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Abstract: A seller has several heterogeneous indivisible items like tickets to sell over time before a deadline. These items become worthless after the deadline. Buyers arrive sequentially and randomly and have their own private valuations over items. Each buyer may acquire more than one item. We formulate this as an incentive compatible revenue maximization problem and characterize optimal allocation policies and derive various properties.

Keywords: Revenue Maximization, Random and Sequential Assignment, Heterogeneity

JEL Codes: C61, D21,D82.

1 Introduction

This paper aims to study a revenue maximization problem of sequentially selling items to randomly arriving buyers with incomplete information and incentives. To be more precise, a seller has a fixed number of heterogeneous indivisible items for sale over a period of time. These items have to be sold before a given deadline. Buyers arrive sequentially and randomly and have their own private valuations over every bundle of those items. Each buyer may acquire several items.[1] Buyers rank all bundles of items in the same fashion[2] but differ in

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[1]This case will be called *multi-item demand*.

[2]In other words, bundles of items are commonly ranked.

values involving their types. They make their decision on what to buy immediately on the moment of arrival. The seller has to decide a price for every bundle of available items at every time in order to maximize her expected revenue over the given period of time. This model is general enough to cover a variety of revenue management problems. For instance, consider the sale problem of tickets in the airline industry. For each flight, the seller has a fixed number of tickets of economy, business and first classes. Tickets have to be sold well in advance before the departure of the flight. Similar problems occur in hotel, food, entertainment and other travel industries. For this, we refer the interested reader to the following excellent survey articles and book by Bitran and Caldentey (2003), Elmaghraby and Keskinocak (2003), Talluri and Van Ryzin (2004), McAfee and Velde (2007), Bergemann and Said (2011), and Gershkov and Moldovanu (2014).

Gershkov and Moldovanu (2009) articulate the difficulty of dealing with such problems: "Assigning an object today means that the valuable option of assigning it in the future-possibly to an agent who values it more–is forgone. On the other hand, since the arrival process of agents is stochastic, and since there is a deadline, the "future" may never materialize. The revenue maximizing policy needs to define prices for each available object such that, at each point in time, this trade-off is optimally taken into account." While the literature has so far focused on an important class of revenue management problems where every buyer demands at most one item,[3] in this paper we attempt to make a step to tackle a wider class of revenue management problems where buyers may demand several items. This is a major open question stated in Gershkov and Moldovanu (2009, p. 183).

Following Gershkov and Moldovanu (2009) we adopt the mechanism design approach of e.g., Hurwicz (1973) and Myerson (1981) to this general class of revenue management problems. We formulate this as a continuous-time incentive compatible revenue maximization problem and characterize optimal allocation policies and derive various properties. As in Gershkov and Moldovanu (2009), we first describe interim, individually rational, non-randomized and Markovian allocation policies and prove in Lemma 1 that such policies are implementable if and only if they are based on a partition of the agents' type space. We then compute the expected revenue for any such implementable allocation policy and show that the revenue-maximizing policy can be obtained by either a recursive dynamic programming or a variational argument. However, in practice, it can be still computationally difficult. To mitigate this issue, we establish an equivalence between our general problem and a canonical one by exploring a transformation. A major consequence of our key Theorem 9 implies that the optimal allocation policies of a general problem with any general positive and Lebesgue integrable arrival function can be obtained from a canonical problem with a simple constant arrival rate. In other words, any general problem is equivalent to a canonical problem with

[3]This is called *the unit demand problem or model* in the literature.

a constant arrival rate. The optimal price at any time t before the deadline for any given subset of available items at the time depends only on one dynamic factor which is determined by the current time t alone with valuations of relevant subsets of items. Our Theorem 9 makes use of an integration transformation lemma 8 which is mathematically interesting on its own.

There is a large volume of literature on revenue or yield management problems. We will briefly review several closely related papers. Other relevant ones can be found in the above mentioned survey articles and book. Derman et al. (1972) are the first to examine a discrete-time model with complete information of agents' valuations on items for sale. In their model a collection of commonly ranked heterogeneous items is assigned to randomly and sequentially arriving unit-demand agents. They analyze dynamic allocation rules to maximize the seller's expected revenue. Albright (1974) extends this model and analysis to a continuous setting with random and sequential arrivals of agents. In a similar but static framework, Mussa and Rosen (1978) investigate a monopolist's price/quality decision under incomplete information but with nonrandom demand. See also Stokey (1979) and Bulow (1982).

