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Abstract: In this paper we provide a quantitative analysis of how wealth may affect economic growth. In the economy, the utility of every individual depends on both consumption and wealth. Exploring a class of specific utility functions in which wealth has a weakening effect on the marginal utility of consumption, we find a closed-form solution of steady-state consumption, capital stock, savings rate, and convergence rate and obtain several novel results of wealth effects on economic growth. We also demonstrate that the new models can be calibrated to fit well with empirical observation.

Key words: Economic growth, wealth effects, savings rate, convergence rate.
JEL classification: C61, O40, O41.

1 Introduction

This paper aims to provide a quantitative analysis of how the spirit of capitalism or the human quest for wealth may have a profound impact on the long-run economic growth and development of a nation. In a famous paper, Lucas (1988) called for theories to explain the diversity of development and growth paths across countries. Besides the dominant force of technology, what else could lead to different growth rates between different countries? even if they may have the same level of technology, the same population growth rate, and the same initial endowments, etc.

In growth models, it is typically assumed that every household's utility function depends only on the level of consumption. In reality, people concern not only about their consumption but also their accumulated wealth, which can be seen as a symbol of their social status. Departing from the conventional approach of growth, Kurz (1968) incorporated

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wealth effects into a neoclassical growth model by defining a general utility function on both consumption and capital. As his function is too general, one could only expect to obtain a few qualitative results. As his main result, Kurz demonstrated that the steady-state capital stock could exceed the modified golden capital stock in the well-known neoclassical RCK growth model; see Ramsey (1928), Cass (1965), and Koopmans (1965). Cole et al. (1992) have interpreted a person's relative wealth as social status that determines how well the person fares in the nonmarket sector concerning their marriage and used this to explain different growth rates across countries. Zou (1994) eloquently argued that Kurz's model with wealth effects captures the essence of the spirit of capitalism stemmed from the classic works of Weber (1958), Smith (1993), and Keynes (1971), among many others. Wealth effect or the spirit of capitalism has been used to address a variety of economic issues, including growth (Fershtman et al. 1996, Corneo and Jeanne 1997, 2001, Smith 1999, Gong and Zou 2002, Pham 2005, Roy 2010, Rehme 2017), savings and consumption (Deaton 1972, 1992, Zou 1995, Carroll 2000, Modigliani and Cao 2004, Luo et al. 2009), asset pricing (Bakshi and Chen 1996, Smith 2001), bubbles (Kamihigashi 2008, Zhou 2016), and housing (Case et al. 2013), etc.

In the literature on growth models with wealth effects, two typical classes of utility functions have been explored. The first class was given by Kurz (1968). As mentioned above, his function is too general to obtain enough insights. The other class concerns additive utility functions of the form in which consumption and wealth or capital stock do not have cross effect. This type has been used by Zou (1994, 1995), Corneo and Jeanne (1997, 2001), Pham (2005), Kamihigashi (2008), and Rehme (2017) among others. In this paper, we explore a class of specific concave utility functions which markedly differ from additive functions and have cross effects of consumption and wealth. For this class of functions, wealth has a weakening effect on the marginal utility of consumption, to reflect a widely observed phenomenon that the consumption of one dollar can be very valuable to a poor person, but to a rich man, it is negligible. This class of utility functions may be seen as a natural extension of the familiar CRRA utility functions and will enable us to obtain clear-cut and rich quantitative results of and insights into economic growth. Variants of this type utility function are used by Abel (1990), Bakshi and Chen (1996), Carroll et. (1997), and Gong and Zou (2002) for their analyses.

We will investigate two different growth models: a Kurz-like neoclassical growth model and a basic endogenous growth model. We focus on the first model as it will allow us to obtain several interesting and novel results together with some familiar ones in the traditional RCK model. Then we briefly discuss the second model which will also offer some new and interesting results. We introduce a new notion of *thrift index* as an indictor to measure an individual's preference for wealth and derive various results based on the index. For the first model, we are able to derive a closed-form solution of steady-state consumption and capital stock. We show that an appropriate degree of preference for wealth can generate a steady-state consumption equal to the golden level. We obtain an optimal thrift index at which the golden level can be achieved. This implies that too high or too low preference for wealth may actually reduce steady-state consumption. Time preference has also a unimodal-shape effect on the steady-state consumption. Too much or too little patience may decrease the steady-state consumption. This is in contrast to what is known about the traditional RCK model: more patience yields more steady-state consumption. We derive a formula of the steady-state saving rate which increases with the thrift index but decreases with the preference for current consumption. We prove that when the rate of technological progress is sufficiently low, the convergence is a decreasing function of the thrift index. Using the benchmark parameters from Barro and Sala-i-Martin (2004), we find that the new model can be calibrated to match empirical observation particularly well.

Wealth effects can be also embedded into endogenous growth models studied by Rommer (1986, 1990), Lucas (1988), Barro (1990), Rebelo (1991), Aghion and Howitt (1992), and Bambi et al. (2014), among many others; see also Barro and Sala-i-Martin (2004) and Acemoglu (2009). To be instructive and keep things simple, we focus on wealth effects in a basic endogenous growth AK model. We obtain a closed-form solution of optimal consumption, capital stock, savings rate and growth rate. We can derive several properties from the closed-form solution. One of these properties shows clearly that nations with different degree of preferences for wealth can have different growth rates even if they have the same technology, the same population growth rate, the same time discount, etc. Another property says that endogenous growth can be still achieved even if the marginal product of capital is smaller than the time discount rate. This property reconfirms a similar one of Zou (1994) and Roy (2007) and shows that a positive impact of wealth effect on growth rate may be a universal conclusion. The results we obtain from the two models offer some fresh insights and could help us have a better understanding of potential effects of wealth on consumption, savings, growth, convergence, and wealth accumulation.

The rest of this paper is organized as follows. Section 2 introduces our Kurz-like neoclassical growth model. Section 3 presents qualitative and quantitative results of the model. Section 4 discusses convergence. Section 5 briefly examines a basic endogenous growth model. Section 6 concludes. All the proofs are deferred to the appendix. Some of these proofs are quite involved and complicated.

2 The Model

We first reconsider the model of Kurz (1968) with an important modification. That is, we introduce a class of new utility functions which will enable us to obtain rich and meaningful qualitative and quantitative results going far beyond his. In the economy, a representative household tries to maximize a discounted utility over an infinite time horizon subject to a dynamic constraint of capital accumulation. In terms of per effective labor, the problem can be formulated as:

$$\max \int_0^{\infty} -c(t)^{-a} k(t)^{-b} e^{-[\rho - n + (a+b)g]t} dt \tag{1}$$

$$\text{s.t. } \dot{k}(t) = f(k) - c(t) - (n + g + \delta)k(t), \quad k(0) = k_0 \tag{2}$$

where $c(t)$ and $k(t)$ are respectively consumption and capital stock at time t, $k_0 > 0$ is the initial capital stock, and $f(k) = k^{\theta}$ is the output function of capital stocks derived from the Cobb-Douglas production function with constant returns to scale. The capital stock $k(t)$ is viewed as the wealth of the household and was first introduced by Kurz (1968) into the RCK model as an important new factor. Here $n > 0$ is the population growth rate, $\delta > 0$ is the capital depreciation rate, $\rho > 0$ is the time discount rate, g is the exogenous growth rate of technology, and $0 < \theta < 1$ is the elasticity of output to capital. The parameters $a > 0$ and $b \geq 0$ represent household's preference for consumption and wealth, so the ratio b/a can be used to reflect the household's relative preference between wealth and consumption or the household's degree of frugality.

It is easy to verify that $f'(k) > 0$, $f''(k) < 0$, $\lim_{k \to \infty} f'(k) = 0$, and $\lim_{k \to 0} f'(k) = \infty$. That is, the output function is strictly increasing and concave and satisfies the Inada conditions.

Note that the household has the utility function $u(c, k) = -c^{-a} k^{-b}$ of consumption c and capital stock k, which satisfies the following conditions:

(i) $u_c = ac^{-a-1}k^{-b} > 0$, $u_{cc} = -a(a+1)c^{-a-2}k^{-b} < 0$, $u_c|_{c=0} = \infty$;

(ii) $u_k = bc^{-a}k^{-b-1} > 0$, $u_{kk} = -b(b+1)c^{-a}k^{-b-2} < 0$, $u_k|_{k=0} = \infty$;

(iii) $u_{ck} = -abc^{-a-1}k^{-b-1} < 0$;

(iv) u is concave in c and k.

(i) says that the marginal utility of consumption is positive and diminishing, and tends to be infinite when the consumption approaches zero. (ii) has a similar interpretation for capital stock. (iii) reflects that capital stock has a weakening effect on the marginal utility of consumption. (iv) is an important qualification for optimality.

The utility function $u(c, k)$ coincides with the CRRA utility function if $b = 0$. The utility function still has reasonable economic interpretations even for the two extreme cases: $\lim_{c \to 0} u(c, k) = -\infty$ and $\lim_{k \to 0} u(c, k) = -\infty$. The former says that no matter how wealthy you are, your utility would be extremely low if you consume almost nothing, while the latter says that no matter how much you consume now, your utility would be extremely low if your wealth approaches zero. At first glance, this latter statement seems to be somewhat counterintuitive since there are people who can live well without any wealth or even with lots of debts. But such a living pattern cannot be sustainable for society as a whole, since no wealth implies an economy without capital, and without capital, there is of course no output for future consumption. This extreme case is like a situation in which all seeds are eaten up by farmer or all cattle are killed by cattleman. Mathematically speaking, the Inada-like conditions for the utility function are important to ensure an interior solution.

Kurz (1968) did not give a specific utility function $u(c, k)$ but introduced a general utility function $u(c, k)$ by requiring it to satisfy $u_c > 0$, $u_{cc} < 0$, $u_k > 0$, and $u_{kk} < 0$, which are similar to (i) and (ii) above. As the function is too general, one could only expect to obtain some qualitative result. His main result is that the steady-state capital stock may exceed the golden capital stock. His claim on multiplicity of steady states was disproved by Bose (1971). Zou (1994) was the first to relate Kurz's model with wealth effects to the spirit of capitalism stemmed from the classic work of Max Weber (1958). He used an additive utility function $u(c, k) = u(c) + v(k)$ in which consumption c and capital stock k do not have cross effect. See also Zou (1995), Corneo and Jeanne (1997, 2001), Pham (2005), Kamihigashi (2008), and Rehme (2017) for the use of similar additive utility functions.

Now we return to the ratio b/a defined above and will use it as an indicator to measure the relative preference between wealth and consumption or the degree of the capitalist spirit. Intuitively, the higher the value of b/a, the more thrifty an individual tends to be and thus more capital can be accumulated. Hence we call b/a the thrift index.

We can also look at the utility function $u(c, k) = -c^{-a}k^{-b}$ from a different perspective. Let $a = \zeta - 1$, $b = \varepsilon(\zeta - 1)$, and $\zeta > 1$, where $\varepsilon = b/a$ is the thrift index. The functions $-c^{-a}k^{-b}$ and $\frac{(ck^{\varepsilon})^{1-\zeta}}{1-\zeta}$ are equivalent in the sense of maximizing (1). If $\varepsilon = 0$, the latter function is reduced to the standard CRRA function $c^{1-\zeta}/(1-\zeta)$, with $1/\zeta$ the elasticity of intertemporary substitution.

In the next section we will discuss how to obtain quantitative results concerning steady-state consumption and capital stock.

3 Qualitative and Quantitative Results

To solve the optimal control problem (1) and (2), we first construct the Hamiltonian[1]

$$H = u(c,k) + \lambda[k^{\theta} - c - (n+g+\delta)k], \tag{3}$$

where λ represents the shadow price of capital. From the first order condition, we have

$$u_c = \lambda. \tag{4}$$

The co-state ODE is

$$\dot{\lambda} = [\rho + (1+(a+b))g + \delta]\lambda - u_k - f'(k)\lambda. \tag{5}$$

Under the condition of $\rho > n$, the transversality condition below is satisfied (see the Appendix):

$$\lim_{t\to\infty} \lambda k e^{-[\rho - n + (a+b)g]t} = 0. \tag{6}$$

The ODE for capital accumulation (2), the ODE for the co-state variable (5), the first order condition (4), and the transversality condition (6) consist of the necessary conditions for the optimal solution. Notice further that $\lambda = u_c = ac^{-(1+a)}k^{-b} > 0$ by (4), and both the instantaneous utility function and the RHS of (2) are jointly concave in c and k. So the necessary conditions are also sufficiency conditions (Sethi, 2000, p55).

Differentiating both side of (4) with respect to t, and dividing both sides of the resulting equality by $\lambda(t)$ lead to

$$\frac{\dot{\lambda}}{\lambda} = \frac{u_{cc}}{u_c}\dot{c} + \frac{u_{ck}}{u_c}\dot{k}. \tag{7}$$

Combining (4), (5), and (7), we obtain the ODE for c

$$\dot{c} = -\frac{u_c}{u_{cc}}[\frac{u_k}{u_c} + \theta k^{\theta-1} - (\rho + \delta + (1+(a+b))g) - b\frac{\dot{k}}{k}]. \tag{8}$$

Then we have the following two equations:

$$\dot{k} = k^{\theta} - c - (n+g+\delta)k, \;\; k(0) = k_0 \tag{9}$$

$$\dot{c} = \frac{c}{1+a}[\frac{bc}{ak} + \theta k^{(\theta-1)} - (\rho + \delta + (1+(a+b))g) - b\frac{\dot{k}}{k}], c(0) = c_0, \tag{10}$$

where c_0 is the initial consumption to be determined. Note that it is impossible to derive a closed-form solution to the highly nonlinear system consisting of ODEs (9) and (10).

