new bids. At the following rounds, the losing bidder will make possible new bids to or withdraw some of his earlier bids from the seller. The auction process continues until no bidder is rejected. When the auction stops, the chosen bids become finally accepted.

We now give a detailed description of the dynamic auction mechanism.

The Dynamic Auction

Initialization: Set $k = 1$ being the first round. Every bidder $i \in M$ decides whether to bid or not. He can make a bid $p_1^i(a) \in \mathbb{Z}_+$ on some item $a \in N_0$ or several bids if he is indifferent to them. Go to the Assigning stage.

Bidding stage: After being offered to make new bids, every provisionally losing bidder i increases at least one of his previous bids by one unit or withdraws some of his previous bids or makes a bid $p_k^i(a) \in \mathbb{Z}_+$ on some item $a \in N_0$ which he has not bid previously. He can do this operation on several items if he is indifferent to them. Any other bidder j keeps his bids unchanged by setting $p_k^j = p_{k-1}^j$. Go to the Assigning stage.

Assigning stage: If a bidder i does not bid on the null item, he is said to be *active* and his price of the null item is set as $p_k^i(0) = -2^{-i}$. Otherwise, bidder i is *inactive* and he must bid $p_k^i(0) = 0$. In this way we have the price system $P_k = (p_k^i)_{i \in M}$ at time k.

Based on the current bidding prices $P_k = (p_k^i)_{i \in M}$, the seller finds an optimal assignment π_k by solving the following maximization problem

$$\max_{\rho \in \mathcal{A}} \sum_{i \in M} p_k^i(\rho(i)). \tag{2}$$

At π_k, bidder i is said to be a *provisional loser*, if he is active and assigned the null item, i.e., $p_k^i(0) = -2^{-i}$ and $\pi_k(i) = 0$. If there is no provisional loser, go to the Final Assignment. Otherwise, the seller asks all provisional losers to submit new bids at next round. Set $k = k + 1$ and go to the Bidding stage.

Final Assignment The auction stops. The auctioneer assigns every bidder $i \in M$ with item $\pi_k(i)$ specified by the current provisional assignment π_k and receives the corresponding payment $p_k^i(\pi_k(i))$ from bidder i for the item.

The proposed auction rules are very intuitive, general, easy to implement, and do not impose any unreasonable restriction on bidders' behavior. In the auction, every bidder can decide whether to bid or not and what items to bid, and can also withdraw bids. The auction has a unique and specific activity rule on the null item. When a bidder $i \in M$ does not bid on the null item, the price of the null item is set to be $p_k^i(0) = -2^{-i}$, depending on the bidder. Otherwise, the bid on the null item is set to be $p_k^i(0) = 0$. This means that every bidder just needs to indicate if he wants to demand a null item or not. The auctioneer will set the bid on the null item. The rule can be easily implemented and can prevent any bidder's flagrant manipulation. More importantly, this is a novel tie-breaking rule and will play an indispensable rule in establishing several basic properties of the auction. Observe that when the auction terminates, any bidder who is assigned a null item must have bid it and will pay nothing.

Notice that the objective function of the problem (2) can be seen as an artificial revenue of the seller before the auction stops, because it contains the artificial price of $p_k^i(0) = -2^{-i}$. However, it will become the true revenue of the seller when the auction stops. The important and novel point of the problem (2) is that it always has a unique set of winners, and its solution is also an optimal solution to the following revenue maximization problem

$$\max_{\rho \in \mathcal{A}} \sum_{i \in M} \hat{p}_k^i(\rho(i)), \tag{3}$$

where $\hat{p}_k^i(0) = 0$ and $\hat{p}_k^i(a) = p_k^i(a)$ for all $i \in M$, k, and $a \in N$. It can be easily understood that an optimal solution of the problem (3) need not be an optimal solution of the problem (2).

When facing the auction, every rational bidder could act sincerely or strategically as long as it serves his interest. Because both valuations and budgets are private information, a manipulative bidder may not necessarily behave honestly according to his true valuations or budget. In the following we will investigate various properties of the auction. When facing the auction, it is best or optimal for every bidder to bid truthfully. In other words, sincere bidding will be a Nash equilibrium of the underlying dynamic auction game with incomplete information. We will also show that when bidders bid sincerely, the auction will find a strongly Pareto-efficient core allocation when bidders are budget constrained, otherwise a Walrasian equilibrium with the minimum equilibrium price vector, thus always yielding an efficient outcome in all circumstances.

We now specify and focus on a class of sincere bidding strategies that can facilitate the bidding process. In such strategies bidders make bids according to their true valuations and budgets. Every bidder $i \in M$ initially sets a target utility $\hat{u}_1^i \in \mathbb{Z}_+$ that the bidder wishes to achieve

$$\hat{u}_1^i \geq \max_{a \in N_0} v^i(a) + m^i.$$

On each round, he will make bids according to this target and also update this target utility by gradually decreasing it. On each round k, for every item $a \in N_0$, the bidder calculates a possible bidding price

$$\hat{p}^i(a|\hat{u}_k^i) = v^i(a) + m^i - \hat{u}_k^i,$$

and makes a bid $p_k^i(a) = \hat{p}^i(a|\hat{u}_k^i)$ on every item $a \in N_0$ if the bidding price $p_k^i(a)$ is nonnegative and does not exceed his budget m^i. Because agents are rational, no bidder will make any nonsense bid such as a negative price for any item or any bid which makes his position worse than his status quo m^i. The seller will not sell any of her items if its price is below 0.

On each subsequent round $k > 1$, if a bidder i is a provisional loser, he will be offered new opportunities to make new bids. He will need to reduce his target utility by a decrement $\min\{d \in \mathbb{Z}_{++} \mid \hat{p}^i(a|\hat{u}_k^i - d) \in [0, m^i] \text{ for some } a \in N_0\}$. This is the minimal integer that leads to new bids. In most cases, the decrement is one. However, when the bidding price of some item reaches the budget, the decrement can be larger than one.

3.2 An Illustrative Example

We illustrate the proposed dynamic auction mechanism and compare it with the well-known DGS auction through the following example. It should be pointed out that the DGS was proposed to deal with the assignment market without budget constrained bidders. If the DGS auction applies to the current example, it starts with prices $p_1(0, a, b) = (0, 0, 0)$ and ends up with $p_7(0, a, b) = (0, 6, 4)$, at which no bidder demands item a.

Example 2. *A seller has two items $\{a, b\}$ for sale. There are four bidders 1, 2, 3, and 4. Each bidder has valuations and a budget as given in Table 2. Observe that bidders are financially constrained.*

Table 2: Valuation and budget

Bidder	$v^i(0)$	$v^i(a)$	$v^i(b)$	Budget m^i
1	0	10	2	5
2	0	10	4	5
3	0	2	7	4
4	0	7	7	3

Table 3 collects the information generated by the current auction mechanism. When a bidder does not make a bid on an item or withdraws a bid on an item, the symbol $-$ will be used.

