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Abstract

In the context of the matching-with-contracts model, we generalize the cumulative offer process to allow for arbitrary subsets of doctors to make proposals in each round. We show that, under a condition on the hospitals' choice functions, the outcome of this generalized cumulative offer process is independent of the sets of doctors making proposals in each round. The flexibility of the resulting model allows it to be used to describe different dynamic processes and their final outcomes.

Keywords: Matching with contracts, cumulative offer mechanism, asynchrony, order independence.
JEL Codes: C78; D44; D47.

1 Introduction

In the domain of matching and discrete allocation problems, very often algorithms that are used as mechanisms or tools to produce allocations share many characteristics with descriptions of processes, involving economic agents, which converge to some outcome.

Gale and Shapley (1962), for example, describe an algorithm—the deferred acceptance (DA)—that produces a matching of students to colleges by simulating a process involving a sequence of applications by students and tentative acceptances and rejections by colleges. The authors use the description both as a *method for producing a stable matching* and as a *method for proving that stable matchings always exist* in the domain that they consider. Dubins and Freedman (1981) and Roth (1982) evaluate the incentives that participants have to reveal their true preferences when that algorithm is used as a *direct mechanism*.

Similarly, when considering a labor market, Crawford and Knoer (1981) and Kelso and Crawford (1982) describe processes in which firms make offers sequentially to workers, adjusting the salaries accordingly, until a stable equilibrium is reached. Hatfield and Milgrom

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(2005) extend the standard matching model to include contracts between doctors and hospitals, generalizing both the college admission and labor market models above, and introducing an algorithm—the cumulative offer process (COP)—that under certain conditions on the preferences of the participants, produces stable matchings. Similarly to DA, the COP involves a sequence of contractual offers by doctors, tentative acceptances, rejections and renegotiations by hospitals.

While not always emphasized in these papers, many of these algorithms could be interpreted as describing dynamic processes that can take place in the real world—students applying to schools and being "waitlisted", firms making offers to workers and adjusting salaries, etc. One obstacle in this interpretation is that the description of these algorithms often involves unrealistic sequences and timing of decisions. In DA, all students who are not held at some college make simultaneous offers. In Crawford and Knoer (1981) and Kelso and Crawford (1982), firms still looking for workers to hire also make offers simultaneously.[1]

In the matching with contracts model, the COP process was defined in Hatfield and Milgrom (2005) similarly to DA, where every doctor who makes an offer does so simultaneously. Hatfield and Kojima (2010), on the other hand, describe the process as involving an offer from a single doctor at a time. Hirata and Kasuya (2014) showed that, under certain conditions, all single-offer COPs (regardless of the order in which the doctors make their offers) induce the same outcome as the simultaneous-offer COP. Hatfield *et al.* (2021) provide an alternative order independence result for single-offer COPs.

In this paper, we extend the COP to allow for arbitrary subsets of doctors to make offers in every period. This generalized COP includes both the single-offer and simultaneous-offer COPs as special cases. We obtain an "order independence" of this offer process, generalizing the result of Hirata and Kasuya (2014) to show that under the same conditions considered by these authors, any arbitrary subset of doctors making offers in each period results in the same outcome (Theorem 1). In section 5 we establish the relation between the conditions used for the order independence results in Hirata and Kasuya (2014) and Hatfield *et al.* (2021).

More than providing an additional family of algorithms for computing stable matchings with contracts, the generalized COP can be used to describe more realistic dynamic processes. For example, it can be used to describe a process in which doctors make offers to hospitals asynchronously, and these process pending proposals asynchronously as well. This includes processes in which, whenever a contract is rejected, the most preferred remaining contract of the rejected doctor could immediately be made available for consideration by another hospital. Such processes can arise in applications that are susceptible to delays in decision making, but can not be described by single-offer or simultaneous-offer COPs. Single-offer COPs require offers to be evaluated in order, one at a time, while the simultaneous-offer COP requires all hospitals to finish evaluating their current offers in each round before any hospital can move on to evaluate the offers in the next round. The generalized COP does not require coordination of this kind and Theorem 1 shows that a lack of synchrony in these decisions is inconsequential to the final outcome. Section 6 concludes with brief descriptions

[1] One notable exception is Roth and Xing (1997), which describes the entry-level market for Clinical Psychologists as a decentralized process involving, among other phases, an asynchronous and stochastic version of the DA algorithm.

of other applications for the model. All proofs can be found in Appendix A.

2 Preliminaries

Let D be a finite set of doctors, let H be a finite set of hospitals, and let $X \subseteq D \times H \times \Theta$ be a finite set of contracts where Θ is a finite set (e.g., wages and job descriptions), with $d \in D$, $h \in H$, $x \in X$, and $\theta \in \Theta$ being their typical elements. For each contract $x \in X$, let $d(x)$ and $h(x)$ denote the doctor and hospital involved in x, respectively. For any $X' \subseteq X$, let $X'_i := \{x \in X' \mid i \in \{d(x), h(x)\}\}$ for every $i \in D \cup H$. For any $X' \subseteq X$, let $d(X') := \bigcup_{x \in X'} \{d(x)\}$ and $h(X') := \bigcup_{x \in X'} \{h(x)\}$. We call $X' \subseteq X$ an *allocation* if $|X'_d| \leq 1$ for each $d \in D$. At an allocation X', each doctor in $D \setminus d(X')$ is assigned the *null contract* $x^{\emptyset}$.

For each doctor $d \in D$, let $\succ_d$ be the doctor's strict preference relation over $X_d \cup \{x^{\emptyset}\}$ and $\mathcal{P}_d$ be the set of all possible strict preference relations over $X_d \cup \{x^{\emptyset}\}$. Let $\mathcal{P}_D := \prod_{d \in D} \mathcal{P}_d$ be the set of all possible preference profiles, with $\succ_D \in \mathcal{P}_D$ being a typical element of $\mathcal{P}_D$. A contract $x \in X_d$ is *acceptable* to doctor d if $x \succ_d x^{\emptyset}$. Let $AC(\succ_d) := \{x \in X_d \mid x \succ_d x^{\emptyset}\}$ be the set of acceptable contracts to a doctor with preference relation $\succ_d$. We assume that $|AC(\succ_d)| \geq 1$ for each $\succ_d \in \mathcal{P}_d$ and each $d \in D$.