[1]To simplify the description, we omit the time t in the following formulas.

Fortunately, we will be able to obtain the steady-state capital stock and consumption and therefore analyze the properties of the economy in the long run.

When the economy is at a steady state, we must have $\dot{k} = 0$ and $\dot{c} = 0$. It follows from (9) and (10) that

$$k^\theta - c - (n + g + \delta)k = 0 \tag{11}$$

$$\frac{c}{1+a}\left[\frac{bc}{ak} + \theta k^{(\theta-1)} - (\rho + \delta + (1 + (a + b))g) - b\frac{\dot{k}}{k}\right] = 0 \tag{12}$$

Now we first use the phase diagram to get an intuitive and quick understanding of the qualitative properties of the steady-state capital and consumption. Then we derive the closed-form formulas of the steady-state capital and consumption and conduct quantitative analysis. The four typical phase diagrams can be illustrated by Figures 1 (a)-(d) and are used to show the qualitative effect of parameter b on the steady state. In each case, the horizontal axis represents capital and the vertical one consumption. The two curves $\dot{c} = 0$ and $\dot{k} = 0$ divide the first quadrant into four regions. The arrows show the direction of the trajectories in different regions.

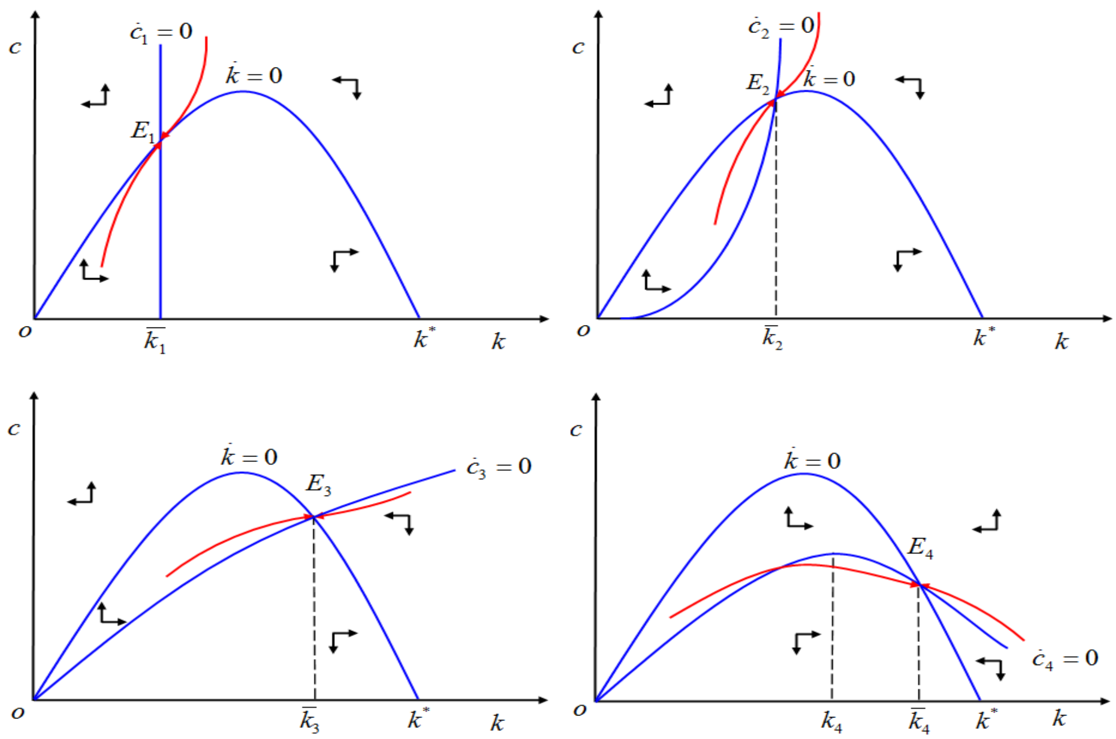

Figure 1: The four cases by the phase plane method

Figure 1 (a) is the case of $b = 0$, which corresponds to the traditional RCK model. The curve of $\dot{c} = 0$ is vertical, and the intersection of the two loci $\dot{c} = 0$ and $\dot{k} = 0$ is the steady state. The steady-state consumption is less than the golden level in this case. The red curve represents the unique saddle path.

Figure 1 (b) represents the case of $0 < b < \theta$. In this case, the curve of $\dot{c} = 0$ is no longer a straight line, but a convex one. The steady capital is higher than the case of $b = 0$. Furthermore, if the value of b is selected properly, the steady-state consumption might reach the golden level, that is, the peak point of the curve $\dot{k} = 0$. Also, there is a unique saddle path.

Figure 1 (c) is a case of $b > \theta$. In this case, the curve of $\dot{c} = 0$ now becomes concave rather than convex. But there is still a unique saddle path.

Figure 1 (d) is also a case of $b > \theta$, but now the value of parameter b is so high that the steady state is moved to the falling part of the curve $\dot{c} = 0$. In this case, there is still a saddle path, but unlike in the previous cases, the saddle path is now not a monotonic shape, but a unimodal shape.

Having discussed the qualitative features of the steady state of the economy, we now move to the quantitative analysis of the steady state. Barring the case of $c = 0$, from (12) we have

$$\frac{bc}{ak} + \theta k^{(\theta-1)} - (\rho + \delta + (1 + (a+b))g) - b\frac{\dot{k}}{k} = 0 \tag{13}$$

Substituting (9) into (13) leads to

$$\frac{bc}{ak} + \theta k^{(\theta-1)} - (\rho + \delta + (1 + (a+b))g) - b\frac{k^{\theta} - c - (n+g+\delta)k}{k} = 0 \tag{14}$$

Solving the algebraic system of equations (11) and (14) yields

$$\begin{aligned} \bar{k} &= \left(\frac{b+a\theta}{b(n+g+\delta)+a[\rho+\delta+(1+(a+b))g]}\right)^{\frac{1}{1-\theta}} \\ \bar{c} &= \frac{a[\rho+\delta+(1+(a+b))g-\theta(n+\delta+g)]}{b+a\theta}\left(\frac{b+a\theta}{b(n+g+\delta)+a[\rho+\delta+(1+(a+b))g]}\right)^{\frac{1}{1-\theta}} \end{aligned} \tag{15}$$

These formulas of steady-state capital and consumption hold the key to deriving quantitative results concerning the model. Observe that as $\rho - n + (a+b)g > 0$ and $0 < \theta < 1$, the steady-state capital and consumption must always be positive.

For convenience, we will frequently write the steady-state capital and consumption in terms of a and the thrift index $x = \frac{b}{a}$, i.e., $\bar{k} = \bar{k}(a, x)$ and $\bar{c} = \bar{c}(a, x)$.

Now we are ready to present several properties of the steady-state capital and consumption.

Proposition 1 *Let* $\psi_g = \frac{\theta(\rho-n+ag)}{(n+g+\delta)(1-\theta)-\theta ag}$ *and* $a^* = \frac{(n+g+\delta)(1-\theta)}{\theta g}$. *Then we have:*

(i) $\frac{\partial \bar{k}(a,x)}{\partial x} > 0$.

(ii) *If* $a < a^*$, *then* $\frac{\partial \bar{c}(a,x)}{\partial x} > 0$ *for* $x < \psi_g$; $\frac{\partial \bar{c}(a,x)}{\partial x} < 0$ *for* $x > \psi_g$.

(iii) *If* $a > a^*$, *then* $\frac{\partial \bar{c}(a,x)}{\partial x} > 0$.

Proof: See the Appendix.

In the proposition ψ_g is the thrift index that generates the golden level of consumption. We may call it *the golden thrift index* and a^* *the critical preference (CP) value* for consumption. Proposition 1 shows the effect of thrift index on the steady-state capital and consumption. The bigger the thrift index, the higher the steady-state capital. But for the consumption, there will be two different cases: If $a < a^*$, the steady-state consumption first increases and then decreases with the thrift index; If $a > a^*$, the steady-state consumption is always increasing with the thrift index. An interesting implication of this proposition is that the seemingly non-consumptive pursuit of wealth may actually lead to a higher steady-state consumption level. At the same time, it is not that, the higher the thrift index, the higher the steady-state consumption, but the moderate thrift index can make it reach the golden level of consumption, i.e., the peak of the curve $\dot{k} = 0$ in Figure 1.

Property (i) is very intuitive, saying that a higher preference for wealth leads to a higher level of steady-state capital stock, and an appropriate frugality can help achieve the golden level of consumption. Properties (ii) and (iii) are not only novel but also somewhat counterintuitive. Generally speaking, households would like to curb their current consumption to save more in order to have a higher consumption in the future. On the contrary, (ii) tells that the future consumption could be reduced due to over saving. This paradox, however, can be understood as follows: On the one hand, an increase in the thrift index increases capital stock by (i), and an increase in capital stock results in a higher output by $f'(k) > 0$. On the other hand, a higher level of capital stock results in a higher depreciation. Roughly speaking, the output increment will dominate if $x < \psi_g$ and the corresponding capital stock is relatively low, i.e., $\bar{k}(a,x) < \bar{k}(a,\psi_g)$, which makes it possible for the householder to consume more in the steady state. In the opposite case, if $x > \psi_g$ and the capital stock is already high enough, i.e., $\bar{k}(a,x) > \bar{k}(a,\psi_g)$, the depreciation may dominate due to the law of diminishing returns to capital and the assumption of linear depreciation, which may lead to a fall in the steady-state consumption. The primary reason for why the householder would like to accept a reduction in steady-state consumption is that what he or she cares about now is not just the amount of consumption, but also the amount of wealth, since both consumption and wealth now affect utility. Of course, a reduction in the steady-state consumption certainly has a negative effect on utility if other things are held fixed. But now the increased steady-state capital stock contributes even a higher increment to utility.

To understand (iii), observe that we have $\bar{k}(a,x) \leq \lim_{x\to\infty} \bar{k}(a,x)$ by (i) and that $\partial\bar{k}/\partial a < 0$ for any $a > 0$. Then for any $a > a^*$, it holds $\bar{k}(a,x) < \bar{k}(a^*,x) \leq \bar{k}(a^*,\infty) = [1/(n+g+\delta+a^*g)]^{1/(1-\theta)} = [\theta/(n+g+\delta)]^{1/(1-\theta)} \triangleq M$. Moreover, the steady state must be on the curve of $\dot{k} = 0$, i.e., $c = k^\theta - (n+g+\delta)k$, which implies $\partial c/\partial k = \theta k^{\theta-1} - (n+g+\delta) > 0$ under the condition of $\bar{k}(a,x) < M$. Combining this with (i), we have $\partial\bar{c}/\partial x = (\partial\bar{c}/\partial\bar{k})(\partial\bar{k}/\partial x) > 0$ for all $a > a^*$. Intuitively, since a represents the preference for current consumption, other things being equal, the greater the value of a, the more likely it is that the household tends to raise the current consumption, and therefore the accumulation of capital can be reduced. When a is sufficiently large, the resulting steady-state capital might be driven to a level that is low enough to ensure $\partial\bar{k}/\partial x > 0$ regardless of how large the thrift index is.

It is well-known that for neoclassical growth models, if two countries have the same level of technology, same population growth rate, same time preference parameter, and same capital depreciation rate, their per capita consumption and per capita capital should converge to the same steady-state level, which is the so-called convergence. In contrast, when wealth effect is considered, as in the current model, this property will change. Given those parameters, different incomes may still occur because of different thrift indexes. Therefore wealth effect could provide a useful channel for explaining income differences in different countries.

Proposition 2 *Let* $\hat{\rho} = \frac{b(1-\theta)(n+g+\delta)}{a\theta} + n - (a+b)g$. *Then we have:*

(i) $\frac{\partial\bar{k}}{\partial\rho} < 0$.

(ii) $\frac{\partial\bar{c}}{\partial\rho} > 0$ *if* $\rho < \hat{\rho}$, *while* $\frac{\partial\bar{c}}{\partial\rho} < 0$ *if* $\rho > \hat{\rho}$.

Proof: See the Appendix.

Proposition 2 (i) says that the steady-state capital is monotonically decreasing in the discount rate. Statement (ii) implies that not the most patient household will be able to achieve the highest steady-state consumption, but the moderately patient household will be able to do so. It can be interpreted in a way similar to the one applied to Proposition 1. On the one hand, an increase in patience, equivalently, a reduction in ρ, tends to curb current consumption, which helps to promote capital accumulation and total output. On the other hand, the marginal return to capital is diminishing but the depreciation grows linearly. When the capital stock exceeds a certain level, its depreciation may outweigh its contribution to output, and thus the steady-state consumption will decline.

It is also worth mentioning that we have

$$\partial\bar{k}/\partial\delta < 0 \text{ and } \partial\bar{c}/\partial\delta < 0 \tag{16}$$

See the Appendix. This means that an increase in depreciation rate will lower capital. As capital decreases, output will fall so will consumption.

Proposition 3 *The steady-state saving rate is* $\bar{s}(a,x) = \frac{(x+\theta)(n+g+\delta)}{x(n+g+\delta)+[\rho+\delta+(1+a(1+x))g]}$, $\frac{\partial \bar{s}}{\partial x} > 0$, *and* $\frac{\partial \bar{s}}{\partial a} \leq 0$.

Proof: See the Appendix.

Proposition 3 says that the steady-state saving rate is an increasing function of the thrift index but a decreasing function of the preference for current consumption.