Observe that on the first round, bidder 1's initial target utility is 16 and does not make any offer. No bidder bids for the null item, so they are all active. Observe that when

Table 3: Illustration of the proposed auction mechanism for Example 2.

k	$(\hat{u}^1_k, \hat{u}^2_k, \hat{u}^3_k, \hat{u}^4_k)$	$p^1_k(0,a,b)$	$p^2_k(0,a,b)$	$p^3_k(0,a,b)$	$p^4_k(0,a,b)$	$\pi_k(0,1,2,3,4)$
1	$(16,15,11,10)$	$(-\frac{1}{2},-,-)$	$(-\frac{1}{4},0,-)$	$(-\frac{1}{8},-,0)$	$(-\frac{1}{16},0,0)$	$(0,0,a,b,0)$
2	$(15,15,11,9)$	$(-\frac{1}{2},0,-)$	$(-\frac{1}{4},0,-)$	$(-\frac{1}{8},-,0)$	$(-\frac{1}{16},1,1)$	$(0,a,0,0,b)$
3	$(15,14,10,9)$	$(-\frac{1}{2},0,-)$	$(-\frac{1}{4},1,-)$	$(-\frac{1}{8},-,1)$	$(-\frac{1}{16},1,1)$	$(0,0,a,b,0)$
4	$(14,14,10,8)$	$(-\frac{1}{2},1,-)$	$(-\frac{1}{4},1,-)$	$(-\frac{1}{8},-,1)$	$(-\frac{1}{16},2,2)$	$(0,a,0,0,b)$
5	$(14,13,9,8)$	$(-\frac{1}{2},1,-)$	$(-\frac{1}{4},2,-)$	$(-\frac{1}{8},-,2)$	$(-\frac{1}{16},2,2)$	$(0,0,a,b,0)$
6	$(13,13,9,7)$	$(-\frac{1}{2},2,-)$	$(-\frac{1}{4},2,-)$	$(-\frac{1}{8},-,2)$	$(-\frac{1}{16},3,3)$	$(0,a,0,0,b)$
7	$(13,12,8,7)$	$(-\frac{1}{2},2,-)$	$(-\frac{1}{4},3,-)$	$(-\frac{1}{8},-,3)$	$(-\frac{1}{16},3,3)$	$(0,0,a,b,0)$
8	$(12,12,8,3)$	$(-\frac{1}{2},3,-)$	$(-\frac{1}{4},3,-)$	$(-\frac{1}{8},-,3)$	$(0,-,-)$	$(0,a,0,b,0)$
9	$(12,11,8,3)$	$(-\frac{1}{2},3,-)$	$(-\frac{1}{4},4,-)$	$(-\frac{1}{8},-,3)$	$(0,-,-)$	$(0,0,a,b,0)$
10	$(11,11,8,3)$	$(-\frac{1}{2},4,-)$	$(-\frac{1}{4},4,-)$	$(-\frac{1}{8},-,3)$	$(0,-,-)$	$(0,a,0,b,0)$
11	$(11,10,8,3)$	$(-\frac{1}{2},4,-)$	$(-\frac{1}{4},5,-)$	$(-\frac{1}{8},-,3)$	$(0,-,-)$	$(0,0,a,b,0)$
12	$(10,10,8,3)$	$(-\frac{1}{2},5,-)$	$(-\frac{1}{4},5,-)$	$(-\frac{1}{8},-,3)$	$(0,-,-)$	$(0,a,0,b,0)$
13	$(10,9,8,3)$	$(-\frac{1}{2},5,-)$	$(-\frac{1}{4},-,0)$	$(-\frac{1}{8},-,3)$	$(0,-,-)$	$(0,a,0,b,0)$
14	$(10,8,8,3)$	$(-\frac{1}{2},5,-)$	$(-\frac{1}{4},-,1)$	$(-\frac{1}{8},-,3)$	$(0,-,-)$	$(0,a,0,b,0)$
15	$(10,7,8,3)$	$(-\frac{1}{2},5,-)$	$(-\frac{1}{4},-,2)$	$(-\frac{1}{8},-,3)$	$(0,-,-)$	$(0,a,0,b,0)$
16	$(10,6,8,3)$	$(-\frac{1}{2},5,-)$	$(-\frac{1}{4},-,3)$	$(-\frac{1}{8},-,3)$	$(0,-,-)$	$(0,a,b,0,0)$
17	$(10,6,7,3)$	$(-\frac{1}{2},5,-)$	$(-\frac{1}{4},-,3)$	$(-\frac{1}{8},-,4)$	$(0,-,-)$	$(0,a,0,b,0)$
18	$(10,5,7,3)$	$(-\frac{1}{2},5,-)$	$(0,-,4)$	$(-\frac{1}{8},-,4)$	$(0,-,-)$	$(0,a,0,b,0)$

the auction moves from the 7th round and the next round, the target utility of bidder 4 decreases from 7 to 3 as his bid reaches his budget. On the 8th round, bidder 4 withdraws his bids on items a and b.

The auction stops at $k = 18$ when there is no provisional loser. On this round, bidder 2 bids on the null item and becomes inactive, while both bidder 1 and bidder 3 are still active. Observe that although both bidders 2 and 3 offer the same bid of 4 on item b, bidder 3 has a higher priority over bidder 2, as bidder 2's bid on the null item is 0 but bidder 3's bid on it is $-\frac{1}{8}$. Clearly, $\pi^* = (0, a, 0, b, 0)$ is the unique optimal assignment. So in the final outcome, bidder 1 gets a and pays 5 and bidder 3 gets b and pays 4, and all others get nothing and pay nothing. It is easy to verify that this is a core allocation.

3.3 Incentive Results

In this subsection we will show that sincerely bidding is a Nash equilibrium of the underlying auction game with incomplete information on valuations and budgets. To establish this result, we need to prove that sincere bidding is optimal for every bidder, provided that all other bidders bid truthfully. To facilitate a better understanding of this result, we will first give an intuitive but informal argument for the basic case of a single item a. To see this, assume that bidder i^* is the unique winner of the item in the proposed dynamic auction when all bidders act truthfully according to their true valuations and budgets. By the auction rule bidder i^* will pay a price proposed by him $p^{i^*}(a)$ which equals the 'second' highest price $p^{j^*}(a) = \max_{j\in M\setminus\{i^*\}} \left(\min\{v^j(a), m^j\}\right)$ for $i^* < j^*$ or equals $p^{j^*}(a) + 1$ otherwise. He will make a loss if he withdraws earlier or if any of his budget and valuation is below the value $p^{i^*}(a)$. Clearly, sincerely bidding is an optimal strategy for bidder i^*. For any other bidder $j \neq i^*$, nothing will change if he acts according to a budget and a valuation of which minimum is still below $p^{i^*}(a)$. Otherwise, he will win the item but make a loss. Clearly, sincerely bidding is also an optimal strategy for any bidder $j \neq i^*$.

Let us now make preparations to establish our general incentive result which requires more sophisticated arguments. We first recall and examine the rule of the Assigning Stage of the proposed auction. On each round k, the seller solves the constrained integer linear programming problem $\max_{\rho\in\mathcal{A}} \sum_{i\in M} p^i_k(\rho(i))$, where $p^i_k(0)$ is negative for each active bidder i. We will consider an equivalent variant of this problem and investigate its properties. Let M^a_k and M^i_k denote the set of active bidders and the set of inactive bidders of round k, respectively. For every active bidder $i \in M^a_k$, we add an increment 2^{-i} on his bidding price vector and obtain an adjusted price vector q^i_k. In this way, we have $q^i_k(0) = 0$. For every inactive bidder $i \in M^i_k$, just let $q^i_k = p^i_k$. Let $Q_k = (q^i_k)_{i\in M}$ be the adjusted price vector profile of round k. Then the seller solves the following constrained integer linear programming problem

$$\max_{\rho \in \mathcal{A}} \sum_{i \in M} q_k^i(\rho(i)) = \max_{\rho \in \mathcal{A}} \sum_{i \in M} p_k^i(\rho(i)) + \sum_{i \in M_k^a} 2^{-i}, \tag{4}$$

which shares the same solution with the original problem (2).