Each hospital $h \in H$ has a choice function $C_h : 2^X \to 2^{X_h}$, such that for any $X' \subseteq X$, $C_h(X') \subseteq X'_h$. For each $h \in H$, C_h chooses at most one contract for each $d \in D$; that is, for any $X' \subseteq X$ and any $h \in H$, $C_h(X')$ is an allocation. Let $C_H = (C_h)_{h \in H}$ be a profile of hospitals' choice functions. For any $X' \subseteq X$ evaluated by hospital h, h's choice function could take into account the set of contracts not involving h, denoted by $X'_{-h} \subseteq X'$. This possibility is, however, ruled out by the common and widely accepted assumption that choice functions satisfy irrelevance of rejected contracts (see Lemma 1 below).

Definition 1 (Aygün and Sönmez (2013)). *Hospital h's choice function C_h satisfies the* ***irrelevance of rejected contracts*** *(IRC) condition if for any $X' \subset X$ and $x \in X$,*[2] *if $x \notin C_h(X' \cup \{x\})$, then $C_h(X') = C_h(X' \cup \{x\})$.*

Note that Definition 1 does not require contract x to involve hospital h.

Lemma 1. *Suppose that each hospital h's choice function satisfies the IRC condition. Then for each $h \in H$ and each $X' \subseteq X$, $C_h(X') = C_h(X'_h)$.*

That is, the IRC condition implies that the choice of a hospital h is only affected by the contracts involving h. Throughout the paper, we assume that all choice functions satisfy the IRC condition.[3]

3 Generalized Offer Process

In this section, we introduce a generalized COP, or the GCOP. It generalizes the two different types of COPs previously considered in the literature: (i) the simultaneous-offer COP evaluating the contracts of all eligible doctors in each step (e.g., Hatfield and Milgrom (2005)),

[2]The original definition in Aygün and Sönmez (2013) requires that $x \in X \setminus X'$. This formulation is equivalent since the statement immediately holds if $x \in X'$.

[3]Hirata and Kasuya (2014) and Hatfield *et al.* (2021) assume the IRC condition.

and (ii) the single-offer COPs evaluating only a single contract at a time (e.g., Hatfield and Kojima (2010)). In the GCOP, an *arbitrary set of eligible doctors* is considered in each step.

Let $AC^0(\succ_d) = AC(\succ_d)$ for each $d \in D$, $UK^0 = \emptyset$, and $X^0 = \emptyset$. The GCOP is defined by the following procedure and finishes in $T \geq 1$ rounds.

Round 1:

– Choose an arbitrary non-empty set of doctors $D^1 \subseteq D$ and identify the most preferred contract, $x_d^1 \in AC^0(\succ_d)$, of each $d \in D^1$ according to $\succ_d$.

Update:

- Let $\tilde{X}^1$ be the set containing x_d^1 for each $d \in D^1$, and let $X^1 = \tilde{X}^1 \cup X^0$.[4]
- For each doctor $d \in D^1$, make x_d^1 unavailable in later rounds;

$$AC^1(\succ_d) = \left\{ \begin{array}{c} AC^0(\succ_d) \setminus \{x_d^1\} \\ AC^0(\succ_d) \end{array} \right\} \text{ if } d \left\{ \begin{array}{c} \in \\ \notin \end{array} \right\} D^1.$$

For each $t \geq 1$, we define X^t and $AC^t(\succ_d)$ recursively. For each $t \geq 1$, X^t is the set of contracts to be considered in round t. For each doctor $d \in D$ and each $t \geq 1$, we call $AC^t(\succ_d)$ the set of *fresh* contacts and $AC(\succ_d) \setminus AC^t(\succ_d)$ the set of *offered* contracts.

– For each hospital $h \in H$, the contracts in $C_h(X^1)$ are reserved.

Update:

- Let $U^1 := \{d \in D \mid AC^1(\succ_d) = \emptyset\}$ be the set of doctors with no fresh contracts for later rounds.
- Let $K^1 := \bigcup_{h \in H} d(C_h(X^1))$ be the set of doctors with offered contacts reserved by hospitals. Their fresh contracts will not be considered in the next round.
- Let $UK^1 := U^1 \cup K^1$ be the set doctors whose fresh contracts may not be reviewed in the next round.

For each doctor $d \in D \setminus UK^1$, no hospital reserves her contract and she has fresh contracts available for later rounds. In other words, $D \setminus UK^1$ is the set of doctors who can be included in D^2. If $UK^1 = D$, the process is complete and stops at $T = 1$. Otherwise, the process moves to the next round.

Round $t \geq 2$:

– Choose an arbitrary non-empty set of doctors $D^t \subseteq D \setminus UK^{t-1}$ and identify the most preferred contract, $x_d^t \in AC^{t-1}(\succ_d)$, of each $d \in D^t$ according to $\succ_d$.

Update:

[4] Note that since $X^0 = \emptyset$, $\tilde{X}^1 \cup X^0 = \tilde{X}^1$. We use this expression to be consistent across different rounds.

- Let $\tilde{X}^t$ be the set containing x_d^t for each $d \in D^t$, and let $X^t := \tilde{X}^t \cup X^{t-1}$.
- For each doctor $d \in D^t$, make x_d^t unavailable in later rounds;

$$AC^t(\succ_d) = \left\{ \begin{array}{c} AC^{t-1}(\succ_d) \setminus \{x_d^t\} \\ AC^{t-1}(\succ_d) \end{array} \right\} \text{ if } d = \left\{ \begin{array}{c} \in \\ \notin \end{array} \right\} D^t.$$

– For each hospital $h \in H$, the contracts in $C_h(X^t)$ are reserved.