Proposition 4 *The optimal saving path* $s(t)$ *satisfies*

$$\dot{s}(t) = [(s(t) - \frac{1}{1+a})\frac{b+a\theta}{a}k(t)^{\theta-1} - m](1-s(t))$$

where $m = \frac{1}{1+a}[b(n+\delta) - (\rho+\delta+(1+a)g)] + \theta(n+g+\delta)$. *Furthermore,*

(i) *If* $m > 0$, *i.e.,* $\bar{s} > \frac{1}{1+a}$,*then* $s(t) > \frac{1}{1+a}$ *and* $\dot{s}(t) > 0$ *for any* $t \in [0,\infty)$;

(ii) *If* $m = 0$, *i.e.,* $\bar{s} = \frac{1}{1+a}$,*then* $s(t) = \frac{1}{1+a}$ *and* $\dot{s}(t) = 0$ *for any* $t \in [0,\infty)$;

(iii) *If* $m < 0$, *i.e.,* $\bar{s} < \frac{1}{1+a}$,*then* $s(t) < \frac{1}{1+a}$ *and* $\dot{s}(t) < 0$ *for any* $t \in [0,\infty)$.

Proof: See the Appendix.

Proposition 4 shows some properties of the optimal saving rate, which involve two major aspects - boundary and monotonicity. The two aspects roughly depict the global dynamic characteristics of the optimal saving rate. Observe that if $b = 0$, the current model reduces to the traditional RCK model. Compared with the RCK model, the current model can be calibrated to better match the empirical observation. To see this point, we use the following widely-used benchmark parameters from Barro and Sala-i-Martin (p.109, 2004): $\delta = 0.05$, $n = 0.01$, $\rho = 0.02$, $g = 0.02$, and $\theta = 0.3$. We have to require $a = 16.5$ if the optimal saving rate needs to remain a constant in the traditional model, i.e., $\bar{s} = 1/(1 + 16.5) \approx 5.71\%$. Then the two values of $a = 16.5$ and $\bar{s} = 5.71\%$ appear to be rather unrealistic and far from the observations - the former is too high and the latter is too low. If we want the path of saving rate to follow an increasing pattern, a has to be increased and then the saving rate would be decreased in the steady sate. This dilemma can be reconciled in the current model. For instance, if we set $b = 0.9$, a will be reduced to 3 which is much lower than 16.5 and the constant saving rate will rise from 5.71% to 25%. In other words, after wealth effect is considered, the saving rate becomes more reasonable and closer to the empirical observation. Table 1 indicates a few possible scenarios for the benchmark parameters. Generally speaking, people view their bank deposits, real estate, and other financial assets as a symbol of their wealth. It is not difficult to imagine or understand

Table 1: Calibration for the benchmark parameters.

b	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
a	16.5	15	13.5	12	10.5	9	7.5	5	4.5	3
$\bar{s}$	0.0571	0.0625	0.0741	0.0769	0.0870	0.1000	0.1176	0.1667	0.1818	0.25

that people's wealth will surely influence their consumption behavior and the performance of the economy as well. In this sense, it is natural that models with wealth effect could better explain or match economic reality than those without wealth effect.

4 Convergence Analysis

This section analyzes the convergence property near the steady states. Notice that (9) and (10) consist of a system of nonlinear differential equations. For convenience, the ODE system is first processed by logarithm before linearized, i.e.,

$$\begin{aligned} \frac{d\ln(k)}{dt} &= e^{(\theta-1)\ln(k)} - e^{\ln(c)-\ln(k)} - (n+g+\delta) \\ \frac{d\ln(c)}{dt} &= \frac{1}{1+a}\{\frac{b(1+a)}{a}e^{\ln(c)-\ln(k)} + (\theta-b)e^{(\theta-1)\ln(k)} - \\ &\quad [(\rho+\delta+(1+(a+b))g - b(n+g+\delta)]\} \end{aligned} \tag{17}$$

Note that the equilibrium point of the above system is the same as $(\bar{k}, \bar{c})$ given by (15). Using the Taylor expansion in (17) around the point $(\bar{k}, \bar{c})$ and ignoring the terms of order higher than 1 gives rise to

$$\begin{bmatrix} \frac{d\ln(k)}{dt} \\ \frac{d\ln(c)}{dt} \end{bmatrix} = \begin{bmatrix} l_1 & l_2 \\ l_3 & l_4 \end{bmatrix} \begin{bmatrix} \ln\frac{k}{\bar{k}} \\ \ln\frac{c}{\bar{c}} \end{bmatrix} \triangleq J_E \begin{bmatrix} \ln\frac{k}{\bar{k}} \\ \ln\frac{c}{\bar{c}} \end{bmatrix}, \tag{18}$$

where

$$\begin{aligned} l_1 &= \frac{1}{b+a\theta}\{(\theta-1)b(n+g+\delta) + a\theta[\rho - n + (a+b)g]\} \\ l_2 &= \frac{a\{\theta(n+\delta+g) - [\rho+\delta+(1+(a+b))g]\}}{b+a\theta} \\ l_3 &= \frac{1}{b+a\theta}\{b\{\theta(n+\delta+g) - [\rho+\delta+(1+(a+b))g]\}+ \\ &\quad \frac{(\theta-b)(\theta-1)\{-b(n+g+\delta) - a[\rho+\delta+(1+(a+b))g]\}}{(1+a)}\} \\ l_4 &= \frac{b\{[\rho+\delta+(1+(a+b))g] - \theta(n+\delta+g)\}}{b+a\theta} \end{aligned} \tag{19}$$

The two eigenvalues r_1 and r_2 of the matrix J_E above can be written as

$$r_1 = \frac{trJ_E - \sqrt{(trJ_E)^2 - 4\,|J_E|}}{2} \text{ and } r_2 = \frac{trJ_E + \sqrt{(trJ_E)^2 - 4\,|J_E|}}{2}$$

where $trJ_E = l_1 + l_4$ is the trace of matrix J_E, and $|J_E| = l_1 l_4 - l_2 l_3$ is the determinant of the matrix $\begin{vmatrix} l_1 & l_2 \\ l_3 & l_4 \end{vmatrix}$. trJ_E and $|J_E|$ are respectively given by

$$\begin{aligned} trJ_E &= \rho - n + (a+b)g \\ |J_E| &= \tfrac{(\theta-1)\{[\rho+\delta+(1+(a+b))g]-\theta(n+\delta+g)\}\{b(n+g+\delta)+a[\rho+\delta+(1+(a+b))g]\}}{(b+a\theta)(1+a)} \end{aligned} \tag{20}$$

Since $\rho > n$ and $0 < \theta < 1$, it holds $trJ_E > 0$ and $|J_E| < 0$. Then the real parts of r_1 and r_2 have opposite signs and therefore $(\bar{k}, \bar{c})$ is a saddle point, as depicted in Figure 1. Assume r_1 and r_2 represent the eigenvalues with a negative real part and a positive real part, respectively. Then the general solution of k to (18) can be described by

$$\ln(k) = \ln(\bar{k}) + d_1 e^{r_1 t} + d_2 e^{r_2 t}, k(0) = k_0 \tag{21}$$

with d_1 and d_2 two parameters to be determined. Since $r_2 > 0$ and because $\ln(k)$ needs to converge to $\ln(\bar{k})$ as $t \to \infty$, we have $d_2 = 0$. Using the initial condition of $k(0) = k_0$ leads to $d_1 = \ln(k_0) - \ln(\bar{k})$. It follows that

$$\ln(k) = \ln(\bar{k}) + (\ln(k_0) - \ln(\bar{k}))e^{r_1 t} \tag{22}$$

Let $\gamma = -r_1$. Then the value of γ can be used to represent the convergence speed near the steady state.

Proposition 5 (i) *The equilibrium solution is a saddle path;* (ii) *If the rate of technological progress is low enough, the convergence rate is decreasing with the thrift index, i.e., $\partial\gamma/\partial x < 0$.*

Proof: See the Appendix.

Property (i) says that the optimal consumption is a saddle path, which is illustrated in Figure 1. Property (ii) says that the convergence rate may be decreased by an increase in the thrift index. This property enables the current model to give a better explanation for the observed convergence speed that appears to be much lower than the theoretical estimation in the traditional model. The observed convergence speed is generally believed to fall into the range of 1.5-3.0%, either from a perspective of different regions inside a country or from a perspective of different countries. If the former benchmark parameters are applied to the traditional model, the convergence rate is $\gamma = (1-\theta)(g+n+\delta) = 5.6\%$, which is significantly higher than the observed level. If we want to reduce the convergence speed to 2%, the parameter θ should be raised to about 0.75. For the current model, if we take $a = b = 6$, the convergence rate will drop from 5.6% to around 3%, which is fairly close to the observed level. Moreover, if θ is allowed to increase to a moderate level of 0.5, the convergence rate will be equal to the observed average level of 2%; see Barro and Sala-i-Martin (2004, p. 59 and p. 496).

5 Endogenous Growth: AK Model

A common feature of many endogenous growth models is that the production technology is increasing or constant returns to scale instead of decreasing returns; see e.g., Rommer (1986, 1990), Lucas (1988), Barro (1990), Rebelo (1991), Aghion and Howitt (1992), and Bambi et al. (2014). In this section, to be instructive and keep things simple, we focus on and investigate wealth effects in a basic AK model. In this model, the utility function is still the same $u(c, k) = -c^{-a}k^{-b}$ with $a > 0$ and $b \geq 0$ as used in the previous sections, but the production function $f(k)$ is changed into Ak, which is characterized by $\theta = 1$. In this case, the output is constant returns to capital as used in Barro (1990) and Rebelo (1991); see also Arrow (1962).

Now substituting $f(k) = Ak$ and $\theta = 1$ into (9) and (10), we have

$$\dot{k} = Ak - c - (n + \delta)k, k(0) = k_0 \tag{23}$$

$$\dot{c} = \frac{c}{1+a}[\frac{bc}{ak} + A\theta - (\rho + \delta) - b\frac{\dot{k}}{k}], c(0) = c_0 \tag{24}$$

where $A > 0$ is the marginal output of capital, the initial optimal consumption $c_0 > 0$ is a constant to be determined, and the other parameters are defined as before. Here, we have assumed the exogenous technological progress rate $g = 0$ because the model itself may generate sustained growth due to constant marginal returns to capital. Although the ODE system of (23) and (24) is nonlinear, its closed-form solution exists and can be given as in the following proposition.

Proposition 6 *The optimal trajectories of capital and consumption are given by*

$$k^*(t) = k_0 \exp(\frac{1}{1+a}(A - \delta - \frac{a}{a+b}\rho - \frac{b}{a+b}n)t) \triangleq k_0 e^{\mu t}, \tag{25}$$

$$c^*(t) = \frac{a}{1+a}(A + \frac{\rho - n}{a+b} - n - \delta)k_0 \exp(\frac{1}{1+a}(A - \delta - \frac{a}{a+b}\rho - \frac{b}{a+b}n)t) \triangleq \eta k_0 e^{\mu t}. \tag{26}$$

Proof: See the Appendix.

Assume further that $\eta > 0$ so as to keep $c(t)$ positive. Proposition 6 shows that when the marginal output of capital is constant, the new model does not have transition process - a property similar to the traditional AK model. But now the quantitative properties of growth rate and saving rate are affected by wealth effect.

Proposition 6 gives the closed-form of the optimal solutions for capital k and consumption c. So we can use them to investigate the quantitative effect of relevant parameters in a clear manner. It is easy to see that the saving rate and the growth rate are given by $s = 1 - \frac{1}{A}\frac{a}{1+a}(A + \frac{\rho-n}{a+b} - n - \delta)$ and $\mu = \frac{1}{1+a}(A - \delta - \frac{a}{a+b}\rho - \frac{b}{a+b}n)$, respectively. Both of them are constant along the entire path. They can be also rewritten as $s = 1 - \frac{1}{A}\frac{a}{1+a}[A + \frac{\rho-n}{a(1+x)} - n - \delta]$ and $\mu = \frac{1}{1+a}[A - \delta - \rho + \frac{x(\rho-n)}{1+x}]$. It is clear that countries with different degree of preferences for wealth or thrift index will have different growth rates even if they have the same technology, the same time discount, the same population growth rate, and the same capital depreciation rate. We can also easily derive the following property.

Corollary 1 $\frac{\partial s}{\partial x} > 0$ *and* $\frac{\partial \mu}{\partial x} > 0$.

The property of $\frac{\partial s}{\partial x} > 0$ in the corollary says that a higher x stimulates more saving and then increases the saving rate on the entire path. We know that under the traditional AK model, the growth rate is negative if $A - \delta - \rho < 0$. Corollary 1 implies that in the current model, even if $A - \delta - \rho < 0$, the growth rate μ could still be positive as long as x is large enough. This is because that a higher x leads to a higher level of capital but the marginal contribution of capital does not diminish.

Proposition 7 *The value function is*

$$J(\bullet) = -\eta^{-(a+1)}\frac{a}{b+a}k_0^{-(a+b)}, \tag{27}$$

with $\eta = \frac{a}{1+a}(A + \frac{\rho-n}{a+b} - n - \delta)$.

Proof: See the Appendix.

It is easy to derive from Proposition 7 that $\partial J/\partial k_0 > 0$, $\partial J/\partial A > 0$, $\partial J/\partial \delta < 0$, $\partial J/\partial n < 0$, and $\partial J/\partial \rho > 0$. Their economic interpretations are obvious. However, it is worth pointing out that the signs for $\partial J/\partial a$ and $\partial J/\partial b$ are not necessarily definite, that is, the optimal objective value may not be a monotonic function of the thrift index. This property indicates that the preference for wealth can be seen as a balance force between present and future. It is shown that when the initial capital stock is low enough, the total utility J increases with the thrift index. This will be good news for a country at a relatively poor stage if the preference for wealth can be strengthened. Though potential factors determining the preference for wealth can be complicated and are not discussed here, we believe that culture, institution, and history should be among the fundamental factors. Take as an example the consumption behavior in the East Asia region such as Japan, South Korea, and China; see Modigliani and Cao (2004). In their culture, the thought of advocation of thrift and opposition of luxury has been influencing for thousands of years. This should be an important invisible force that has kept their savings rate fairly high.