Let $\alpha = \left(\alpha(a)\right)_{a \in N_0} \in \mathbb{R}_+^{|N|}$ with $\alpha(0) = 0$ be an optimal solution of the dual of the problem (4). Each component $a \in N_0$ of this solution gives a post price $\alpha(a)$ for item a. Given the post price vector α and the adjusted price vector profile Q_k, let $\beta_\alpha = \left(\beta_\alpha(i|Q_k)\right)_{i \in M} \in \mathbb{R}_+^{|M|}$ be an extra bidding price vector such that $\beta_\alpha(i|Q_k) = \max\left\{\max_{a \in N}\left(q_k^i(a) - \alpha(a)\right), 0\right\}$ for each $i \in M$. The demand set of bidder i at this round is defined by

$$D_\alpha(i|Q_k) = \left\{a \in N_0 \mid q_k^i(a) - \alpha(a) = \beta_\alpha(i|Q_k)\right\}.$$

Let $D_\alpha(R|Q_k) = \bigcup_{i \in R} D_\alpha(i|Q_k)$ denote the set of items demanded by a group R of bidders, i.e., $R \subseteq M$. Similarly, let $D_\alpha^{-1}(a|Q_k) = \{i \in M \mid a \in D_\alpha(i|Q_k)\}$ be the set of bidders who demands item $a \in N$ and $D_\alpha^{-1}(S|Q_k) = \bigcup_{a \in S} D_\alpha^{-1}(a|Q_k)$ the set of bidders who demand at least one item from the set $S \subseteq N$.

Define $M^+(\alpha|Q_k) = \{i \in M \mid \beta_\alpha(i|Q_k) > 0\}$ and $N^+(\alpha|Q_k) = \{a \in N \mid \alpha(a) > 0\}$. A set of bidders $R \subseteq M^+(\alpha|Q_k)$ is *under-supplied* if $|R| > |D_\alpha(R|Q_k)|$. In this case, the set of items $D_\alpha(R|Q_k)$ is called *over-demanded*. A set of items $S \subseteq N^+(\alpha|Q_k)$ is *under-demanded* if $|S| > D_\alpha^{-1}(S|Q_k)|$. If there is neither under-supply nor under-demand, the post price vector α is said to be balanced, and there exists an assignment, π_k, at which every item with positive post price is assigned to a bidder who demands it and every bidder with positive extra bidding price is assigned an item in his demand set. If so, we have $\sum_{i \in M} q_k^i(\pi_k(i)) = \sum_{a \in N} \alpha(a) + \sum_{i \in M} \beta_\alpha(i|Q_k)$ by the fundamental duality theorem (Schrijver, 1986).

It follows from Shapley and Shubik (1971) and Gul and Stacchetti (1999) that the set of balanced post prices forms a nonempty complete lattice. Specifically, let α_k denote the maximum balanced price vector under Q_k. The following two properties of the maximum balanced price vectors will be used to establish our incentive result.

The first result shows that $\alpha_k(a)$ is the threshold of round k for every item $a \in N$, so bidder i provisionally wins if and only if there is at least one item $a \in N$ such that $q_k^i(a) \geq \alpha_k(a)$.

Lemma 1. *If i is a provisional loser at round k, then $q_k^i(a) < \alpha_k(a)$ for all $a \in N$.*

Proof. Observe that if $q_k^i(a^*) > \alpha_k(a^*)$ for some $a^* \in N$, then $\beta_{\alpha_k}(i|Q_k) > 0$ implies that bidder i should win at the provisional assignment π_k.

Suppose that $\alpha_k(a^*) = q_k^i(a^*)(= p_k^i(a^*) + 2^{-i})$ for some $a^* \in N$. Recall that a provisional loser i is active. Here we need to introduce an additional concept. A set R of bidders with $R \subseteq M^+(\alpha|Q_k)$ is called *weakly under-supplied* if $|R| = |D_\alpha(R|Q_k)|$. By Theorem 2 of Mishra and Talman (2010), there is no weakly under-supplied set at the maximum balanced price vector. Let $S = \{a \in N \mid \alpha_k(a) = \ell + 2^{-i} \text{ for some integer } \ell\}$. Clearly, S contains a^*. Let $R = \{j \in M \mid \pi_k(j) \in S\}$. For every bidder $j \in R$, that $j \neq i$ implies $\beta_{\alpha_k}(j|Q_k) = q_k^j(\pi_k(j)) - \ell - 2^{-i} > 0$ and $j \in M^+(\alpha_k|Q_k)$. For every $b \in D_{\alpha_k}(j|Q_k)$, $\alpha_k(b) = q_k^j(b) - \beta_{\alpha_k}(j|Q_k) = p_k^j(b) - p_k^j(\pi_k(j)) + \ell + 2^{-i}$ implies that $b \in S$ and thus $D_{\alpha_k}(j|Q_k) \subseteq S$. In summary, we have $R \subseteq M^+(\alpha_k|Q_k)$ and $|R| = |D_{\alpha_k}(R|Q_k)| = |S|$. This contradicts the fact that there is no weakly under-supplied set at the maximum balanced price vector α_k. □

The next lemma says that the threshold vector α_k monotonically increases with the time k.

Lemma 2. *The maximum balanced price vector α_k weakly increases with the time k, i.e., $\alpha_k \leq \alpha_{k+1}$ for all k.*

Proof. Since α_k is a balanced post price under Q_k, there is no under-demand at α_k. Specifically, every item $a \in N^+(\alpha_k|Q_k)$ is demanded at least by a provisional winner i with $\pi_k(i) = a$. On round $k+1$, the provisional winners of round k keep their bidding prices unchanged, so there is no under-demand at α_k under Q_{k+1}. By Theorem 2 of Mishra and Talman (2010) we know that there exists a balanced post price vector α'_{k+1} for Q_{k+1} such that $\alpha'_{k+1} \geq \alpha_k$. Since α_{k+1} is the maximum balanced post price vector for Q_{k+1}, we have $\alpha_{k+1} \geq \alpha_k$. □

An immediate implication of the above two lemmas is that if bidder i bids $p^i_k(a)$ and provisionally loses on round k, then he cannot win item a on any latter round $k' > k$ by repeating the same bid $p^i_{k'}(a) = p^i_k(a)$.

We are now ready to establish the first major result of this paper which shows that in the face of the proposed dynamic auction, it is an optimal strategy for every bidder to bid truthfully.