Update:

- Let $U^t := \{d \in D \mid AC^t(\succ_d) = \emptyset\}$,
- let $K^t := \bigcup_{h \in H} d(C_h(X^t))$, and
- let $UK^t := U^t \cup K^t$.

If $UK^t = D$, the process is complete and stops at $T = t$. Otherwise, the process moves to the next round.

If $D^t = D \setminus UK^{t-1}$ for each $t \in \{1, \ldots, T\}$, the process corresponds to the simultaneous-offer COP. If $|D^t| = 1$ for each $t \in \{1, \ldots, T\}$, the process corresponds to a single-offer COP. We call the resulting set of contracts reserved by hospitals, $\bigcup_{h \in H} C_h(X^T)$, an *outcome*. We say that two GCOPs are *outcome-equivalent* if they have the same outcome.

4 Generalized Order Independence

Hirata and Kasuya (2014) compared single-offer COPs and the simultaneous-offer COP and showed that (i) any two single-offer COPs are outcome-equivalent and (ii) any single-offer COP and the simultaneous-offer COP are outcome-equivalent, assuming that the choice functions of all hospitals satisfy the IRC and bilateral substitutes conditions. This means that the order in which contracts are considered does not affect the outcome of single-offer COPs, and that all single-offer COPs induce the same outcome as the simultaneous-offer COP. To achieve these results, the bilateral substitutes condition is only used to ensure that the following condition holds.[5]

Definition 2. *Hospital h's choice function C_h satisfies the **Hirata-Kasuya** (HK) condition if for any $d, d' \in D$ with $d \neq d'$, any $x \in X_d$, and any $X' \subseteq X$ with $d, d' \notin d(C_h(X'))$, $d' \notin d(C_h(X' \cup \{x\}))$.*

We generalize the results in Hirata and Kasuya (2014) by showing that all GCOPs are outcome-equivalent, assuming that choice functions satisfy the HK condition. This means that the set of contracts considered in each round of a GCOP has no impact on its outcome and that all GCOPs induce the same outcome as the simultaneous-offer COP. To establish this, we show that for any GCOP, there always exists a single-offer COP which replicates it. Combined with the outcome equivalence of single-offer COPs, the generalized order independence result immediately follows.

Theorem 1. *Given $\succ_D$ and C_H, if C_h satisfies the HK condition for each hospital $h \in H$, then all GCOPs are outcome-equivalent.*

[5] Stated as a Lemma in Hirata and Kasuya (2014).

5 Relation with Hatfield *et al.* (2021)

The order independence results of Hirata and Kasuya (2014) rely on their Theorem 1, which shows that the single-offer process is outcome-equivalent under the HK condition, also shown as Lemma 3 in Appendix A. Hatfield *et al.* (2021) also provided a condition (discussed in Appendix B) under which the outcome-equivalence of single-offer processes (Proposition 3 in Hatfield *et al.* (2021)) is established.[6] Hatfield *et al.* (2021) (footnote 43) state that this result generalizes that of Hirata and Kasuya (2014), since their condition is weaker than the bilateral substitutability condition that Hirata and Kasuya (2014)'s outcome-equivalence result relies on.

One natural question is the following: can the outcome-equivalence result in Theorem 1 be established under the condition from Hatfield *et al.* (2021)?

Two points. First, the condition from Hatfield *et al.* (2021)) is applicable only to single-offer processes, and thus is not readily applicable to the GCOP. Second, as we observed above, the results in Hirata and Kasuya (2014) rather rely on their Lemma (Definition 2), which is implied by the bilateral substitutability condition (in addition to the IRC condition, which is assumed throughout). The relationship between the condition in Hirata and Kasuya (2014) (Definition 2) and that in Hatfield *et al.* (2021) is still unclear.

To address the question above, we first provide a condition, which we term the Hatfield-Kominers-Westkamp (HKW) condition. The HKW condition is a natural extension of the original condition from Hatfield *et al.* (2021) called *observable substitutability across doctors* since the HKW condition not only coincides with the condition of observable substitutability across doctors within the framework of Hatfield *et al.* (2021)), but is also applicable to the GCOP.[7] We then compare the HK and HKW conditions. We will show below that the HK and HKW conditions are indeed identical, and thus the condition of observable substitutability across doctors from Hatfield *et al.* (2021) and the HK condition are identical in the framework of Hatfield *et al.* (2021). This also implies that the HK and the HKW conditions can be used interchangeably in the current framework. We first provide the Hatfield-Kominers-Westkamp (HKW) condition.

Definition 3. *Hospital h's choice function C_h satisfies the* ***Hatfield-Kominers-Westkamp*** *(HKW) condition if for any $X' \subseteq X$, any $d \notin d(C_h(X'))$ and any $x \in (X_d \cap X_h) \setminus X'_d$, $x' \in R_h(X') \setminus R_h(X' \cup \{x\})$ implies $d(x') \in d(C_h(X'))$.*

Note that while the HKW condition has $d(x') \in d(C_h(X'))$ with $x' \in R_h(X') \setminus R_h(X' \cup \{x\})$, the HK condition has $d' \notin d(C_h(X'))$ with $d' \notin d(C_h(X' \cup \{x\})$. We now show that the two conditions in Definitions 2 and 3 are equivalent.

Proposition 1. *Hospital h's choice function C_h satisfies the HK condition if and only if it satisfies the HKW condition.*

[6] Hatfield *et al.* (2021) also assume the IRC condition.

[7] Appendix B describes the framework as well as the condition in Hatfield *et al.* (2021).