6 Conclusion

In this paper we provided a detailed quantitative analysis of wealth effects on the long-run consumption, capital, savings, growth, and convergence. We examined a Kurz-like neoclassical growth model and a basic endogenous growth model. In the economy, utility of the representative household depends on both consumption and capital. We explored a class of specific concave utility functions of a multiplicative form of consumption and capital or wealth. The utility function of this type not only possesses a natural property that wealth has a weakening effect on the marginal utility of consumption, but also allows us to derive explicit steady-state or optimal capital and consumption. So the quantitative properties of interested variables, such as savings rate, convergence rate, and growth rate can be studied in a transparent way.

In our analysis, we introduced the notion of thrift index to measure the degree of the household's preference for wealth and obtained various results based on the index. For the first model we derived an explicit solution of steady-state consumption, capital stock, savings rate, and convergence rate. It was shown that a moderate preference for wealth can achieve the golden level steady-state consumption. Too high or too low preference for wealth may reduce steady-state consumption. Time preference also has similar effects on the steady-state consumption. We further demonstrated that the model could be calibrated to fit very well with empirical observation. For the second endogenous growth model a closed-form solution of optimal consumption, capital stock, savings rate and growth rate was obtained. We offered a plausible explanation as to why different countries with different thrift indices could have different growth rates even if they may have the same technology, the same population growth rate, and the same time discount. It was also proved that even if the time discount rate is greater than the marginal product of capital, sustained growth can be still generated.

The current study has provided several interesting and novel results, going beyond what is known in the literature. We hope this study can improve our understanding of why and how wealth effects may influence long-run consumption, wealth accumulation, savings, growth, and convergence.

Appendices

Proof of the Transversality Condition (6)**:** It follows from (4) and (5) that

$$\frac{\dot{\lambda}}{\lambda} = [\rho + (1 + (a + b))g + \delta] - \frac{b}{a}\frac{c}{k} - f'(k). \tag{28}$$

We need to prove $\lim_{t\to\infty}\{\frac{\dot{\lambda}}{\lambda} + \frac{\dot{k}}{k} - [\rho - n + (a + b)g]\} < 0$. By $\lim_{t\to\infty}\frac{\dot{k}}{k} = 0$ and (28), it suffices to show

$$\lim_{t\to\infty}[(n + g + \delta) - \frac{b}{a}\frac{c}{k} - f'(k)] = (n + g + \delta) - \frac{b}{a}\frac{\bar{c}}{\bar{k}} - f'(\bar{k}) < 0. \tag{29}$$

From (15) we have

$$\frac{\bar{c}}{\bar{k}} = \frac{a[\rho + \delta + (1 + (a + b))g - \theta(n + \delta + g)]}{b + a\theta}, \tag{30}$$

$$f'(\bar{k}) = \theta\bar{k}^{\theta-1} = \theta\frac{b(n + g + \delta) + a[\rho + \delta + (1 + (a + b))g]}{b + a\theta}. \tag{31}$$

Combining (29)-(31), we have

$$\begin{aligned}
&\theta\tfrac{b(n+g+\delta)+a[\rho+\delta+(1+(a+b))g]}{b+a\theta} + \tfrac{b[\rho+\delta+(1+(a+b))g-\theta(n+\delta+g)]}{b+a\theta} > n + g + \delta \Leftrightarrow \\
&b\theta(n + g + \delta) + a\theta[\rho + \delta + (1 + (a + b))g] + b[\rho + \delta + (1 + (a + b))g- \\
&\theta(n + \delta + g)] > (b + a\theta)(n + g + \delta) \Leftrightarrow \rho + (a + b)g > n
\end{aligned} \tag{32}$$

By assumption of $\rho > n$, it is easy to see that (32) holds. □

Proof of Proposition 1: We first prove (i). Recall $x = b/a$. By (15), we have

$$\bar{k}(a, x) = (\frac{x + \theta}{x(n + g + \delta) + [\rho + \delta + (1 + a(1 + x))g]})^{\frac{1}{1-\theta}} \triangleq (h(a, x))^{\frac{1}{1-\theta}}. \tag{33}$$

Then

$$\begin{aligned}
\tfrac{\partial h(a,x)}{\partial x} &\triangleq x(n + g + \delta) + [\rho + \delta + (1 + a(1 + x))g] - (x + \theta)[n + g + \delta + ag] \\
&= [\rho + \delta + (1 + a(1 + x))g] - \theta[n + g + \delta + ag] - axg \\
&= \rho + \delta + g + ag + axg - \theta n - \theta g - \theta\delta - \theta ag - axg \\
&= \rho - \theta n + (1 - \theta)\delta + g + ag - \theta g - \theta ag \\
&> g + ag - \theta g - \theta ag = g(1 + a)(1 - \theta) > 0
\end{aligned} \quad . \tag{34}$$

As the monotonic property of $\bar{k}(a, x)$ w.r.t. x is the same as $h(a, x)$ is w.r.t. x, $\frac{\partial\bar{k}(a,x)}{\partial x} > 0$.

We now turn to (ii) and (iii). Consider

$$\begin{aligned}
\bar{c}(a, x) \quad = \quad &\tfrac{\rho+\delta+(1+a(1+x))g-\theta(n+\delta+g)}{x+\theta}\times \\
&\{\tfrac{x+\theta}{x(n+g+\delta)+[\rho+\delta+(1+a(1+x))g]}\}^{\frac{1}{1-\theta}} \triangleq m(a, x)\bar{k}(a, x)
\end{aligned} \tag{35}$$

Then we have

$$\begin{aligned} \frac{\partial m(a,x)}{\partial x} &= \frac{ag(x+\theta)-[\rho+\delta+(1+a(1+x))g-\theta(n+\delta+g)]}{(x+\theta)^2} \\ &= \frac{ag\theta-\rho-\delta-g-ag+\theta(n+\delta+g)}{(x+\theta)^2} \\ &= \frac{-ag(1-\theta)-(1-\theta)\delta-(1-\theta)g-(\rho-\theta n)}{(x+\theta)^2} \end{aligned} \tag{36}$$

and

$$\begin{aligned} \frac{\partial \bar{k}}{\partial x} &= \frac{1}{1-\theta}\left(\frac{x+\theta}{x(n+g+\delta)+[\rho+\delta+(1+a(1+x))g]}\right)^{\frac{\theta}{1-\theta}} \times \\ & \frac{\rho+\delta+g+ag-\theta n-\theta g-\theta\delta-\theta ag}{\{x(n+g+\delta)+[\rho+\delta+(1+a(1+x))g]\}^2} \end{aligned} \tag{37}$$

It follows that

$$\begin{aligned} \frac{\partial \bar{c}(a,x)}{\partial x} &= \frac{\partial m(a,x)}{\partial x}\bar{k}(a,x) + m(a,x)\frac{\partial \bar{k}(a,x)}{\partial x} \\ &= \frac{ag\theta-[\rho-\theta n+ag+(1-\theta)\delta+(1-\theta)g)]}{(x+\theta)^2}\left(\frac{x+\theta}{x(n+g+\delta)+[\rho+\delta+(1+a(1+x))g]}\right)^{\frac{1}{1-\theta}} \\ & +\frac{\rho+\delta+(1+a(1+x))g-\theta(n+\delta+g)}{x+\theta}\frac{1}{1-\theta}\left(\frac{x+\theta}{x(n+g+\delta)+[\rho+\delta+(1+a(1+x))g]}\right)^{\frac{\theta}{1-\theta}} \times \\ & \frac{\rho+\delta+g+ag-\theta n-\theta g-\theta\delta-\theta ag}{\{x(n+g+\delta)+[\rho+\delta+(1+a(1+x))g]\}^2} \\ &= \left(\frac{x+\theta}{x(n+g+\delta)+[\rho+\delta+(1+a(1+x))g]}\right)^{\frac{1}{1-\theta}}\Big\{\frac{ag\theta-[\rho-\theta n+ag+(1-\theta)\delta+(1-\theta)g)]}{(x+\theta)^2} \\ & +\frac{\rho+\delta+(1+a(1+x))g-\theta(n+\delta+g)}{x+\theta}\frac{1}{1-\theta}\left(\frac{x+\theta}{x(n+g+\delta)+[\rho+\delta+(1+a(1+x))g]}\right)^{-1} \times \\ & \frac{\rho+\delta+g+ag-\theta n-\theta g-\theta\delta-\theta ag}{\{x(n+g+\delta)+[\rho+\delta+(1+a(1+x))g]\}^2}\Big\} \end{aligned} \tag{38}$$

By

$$\left(\frac{x+\theta}{x(n+g+\delta)+[\rho+\delta+(1+a(1+x))g]}\right)^{\frac{1}{1-\theta}} > 0 \tag{39}$$

the sign of $\frac{\partial \bar{c}(a,x)}{\partial x}$ is the same as the sign of the following formula:

$$\begin{aligned} & \frac{ag\theta-[\rho-\theta n+ag+(1-\theta)\delta+(1-\theta)g)]}{(x+\theta)^2} \\ & +\frac{\rho+\delta+(1+a(1+x))g-\theta(n+\delta+g)}{x+\theta}\frac{1}{1-\theta}\frac{x(n+g+\delta)+[\rho+\delta+(1+a(1+x))g]}{x+\theta} \times \\ & \frac{\rho+\delta+g+ag-\theta n-\theta g-\theta\delta-\theta ag}{\{x(n+g+\delta)+[\rho+\delta+(1+a(1+x))g]\}^2} \\ & = \frac{ag\theta-[\rho-\theta n+ag+(1-\theta)\delta+(1-\theta)g)]}{(x+\theta)^2} \\ & +\frac{[\rho+\delta+(1+a(1+x))g-\theta(n+\delta+g)]}{(x+\theta)^2}\frac{1}{1-\theta}\frac{\rho+\delta+g+ag-\theta n-\theta g-\theta\delta-\theta ag}{x(n+g+\delta)+[\rho+\delta+(1+a(1+x))g]} \end{aligned} \tag{40}$$

The sign of (40) is determined by the following formula:

$$\begin{aligned} & ag\theta-[\rho-\theta n+ag+(1-\theta)\delta+(1-\theta)g)] \\ & +\frac{1}{1-\theta}\frac{[\rho+\delta+(1+a(1+x))g-\theta(n+\delta+g)][\rho+\delta+g+ag-\theta n-\theta g-\theta\delta-\theta ag]}{x(n+g+\delta)+[\rho+\delta+(1+a(1+x))g]} \end{aligned} \tag{41}$$

(41) can be rewritten as

$$\begin{aligned} & \frac{(1-\theta)\{ag\theta-[\rho-\theta n+ag+(1-\theta)\delta+(1-\theta)g)]\}\{x(n+g+\delta)+[\rho+\delta+(1+a(1+x))g]\}}{(1-\theta)\{x(n+g+\delta)+[\rho+\delta+(1+a(1+x))g]\}} \\ & +\frac{1}{1-\theta}\frac{[\rho+\delta+(1+a(1+x))g-\theta(n+\delta+g)][\rho+\delta+g+ag-\theta n-\theta g-\theta\delta-\theta ag]}{x(n+g+\delta)+[\rho+\delta+(1+a(1+x))g]} \end{aligned} \tag{42}$$

whose sign is determined by the following formula:

$$\begin{aligned}
&(1-\theta)\{ag\theta-[\rho-\theta n+ag+(1-\theta)\delta+(1-\theta)g)]\}\times\\
&\ \{x(n+g+\delta)+[\rho+\delta+(1+a(1+x))g]\}\\
&\ +[\rho+\delta+(1+a(1+x))g-\theta(n+\delta+g)][\rho+\delta+g+ag-\theta n-\theta g-\theta\delta-\theta ag]\\
&\triangleq c_1\{x(n+g+\delta)+[\rho+\delta+(1+a(1+x))g]\}+\\
&\ c_2[\rho+\delta+(1+a(1+x))g-\theta(n+\delta+g)]\\
&=c_1\{x(n+g+\delta+ag)+[\rho+\delta+(1+a)g]\}+\\
&\ c_2[agx+\rho+\delta+(1+a)g-\theta(n+\delta+g)]\\
&=x(n+g+\delta+ag)c_1+[\rho+\delta+(1+a)g]c_1+agc_2x+\\
&\ [\rho+\delta+(1+a)g-\theta(n+\delta+g)]c_2\\
&=x[(n+g+\delta+ag)c_1+agc_2]+[\rho+\delta+(1+a)g]c_1+\\
&\ [\rho+\delta+(1+a)g-\theta(n+\delta+g)]c_2
\end{aligned}\tag{43}$$

Simplifying the above formula gives

$$\begin{aligned}
&[(n+g+\delta+ag)c_1+agc_2]x+\\
&\ \{[\rho+\delta+(1+a)g]c_1+[\rho+\delta+(1+a)g-\theta(n+\delta+g)]c_2\}\\
&=-\{(n+g+\delta+ag)(1-\theta)-ag\}\times\\
&\ [\rho-\theta n+(1-\theta)\delta+(1-\theta)g+ag(1-\theta)]x\\
&\ +\theta(\rho-n+ag)[\rho-\theta n+(1-\theta)\delta+(1-\theta)g+ag(1-\theta)]
\end{aligned}\tag{44}$$

By $\rho-\theta n+(1-\theta)\delta+(1-\theta)g+ag(1-\theta)>0$, the sign of $\frac{\partial\bar{c}}{\partial x}$ is the same as the sign of the following formula:

$$[\theta ag-(n+g+\delta)(1-\theta)]x+\theta(\rho-n+ag).\tag{45}$$

It is easy to see that if $a\geq\frac{(n+g+\delta)(1-\theta)}{\theta g}$, the sign of (45) is always positive, i.e., $\frac{\partial\bar{c}(a,x)}{\partial x}>0$. But for $a<\frac{(n+g+\delta)(1-\theta)}{\theta g}$, there exists

$$\psi_g=\frac{\theta(\rho-n+ag)}{(n+g+\delta)(1-\theta)-\theta ag}\tag{46}$$

such that $\frac{\partial\bar{c}(a,x)}{\partial x}>0$ for $x\in(0,\psi_g)$ and $\frac{\partial\bar{c}(a,x)}{\partial x}<0$ for $x\in(\psi_g,\infty)$. □

Proof of Proposition 2: Using (15) it is easy to prove $\frac{\partial\bar{k}}{\partial\rho}<0$.