Theorem 1. *Sincerely bidding by every bidder is a Nash equilibrium of the dynamic auction game with incomplete information.*

Proof. Take an arbitrary bidder $i_0 \in M$ and assume that all bidders but i_0 always bid sincerely. If i_0 also bids sincerely, we use $(P_k, Q_k, \alpha_k, \pi_k, r_k)_{1 \leq k \leq K}$ to describe the truthful auction process. Suppose i_0 can manipulate the auction and get a better outcome. We use $(\tilde{P}_{\tilde{k}}, \tilde{Q}_{\tilde{k}}, \tilde{\alpha}_{\tilde{k}}, \tilde{\pi}_{\tilde{k}}, \tilde{r}_{\tilde{k}})_{1 \leq \tilde{k} \leq \tilde{K}}$ to describe the manipulated auction process.

Let $a_0 = \tilde{\pi}_{\tilde{K}}(i_0)$. i_0 strictly prefers $(\tilde{\pi}_{\tilde{K}}, \tilde{r}_{\tilde{K}})$ to (π_K, r_K) and thus to $(0, m^i)$. So up to some round $k \leq K$ of the truthful auction (in which i_0 also bids sincerely), i_0 proposes the offer $p^{i_0}_k(a_0) = \tilde{p}^{i_0}_{\tilde{K}}(a_0)$ but is rejected and is required to make new bids on next round. On round k, he must not have bid on the null item and thus $q^i_k = p^i_k + 2^{-i}$. We now compare the two outcomes $(\tilde{\pi}_{\tilde{K}}, \tilde{r}_{\tilde{K}})$ and (π_k, r_k).

If item a_0 is not assigned at π_k, then the assignment which assigns a_0 to i_0 and all other items to the bidders as π_k does would yield a higher value for the optimal problem $\max_{\rho \in \mathcal{A}} \sum_{i \in M} q^i_k(\rho(i))$ than π_k does. Thus a_0 should be assigned to some bidder at π_k. Let i_1 be the bidder such that $\pi_k(i_1) = a_0$. If i_1 (weakly) prefers $(\tilde{\pi}_{\tilde{K}}, \tilde{r}_{\tilde{K}})$ to (π_k, r_k), then $a_1 = \tilde{\pi}_{\tilde{K}}(i_1) \neq 0$. If item a_1 is not sold at (π_k, r_k), then the assignment which assigns a_0

to i_0, assigns a_1 to i_1, and assigns all other items to the bidders as π_k does, would yield a higher value for the optimal problem $\max_{\rho \in \mathcal{A}} \sum_{i \in M} q_k^i(\rho(i))$ than π_k does. So a_1 should be assigned at (π_k, r_k) and let i_2 denote the bidder such that $\pi_k(i_2) = a_1$. If i_2 (weakly) prefers $(\tilde{\pi}_{\tilde{K}}, \tilde{r}_{\tilde{K}})$ to (π_k, r_k), we can repeat the same argument and define i_3 as the bidder such that $\pi_k(i_3) = a_2 = \tilde{\pi}_{\tilde{K}}^{-1}(i_2)$, and so on. i_0 gets 0 at π_k and gets item a_0 at $\tilde{\pi}_{\tilde{K}}$, so there is at least one bidder who gets an item at π_k but gets 0 at $\tilde{\pi}_{\tilde{K}}$. By repeating the above argument, we can always find a bidder, say i_L, who strictly prefers (π_k, r_k) to $(\tilde{\pi}_{\tilde{K}}, \tilde{r}_{\tilde{K}})$. Similarly, let $a_L = \tilde{\pi}_{\tilde{K}}(i_L)$. In this case, a_L may be 0.

	i_0	i_1	i_2	$\cdots$	i_L
$\tilde{\pi}_{\tilde{K}}(\cdot)$	a_0	a_1	a_2	$\cdots$	a_L
$\pi_k(\cdot)$	$\emptyset$	a_0	a_1	$\cdots$	a_{L-1}

That bidder i_0 loses at (π_k, r_k) but wins a_0 at $(\tilde{\pi}_{\tilde{K}}, \tilde{r}_{\tilde{K}})$ implies that $\alpha_k(a_0) > q_k^{i_0}(a_0) = \tilde{q}_{\tilde{K}}^{i_0}(a_0) \geq \tilde{\alpha}_{\tilde{K}}(a_0)$. Let's show that $\alpha_k(a_{\ell-1}) > \tilde{\alpha}_{\tilde{K}}(a_{\ell-1})$ implies that $\alpha_k(a_\ell) > \tilde{\alpha}_{\tilde{K}}(a_\ell)$ for all $\ell \in \{1, \ldots, L-1\}$.

Consider the following two cases. Case 1: bidder i_ℓ strictly prefers $(\tilde{\pi}_{\tilde{K}}, \tilde{r}_{\tilde{K}})$ to (π_k, r_k). On some round $k' < k$ of the truthful auction process, i_ℓ proposes $p_{k'}^{i_\ell}(a_\ell) = \tilde{p}_{\tilde{K}}^{i_\ell}(a_\ell)$ but is rejected and required to submit new bids. This implies that $\alpha_k(a_\ell) \geq \alpha_{k'}(a_\ell) > q_{k'}^{i_\ell}(a_\ell) = \tilde{q}_{\tilde{K}}^{i_\ell}(a_\ell) \geq \tilde{\alpha}_{\tilde{K}}(a_\ell)$. Case 2: bidder i_ℓ is indifferent to the two allocations $(\tilde{\pi}_{\tilde{K}}, \tilde{r}_{\tilde{K}})$ and (π_k, r_k). Then we have $q_k^{i_\ell}(a_\ell) = \tilde{q}_{\tilde{K}}^{i_\ell}(a_\ell)$ and $q_k^{i_\ell}(a_{\ell-1}) = \tilde{q}_{\tilde{K}}^{i_\ell}(a_{\ell-1})$. On round k of the truthful auction process, that i_ℓ wins $a_{\ell-1}$ implies that $a_{\ell-1} \in D_{\alpha_k}(i_\ell | Q_k)$ and thus $q_k^{i_\ell}(a_{\ell-1}) - \alpha_k(a_{\ell-1}) \geq q_k^{i_\ell}(a_\ell) - \alpha_k(a_\ell)$. Similarly, on round $\tilde{K}$ of the manipulated auction process, that i_ℓ wins a_ℓ implies that $a_\ell \in D_{\tilde{\alpha}_{\tilde{K}}}(i_\ell | \tilde{Q}_{\tilde{K}})$ and thus $\tilde{q}_{\tilde{K}}^{i_\ell}(a_\ell) - \tilde{\alpha}_{\tilde{K}}(a_\ell) \geq \tilde{q}_{\tilde{K}}^{i_\ell}(a_{\ell-1}) - \tilde{\alpha}_{\tilde{K}}(a_{\ell-1})$. In summary, we have

$$\alpha_k(a_\ell) \geq \alpha_k(a_{\ell-1}) + \left(q_k^{i_\ell}(a_\ell) - q_k^{i_\ell}(a_{\ell-1})\right) > \tilde{\alpha}_{\tilde{K}}(a_{\ell-1}) + \left(\tilde{q}_{\tilde{K}}^{i_\ell}(a_\ell) - \tilde{q}_{\tilde{K}}^{i_\ell}(a_{\ell-1})\right) \geq \tilde{\alpha}_{\tilde{K}}(a_\ell).$$