6 Discussion

In this paper, we extended the cumulative offer process to allow for arbitrary subsets of doctors to make proposals at any time, and show that, when hospitals' choice functions satisfy the HK condition, the outcome does not depend on the order and sets of doctors making these proposals at any time. In addition to providing an alternative family of algorithms for the COP, we argue that the model becomes general enough to be able to represent more realistic dynamic processes. In this concluding section, we provide examples of processes that can be modeled as instances of the generalized cumulative offer process. These highlight the flexibility that the model provides to represent many dynamic matching processes.

The key characteristics that a matching process must have to be modeled as a GCOP, in addition to the assumptions about doctors and hospitals preferences we introduced in Section 2, are that (i) doctors can have at most one proposal being held by a hospital at any time, and (ii) the process only ends when there are no doctors waiting to make another proposal or proposals not yet processed by the hospitals.

Doctors and Hospitals working asynchronously Doctors can, each one independently and at any time, make a proposal to a hospital, including the contractual terms that they desire. Similarly, hospitals can, each one independently and at any time, process the pending proposals, holding some of these offers, renegotiating some of them, and rejecting others. The process ends when no doctor wants to make some additional proposal and all hospitals processed their pending proposals.

Doctors arrive at different times The market starts, initially, with a subset of the doctors, who can start making their proposals, which are processed accordingly by the hospitals. As new doctors arrive, they can make their proposals and hospitals process them.[8]

Limited communication bandwidth The communication channels are such that, in each period, at most K proposals can be made.

Statistical reduction of interactions in university admissions Based on historical data, university entrance administrators design the sets of students who are called for making applications in each period such that the total number of times students are called to make a new proposal is reduced. The identities of the students who are called at each period can be dynamically determined not only on the basis of the historical data, but on the proposals that students make in each period.[9]

It is important, however, to emphasize that we are not claiming that our results say anything about whether we can predict the behavior of strategic agents in these scenarios to

[8] Notice that a model in which hospitals can also arrive at different times would involve less appealing assumptions: doctors with fresh contracts involving hospitals that have not yet arrived would wait for that to happen before making their next proposal.

[9] A simple motivating example is the case in which students have highly correlated preferences over universities. By asking first the high-grading students to make their proposals, and providing definite information about entry requirements to the students, those with lower grades might skip unnecessary proposals.

be "truthful". That is, it might be that for some of these scenarios some doctors would be better off by making proposals that don't simply follow their preferences, as described in the GCOP. It is outside of the scope of this paper to evaluate the incentives of doctors and/or hospitals in these dynamic processes. Our results show that when the agents involved make proposals following their preferences over contracts, all of these processes will converge to the same outcome as the COP. Therefore, if the hospitals' (or universities') choice functions satisfy bilateral substitutes, the outcome will be stable.

It is known that under certain additional conditions on the hospitals' choice functions, a *direct revelation mechanism* that produces the same outcome as the GCOP is strategy-proof for doctors (Hatfield and Kojima, 2010; Hatfield and Kominers, 2015; Hatfield *et al.*, 2021). This fact might in some cases be combined with a model that represents the game induced in the participants by these dynamic processes, obtaining equilibrium predictions for these outcomes (see, for example, Mackenzie and Zhou (2020) and Bó and Hakimov (2016)). Theorem 1 would likely have an important role in these analyses.

References

Aygün, O. and Sönmez, T. (2013). Matching with Contracts: Comment. *American Economic Review*, **103** (5), 2050–2051.

Bó, I. and Hakimov, R. (2016). The iterative deferred acceptance mechanism, Working Paper.

Crawford, V. P. and Knoer, E. M. (1981). Job Matching with Heterogeneous Firms and Workers. *Econometrica*, **49** (2), 437.

Dubins, L. E. and Freedman, D. A. (1981). Machiavelli and the Gale-Shapley Algorithm. *The American Mathematical Monthly*, **88** (7), 485–494.

Gale, D. and Shapley, L. S. (1962). College Admissions and the Stability of Marriage. *The American Mathematical Monthly*, **69** (1), 9–15.

Hatfield, J. W. and Kojima, F. (2010). Substitutes and stability for matching with contracts. *Journal of Economic Theory*, **145** (5), 1704–1723.

— and Kominers, S. D. (2015). Hidden Substitutes. In *Proceedings of the Sixteenth ACM Conference on Economics and Computation*, Portland Oregon USA: ACM, pp. 37–37.

—, — and Westkamp, A. (2021). Stability, Strategy-Proofness, and Cumulative Offer Mechanisms. *The Review of Economic Studies*, **88** (3), 1457–1502.

— and Milgrom, P. R. (2005). Matching with Contracts. *American Economic Review*, **95** (4), 913–935.

Hirata, D. and Kasuya, Y. (2014). Cumulative offer process is order-independent. *Economics Letters*, **124** (1), 37–40.

Kelso, A. S. and Crawford, V. P. (1982). Job Matching, Coalition Formation, and Gross Substitutes. *Econometrica*, **50** (6), 1483.

Mackenzie, A. and Zhou, Y. (2020). Menu mechanisms, Working Paper.

Roth, A. and Xing, X. (1997). Turnaround Time and Bottlenecks in Market Clearing: Decentralized Matching in the Market for Clinical Psychologists. *Journal of Political Economy*, **105** (2), 284–329.

Roth, A. E. (1982). The Economics of Matching: Stability and Incentives. *Mathematics of Operations Research*, **7** (4), 617–628.