Next, let's see the sign of $\frac{\partial\bar{c}}{\partial\rho}$. Rewrite the expression for $\bar{c}$ as

$$\bar{c}=\frac{a}{b+a\theta}[\rho+\delta+(1+(a+b))g-\theta(n+\delta+g)](\frac{b+a\theta}{b(n+g+\delta)+a[\rho+\delta+(1+(a+b))g]})^{\frac{1}{1-\theta}}.\tag{47}$$

Then we have

$$
\begin{aligned}
\frac{\partial \bar{c}}{\partial \rho} &= \tfrac{a}{b+a\theta}\{(\tfrac{b+a\theta}{b(n+g+\delta)+a[\rho+\delta+(1+(a+b))g]})^{\frac{1}{1-\theta}} + \tfrac{1}{1-\theta}[\rho+\delta+(1+(a+b))g-\theta(n+\delta+g)]\times \\
&\quad (\tfrac{b+a\theta}{b(n+g+\delta)+a[\rho+\delta+(1+(a+b))g]})^{\frac{1}{1-\theta}-1} \tfrac{-a(b+a\theta)}{[b(n+g+\delta)+a[\rho+\delta+(1+(a+b))g]]^2}\} \\
&= \tfrac{a}{b+a\theta}\{(\tfrac{b+a\theta}{b(n+g+\delta)+a[\rho+\delta+(1+(a+b))g]})^{\frac{1}{1-\theta}} - \tfrac{1}{1-\theta}[\rho+\delta+(1+(a+b))g-\theta(n+\delta+g)]\times \\
&\quad (\tfrac{b+a\theta}{b(n+g+\delta)+a[\rho+\delta+(1+(a+b))g]})^{\frac{\theta}{1-\theta}} \tfrac{a(b+a\theta)}{[b(n+g+\delta)+a[\rho+\delta+(1+(a+b))g]]^2}\} \\
&= \tfrac{a}{b+a\theta}(\tfrac{b+a\theta}{b(n+g+\delta)+a[\rho+\delta+(1+(a+b))g]})^{\frac{1}{1-\theta}}\{1-\tfrac{1}{1-\theta}[\rho+\delta+(1+(a+b))g-\theta(n+\delta+g)]\times \\
&\quad \{b(n+g+\delta)+a[\rho+\delta+(1+(a+b))g]\}\tfrac{a}{[b(n+g+\delta)+a[\rho+\delta+(1+(a+b))g]]^2}\}\times \\
&= \tfrac{a}{b+a\theta}(\tfrac{b+a\theta}{b(n+g+\delta)+a[\rho+\delta+(1+(a+b))g]})^{\frac{1}{1-\theta}}\{1-\tfrac{a}{1-\theta}\tfrac{[\rho+\delta+(1+(a+b))g-\theta(n+\delta+g)]}{b(n+g+\delta)+a[\rho+\delta+(1+(a+b))g]}\} \\
&= \tfrac{a}{b+a\theta}(\tfrac{b+a\theta}{b(n+g+\delta)+a[\rho+\delta+(1+(a+b))g]})^{\frac{1}{1-\theta}}\times \\
&\quad \tfrac{(1-\theta)\{b(n+g+\delta)+a[\rho+\delta+(1+(a+b))g]\}-a[\rho+\delta+(1+(a+b))g-\theta(n+\delta+g)]}{(1-\theta)\{b(n+g+\delta)+a[\rho+\delta+(1+(a+b))g]\}}
\end{aligned}
$$

The sign of the above formula depends on the sign of the third term of the numerator on the RHS. That is,

$$
\begin{aligned}
&\tfrac{\partial c}{\partial \rho} > 0 \Leftrightarrow (1-\theta)\{b(n+g+\delta)+a[\rho+\delta+(1+(a+b))g]\}- \\
&\quad a[\rho+\delta+(1+(a+b))g-\theta(n+\delta+g)] > 0 \Leftrightarrow \\
&b(1-\theta)(n+g+\delta)+a(1-\theta)[\rho+\delta+(1+(a+b))g]- \\
&\quad a[\rho+\delta+(1+(a+b))g]+a\theta(n+\delta+g) > 0 \Leftrightarrow \\
&b(1-\theta)(n+g+\delta)+[a(1-\theta)-a][\rho+\delta+(1+(a+b))g]+a\theta(n+\delta+g) > 0 \Leftrightarrow \\
&b(1-\theta)(n+g+\delta)-a\theta[\rho+\delta+(1+(a+b))g]+a\theta(n+\delta+g) > 0 \Leftrightarrow \\
&b(1-\theta)(n+g+\delta)-a\theta[\rho+(a+b)g-n] > 0
\end{aligned} \tag{48}
$$

Rearranging the above formula leads to

$$
\begin{aligned}
\tfrac{\partial c}{\partial \rho} > 0 &\Leftrightarrow b(1-\theta)(n+g+\delta)-a\theta\rho-a\theta(a+b)g+a\theta n > 0 \\
&\Leftrightarrow a\theta\rho < b(1-\theta)(n+g+\delta)-a\theta(a+b)g+a\theta n \\
&\Leftrightarrow \rho < \tfrac{b(1-\theta)(n+g+\delta)-a\theta(a+b)g+a\theta n}{a\theta} \\
&\Leftrightarrow \rho < \tfrac{b(1-\theta)(n+g+\delta)}{a\theta}-(a+b)g+n \triangleq \hat{\rho}
\end{aligned} \tag{49}
$$

It follows from (49) that $\frac{\partial \bar{c}}{\partial \rho} > 0$ for $n-(a+b)g < \rho < \hat{\rho}$ and $\frac{\partial \bar{c}}{\partial \rho} < 0$ for $\rho > \hat{\rho}$. □

Proof of the Formula (16)**:** It follows immediately from (15) that $\frac{\partial \bar{k}}{\partial \delta} < 0$.

Let

$$
\begin{aligned}
c &= \tfrac{a}{b+a\theta}[\rho+\delta+(1+(a+b))g-\theta(n+\delta+g)](\tfrac{b+a\theta}{b(n+g+\delta)+a[\rho+\delta+(1+(a+b))g]})^{\frac{1}{1-\theta}} \\
&\triangleq \tfrac{a}{b+a\theta}h(\delta)
\end{aligned} \tag{50}
$$

Then we have

$$
\begin{aligned}
\frac{\partial h}{\partial \delta} &= (1-\theta)(\tfrac{b+a\theta}{b(n+g+\delta)+a[\rho+\delta+(1+(a+b))g]})^{\frac{1}{1-\theta}} \\
&\quad +\tfrac{1}{1-\theta}[\rho+\delta+(1+(a+b))g-\theta(n+\delta+g)](\tfrac{b+a\theta}{b(n+g+\delta)+a[\rho+\delta+(1+(a+b))g]})^{\frac{1}{1-\theta}-1} \\
&\quad (-1)\tfrac{(b+a)(b+a\theta)}{\{b(n+g+\delta)+a[\rho+\delta+(1+(a+b))g]\}^2} \\
&= -(1-\theta)(\tfrac{b+a\theta}{b(n+g+\delta)+a[\rho+\delta+(1+(a+b))g]})^{\frac{1}{1-\theta}}\{-1+\tfrac{b+a}{(1-\theta)^2}\tfrac{\rho+\delta+(1+(a+b))g-\theta(n+\delta+g)}{b(n+g+\delta)+a[\rho+\delta+(1+(a+b))g]}\}
\end{aligned} \tag{51}
$$

Notice that the sign of $\frac{\partial h}{\partial \delta}$ is opposite to the following formula:

$$\begin{aligned}
&(1-\theta)^2\{-b(n+g+\delta)-a[\rho+\delta+(1+(a+b))g]\}+\\
&\ (a+b)[\rho+\delta+(1+(a+b))g-\theta(n+\delta+g)]\\
&=-(1-\theta)^2 b(n+g+\delta)-(1-\theta)^2 a[\rho+\delta+(1+(a+b))g]+\\
&\ b[\rho+\delta+(1+(a+b))g-\theta(n+\delta+g)]+\\
&\ a[\rho+\delta+(1+(a+b))g]-a\theta(n+\delta+g)\\
&=-(1-\theta)^2 b(n+g+\delta)+b[\rho+\delta+(1+(a+b))g]-b\theta(n+\delta+g)-\\
&\ (1-\theta)^2 a[\rho+\delta+(1+(a+b))g]+a[\rho+\delta+(1+(a+b))g]-a\theta(n+\delta+g)\\
&=-b(n+g+\delta)+2\theta b(n+g+\delta)-\theta^2 b(n+g+\delta)+\\
&\ b[\rho+\delta+(1+(a+b))g]-b\theta(n+\delta+g)-\\
&\ (1-\theta)^2 a[\rho+\delta+(1+(a+b))g]+a[\rho+\delta+(1-(\alpha+\beta))g]-a\theta(n+\delta+g)\\
&=-b(n+g+\delta)+\theta b(n+g+\delta)-\theta^2 b(n+g+\delta)+b[\rho+\delta+(1+(a+b))g]-\\
&\ (1-\theta)^2 a[\rho+\delta+(1+(a+b))g]+a[\rho+\delta+(1+(a+b))g]-a\theta(n+\delta+g)\\
&=\theta b(1-\theta)(n+g+\delta)+b[\rho-n+(a+b)g]-\\
&\ (1-\theta)^2 a[\rho+\delta+(1+(a+b))g]+a[\rho+\delta+(1+(a+b))g]-a\theta(n+\delta+g)\\
&=\theta b(1-\theta)(n+g+\delta)+b[\rho-n+(a+b)g]-\\
&\ a[\rho+\delta+(1+(a+b))g]+2a\theta[\rho+\delta+(1+(a+b))g]-\\
&\ \theta^2 a[\rho+\delta+(1+(a+b))g]+a[\rho+\delta+(1+(a+b))g]-a\theta(n+\delta+g)\\
&=\theta b(1-\theta)(n+g+\delta)+b[\rho-n+(a+b)g]+\\
&\ 2a\theta[\rho+\delta+(1+(a+b))g]-\theta^2 a[\rho+\delta+(1+(a+b))g]-a\theta(n+\delta+g)\\
&=\theta b(1-\theta)(n+g+\delta)+b[\rho-n-(\alpha+\beta)g]+\\
&\ a\theta[\rho+\delta+(1+(a+b))g]-\theta^2 a[\rho+\delta+(1+(a+b))g]+a\theta[\rho-n+(a+b)g]\\
&=\theta b(1-\theta)(n+g+\delta)+b[\rho-n+(a+b)g]+\\
&\ a\theta(1-\theta)[\rho+\delta+(1+(a+b))g]+a\theta[\rho-n+(a+b)g]\\
&>0
\end{aligned} \tag{52}$$

Then $\frac{\partial c}{\partial \delta}=\frac{a}{b+a\theta}\frac{\partial h}{\partial \delta}<0$. □

Proof of Proposition 3: The steady-state saving rate $\bar{s}$ can be expressed as $\bar{s}=1-\bar{c}/f(\bar{k})$. Substituting $\bar{k}$ and $\bar{c}$ in (15) into the RHS of $\bar{s}$ leads to

$$\begin{aligned}
\bar{s}&=1-\frac{\bar{c}}{\bar{k}^{\theta}}\\
&=1-\frac{\frac{a[\rho+\delta+(1+(a+b))g-\theta(n+\delta+g)]}{b+a\theta}\left(\frac{b+a\theta}{b(n+g+\delta)+a[\rho+\delta+(1+(a+b))g]}\right)^{\frac{1}{1-\theta}}}{\left(\frac{b+a\theta}{b(n+g+\delta)+a[\rho+\delta+(1+(a+b))g]}\right)^{\frac{\theta}{1-\theta}}}\\
&=1-\frac{a[\rho+\delta+(1+(a+b))g-\theta(n+\delta+g)]}{b+a\theta}\left(\frac{b+a\theta}{b(n+g+\delta)+a[\rho+\delta+(1+(a+b))g]}\right)^{\frac{1}{1-\theta}-\frac{\theta}{1-\theta}}\\
&=1-\frac{a[\rho+\delta+(1+(a+b))g-\theta(n+\delta+g)]}{b+a\theta}\left(\frac{b+a\theta}{b(n+g+\delta)+a[\rho+\delta+(1+(a+b))g]}\right)\\
&=1-\frac{a[\rho+\delta+(1+(a+b))g-\theta(n+\delta+g)]}{b(n+g+\delta)+a[\rho+\delta+(1+(a+b))g]}\\
&=\frac{b(n+g+\delta)+a[\rho+\delta+(1+(a+b))g]-a[\rho+\delta+(1+(a+b))g]+a\theta(n+\delta+g)}{b(n+g+\delta)+a[\rho+\delta+(1+(a+b))g]}\\
&=\frac{(b+a\theta)(n+g+\delta)}{b(n+g+\delta)+a[\rho+\delta+(1+(a+b))g]}
\end{aligned}. \tag{53}$$