By induction, we have $\alpha_k(a_{L-1}) > \tilde{\alpha}_{\tilde{K}}(a_{L-1})$. Since i_L strictly prefers (π_k, r_k) to $(\tilde{\pi}_{\tilde{K}}, \tilde{r}_{\tilde{K}})$, then on some round $\tilde{k} \leq \tilde{K}$ of the manipulated auction process i_L makes the bid $\tilde{p}^{i_L}_{\tilde{k}}(a_{L-1}) = p^{i_L}_k(a_{L-1})$ but is rejected and required to make new bids. We have $\tilde{\alpha}_{\tilde{K}}(a_{L-1}) \geq \tilde{\alpha}_{\tilde{k}}(a_{L-1}) > \tilde{q}^{i_L}_{\tilde{k}}(a_{L-1}) = q^{i_L}_k(a_{L-1}) \geq \alpha_k(a_{L-1})$, yielding a contradiction. □

3.4 Core and Equilibrium Properties

In the previous section we have proved that sincere bidding is an optimal strategy for every bidder in the face of the proposed dynamic auction. In this subsection, we will explore other important properties of the auction in the environment where all bidders bid sincerely. For the auction model $((v^i, m^i), i \in M, N)$, let K denote the last round of the dynamic auction. The final assignment is π_K, and the corresponding income distribution is $r^0 = \sum_{i \in M} p^i_K(\pi_K(i))$ for the seller, and $r^i = m^i - p^i_K(\pi_K(i))$ for every bidder $i \in M$. Bidder i is said to be a loser if he is assigned the null item $\pi_K(i) = 0$; otherwise, he is a winner. Let (π_K, r) be the final outcome generated by the auction.

Lemma 3. *The outcome (π_K, r) generated by the proposed auction is individually rational and gives every bidder $i \in M$ his target utility $\hat{u}^i_K$ and the seller her highest revenue under P_K.*

Proof. Observe that bidder i receives item $a \in N_0$ only if he bids on a, that is $p^i_K(a) = v^i(a) - \hat{u}^i_K + m^i$. So bidder i's utility is $u^i(\pi_K, r) = v^i(a) + m^i - p^i_K(a) = \hat{u}^i_K$. Once a bidder's target $\hat{u}^i_k$ equals his budget m^i, then $\hat{p}^i(0|\hat{u}^i_k) = 0$ implies that he bids on the null item and becomes inactive. An inactive bidder cannot be a provisional loser and therefore would not make any new bid, so $\hat{u}^i_k \geq m^i$.

Suppose there is another assignment $\rho \in \mathcal{A}$ that gives the seller a higher revenue under P_K such that $\sum_{i \in M:\rho(i) \neq 0} p^i_K(\rho(i)) \geq r^0 + 1$. Then we have

$$\begin{aligned}\sum_{i \in M} p^i_K(\rho(i)) &= \sum_{i \in M:\rho(i) \neq 0} p^i_K(\rho(i)) + \sum_{i \in M:\rho(i)=0} (-2^{-i}) \\ &> \sum_{i \in M:\rho(i) \neq 0} p^i_K(\rho(i)) - 1 \geq \sum_{i \in M} p^i_K(\pi_K(i)),\end{aligned}$$

which contradicts the fact that π_K is an optimal solution to the problem (2) on the last round K. Since the no-sale assignment is feasible and gives the seller the utility of zero, the seller's optimal choice guarantees her rationality. □

The next result states that the outcome generated by the auction is in the core and it is a strongly Pareto efficient allocation.

Theorem 2. *The outcome (π_K, r) generated by the proposed auction is in the core and strongly Pareto efficient.*

Proof. We first extend every bidder i's price vector on the last round by defining

$$\tilde{p}^i_K(a) = \begin{cases} m^i, & \text{if } \hat{p}^i_K(a|\hat{u}^i_K) > m^i; \\ p^i_K(a), & \text{otherwise.} \end{cases}$$

Using a definition similar to the one in Section 3.3, we define the extended bidding price by letting $\tilde{q}^i_K(a) = \tilde{p}^i_K(a) + 2^{-i}$ for every active bidder $i \in M^a_K$, and $\tilde{q}^i_K(a) = \tilde{p}^i_K(a)$ for every inactive bidder $i \in M^a_K$. Here we show that π_K also solves

$$\max_{\rho \in \mathcal{A}} \sum_{i \in M} \tilde{q}^i_K(\rho(i)), \tag{5}$$

We use $(P_k, Q_k, \alpha_k, \pi_k, r_k)_{1 \leq k \leq K}$ to describe the auction process. For every bidder $i \in M$ and every item $a \in N$ such that $\hat{p}^i_K(a|\hat{u}^i_K) > m^i$, i must bid $p^i_k(a) = \hat{p}^i_k(a|\hat{u}^i_k) = m^i$ on some round $k < K$ and must be a provisional loser. By Lemma 1 and Lemma 2, we have $\alpha_K(a) \geq \alpha_k(a) > m^i + 2^{-i}$ and thus $a \notin D_{\alpha_K}(i|\tilde{Q}_K)$. So $D_{\alpha_K}(i|Q_K)$ coincides with $D_{\alpha_K}(i|\tilde{Q}_K)$. At the extended optimal problem (5), α_K is balanced and π_K is a solution. Similar to the conclusion of Lemma 3, π_K maximizes seller's revenue under $\tilde{P}_K$.

We now prove that the outcome (π_K, r) is a core allocation. By Lemma 3, (π_K, r) is individually rational. Suppose to the contrary that (π_K, r) is not in the core, then there exist a coalition S consisting of the seller and at least one bidder and an allocation (ρ^S, τ) such that $u^i(\rho^S, \tau) > u^i(\pi_K, r)$ for all $i \in S$. For every bidder in the coalition $i \in S \setminus \{0\}$, he wins $\rho^S(i) \neq 0$ and prefers $u^i(\rho^S, \tau)$ to his target utility of round K, i.e., $u^i(\rho^S, \tau) = v^i(\rho^S(i)) + \tau^i > u^i(\pi_K, r) = \hat{u}^i_K$. This implies that he sets a higher possible price on item $\rho^S(i)$, that is $\hat{p}^i_K(\rho^S(i)|\hat{u}^i_K) = v^i(\rho^S(i)) + m^i - \hat{u}^i_K > m^i - \tau^i$ and $\tilde{p}^i_K(\rho^S(i)) \geq m^i - \tau^i$. For the seller, we have

$$\sum_{i \in M: \rho(i) \neq 0} \tilde{p}^i_K(\rho^S(i)) \geq \sum_{i \in S \setminus \{0\}} (m^i - \tau^i) = u^0(\rho^S, \tau) > u^0(\pi_K, r) = \sum_{i \in M: \pi_K(i) \neq 0} \tilde{p}^i_K(\pi_K(i)).$$

It contradicts the fact that π_K maximizes the seller's revenue under $\tilde{P}_K$. Thus, (π_K, r) is in the core.