A Proofs

A.1 Lemma 1

Proof. Consider any $h \in H$ and any $X' \subseteq X$. Let $X^1 = X'_{-h} \setminus \{x_1\}$ for some $x_1 \in X'_{-h}$. Since $C_h(X') \subseteq X'_h$ by definition, $x_1 \in X'_{-h}$ implies that $x_1 \notin C_h(X')$. By IRC, we then have that $C_h(X') = C_h(X'_h \cup X'_{-h}) = C_h(X'_h \cup X^1)$. If $X^1 = \emptyset$, then $C_h(X') = C_h(X'_h \cup X^1) = C_h(X'_h)$. If $X^1 \neq \emptyset$, let $X^2 = X^1 \setminus \{x_2\}$ for some $x_2 \in X^1$. By the same logic as above, IRC requires that $C_h(X') = C_h(X'_h \cup X^1) = C_h(X'_h \cup X^2)$. Repeating this argument yields $C_h(X') = C_h(X'_h \cup X^1) = \cdots = C_h(X'_h \cup X^{|X'_{-h}|}) = C_h(X'_h)$. The final equality follows from the fact that $X^{|X'_{-h}|} = \emptyset$. ♣

A.2 Theorem 1

Proof. Fix $\succ_D$ and C_H. Take a COP requiring T rounds of iteration and its corresponding $\tilde{X}^t$, for each $t \in \{1, \ldots, T\}$. Arrange the elements of $\tilde{X}^1 \ldots \tilde{X}^T$ as follows.

$$\underbrace{x^1, \ldots, x^{|X^1|}}_{\tilde{X}^1}, \underbrace{x^{|X^1|+1}, \ldots, x^{|X^2|}}_{\tilde{X}^2}, \ldots, \underbrace{x^{|X^{T-1}|+1}, \ldots, x^{|X^T|}}_{\tilde{X}^T}$$

The internal ordering of the elements within each $\tilde{X}^t$ is arbitrary. In order to make use of Hirata and Kasuya's order independence result, we need to show that the sequence of contracts above represents a single-offer COP.

Definition 4 (Hirata and Kasuya (2014, Definition 4))**.** *Given $\succ_D$ and C_H, a finite sequence of contracts $(x^t)_{t=1}^T$* ***represents a single-offer COP*** *if the following conditions are satisfied:*

(1) For each $t \in \{1, \ldots, T\}$ and $h \in H$, $d(x^t) \notin d(C_h(\{x^1, \ldots, x^{t-1}\}))$.[10]

(2) For each $t \in \{1, \ldots, T\}$ and $x \in X$, if $d(x^t) = d(x)$ and $x \succ_{d(x)} x^t$, there exists $\tau < t$ such that $x = x^\tau$.

(3) For each $d \in D$, either (i) $d \in d(C_h(\{x^1, \ldots, x^T\}))$ for some $h \in H$ or (ii) $AC(\succ_d) \subseteq \{x^1, \ldots, x^T\}$.[11]

Note that, by construction, no doctors involved in the contracts in $\tilde{X}^t$ are reserved in round $t-1$ of the GCOP. That is, $d(x) \notin C_h(X^{t-1})$ for all $x \in \tilde{X}^t$. Furthermore, $d(x) \neq d(x')$ for all $x, x' \in \tilde{X}^t$ such that $x \neq x'$. This implies that the sequence of contracts satisfies condition (1) in Definition 4. We construct a single-offer COP within our framework which has the sequence of contracts shown above and demonstrate that it satisfies condition (2) and (3) as well.

To differentiate the single-offer COP from the original COP, we use y, Y and $\tilde{Y}$ for the single-offer COP instead of x, X and $\tilde{X}$. Furthermore, we use $\bar{AC}$, $\bar{D}$, $\bar{U}$, $\bar{K}$ and $\bar{UK}$ for the

[10] This corresponds to the notion of observability in Hatfield *et al.* (2021).

[11] Note that (i) and (ii) are not mutually exclusive. There may exist doctor d whose least-preferred contract in $AC(\succ_d)$ is accepted in round T.

single-offer COP instead of AC, D, U, K and UK. We call the sequence corresponding to $\tilde{X}^t$ the $\tilde{X}^t$*-sequence.*

Suppose that the two COPs described above have considered the same set of contracts up until the end of the $\tilde{X}^{t-1}$-sequence. That is, the set of contracts that have been considered in the original COP after $t-1$ steps equals the set of contracts that have been considered in the single-offer COP after $|X^{t-1}|$ steps; $X^{t-1} = Y^{|X^{t-1}|}$ and $AC^{t-1}(\succ_d) = \bar{AC}^{|X^{t-1}|}(\succ_d)$ for all $d \in D$. We will show that the two COPs will then also have considered the same set of contracts by the end of the $\tilde{X}^t$-sequence. By induction, we can then conclude that $X^T = Y^{|X^T|}$, implying outcome-equivalence.

Base case: By construction, $X^0 = Y^{|X^0|} = Y^0 = \emptyset$ and $AC^0(\succ_d) = \bar{AC}^{|X^0|}(\succ_d) = \bar{AC}^0(\succ_d) = AC(\succ_d)$ for all $d \in D$.

Induction hypothesis: Assume that there exists some t such that $X^{t-1} = Y^{|X^{t-1}|}$ and $AC^{t-1}(\succ_d) = \bar{AC}^{|X^{t-1}|}(\succ_d)$ for all $d \in D$.

Induction step: We will now demonstrate that $X^t = Y^{|X^t|}$ and $AC^t(\succ_d) = \bar{AC}^{|X^t|}(\succ_d)$ for all $d \in D$. Consider the $\tilde{X}^t$-sequence. The following argument applies to the $\tilde{X}^1$-sequence by taking $X^0 = \emptyset$.

$$\dots x^{|X^{t-1}|}, \underbrace{x^{|X^{t-1}|+1}, \dots, x^{|X^t|}}_{\tilde{X}^t}, \dots$$

Take $\tau \in \{1, \dots, |\tilde{X}^t|\}$ and let (i) the set of doctors whose contracts are considered in rounds $|X^{t-1}|+1$ through $|X^{t-1}|+\tau$ in the single-offer COP be $\tilde{D}^\tau \subseteq D^t$ with $|\tilde{D}^\tau| = \tau$ and $\tilde{D}^0 = \emptyset$, and (ii) $\hat{D}^\tau := D^t \setminus \tilde{D}^\tau$.