Recall $x = b/a$. Then we can write $\bar{s}$ as

$$\bar{s} = \frac{(x+\theta)(n+g+\delta)}{x(n+g+\delta)+[\rho+\delta+(1+a(1+x))g]}. \tag{54}$$

It is easy to see that $\frac{\partial \bar{s}}{\partial a} < 0$ if $g \neq 0$. At the same time, it follows from (54) that

$$\begin{aligned}
\frac{\partial \bar{s}}{\partial x} &= \frac{(n+g+\delta)\{x(n+g+\delta)+[\rho+\delta+(1+a(1+x))g]\}-(x+\theta)(n+g+\delta)[(n+g+\delta)+ag]}{\{x(n+g+\delta)+[\rho+\delta+(1+a(1+x))g]\}^2} \\
&= \frac{(n+g+\delta)[\rho+\delta+(1+a(1+x))g]-xag(n+g+\delta)-\theta(n+g+\delta)(n+g+\delta)-ag\theta(n+g+\delta)}{\{x(n+g+\delta)+[\rho+\delta+(1+a(1+x))g]\}^2} \\
&= \frac{(n+g+\delta)[\rho+\delta+(1+a(1+x))g-xag-\theta(n+g+\delta)-ag\theta]}{\{x(n+g+\delta)+[\rho+\delta+(1+a(1+x))g]\}^2} \\
&= \frac{(n+g+\delta)\{\rho+\delta+(1+a(1+x))g-\theta n-\theta g-\theta\delta-ag\theta-xag\}}{\{x(n+g+\delta)+[\rho+\delta+(1+a(1+x))g]\}^2} \\
&= \frac{(n+g+\delta)\{\rho-\theta n+(1-\theta)\delta+(1-\theta)g+a(1+x)g-ag\theta-xag\}}{\{x(n+g+\delta)+[\rho+\delta+(1+a(1+x))g]\}^2} \\
&= \frac{(n+g+\delta)\{\rho-\theta n+(1-\theta)\delta+(1-\theta)g+ag-ag\theta\}}{\{x(n+g+\delta)+[\rho+\delta+(1+a(1+x))g]\}^2}
\end{aligned} \tag{55}$$

Clearly $\partial\bar{s}/\partial x > 0$ as $\rho - \theta n + (1-\theta)\delta + (1-\theta)g + ag(1-\theta) > 0$. □

Proof of Proposition 4: Let z denote the ratio of consumption to output, i.e., $z = \frac{c}{f}$. Differentiating its both sides w.r.t. time gives

$$\frac{\dot{z}}{z} = \frac{\dot{c}}{c} - \frac{f'(k)}{f}\dot{k} \tag{56}$$

Recall that $\dot{k}$ and $\dot{c}$ satisfy ODE's (9) and (10), i.e.,

$$\dot{k} = k^\theta - c - (n+g+\delta)k, \tag{57}$$

$$\dot{c} = \frac{c}{1+a}\{\frac{bc}{ak} + \theta k^{(\theta-1)} - (\rho+\delta+(1+(a+b))g) - b[k^{\theta-1} - \frac{c}{k} - (n+g+\delta)]\} \tag{58}$$

It follows from (57) and (58) that

$$\frac{\dot{k}}{k} = k^{\theta-1} - \frac{c}{k} - (n+g+\delta) \tag{59}$$

$$\frac{\dot{c}}{c} = \frac{1}{1+a}\{(\frac{b}{a}+b)\frac{c}{k} + (\theta-b)k^{\theta-1} + [b(n+\delta) - (\rho+\delta+(1+a)g)]\} \tag{60}$$

Observe that $\frac{f'(k)}{f(k)} = \frac{\theta k^{\theta-1}}{k^\theta} = \theta\frac{1}{k}$. Then, substituting this result and (60) into (56) together with (59) yields

$$
\begin{aligned}
\frac{\dot z}{z} &= \frac{\dot c}{c} - \frac{f'(k)}{f}\dot k = \frac{\dot c}{c} - \frac{\theta k^{\theta-1}}{k^\theta}\dot k = \frac{\dot c}{c} - \theta\frac{\dot k}{k} \\
&= \tfrac{1}{1+a}\{(\tfrac{b}{a}+b)\tfrac{c}{k} + (\theta-b)k^{\theta-1} + [b(n+\delta) - (\rho+\delta+(1+a)g)]\} - \\
&\quad \theta\{k^{\theta-1} - \tfrac{c}{k} - (n+g+\delta)\} \\
&= \tfrac{1}{1+a}\{(\tfrac{b}{a}+b)\tfrac{c}{f(k)}\tfrac{f(k)}{k} + (\theta-b)k^{\theta-1} + [b(n+\delta) - (\rho+\delta+(1+a)g)]\} - \\
&\quad \theta\{k^{\theta-1} - \quad \tfrac{c}{f(k)}\tfrac{f(k)}{k} - (n+g+\delta)\} \\
&= \tfrac{1}{1+a}\{(\tfrac{b}{a}+b)\tfrac{f(k)}{k}z + (\theta-b)k^{\theta-1} + [b(n+\delta) - (\rho+\delta+(1+a)g)]\} - \\
&\quad \theta\{k^{\theta-1} - \tfrac{f(k)}{k}z - (n+g+\delta)\} \\
&= \tfrac{1}{1+a}\{(\tfrac{b}{a}+b)k^{\theta-1}z + (\theta-b)k^{\theta-1} + [b(n+\delta) - (\rho+\delta+(1+a)g)]\} - \\
&\quad \theta\{k^{\theta-1} - k^{\theta-1}z - (n+g+\delta)\} \\
&= \tfrac{1}{1+a}[(\tfrac{b}{a}+b)z + (\theta-b)]k^{\theta-1} + \tfrac{1}{1+a}[b(n+\delta) - (\rho+\delta+(1+a)g)] - \\
&\quad \theta(1-z)k^{\theta-1} + \theta(n+g+\delta) \\
&= \{\tfrac{1}{1+a}[(\tfrac{b}{a}+b)z + (\theta-b)] - \theta(1-z)\}k^{\theta-1} + \\
&\quad \tfrac{1}{1+a}[b(n+\delta) - (\rho+\delta+(1+a)g)] + \theta(n+g+\delta) \\
&= \{\tfrac{1}{1+a}(\tfrac{b}{a}+b)z + \tfrac{1}{1+a}(\theta-b) + \theta(z-1)\}k^{\theta-1} + \\
&\quad \tfrac{1}{1+a}[b(n+\delta) - (\rho+\delta+(1+a)g)] + \theta(n+g+\delta) \\
&= \{[\tfrac{1}{1+a}(\tfrac{b}{a}+b) + \theta]z - [\theta - \tfrac{1}{1+a}(\theta-b)]\}k^{\theta-1} + \\
&\quad \tfrac{1}{1+a}[b(n+\delta) - (\rho+\delta+(1+a)g)] + \theta(n+g+\delta) \\
&= \{[\tfrac{1}{1+a}(\tfrac{b}{a}+b) + \theta]z - \tfrac{\theta(1+a)+b-\theta}{1+a}\}k^{\theta-1} + \\
&\quad \tfrac{1}{1+a}[b(n+\delta) - (\rho+\delta+(1+a)g)] + \theta(n+g+\delta) \\
&= \{(\tfrac{b}{a}+\theta)z - \tfrac{b+a\theta}{1+a}\}k^{\theta-1} + \\
&\quad \tfrac{1}{1+a}[b(n+\delta) - (\rho+\delta+(1+a)g)] + \theta(n+g+\delta) \\
&= \{\tfrac{b+a\theta}{a}z - \tfrac{b+a\theta}{1+a}\}k^{\theta-1} + \\
&\quad \tfrac{1}{1+a}[b(n+\delta) - (\rho+\delta+(1+a)g)] + \theta(n+g+\delta) \\
&= (z - \tfrac{a}{1+a})\tfrac{b+a\theta}{a}k^{\theta-1} + \tfrac{1}{1+a}[b(n+\delta) - (\rho+\delta+(1+a)g)] + \\
&\quad \theta(n+g+\delta)
\end{aligned} \tag{61}
$$

If $b = 0$, it becomes

$$
\frac{\dot z}{z} = (z - \frac{a}{1+a})\theta k^{\theta-1} + \theta(n+g+\delta) - \frac{1}{1+a}(\rho+\delta+(1+a)g) \tag{62}
$$

Setting $1 + a = \theta$, then (62) is identical to one for the traditional model (see Barro and Sala-i-Martin, 2004). In general, we can write (61) in the following succinct form:

$$
\frac{\dot z}{z} = (z - m_1)\frac{b+a\theta}{a}k^{\theta-1} + m_2, \tag{63}
$$

where $m_1 = \frac{a}{1+a} > 0$ and $m_2 = \frac{1}{1+a}[b(n+\delta) - (\rho+\delta+(1+a)g)] + \theta(n+g+\delta)$. Substituting $s = 1 - z$ and its derivative $\dot s = 1 - \dot z$ into (63), we obtain the ODE for the optimal saving rate:

$$\dot{s}(t) = [(s(t) - \frac{1}{1+a})\frac{b+a\theta}{a}k(t)^{\theta-1} - m_2](1-s(t)). \tag{64}$$

Let's first consider the case (i) of the proposition, i.e., $m_2 > 0$. In this case, if there exists a time $\tau > 0$ such that $z(\tau) \geq m_1$, then by (63), we have $\dot{z}(t) > 0$ for all $t \in (\tau, \infty)$. Hence $z(t) > z(\tau) > 0$ for all $t \in (\tau, \infty)$. Notice again that for all $t \in (\tau, \infty)$ it holds $k(t) < \bar{k}$. Then $\frac{\dot{z}(t)}{z(t)} = (z(t) - m_1)\frac{b+a\theta}{a}k(t)^{\theta-1} + m_2 > (z(\tau) - m_1)\frac{b+a\theta}{a}\bar{k}^{\theta-1} + m_2 \triangleq \xi_1 > 0$ for all $t \in [\tau, \infty)$. So $z(t) > z(\tau)e^{\xi_1(t-\tau)}$ for all $t \in (\tau, \infty)$. It is easy to see that $z(t) = z(\tau)e^{\xi_1(t-\tau)} > 1$ for $t > \tau - \frac{\ln z(\tau)}{\xi_1}$, contradicting the condition of $z(t) < 1$. Therefore, for any $t \in [0, \infty)$, $z(t) \geq m_1$ is impossible if $m_2 > 0$. In other words, we must have $z(t) < m_1$ in this situation, or equivalently, $s(t) > 1 - \frac{a}{1+a} = \frac{1}{1+a}$ for all $t \in [0, \infty)$.

Next, consider (ii), i.e., $m_2 = 0$. Then (63) becomes $\frac{\dot{z}(t)}{z(t)} = (z(t) - m_1)\frac{b+a\theta}{a}k(t)^{\theta-1}$. This can be broken down into two subcases: First, the case in which $z(\tau) > m_1$ for some $\tau > 0$; Second, the case in which $z(\tau) \leq m_1$ for some $\tau > 0$. In the first subcase, by $z(\tau) > m_1$, it is easy to see that $\dot{z}(t) > 0$ for all $t \in (\tau, \infty)$. Then $z(t) > z(\tau) > 0$ for $\forall t \in (\tau, \infty)$. Notice that for all $t \in (\tau, \infty)$, $k(t) < \bar{k}$ and thus $(z(t) - m_1)\frac{b+a\theta}{a}k(t)^{\theta-1} > (z(\tau) - m_1)\frac{b+a\theta}{a}\bar{k}^{\theta-1}$ for all $t \in [\tau, \infty)$. Let $(z(\tau) - m_1)\frac{b+a\theta}{a}\bar{k}^{\theta-1} \triangleq \xi_2 > 0$. Then $\frac{\dot{z}(t)}{z(t)} > \xi_2$ and $z(t) > z(\tau)e^{\xi_2(t-\tau)}$ for all $t \in (\tau, \infty)$. When $t > \tau - \frac{\ln z(\tau)}{\xi_2}$, $z(t) = z(\tau)e^{\xi_2(t-\tau)} > 1$, which contradicts the assumption $z(t) < 1$. So, under the condition of $m_2 = 0$, for any $t \in [0, \infty)$, $z(t) > m_1$ is impossible. Similarly, we can prove that for any $t \in [0, \infty)$, $z(t) < m_1$ is also impossible. Consequently, we have that if $m_2 = 0$, then $z(t) = m_1$, e.g., $s(t) = 1 - z(t) = \frac{1}{1+a}$.

(iii) The third case of $m_2 < 0$ can be proved similarly, i.e., $s(t) < \frac{1}{1+a}$ if $m_2 < 0$.