We next show that (π_K, r) is strongly Pareto efficient. Suppose to the contrary that it is Pareto dominated by an outcome (ρ, τ) such that $u^i(\rho, \tau) \geq u^i(\pi_K, r)$ for all $i \in M_0$ with at least one strict inequality. Because every bidder is (weakly) better off at (ρ, τ), as in the above proof, every bidder $i \in M$ sets target utility $\hat{u}^i_K \leq v^i(\rho(i)) + \tau^i$ and $\tilde{p}^i_K(\rho(i)) \geq m^i - \tau^i$ if $\rho(i) \neq 0$. Since the seller is also weakly better off, we have

$$\sum_{i \in M:\, \rho(i) \neq 0} \tilde{p}^i_K(\rho(i)) \geq \sum_{i \in M:\, \rho(i) \neq 0} (m^i - \tau^i) = u^0(\rho, \tau) \geq u^0(\pi_K, r) = \sum_{i \in M:\, \pi_K(i) \neq 0} \tilde{p}^i_K(\pi_K(i)).$$

Recall that π_K maximizes the seller's revenue under $\tilde{P}_K$. So the seller is indifferent to the two outcomes (π_K, r) and (ρ, τ), and every bidder $i \in M$ sets $\tilde{p}^i_K(\rho(i)) = m^i - \tau^i$.

Then there should be at least one bidder, say bidder j, who strictly prefers (ρ, τ) to (π_K, r). If $\tilde{p}^j_K(\rho(j)) = \hat{p}^j_K(\rho(j)|\hat{u}^j_K)$, then j also gets his target utility at (ρ, τ) and thus is not strictly better off. The only possibility is that $\hat{p}^j_K(\rho(j)|\hat{u}^j_K) > \tilde{p}^j_K(\rho(j)) = m^j$. It means that, on some round $k < K$, j must have bid $p^j_k(\rho(j)) = m^j$ but has been rejected. By Lemma 1 and Lemma 2, we have $\alpha_K(\rho(j)) \geq \alpha_k(\rho(j)) > m^j + 2^{-j}$ and thus $\rho(j) \notin D_{\alpha_K}(j|\tilde{Q}_K)$. The assignment ρ which assigns bidder j an item not in his demand set under a balanced post price is not an optimal solution of the problem (5). We have

$$\sum_{i \in M} \tilde{p}^i_K(\rho(i)) < \sum_{i \in M} \tilde{p}^i_K(\pi_K(i)) \quad \Rightarrow \quad \sum_{i \in M:\, \rho(i)=0} \tilde{p}^i_K(\rho(i)) < \sum_{i \in M:\, \pi_K(i)=0} \tilde{p}^i_K(\pi_K(i)) = 0.$$

There must be an active bidder $i \in M^a_K$ who gets an null item $\rho(i) = 0$ and pays 0 at (ρ, τ). However, the active bidder i wins $\hat{u}^i_K > m^i$ at (π_K, r). This contradicts the hypothesis that (ρ, τ) Pareto dominates (π_K, r). □

We have shown that the proposed auction can always find a strongly Pareto efficient core allocation. This does not mean that the auction can guarantee to locate a strong core allocation even if it exists. This does not contradicts Theorem 3 below, which says that the proposed auction can always find a strong core allocation when no bidder is financially constrained.

We know that when bidders face budget constraints, we can guarantee to find a core and strongly Pareto efficient allocation but we cannot expect to have a strong core allocation and therefore we have to accept some loss of market efficiency. This raises an important question whether the auction can find a strong core allocation when bidders are not budget constrained. Our next result establishes the equivalence between the core and the strong core when bidders are not budget constrained.

Lemma 4. *When no bidder is budget constrained, an allocation (π, r) is in the core if and only if it is in the strong core.*

Proof. The 'if' part is obvious, so we prove the 'only if' part. Suppose to the contrary that (π, r) is in the core but not in the strong core. By definition, (π, r) is individually rational and cannot be blocked by any single agent. Then there would exist a viable coalition $S \subseteq M_0$ and an implementable pair (ρ^S, τ) such that $u^i(\rho^S, \tau) \geq u^i(\pi, r)$ for all $i \in S$ with at least one strict inequality.

Let $j \in S$ be one of the agents being strictly improved, i.e. $u^j(\rho^S, \tau) > u^j(\pi, r)$. Define $\Delta = u^j(\rho^S, \tau) - u^j(\pi, r) > 0$. Define a new income distribution $\tilde{\tau}$ by

$$\tilde{\tau}^i = \begin{cases} \tau^j - \frac{|S|-1}{|S|}\Delta, & \text{if } i = j; \\ \tau^i + \frac{1}{|S|}\Delta, & \text{if } i \in S \setminus \{j\}; \\ \tau^i, & \text{if } i \in M \setminus S. \end{cases}$$

For every $i \in M \setminus S$, $\tilde{\tau}^i = \tau^i = m^i$ is feasible. For every $i \in S \setminus \{j\}$, $\tilde{\tau}^i > \tau^i \geq 0$ is also feasible. Let's consider the feasibility for agent j. If j is a bidder, then we have

$$\begin{aligned} \tau^j - \frac{|S|-1}{|S|}\Delta &= v^j(\pi(j)) + r^j - v^j(\rho^S(j)) + \frac{1}{|S|}\Delta \\ &\geq m^j - v^j(\rho^S(j)) + \frac{1}{|S|}\Delta \geq \frac{1}{|S|}\Delta. \end{aligned}$$

The first inequality is because (π, r) is individual rationality such that $v^j(\pi(j)) + r^j \geq m^i$; the second inequality is because bidder j is not budget constrained such that $m^j - v^j(\rho^S(j)) \geq 0$. If j is the seller, we have a similar condition

$$\begin{aligned} \tau^0 - \frac{|S|-1}{|S|}\Delta &= v^0(\pi(0)) + r^0 - v^j(\rho^S(0)) + \frac{1}{|S|}\Delta \\ &\geq v^0(N) - v^0(\rho^S(0)) + \frac{1}{|S|}\Delta \geq \frac{1}{|S|}\Delta. \end{aligned}$$

The first inequality is because (π, r) is individually rational for the seller; the second inequality is attributed to monotonicity of the seller's values. Therefore $\tilde{\tau}^j > 0$ is also feasible for agent j.

Then consider the implementable allocation $(\rho^S, \tilde{\tau})$. For agent j, we have $u^j(\rho^S, \tilde{\tau}) = u^j(\rho^S, \tau) - \frac{|S|-1}{|S|}\Delta = u^j(\pi, r) + \frac{1}{|S|}\Delta$. For any other agent $i \in S \setminus \{j\}$, we have $u^i(\rho^S, \tilde{\tau}) = u^i(\rho^S, \tau) + \frac{1}{|S|}\Delta \geq u^i(\pi, r) + \frac{1}{|S|}\Delta$. This means that (π, r) is blocked by the coalition S with the allocation $(\rho^S, \tilde{\tau})$, yielding a contradiction. □

As an implication of Theorem 2 and Lemma 4, the proposed auction will always generate a strong core allocation if bidders can afford to pay up to their valuations. Every strong core allocation must be strongly Pareto efficient. When no bidder is budget constrained, every strongly Pareto efficient allocation must be fully efficient and thus every strong core allocation must be fully efficient. In summary, we have the next result.

Theorem 3. *When no bidder is budget constrained, the outcome (π_K, r) generated by the proposed auction is in a strong core and thus strongly Pareto efficient and fully efficient.*

The following theorem states that when no bidder is budget constrained, the proposed auction will find a Walrasian equilibrium with the minimum equilibrium price vector. This implies that the proposed auction can recover the classic results of Leonard (1983) and Demange et al. (1986) for the assignment market without budget constraints that the VCG payment vector coincides with the minimum Walrasian equilibrium price vector and any auction for finding it must be strategy-proof for bidders.