Round $|X^{t-1}|+\tau$: By Lemma 2 below, any $d' \in \hat{D}^{\tau-1}$ is available in round $|X^{t-1}|+\tau$. Take an arbitrary doctor $d' \in \hat{D}^{\tau-1}$.

D1: Identify doctor d''s most preferred contract in $\bar{AC}^{|X^{t-1}|+\tau-1}(\succ_{d'})$, $y_{d'}$.

Update: Let

- $\tilde{D}^\tau = \tilde{D}^{\tau-1} \cup \{d'\}$,
- $\tilde{Y}^{|X^{t-1}|+\tau} = \{y_{d'}\}$ and $Y^{|X^{t-1}|+\tau} = \tilde{Y}^{|X^{t-1}|+\tau} \cup Y^{|X^{t-1}|+\tau-1}$, and
- $\bar{AC}^{|X^{t-1}|+\tau}(\succ_{d'}) = \bar{AC}^{|X^{t-1}|+\tau-1}(\succ_{d'}) \setminus \{y_{d'}\}$ and $\bar{AC}^{|X^{t-1}|+\tau}(\succ_d) = \bar{AC}^{|X^{t-1}|+\tau-1}(\succ_d)$ for each $d \neq d'$. Note that $\bar{AC}^{|X^{t-1}|+\tau}(\succ_{d'}) = AC^t(\succ_{d'})$.

H1: Let $h(y_{d'}) = h'$. Hospital h' reserves $y_{d'}$ if $y_{d'} \in C_{h'}(Y^{|X^{t-1}|+\tau})$.

By varying the values of t and τ, the process above describes any round in the single-offer COP. Note that $\bar{AC}^{|X^0|}(\succ_d) = AC(\succ_d)$ for all $d \in D$ and that, by construction, any $x \in \bar{AC}^{|X^0|}(\succ_d) \setminus \bar{AC}^{|X^t|+\tau}(\succ_d)$ has been considered in rounds 1 through $|X^t| + \tau$. Furthermore, since $\bar{AC}^{|X^0|}(\succ_d)$ is $d(x^{|X^{t-1}|+\tau})$'s most preferred contract in $\bar{AC}^{|X^t|+\tau}(\succ_d)$, condition

(2) in Definition 4 is also satisfied.

Update: Let

- $\bar{U}^{|X^{t-1}|+\tau} = \{d \in D \mid \bar{AC}^{|X^{t-1}|+\tau}(\succ_d) = \emptyset\}$,
- $\bar{K}^{|X^{t-1}|+\tau} = \bigcup_{h \in H} d(C_h(Y^{|X^{t-1}|+\tau}))$, and
- $\bar{UK}^{|X^{t-1}|+\tau} = \bar{U}^{|X^{t-1}|+\tau} \cup \bar{K}^{|X^{t-1}|+\tau}$.

The following result shows that every remaining doctor $d \in \hat{D}^\tau$ is available for the next round in the single-offer COP.

Lemma 2. *For any remaining doctor $d \in \hat{D}^\tau$, $d \notin \bar{UK}^{|X^{t-1}|+\tau}$.*

Proof. Consider any contract y for which $d(y) = d \in \hat{D}^\tau$ and note that $d \notin \bar{U}^{|X^{t-1}|+\tau}$, since $\bar{AC}^{|X^{t-1}|}(\succ_d) = \bar{AC}^{|X^{t-1}|+\tau}(\succ_d)$ and $d \notin \bar{U}^{|X^{t-1}|}$ as $d \in D^t$. Furthermore, $d \notin \bigcup_{h \in H} d(C_h(Y^{|X^{t-1}|}))$ for all $d \in D^t$ by construction. Suppose $d \notin \bigcup_{h \in H} d(C_h(Y^{|X^{t-1}|+\sigma})) = \bar{K}^{|X^{t-1}|+\sigma}$ for all $d \in \hat{D}^\sigma$. By the observation above, the statement holds for $\sigma = 0$, since $\hat{D}^0 = D^t$. Note that $Y^{|X^{t-1}|+\sigma+1} = Y^{|X^{t-1}|+\sigma} \cup \tilde{Y}^{|X^{t-1}|+\sigma+1}$ and $\tilde{Y}^{|X^{t-1}|+\sigma+1}$ is a singleton. Since all hospitals satisfy the HK condition, $d \notin d(C_h(Y^{|X^{t-1}|+\sigma} \cup \tilde{Y}^{|X^{t-1}|+\sigma+1})) = d(C_h(Y^{|X^{t-1}|+\sigma+1}))$ for all $d \in \hat{D}^{\sigma+1} = \hat{D}^\sigma \setminus \{d(\tilde{Y}^{|X^{t-1}|+\sigma+1})\}$ and all $h \in H$. Thus, $d \notin \bigcup_{h \in H} d(C_h(Y^{|X^{t-1}|+\sigma+1})) = \bar{K}^{|X^{t-1}|+\sigma+1}$ for all $d \in \hat{D}^{\sigma+1}$. That is, if $d \notin \bar{K}^{|X^{t-1}|+\sigma}$ for all $d \in \hat{D}^\sigma$, then $d \notin \bar{K}^{|X^{t-1}|+\sigma+1}$ for all $d \in \hat{D}^{\sigma+1}$ as well. Since $d \notin \bar{K}^{|X^{t-1}|}$ for all $d \in \hat{D}^0$ by construction, it follows that $d \notin \bar{K}^{|X^{t-1}|+\tau}$ for all $d \in \hat{D}^\tau$ by induction. Since $\bar{UK}^{|X^{t-1}|+\tau} = \bar{U}^{|X^{t-1}|+\tau} \cup \bar{K}^{|X^{t-1}|+\tau}$, $d \notin \bar{UK}^{|X^{t-1}|+\tau}$ for all $d \in \hat{D}^\tau$. ♣