Finally, let's see the property of monotonicity of $s(t)$. Denote $\gamma_z(t) = \frac{\dot{z}(t)}{z(t)}$. Then differentiating both sides of (63) leads to

$$\dot{\gamma}_z = \frac{b+a\theta}{a}k^{\theta-1}[\gamma_z z + (z - m_1)(\theta - 1)\frac{\dot{k}}{k}] \tag{65}$$

If $m_2 > 0$, i.e., $z < m_1$ and then $(z - m_1)(\theta - 1)\frac{\dot{k}}{k} > 0$. Now if $\gamma_z(\tau) > 0$ for some $\tau > 0$, $\dot{\gamma}_z(t) > 0$ for all $t \in (\tau, \infty)$. So $\gamma_z(t)$ is positive and increasing in time $t \in (\tau, \infty)$. Similar to the above analysis for the boundary of z, $z(t)$ must go infinite in this case, which is again a contradiction. So we must have $\dot{\gamma}_z(t) < 0$, i.e., $z(t)$ is monotonically decreasing, or equivalently, $s(t) = 1 - z(t)$ is monotonically increasing. Likewise, we can prove $s(t)$ is decreasing in time if $m_2 < 0$. $\square$

Proof of Proposition 5: Writing (9) and (10) in terms of logarithm, we have

$$\begin{aligned} &\tfrac{d\ln(k)}{dt} = e^{(\theta-1)\ln(k)} - e^{\ln(c)-\ln(k)} - (n+g+\delta) \\ &\tfrac{d\ln(c)}{dt} = \tfrac{1}{1+a}\{\tfrac{b(1+a)}{a}e^{\ln(c)-\ln(k)} + (\theta-b)e^{(\theta-1)\ln(k)} - \\ &\qquad [(\rho+\delta+(1+(a+b))g - b(n+g+\delta)]\} \end{aligned} \tag{66}$$

Letting $\ln(k(t)) = x(t)$ and $\ln(c(t)) = y(t)$, we can write (66) as

$$\begin{aligned}
&\dot{x} = e^{(\theta-1)x} - e^{y-x} - (n+g+\delta) \stackrel{\Delta}{=} f(x,y) \\
&\dot{y} = \tfrac{1}{1+a}\{\tfrac{b(1+a)}{a}e^{y-x} + (\theta-b)e^{(\theta-1)x} - \\
&\qquad [(\rho+\delta+(1+(a+b))g - b(n+g+\delta)]\} \stackrel{\Delta}{=} g(x,y)
\end{aligned} \tag{67}$$

Furthermore, we have

$$\begin{aligned}
&f_x = (\theta-1)e^{(\theta-1)x} + e^{y-x} \\
&f_y = -e^{y-x} \\
&g_x = \tfrac{1}{1+a}[-\tfrac{b(1+a)}{a}e^{y-x} + (\theta-b)(\theta-1)e^{(\theta-1)x}] \\
&g_y = \tfrac{b}{a}e^{y-x}
\end{aligned} \tag{68}$$

Substituting $\bar{k}$ and $\bar{c}$ in (15) into (68) leads to

$$\begin{aligned}
&f_x|_{(\bar{k},\bar{c})} \stackrel{\Delta}{=} l_1 \\
&\qquad = (\theta-1)(\tfrac{b+a\theta}{b(n+g+\delta)+a[\rho+\delta+(1+(a+b))g]})^{\frac{\theta-1}{1-\theta}} + \\
&\qquad\qquad \frac{[\frac{a[\rho+\delta+(1+(a+b))g-\theta(n+\delta+g)]}{b+a\theta}](\frac{b+a\theta}{b(n+g+\delta)+a[\rho+\delta+(1+(a+b))g]})^{\frac{1}{1-\theta}}}{(\frac{b+a\theta}{b(n+g+\delta)+a[\rho+\delta+(1+(a+b))g]})^{\frac{1}{1-\theta}}} \\
&\quad = (\theta-1)\tfrac{b(n+g+\delta)+a[\rho+\delta+(1+(a+b))g]}{b+a\theta} + \tfrac{a[\rho+\delta+(1+(a+b))g-\theta(n+\delta+g)]}{b+a\theta} \\
&\quad = \tfrac{1}{b+a\theta}\{(\theta-1)\{b(n+g+\delta)+ \\
&\quad\ \ a[\rho+\delta+(1+(a+b))g]\} + a[\rho+\delta+(1+(a+b))g - \theta(n+\delta+g)]\} \\
&\quad = \tfrac{1}{b+a\theta}\{\theta\{b(n+g+\delta)+a[\rho+\delta+(1+(a+b))g]\} - b(n+g+\delta) - \\
&\quad\ \ a[\rho+\delta+(1+(a+b))g] + a[\rho+\delta+(1+(a+b))g - \theta(n+\delta+g)]\} \\
&\quad = \tfrac{1}{b+a\theta}\{\theta b(n+g+\delta) + a\theta[\rho+(1+(a+b))g] - b(n+g+\delta) - a\theta(n+g)\} \\
&\quad = \tfrac{1}{b+a\theta}\{\theta b(n+g+\delta) - b(n+g+\delta) + a\theta[\rho+(1+(a+b))g] - a\theta(n+g)\} \\
&\quad = \tfrac{1}{b+a\theta}\{(\theta-1)b(n+g+\delta) + a\theta[\rho-n+(a+b)g]\}
\end{aligned} \tag{69}$$

$$\begin{aligned}
&f_y|_{(\bar{k},\bar{c})} \stackrel{\Delta}{=} l_2 \\
&= -\frac{[\frac{a[\rho+\delta+(1+(a+b))g-\theta(n+\delta+g)]}{b+a\theta}](\frac{b+a\theta}{b(n+g+\delta)+a[\rho+\delta+(1+(a+b))g]})^{\frac{1}{1-\theta}}}{(\frac{b+a\theta}{b(n+g+\delta)+a[\rho+\delta+(1+(a+b))g]})^{\frac{1}{1-\theta}}} \\
&= -\tfrac{a[\rho+\delta+(1+(a+b))g-\theta(n+\delta+g)]}{b+a\theta} \\
&= \tfrac{a\{\theta(n+\delta+g)-[\rho+\delta+(1+(a+b))g]\}}{b+a\theta}
\end{aligned} \tag{70}$$

$$\begin{aligned}
&g_x|_{(\bar{k},\bar{c})} \stackrel{\Delta}{=} l_3 \\
&= \tfrac{1}{1+a}\{\tfrac{b(1+a)}{a}\tfrac{a\{\theta(n+\delta+g)-[\rho+\delta+(1+(a+b))g]\}}{b+a\theta} + (\theta-b)(\theta-1)\tfrac{b(n+g+\delta)+a[\rho+\delta+(1+(a+b))g]}{b+a\theta}\} \\
&= \{\tfrac{b\{\theta(n+\delta+g)-[\rho+\delta+(1+(a+b))g]\}}{b+a\theta} + \tfrac{(\theta-b)(\theta-1)}{(1+a)}\tfrac{b(n+g+\delta)+a[\rho+\delta+(1+(a+b))g]}{b+a\theta}\} \\
&= \tfrac{1}{b+a\theta}(b(\theta(n+\delta+g) - (\rho+\delta+(1+(a+b))g)) + \\
&\qquad \tfrac{(\theta-b)(\theta-1)(b(n+g+\delta)+a(\rho+\delta+(1+(a+b))g))}{(1+a)})
\end{aligned} \tag{71}$$

and

$$
\begin{aligned}
g_y|_{(\bar{k},\bar{c})} &\triangleq l_4 \\
&= -\frac{b}{a}\frac{a\{\theta(n+\delta+g)-[\rho+\delta+(1+(a+b))g]\}}{b+a\theta} \\
&= \frac{b\{[\rho+\delta+(1+(a+b))g]-\theta(n+\delta+g)\}}{b+a\theta}
\end{aligned}
\tag{72}
$$

We can calculate the trace and determinant of the matrix J_E:

$$
\begin{aligned}
&trJ_E = l_1 + l_4 \\
&= \frac{1}{b+a\theta}\{(\theta-1)b(n+g+\delta) + a\theta[\rho - n + (a+b)g]\} + \\
&\quad \frac{b\{[\rho+\delta+(1+(a+b))g]-\theta(n+\delta+g)\}}{b+a\theta} \\
&= \frac{(\theta-1)b(n+g+\delta)+a\theta[\rho-n+(a+b)g]+b\{[\rho+\delta+(1+(a+b))g]-\theta(n+\delta+g)\}}{b+a\theta} \\
&= \frac{\theta b(n+g+\delta)-b(n+g+\delta)+a\theta[\rho-n+(a+b)g]+b[\rho+\delta+(1+(a+b))g]-b\theta(n+\delta+g)}{b+a\theta} \\
&= \frac{-bn+a\theta[\rho-n+(a+b)g]+b[\rho+(a+b)g]}{b+a\theta} = \frac{a\theta[\rho-n+(a+b)g]+b[\rho-n+(a+b)g]}{b+a\theta} \\
&= \rho - n + (a+b)g
\end{aligned}
\tag{73}
$$

and

$$
\begin{aligned}
|J_E| &= l_1 l_4 - l_2 l_3 \\
&= \tfrac{1}{b+a\theta}\{(\theta-1)b(n+g+\delta)+a\theta[\rho-n+(a+b)g]\}\tfrac{b\{[\rho+\delta+(1+(a+b))g]-\theta(n+\delta+g)\}}{b+a\theta}- \\
&\quad \tfrac{a\{\theta(n+\delta+g)-[\rho+\delta+(1+(a+b))g]\}}{b+a\theta}\tfrac{1}{b+a\theta}\{b\{\theta(n+\delta+g)-[\rho+\delta+(1+(a+b))g]\}+ \\
&\quad \tfrac{(\theta-b)(\theta-1)\{b(n+g+\delta)+a[\rho+\delta+(1+(a+b))g]\}}{(1+a)}\} \\
&= \tfrac{1}{(b+a\theta)^2}\{(\theta-1)b(n+g+\delta)+a\theta[\rho-n+(a+b)g]\}b\{[\rho+\delta+ \\
&\quad (1+(a+b))g]-\theta(n+\delta+g)\}-\tfrac{1}{(b+a\theta)^2}a\{\theta(n+\delta+g)-[\rho+\delta+ \\
&\quad (1+(a+b))g]\}b\{\theta(n+\delta+g)-[\rho+\delta+(1+(a+b))g]\}- \\
&\quad \tfrac{a\{\theta(n+\delta+g)-[\rho+\delta+(1+(a+b))g]\}(\theta-b)(\theta-1)\{b(n+g+\delta)+a[\rho+\delta+(1+(a+b))g\}}{(1+a)(b+a\theta)^2} \\
&= \tfrac{1}{(b+a\theta)^2}\{\{(\theta-1)b(n+g+\delta)+a\theta[\rho-n+(a+b)g]\}b\{[\rho+\delta+ \\
&\quad (1+(a+b))g]-\theta(n+\delta+g)\}-a\{\theta(n+\delta+g)-[\rho+\delta+ \\
&\quad (1+(a+b))g]\}b\{\theta(n+\delta+g)-[\rho+\delta+(1+(a+b))g]\}\}- \\
&\quad \tfrac{a\{\theta(n+\delta+g)-[\rho+\delta+(1+(a+b))g]\}(\theta-b)(\theta-1)\{b(n+g+\delta)+a[\rho+\delta+(1+(a+b))g]\}}{(1+a)(b+a\theta)^2} \\
&= \tfrac{1}{(b+a\theta)^2}\{\{(\theta-1)b(n+g+\delta)+a\theta[\rho-n+(a+b)g]\}b\{[\rho+\delta+ \\
&\quad (1+(a+b))g]-\theta(n+\delta+g)\}+a\{\theta(n+\delta+g)-[\rho+\delta+ \\
&\quad (1+(a+b))g]\}b\{[\rho+\delta+(1+(a+b))g]-\theta(n+\delta+g)\}\}- \\
&\quad \tfrac{a\{\theta(n+\delta+g)-[\rho+\delta+(1+(a+b))g]\}(\theta-b)(\theta-1)\{b(n+g+\delta)+a[\rho+\delta+(1+(a+b))g]\}}{(1+a)(b+a\theta)^2} \\
&= \tfrac{b\{[\rho+\delta+(1+(a+b))g]-\theta(n+\delta+g)\}}{(b+a\theta)^2}\{\{(\theta-1)b(n+g+\delta)+a\theta[\rho-n+(a+b)g]\}+ \\
&\quad a\{\theta(n+\delta+g)-[\rho+\delta+(1+(a+b))g]\}\} \\
&\quad -\tfrac{a\{\theta(n+\delta+g)-[\rho+\delta+(1+(a+b))g]\}(\theta-b)(\theta-1)\{b(n+g+\delta)+a[\rho+\delta+(1+(a+b))g]\}}{(1+a)(b+a\theta)^2} \\
&= \tfrac{b\{[\rho+\delta+(1+(a+b))g]-\theta(n+\delta+g)\}(\theta-1)\{b(n+g+\delta)+a[\rho+\delta+(1+(a+b))g]\}}{(b+a\theta)^2} \\
&\quad -\tfrac{a\{\theta(n+\delta+g)-[\rho+\delta+(1+(a+b))g]\}(\theta-b)(\theta-1)\{b(n+g+\delta)+a[\rho+\delta+(1+(a+b))g]\}}{(1+a)(b+a\theta)^2} \\
&= \tfrac{\{[\rho+\delta+(1+(a+b))g]-\theta(n+\delta+g)\}\{b(n+g+\delta)+a[\rho+\delta+(1+(a+b))g]\}(\theta-1)}{(b+a\theta)^2}b \\
&\quad +\tfrac{\{[\rho+\delta+(1+(a+b))g]-\theta(n+\delta+g)\}\{b(n+g+\delta)+a[\rho+\delta+(1+(a+b))g]\}(\theta-1)}{(b+a\theta)^2}\tfrac{(\theta-b)a}{(1+a)} \\
&= \tfrac{(\theta-1)\{[\rho+\delta+(1+(a+b))g]-\theta(n+\delta+g)\}\{b(n+g+\delta)+a[\rho+\delta+(1+(a+b))g]\}}{(b+a\theta)^2}\{b+\tfrac{a(\theta-b)}{1+a}\} \\
&= \tfrac{(\theta-1)\{[\rho+\delta+(1+(a+b))g]-\theta(n+\delta+g)\}\{b(n+g+\delta)+a[\rho+\delta+(1+(a+b))g]\}}{(b+a\theta)(1+a)}
\end{aligned}
\tag{74}
$$