Theorem 4. *When no bidder is budget constrained, the proposed auction will find a Walrasian equilibrium with the minimum equilibrium price vector.*

Proof. Because no bidder is budget constrained, we have $m^i \geq \max_{a \in N} v^i(a)$ for all $i \in M$. It is well-known from Shapley and Shubik (1971) and Leonard (1983) that there exists a unique minimum Walrasian equilibrium price vector, which corresponds to the VCG payment vector. Let (π^*, p^*) be the minimum price Walrasian equilibrium. The utility every bidder $i \in M$ achieves at (π^*, p^*) is $u^i(\pi^*, p^*) = v^i(\pi^*(i)) + m^i - p^*(\pi^*(i))$. We use $(P_k, Q_k, \alpha_k, \pi_k, r_k)_{1 \leq k \leq K}$ to describe the auction process. The outcome of the proposed auction is (π_K, r_K). Define the price vector $p_K = \big(p_K(a)\big)_{a\in N}$ which specifies a price for every item $a \in N$ as $p_K(a) = p^i_K(a)$ if $a \in \pi(i)$. Note that if a is unsold, then $a \in \pi_K(0)$ implies that $p_K(a) = p^0_K(a) = 0$. Let's prove that $p^* = p_K$.

We first prove that, on round k, if bidder sets his target utility as $\hat{u}^i_k = u^i(\pi^*, p^*)$, then $p^i_k(a) = p^*(a)$ for every $a \in D_{p^*}(i|v^i)$ and $p^i_k(a) < p^*(a)$ for every $a \notin D_{p^*}(i|v^i)$. Since bidder i is not budget constrained, his bidding price is $p^i_k(a) = v^i(a) + m^i - \hat{u}^i_k$. If $a \in D_{p^*}(i|v^i)$, we have $v^i(a) + m^i - p^*(a) = u^i(\pi^*, p^*)$ and thus $p^i_k(a) = p^*(a)$. If $a \notin D_{p^*}(i|v^i)$, we have $v^i(a) + m^i - p^*(a) < u^i(\pi^*, p^*)$ and thus $p^i_k(a) < p^*(a)$. Furthermore, if $\hat{u}^i_k > u^i(\pi^*, p^*)$, then $p^i_k(a) < p^*(a)$ for each $a \in N$.

Next we prove that if bidder i sets his target utility as $\hat{u}^i_k = u^i(\pi^*, p^*)$ and any other bidder $j \neq i$ sets target utility as $\hat{u}^j_k \geq u^j(\pi^*, p^*)$, then i cannot be a provisional loser on round k. We need to consider the following two cases. Case 1: if $\hat{u}^i_k = u^i(\pi^*, p^*) = m^i$, then i must have bid on the null item and could not be a provisional loser. Case 2: $\hat{u}^i_k = u^i(\pi^*, p^*) > m^i$. Suppose i is a provisional loser. That $\hat{u}^i_k > m^i$ implies that i is active and must have not bid on the null item. This means that $\pi^*(i) \neq 0$. Let $a = \pi^*(i)$. Then we have $p^i_k(a) = p^*(a)$. If item a is unsold at π_k, then the assignment which assigns a to i and all other items to the bidders as π_k does would yield a higher value than π_k does. So a should be assigned to some bidder at π_k. Let j_1 be the bidder such that $\pi_k(j_1) = a$. At π_k, j_1 wins a and i loses, we have $q^{j_1}_k(a) \geq \alpha_k(a) > q^i_k(a) = p^*(a) + 2^{-i}$. If $\hat{u}^{j_1}_k > u^{j_1}(\pi^*, p^*)$, then $p^{j_1}_k(a) < p^*(a)$, which contradict the above inequality. If $\hat{u}^{j_1}_k = u^{j_1}(\pi^*, p^*) = m^{j_1}$, then j_1 must have bid on the null item and thus $q^{j_1}_k(a) = p^{j_1}_k(a) = p^*(a)$, which also contradicts the above inequality. The only possibility is that $\hat{u}^{j_1}_k = u^{j_1}(\pi^*, p^*) > m^{j_1}$. Analogously, we can have $\pi^*(j_1) \neq 0$ and let $b_1 = \pi^*(j_1)$. Then b_1 must have assigned to some bidder at π_k, so let j_2 be the bidder such that $\pi_k(j_2) = b_1$. Lemma 1 implies that $p^{j_2}_k(b_1) + 2^{-j_2} \geq \alpha_k(b_1) > p^*(b_1) + 2^{-j_1}$ and thus $\hat{u}^{j_2}_k = u^{j_2}(\pi^*, p^*) > m^{j_2}$. We can repeat the same argument for j_2 and so on. As the number of bidders and items is finite, so it is impossible.

We can conclude that during the auction process, no bidder will reduce his target below his utility at (π^*, p^*). When the auction ends, we have $\hat{u}^i_K \geq u^i(\pi^*, p^*)$ and $p^i_K(a) \leq p^*(a)$ for all $i \in M$ and $a \in N$. So $p_K \leq p^*$.

Next we will show that $p_K(a) \geq p^i_K(a)$ for all $a \in N$ and $i \in M$. Suppose to the contrary that there are a bidder i_1 and an item a_0 such that $p_K(a_0) < p^{i_1}_K(a_0)$. Let k be the last round on which i_1 was asked to make new bids, then i_1 must have made his final bids on next round (i.e., $p^{i_1}_{k+1} = p^{i_1}_K$). So we have $p^{i_1}_k(a) = p^{i_1}_K(a) - 1$ for every $a \in N$.

Let $a_1 = \pi_K(i_1)$. First note that $a_1 \neq a_0$. Otherwise $p_K(a_0) = p^{i_1}_K(a_0)$ yields a contradiction. If $a_1 = 0$, then the assignment which assigns a_0 to i_1 and all other items to the bidders as π_K does would yield a higher utility to the seller than π_K does. So $a_1 \neq 0$. On round k, i_1 demands a_1 so a_1 must have been assigned to some other bidder at π_k. Let i_2 be the bidder such that $\pi_k(i_2) = a_1$, and let $a_2 = \pi_K(i_2)$. If $a_2 \neq 0$, then a_2 must have been assigned to some bidder at π_k. Otherwise, the assignment which assigns a_1 to i_1, a_2 to i_2, and all other items to the bidders as π_k does would yield a higher utility for the seller. Let i_3 be the bidder such that $\pi_k(i_3) = a_2$, and let $a_3 = \pi_K(i_3)$. Repeat this process until a bidder i_L is found such that $a_L = \pi_K(i_L) = 0$. Let $R = \{i_1, \ldots, i_L\}$.

	i_1	i_2	$\cdots$	i_L
$\pi_K(\cdot)$	a_1	a_2	$\cdots$	0
$\pi_k(\cdot)$	0	a_1	$\cdots$	a_{L-1}

Without loss of generality, we may assume that $p^{i_\ell}_K(a_{\ell-1}) \leq p_K(a_{\ell-1})$ for all $i_\ell \in R \setminus \{i_1\}$. If the assumption is not true, then there will be a bidder $i_{\hat{\ell}} \in R \setminus \{i_1\}$ such that $p^{i_{\hat{\ell}}}_K(a_{\hat{\ell}-1}) > p_K(a_{\hat{\ell}-1})$. We can rewrite $R = \{i_{\hat{\ell}}, i_{\hat{\ell}+1}, \ldots, i_L\}$ by relabelling $i_1 = i_{\hat{\ell}}$, $i_2 = i_{\hat{\ell}+1}$, $\ldots$, and so on. We can find a contradiction to the relabelled $p^{i_1}_K(a_0) > p_K(a_0)$ as follows. There are two cases that need to be considered.