The set of doctors that can be considered in the next round is given by $D \setminus \bar{UK}^{|X^{t-1}|+\tau}$, where $\hat{D}^\tau \subseteq D \setminus \bar{UK}^{|X^{t-1}|+\tau}$ by Lemma 2. That is, at round $|X^{t-1}| + \tau$ for any $\tau \in \{1, \ldots, |\tilde{X}^t|\}$, all of the doctors in D^t whose contracts in $\tilde{X}^t$ were not considered in rounds $|X^{t-1}| + 1$ through $|X^{t-1}| + \tau$ are still available for consideration in round $|X^{t-1}| + \tau + 1$. In each round from $|X^{t-1}| + 1$ to $|X^{t-1}| + \tau$ a new unique contract in $\tilde{X}^t$ is considered. By letting $\tau = |\tilde{X}^t|$, this implies that all contracts in $\tilde{X}^t$ have been considered in round $|X^{t-1}| + |\tilde{X}^t| = |X^t|$ of the single-offer COP. Thus, $Y^{|X^t|} = Y^{|X^{t-1}|} \cup \tilde{X}^t$. Since $X^t = X^{t-1} \cup \tilde{X}^t$, the induction hypothesis then implies that $AC^t(\succ_d) = \bar{AC}^{|X^t|}(\succ_d)$ for all $d \in D$. Furthermore, since $X^{t-1} = Y^{|X^{t-1}|}$, it follows that $X^t = Y^{|X^t|}$ as $X^t = X^{t-1} \cup \tilde{X}^t = Y^{|X^{t-1}|} \cup \tilde{X}^t = Y^{|X^t|}$. This concludes the induction step.

By induction, we have shown that, for all $t \leq T$, $X^t = Y^{|X^t|}$ and $AC^t(\succ_d) = \bar{AC}^{|X^t|}(\succ_d)$ for all $d \in D$. Thus, the original COP terminates in round T and the single-offer COP terminates in round $|X^T|$, where $X^T = Y^{|X^T|}$. Consequently, $C_h(X^T) = C_h(Y^{|X^T|})$ for each $h \in H$. This implies that $\bigcup_{h \in H} C_h(X^T) = \bigcup_{h \in H} C_h(Y^{|X^T|})$. In other words, the original COP and the single-offer COP are outcome-equivalent.

Since $X^T = Y^{|X^T|}$, the GCOP and the single-offer COP have considered the same contracts in rounds T and $|X^T|$ of the GCOP and single-offer COP, respectively. The GCOP terminates in round T where $UK^T = D$. This means that $d \in U^T \cup K^T$ for all $d \in D$.

(a) If $d \in U^T$, then $AC^T(\succ_d) = \bar{AC}^{|X^T|}(\succ_d) = \emptyset$. Note that $AC^0(\succ_d) = \bar{AC}^{|X^0|}(\succ_d) = AC(\succ_d)$ for each $d \in D$ and that the set of contracts involving d that have been considered in rounds 1 through $|X^t| + \tau$ in the single-offer COP is given by $AC(\succ_d) \backslash \bar{AC}^{|X^t|+\tau}(\succ_d)$. $\bar{AC}^{|X^T|}(\succ_d) = \emptyset$ implies that $AC(\succ_d) \backslash \bar{AC}^{|X^T|}(\succ_d) = \bar{AC}(\succ_d)$. In other words, all contracts in $AC(\succ_d)$ have been considered in round $|X^T|$ of the single-offer COP. This means that $AC(\succ_d) \precsim Y^{|X^T|}$ for all $d \in D$.

(b) If $d \in K^T$, then $d \in \bigcup_{h \in H} d(C_h(X^T)) = \bigcup_{h \in H} d(C_h(Y^{|X^T|}))$.

Both cases (a) and (b) jointly imply that condition (3) in Definition 4 is satisfied. Since conditions (1), (2) and (3) in Definition 4 are satisfied, the sequence of contracts considered here represents a single-offer COP.

Lemma 3. *(Hirata and Kasuya (2014, Theorem 1)) Suppose that two sequences of contracts represent some COPs at $\succ_D$ and C_H. If every C_h satisfies the HK and IRC conditions, then they induce the same set of contracts as their outcome.*[12]

We have demonstrated that for any GCOP and any $\succ_D$ and C_H, there exists some sequence of contracts that represents a single-offer COP that induces the same outcome. By Lemma 3, all sequences of contracts that represent a single-offer COP are outcome equivalent at $\succ_D$ and C_H. This implies that all GCOPs are outcome equivalent at $\succ_D$ and C_H. ♣

A.3 Proposition 1

Proof. [HK implies HKW] Suppose that C_h violates the HKW condition. This implies that there exist (i) $X' \subseteq X$, (ii) $d \notin d(C_h(X'))$, (iii) $x \in (X_d \cap X_h) \setminus X'_d$ and (iv) $x' \in R_h(X') \setminus R_h(X' \cup \{x\})$ such that $d(x') \notin d(C_h(X'))$. Note that $x' \in R_h(X') \setminus R_h(X' \cup \{x\})$ implies (i) $x' \notin C_h(X')$, and (ii) $x' \notin R_h(X' \cup \{x\})$ and thus $x' \in C_h(X' \cup \{x\})$.

- Suppose $d = d(x')$. Since C_h chooses an allocation and allocations cannot contain more than one contract per doctor, $x' \in C_h(X' \cup \{x\})$ implies $x \notin C_h(X' \cup \{x\})$. The combination of $x' \notin C_h(X')$ and $x' \in C_h(X' \cup \{x\})$ implies $C_h(X') \neq C_h(X' \cup \{x\})$. However, since $x \notin C_h(X' \cup \{x\})$, the IRC condition requires that $C_h(X') = C_h(X' \cup \{x\})$. This is a contradiction. Thus, $d \neq d(x')$.
- Suppose instead that $d \neq d(x')$. We have $X' \subseteq X$ with $d, d(x') \notin d(C_h(X'))$. Since $x' \in C_h(X' \cup \{x\})$, it follows that $d(x') \in d(C_h(X' \cup \{x\}))$. This violates the HK condition.