In deriving the result of the above 7th equality, we have used

$$
\begin{aligned}
&\{(\theta-1)b(n+g+\delta)+a\theta[\rho-n+(a+b)g]\}+ \\
&\qquad a\{\theta(n+\delta+g)-[\rho+\delta+(1+(a+b))g]\} \\
&= (\theta-1)b(n+g+\delta)+a\theta[\rho-n+(a+b)g]+ \\
&\qquad a\theta(n+\delta+g)-a[\rho+\delta+(1+(a+b))g] \\
&= (\theta-1)b(n+g+\delta)+a\theta[\rho+\delta+(1+(a+b))g]-a[\rho+\delta+(1+(a+b))g] \\
&= (\theta-1)b(n+g+\delta)+(\theta-1)a[\rho+\delta+(1+(a+b))g] \\
&= (\theta-1)\{b(n+g+\delta)+a[\rho+\delta+(1+(a+b))g]\}
\end{aligned}
\ . \tag{75}
$$

Then

$$
\begin{array}{l}
-\left|J_E\right| = \frac{(1-\theta)\{[\rho+\delta+(1+(a+b))g]-\theta(n+\delta+g)\}}{(b+a\theta)(1+a)}\times \\
\quad \{b(n+g+\delta)+a[\rho+\delta+(1+(a+b))g]\}
\end{array}
\tag{76}
$$

In terms of $x = b/a$ and a, (76) can be formulated as

$$
\begin{array}{l}
-\left|J_E\right| = \frac{1-\theta}{1+a}\frac{\{[\rho+\delta+(1+a(1+x))g]-\theta(n+\delta+g)\}\{[\rho+\delta+(1+a(1+x))g]+x(n+g+\delta)\}}{x+\theta} \\
\quad = \frac{1-\theta}{1+a}\frac{[\rho+\delta+(1+a(1+x))g]^2+(x-\theta)(n+\delta+g)[\rho+\delta+(1+a(1+x))g]-\theta x(n+g+\delta)^2}{x+\theta}
\end{array}.
\tag{77}
$$

Using $\frac{f'(k)}{f(k)} = \frac{\theta k^{\theta-1}}{k^\theta} = \theta\frac{1}{k}$ and differentiating (77) we have

$$
\begin{array}{l}
\frac{\partial(-|J_E|)}{\partial x} = \frac{1-\theta}{1+a}\frac{(x+\theta)G}{(x+\theta)^2} \\
\quad +\frac{1-\theta}{1+a}\frac{-[\rho+\delta+(1+a(1+x))g]^2-(x-\theta)(n+\delta+g)[\rho+\delta+(1+a(1+x))g]+\theta x(n+g+\delta)^2}{(x+\theta)^2}
\end{array}
\tag{78}
$$

where

$$
\begin{array}{l}
G = 2ag[\rho+\delta+(1+a(1+x))g]+[\rho+\delta+(1+a(1+x))g](n+\delta+g) \\
\quad +ag(x-\theta)(n+\delta+g)-\theta(n+g+\delta)^2
\end{array}
\tag{79}
$$

Now the sign of (78) is the same as the sign of the following formula:

$$
\begin{array}{l}
2ag(x+\theta)[\rho+\delta+(1+a(1+x))g]+(x+\theta)[\rho+\delta+(1+a(1+x))g](n+\delta+g)+ \\
ag(x^2-\theta^2)(n+\delta+g)-\theta(x+\theta)(n+g+\delta)^2-[\rho+\delta+(1+a(1+x))g]^2- \\
(x-\theta)(n+\delta+g)[\rho+\delta+(1+a(1+x))g]+\theta x(n+g+\delta)^2 \\
= 2ag(x+\theta)[\rho+\delta+(1+a(1+x))g]+\theta[\rho+\delta+(1+a(1+x))g](n+\delta+g)+ \\
x[\rho+\delta+(1+a(1+x))g](n+\delta+g)+ag(x^2-\theta^2)(n+\delta+g)- \\
\theta x(n+g+\delta)^2-\theta^2(n+g+\delta)^2-[\rho+\delta+(1+a(1+x))g]^2- \\
x(n+\delta+g)[\rho+\delta+(1+a(1+x))g]+ \\
\theta(n+\delta+g)[\rho+\delta+(1+a(1+x))g]+\theta x(n+g+\delta)^2 \\
= 2ag(x+\theta)[\rho+\delta+(1+a(1+x))g]+ag(x^2-\theta^2)(n+\delta+g)-(\rho+\delta+ \\
(1+a(1+x))g)^2-\theta^2(n+g+\delta)^2+2\theta(n+\delta+g)[\rho+\delta+(1+a(1+x))g] \\
= 2ag(x+\theta)[\rho+\delta+(1+a(1+x))g]+ag(x^2-\theta^2)(n+\delta+g)- \\
\{[\rho+\delta+(1+a(1+x))g]-\theta(n+g+\delta)\}^2 \triangleq \Lambda
\end{array}
\tag{80}
$$

It follows from (80) that $\frac{\partial(-|J_E|)}{\partial x} < 0$ if and only if $\Lambda < 0$. It is easy to see that $\Lambda < 0$ if g is small enough. Now let $f(x) = \mathrm{tr}J_E$ and $h(x) = -\left|J_E\right|$. Then the convergence speed can be written as

$$
\gamma(x) = -r_1(x) = \frac{\sqrt{f(x)^2+4h(x)}-f(x)}{2}.
\tag{81}
$$

Differentiating $\gamma(x)$ leads to

$$
\gamma'(x) = \frac{1}{2}\left(\frac{f(x)f'(x)+2h'(x)}{\sqrt{f(x)^2+4h(x)}}-f'(x)\right)
\tag{82}
$$

By (73), we have $f(x) = \rho - n + a(1+x)g$. Then $f'(x) = ag$ and $f(x)f'(x) = [\rho - n + a(1+x)g]ag$. We know from (82) that both $\gamma'(x)$ and $h'(x)$ have the same sign if g is sufficiently small. Hence $\gamma'(x) < 0$ for sufficiently small g. In other words, the convergence speed is decreasing in the thrift index x.

Because the two eigenvalues have opposite signs by $tr J_E > 0$ and $|J_E| < 0$, the optimal solution must be a saddle path. □

Proof of Proposition 6: Let us guess that the optimal consumption function $c^*(t)$ is a linear function of $k^*(t)$,[2] e.g., $c^*(t) = \eta k^*(t)$, with η a positive number to be determined. Obviously, $\frac{\dot{c}^*(t)}{c^*(t)} = \frac{\dot{k}^*(t)}{k^*(t)}$. The growth rate μ can be obtained by using (24) and $f(k) = Ak$. It is $\mu = A - \eta - n - \delta$. Substituting $\frac{c^*(t)}{k^*(t)} = \eta$ into (23) and (24) gives rise to

$$A - \eta - n - \delta = \frac{b\eta}{a} + \frac{-b(A-n-\delta) + (A-\rho-\delta)}{1+a} \tag{83}$$

By (83) and the μ's expression, we have

$$\begin{aligned} \eta &= \tfrac{a}{1+a}(A + \tfrac{\rho-n}{a+b} - n - \delta) \\ \mu &= \tfrac{1}{1+a}(A - \delta - \tfrac{a}{a+b}\rho - \tfrac{b}{a+b}n) \end{aligned} \tag{84}$$

Then we have

$$k^*(t) = k_0 e^{\mu t} \text{ and } c^*(t) = \eta k_0 e^{\mu t} \tag{85}$$

Using (4) we have $\lambda(t) = ac^*(t)^{-(a+1)}k^*(t)^{-b}$. Combining this with $c^*(t)$=$\eta k^*(t)$ gives $\lambda(t)k^*(t) = a\eta^{-(a+1)}k_0^{-(a+b)}e^{-(a+b)gt}$. Therefore we have

$$\lim_{t\to\infty} \lambda(t)k(t)e^{-(\rho-n)t} = \lim_{t\to\infty}\{a\eta^{-(a+1)}k_0^{-(a+b)}e^{-\frac{(a+b)\eta t}{a}}\} = 0 \tag{86}$$

□

Observe from (86) that as long as $\eta > 0$, the transversality condition holds.

Proof of Proposition 7: Using (85) with the objective function of (1), we have

$$\begin{aligned} J(*) &= -\int_0^\infty c(t)^{-a}k(t)^{-b}e^{-(\rho-n)t}dt \\ &= -\int_0^\infty (\eta k_0 e^{\mu t})^{-a}(k_0 e^{\mu t})^{-b}e^{-(\rho-n)t}dt \\ &= -\int_0^\infty \eta^{-a}k_0^{-(a+b)}e^{-[(\rho-n)+(a+b)\mu]t}dt \\ &= -\int_0^\infty \eta^{-a}k_0^{-(a+b)}e^{-\frac{(a+b)\eta}{a}t}dt \\ &= -\eta^{-(1+a)}\tfrac{a}{a+b}k_0^{-(a+b)} \end{aligned} \tag{87}$$

[2] We can also solve this problem by transforming it into a Ricatti ODE, but it is more complicated than the approach of guess-verification adopted here.

Substituting η above into (87) yields

$$J = -[\frac{a}{1+a}(A + \frac{\rho - n}{a+b} - n - \delta)]^{-(1+a)} \frac{a}{a+b} k_0^{-(a+b)}$$

By differentiating J with respect to b we obtain

$$\begin{aligned}
\tfrac{\partial J}{\partial b} &= (1+a)[\tfrac{a}{1+a}(A + \tfrac{\rho-n}{a+b} - n - \delta)]^{-(2+a)}[\tfrac{a(\rho-n)}{1+a}\tfrac{-1}{(a+b)^2}]\tfrac{a}{a+b}k_0^{-(a+b)} - \\
&\quad [\tfrac{a}{1+a}(A + \tfrac{\rho-n}{a+b} - n - \delta)]^{-(1+a)}[\tfrac{-a}{(a+b)^2}k_0^{-(a+b)} - ak_0^{-(a+b+1)}] \\
&= -(1+a)[\tfrac{a}{1+a}(A + \tfrac{\rho-n}{a+b} - n - \delta)]^{-(2+a)}[\tfrac{a(\rho-n)}{1+a}\tfrac{1}{(a+b)^2}]\tfrac{a}{a+b}k_0^{-(a+b)} + \\
&\quad [\tfrac{a}{1+a}(A + \tfrac{\rho-n}{a+b} - n - \delta)]^{-(1+a)}[\tfrac{1}{(a+b)^2} + k_0^{-1}]ak_0^{-(a+b)} \\
&= [\tfrac{a}{1+a}(A + \tfrac{\rho-n}{a+b} - n - \delta)]^{-(1+a)}\{-(1+a)[\tfrac{a}{1+a}(A + \tfrac{\rho-n}{a+b} - n - \delta)]^{-1} \times \\
&\quad [\tfrac{a(\rho-n)}{1+a}\tfrac{1}{(a+b)^2}]\tfrac{a}{a+b}k_0^{-(a+b)} + [\tfrac{1}{(a+b)^2} + k_0^{-1}]ak_0^{-(a+b)}\} \\
&= [\tfrac{a}{1+a}(A + \tfrac{\rho-n}{a+b} - n - \delta)]^{-(1+a)}ak_0^{-(a+b)}\{-(1+a)[\tfrac{a}{1+a}(A + \tfrac{\rho-n}{a+b} - n - \delta)]^{-1} \times \\
&\quad [\tfrac{a(\rho-n)}{1+a}\tfrac{1}{(a+b)^2}]\tfrac{1}{a+b} + \tfrac{1}{(a+b)^2} + k_0^{-1}\}
\end{aligned} \tag{88}$$

It is easy to see that

$$\begin{aligned}
&\tfrac{\partial J}{\partial b} > 0 \\
&\Leftrightarrow -(1+a)[\tfrac{a}{1+a}(A + \tfrac{\rho-n}{a+b} - n - \delta)]^{-1}[\tfrac{a(\rho-n)}{1+a}\tfrac{1}{(a+b)^2}]\tfrac{1}{a+b} + \tfrac{1}{(a+b)^2} + k_0^{-1} > 0 \\
&\Leftrightarrow -\tfrac{1+a}{A+\frac{\rho-n}{a+b}-n-\delta}\tfrac{\rho-n}{(a+b)^3} + \tfrac{1}{(a+b)^2} + k_0^{-1} > 0 \\
&\Leftrightarrow k_0^{-1} > \tfrac{1}{(a+b)^2}[\tfrac{(1+a)(\rho-n)}{\rho-n+(a+b)(A-n-\delta)} + 1] \\
&\Leftrightarrow k_0 < (a+b)^2[\tfrac{(1+a)(\rho-n)}{\rho-n+(a+b)(A-n-\delta)} + 1]^{-1}
\end{aligned} \quad . \tag{89}$$

Observe that $J(a,b) = J(a, a\frac{b}{a}) \triangleq J(a, ax)$. Then $\frac{\partial J}{\partial x} = \frac{\partial J}{\partial b}\frac{\partial b}{\partial x} = a\frac{\partial J}{\partial b}$, i.e., $\frac{\partial J}{\partial x}$ and $\frac{\partial J}{\partial b}$ have the same sign. This implies that if the initial capital stock is low enough, the total utility may increase with the thrift index, and vice versa. $\square$

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