Case 1: $p^{i_\ell}_K(a_{\ell-1}) = p_K(a_{\ell-1})$ for all $i_\ell \in R \setminus \{i_1\}$. Recall that $p^{i_1}_K(a_0) > p_K(a_0)$. If

$a_\ell \neq a_0$ for all $i_\ell \in R$, then the assignment ρ_K given by

$$\rho_K(i) = \begin{cases} a_{\ell-1}, & \text{if } i = i_\ell \in R; \\ 0, & \text{if } \pi_K(i) = a_0; \\ \pi_K(i), & \text{otherwise}; \end{cases}$$

would yield a higher utility to the seller than π_K does in round K. If $a_{\hat{\ell}} = a_0$ for some $\hat{\ell}$, then the assignment ρ'_K given by

$$\rho'_K(i) = \begin{cases} a_{\ell-1}, & i \in \{i_1, \ldots, i_{\hat{\ell}}\}; \\ \pi_K(i), & \text{otherwise.} \end{cases}$$

would yield a higher utility to the seller than π_K does in round K. It is impossible.

Case 2: there exists at least one bidder $i_{\hat{\ell}} \in R \setminus \{i_1\}$ such that $p_K^{i_{\hat{\ell}}}(a_{\hat{\ell}-1}) \leq p_K(a_{\hat{\ell}-1}) - 1$. Consider the assignment ρ_k given by

$$\rho_k(i) = \begin{cases} a_\ell, & \text{if } i = i_\ell \in R \setminus \{i_L\}; \\ 0, & \text{if } i = i_L; \\ \pi_k(i), & \text{otherwise.} \end{cases}$$

On round k, π_k solves $\max_\rho \sum_{i\in M} q_k^i(\rho(i))$. Then we have

$$\begin{aligned}
0 &\le \sum_{i\in M} q_k^i(\pi_k(i)) - \sum_{i\in M} q_k^i(\rho_k(i)) \\
&= q_k^{i_L}(a_{L-1}) - q_k^{i_1}(a_1) + \sum_{\ell=2}^{L-1}\Big(q_k^{i_\ell}(a_{\ell-1}) - q_k^{i_\ell}(a_\ell)\Big) \\
&\le p_K^{i_L}(a_{L-1}) - \Big(p_K^{i_1}(a_1) - 1 + 2^{-i}\Big) + \sum_{\ell=2}^{L-1}\Big(p_K^{i_\ell}(a_{\ell-1}) - p_K^{i_\ell}(a_\ell)\Big) \\
&\le p_K(a_{L-1}) - (p_K(a_1) - 1 + 2^{-i_1}) + \sum_{\ell=2}^{L-1}(p_K(a_{\ell-1}) - p_K(a_\ell)) - 1 \\
&= -2^{-i_1},
\end{aligned}$$

which yields a contradiction again. Look at the inequality at the third row. For bidder i_L, if $p_k^{i_L}(a_{L-1}) \le p_K^{i_L}(a_{L-1}) - 1$ then $q_k^{i_L}(a_{L-1}) = p_k^{i_L}(a_{L-1}) + 2^{-i_L} < p_K^{i_L}(a_{L-1})$. If $p_k^{i_L}(a_{L-1}) = p_K^{i_L}(a_{L-1})$, then i_L makes the same bids on round k as on round K. Note that he is inactive and is assigned the null item 0 on the final round. He is also inactive on round k, i.e., $q_k^{i_L}(a_{L-1}) = p_k^{i_L}(a_{L-1})$. For bidder i_1, we have $p_k^{i_1}(a_1) = p_K^{i_1}(a_1) - 1$ by the definition of round k. For every bidder $i_\ell \in R \setminus \{i_1, i_L\}$, he is not budget constrained and we must have $q_k^{i_\ell}(a_{\ell-1}) - q_k^{i_\ell}(a_\ell) = p_k^{i_\ell}(a_{\ell-1}) - p_k^{i_\ell}(a_\ell) = p_K^{i_\ell}(a_{\ell-1}) - p_K^{i_\ell}(a_\ell)$. The inequality at the fourth row is because the assumption that there is at least one bidder $i_{\hat\ell} \in R \setminus \{i_1\}$ such that $p_K^{i_{\hat\ell}}(a_{\hat\ell-1}) \le p_K(a_{\hat\ell-1}) - 1$. Note that $p_K(a_{\ell-1}) = p_K^{i_\ell}(a_{\ell-1})$ by the definition of p_K.

Finally we prove that (π_K, p_K) is a Walrasian equilibrium. For every $i \in M$, if $\pi_K(i) = a$, then $p_K(a) = p_K^i(a)$ and $v^i(a) + m^i - p_K(a) = \hat u_K^i$. For every other item $b \in N \setminus \{a\}$, we have $\hat u_K^i = v^i(a) + m^i - p_K^i(b) \ge v^i(b) + m^i - p_K(b)$ for $p_K(b) \ge p_K^i(b)$. So $a \in D^i(p_K|v^i)$. Since p^* is the minimum Walrasian equilibrium price, we must have $p^* \le p_K$.

In conclusion, we have $p_K = p^*$. □

4 Concluding Remarks

In this paper we have examined an auction model, in which multiple heterogeneous indivisible items are sold to a group of budget constrained bidders. Every bidder acquires at most one item and has private valuations on those items. His budget could be so tight that he may not be able to pay up to his valuation. Besides valuations, his budget is also his private information. In this market competitive equilibrium is not guaranteed to exist due to budget constraints. So we had to invoke the more general solution-the core instead of the competitive equilibrium as a tool for our dynamic auction design. Importantly and also practically, bidders are not assumed to behave as price-takers and may therefore bid strategically. We have proposed an ascending auction in which bidders determine their own bids and can also withdraw their bids. We have shown that the proposed auction can always induce bidders to bid honestly and lead to a strongly Pareto efficient core allocation when bidders are budget constrained, otherwise a Walrasian equilibrium with the minimum equilibrium price vector. More precisely, sincere bidding is proved to be a Nash equilibrium in the dynamic auction game with incomplete information.

We hope the current study will provide a necessary and useful basis for examining more challenging and more practical resource allocation problems involving indivisibilities, heterogeneity in preferences and shortage of financial resources. For instance, each agent may acquire several items not just a single item and goods can be substitutes or complements. Several efficient dynamic auctions have been proposed for such general market models without budget constraints; see e.g., Kelso and Crawford (1982), Gul and Stacchetti (2000), Milgrom (2000), Ausubel (2004, 2006), Hatfield and Milgrom (2005), Perry and Reny (2005), Mishra and Parkes (2007), and Sun and Yang (2009, 2014). The first important open question is how to design both efficient and incentive compatible dynamic auctions for substitutes. Another important question is how to deal with the auction design problem in the interdependent value setting under budget constraints. For the settings without budget constraints, we refer to Milgrom and Weber (1982) on auction for a single item and Perry and Reny (2002, 2005) and Ausubel (2004) for homogeneous goods.

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