Thus, a violation of the HKW condition implies that the HK condition is violated, or equivalently, the HK condition implies the HKW condition.

[HKW implies HK] Suppose that C_h violates the HK condition. Then there exist (i) $d, d' \in D$ with $d \neq d'$, (ii) $x \in X_d$, and (iii) $X' \subseteq X$ with $d, d' \notin d(C_h(X'))$ such that $d' \in d(C_h(X' \cup \{x\}))$.

[12] Theorem 1 in Hirata and Kasuya (2014) uses the bilateral substitutability condition rather than the HK condition. However, bilateral substitutability is only used to ensure that the HK condition is satisfied in their proof. Their Theorem 1 can therefore be rephrased as in Lemma 3.

- If $x \in X'_d$, this leads to a contradiction since it implies $X' \cup \{x\} = X'$, while $d' \in d(C_h(X' \cup \{x\})) \setminus d(C_h(X'))$. Thus, $x \in X_d \setminus X'_d$.
- Since $d' \notin d(C_h(X'))$ and $d' \in d(C_h(X' \cup \{x\}))$, there must exist some $x' \in X'_{d'}$ such that $x' \in R_h(X') \setminus R_h(X' \cup \{x\})$.[13] Then the HKW condition is violated, since $d' \notin d(C_h(X'))$.

Thus, a violation of the HK condition implies that the HKW condition is violated, or equivalently, the HKW condition implies the HK condition. ♣

B Observable Substitutability across Doctors

We first provide the framework of Hatfield *et al.* (2021). Hatfield *et al.* (2021) define an *offer process for* h as a finite sequence of distinct contracts $(x^1, \dots, x^m)$, where $x^\tau \in X_h$ for all $\tau \in \{1, \dots, m\}$. An offer process for h, $(x^1, \dots, x^m)$, is *observable* if $d(x^\tau) \notin d(C_h(\{x^1, \dots, x^{\tau-1}\}))$ for all $\tau \in \{1, \dots, m\}$. In other words, the doctor involved in the τth contract is not involved in any of the contracts chosen by h when the first $\tau - 1$ contracts are considered.

Let $\vdash$ represent a strict ordering of the elements of X determining which contract is considered in each round. Given $\succ_D$ and $\vdash$, the single-offer COP in Hatfield *et al.* (2021) is defined by the following procedure: First, let $A^0 := \emptyset$ be the set of contracts available to hospitals.

Round $t \geq 1$: Consider the following set:

$$U^t := \left\{ x \in X \setminus A^{t-1} \;\middle|\; \begin{array}{l} d(x) \notin d(C_h(A^{t-1})) \text{ for all } h \in H, \text{ and} \\ \nexists x' \in (X_{d(x)} \setminus A^{t-1}) \cup \{x^\emptyset\} \text{ such that } x' \succ_{d(x)} x \end{array} \right\}$$

If $U^t = \emptyset$, the process is complete and stops. Otherwise, let $\tilde{x}$ be the highest-ranked element of U^t according to $\vdash$, and let $A^t := A^{t-1} \cup \{\tilde{x}\}$. Identify $C_h(A^t)$ for all $h \in H$ and move to the next round.

Note that only one contract is considered in each round and that the first condition implies that the process is observable. Similarly, the resulting offer process for any single-offer COP in our framework is observable since the fresh contracts of doctors with reserved contracts in round $t - 1$ are not considered in round t.

For each hospital h, let $R_h(X') := X'_h \setminus C_h(X')$ be the contracts in X'_h rejected by h. (Hatfield *et al.*, 2021, Proposition 3) show that under the following condition, any two single-offer COPs lead to the same outcome.

Definition 5 (Hatfield, Kominers, and Westkamp (2021, Definition 8))**.** *Hospital h's choice function, C_h, is* ***observably substitutable across doctors*** *if, for any observable offer process $(x^1, \dots, x^m)$ for h, $x \in R_h(\{x^1, \dots, x^{m-1}\}) \setminus R_h(\{x^1, \dots, x^m\})$ implies $d(x) \in d(C_h(\{x^1, \dots, x^{m-1}\}))$.*

[13] Note that $x' \neq x$ since $d(x') = d' \neq d = d(x)$.

In other words, if $x \in \{x^1, \ldots, x^{m-1}\}$ – that is, it was considered by hospital h in an earlier round – is (i) not reserved when the offer process for h involves $m-1$ contracts and (ii) then reserved when the m-th contract is added to the offer process for h, there exists another contract, $\tilde{x} \neq x$, for the corresponding doctor, $d(x) = d(\tilde{x})$, which is reserved in the offer process for h involving $m-1$ contracts. Note that this implies (i) that $\tilde{x} \notin C_h(\{x^1, \ldots x^m\})$ since $x \in C_h(\{x^1, \ldots x^m\})$ and (ii) that $d(x^m) \neq d(x)$ since the offer process is observable.

While observable substitutability across doctors only imposes structure to observable offer processes in Hatfield *et al.* (2021), the HKW condition imposes an analogous requirement for any subset of contracts. As such, the HKW condition is technically stronger than observable substitutability across doctors when considering GCOPs. However, the conditions are equivalent when focusing on single-offer COPs.

The following is the outcome-equivalence result for single-offer COPs in Hatfield *et al.* (2021).

Proposition 2 (Hatfield, Kominers, and Westkamp (2021, Proposition 3))**.** *If C_h is observably substitutable across doctors for each $h \in H$, for any $\succ_D$ and any two orderings $\vdash$ and $\vdash'$, the outcome with $\vdash$ is identical to the outcome with $\vdash'$.*