In contrast to the early literature, Gershkov and Moldovanu (2009, 2010) and Pai and Vohra (2013) adopt a general approach of mechanism design to revenue management problems. Gershkov and Moldovanu (2009) study a continuous-time model in which unit demand buyers arrive randomly and sequentially before a deadline and have a common ranking over several heterogeneous items but possess private values over the items. They formulate it as a continuous-time incentive compatible problem and derive revenue-maximizing implementable allocation schemes and explore the variational approach to the setting. In a similar setting, Gershkov and Moldovanu (2010) show that the dynamically efficient allocation rule of Albright (1974) is implementable by a dynamic version of the classic VCG mechanism. Dizdar et al. (2011) extend the model and analysis of Gershkov and Moldovanu (2009) to a more general setting where each buyer is allowed to have a private two-dimensional type that describes his capacity requirement and his willingness to pay per unit of capacity. Types are two-dimensional in contrast to the one-dimensional case of Gershkov and Moldovanu (2009). Pai and Vohra (2013) consider the problem of a monopolist's optimally selling multiple units of an indivisible good to randomly arriving buyers over a discrete, finite time horizon T. Every buyer demands only one unit and privately knows his value for a unit, his arrival time, and his deadline before T. They propose an optimal dynamic auction which uses only a simple index rule. This index rule can be seen as a dynamic analogue of Myerson's static auction (1981). They investigate under what circumstances the index rule is optimal.

The paper is organized as follows. Section 2 presents the model. Section 3 introduces basic concepts, allocation policies and derives the expected revenue generated by the policies.

Section 4 establishes the main results. Sections 5 and 6 discuss two special cases of our current model: the case of multiple identical items in which buyers may demand several items, and the unit demand case of multiple heterogenous, commonly ranked items. The former case has not been investigated previously while the latter has been widely studied in the literature. Section 7 concludes.

2 The Model

A seller (she) has n heterogeneous indivisible items such as tickets and hotel rooms for sale. Let $N = \{1, 2, \ldots, n\}$ be the set of these n items. Consumers arrive according to a (possibly non-homogeneous) Poisson process with a strictly positive Lebesgue integrable intensity function $\lambda(t)$ for t in a known finite time-interval $[0, T]$. Items have to be sold before the deadline T after which they become worthless,like tickets. Every arriving agent (he) may acquire any bundle of items which are still available upon his arrival. It is assumed that arriving agents are impatient and rational. Therefore every agent decides immediately on the moment t of arrival and makes a choice which maximizes his utility over all subsets of M, where M is the set of items that have not yet been sold at time t. Suppose that the seller has a price $P_M(S, t)$ for each available bundle $S \subseteq M$ at time t. Then for every agent j arriving at time t, he will get a net utility given by $u^j_{(M,t)}(S) = v^j(S) - p_M(S, t)$ if he buys bundle S and pays its price $p_M(S, t)$, where $v^j(S)$ is his private value of the bundle. He will choose an available bundle at time that maximizes his net utility. It is natural to assume that $v^j(\emptyset) = 0$, $p_M(\emptyset, t) = 0$ and thus $u^j_{(M,t)}(\emptyset) = 0$.

Each buyer j knows her own value function $v^j : 2^N \mapsto \mathbb{R}$ which is given by $v^j(S) = B_j v(S)$ for every $S \subseteq N$, where $v : 2^N \to \mathbb{R}_+$ is a commonly known nonnegative value function with $v(\emptyset) = 0$. In other words, all buyers rank all bundles in a similar way but can differ in their values. The variable B_j is a random variable and represents the type of agent j. The realization of B_j is the private information of agent j but the B_j is independently and identically distributed with a commonly known distribution function F with $F(0) = 0$. Moreover, omitting the reference to a particular agent j we denote by B the random variable for which $P(B \leq x) = F(x)$ for all $x \in \mathbb{R}$ and let Ω denote the sample space for random variable B. We can immediately identify an event $\beta(K)$ with any interval K of real numbers by defining $\beta(K) = \{\omega \in \Omega \mid B(\omega) \in K\}$.

We shall examine the problem of how the seller should determine prices $p_{(M,t)}(S)$ for all bundles of items $S \subseteq M \subseteq N$ at every time $t \in [0, T]$ for all nonempty $S \subseteq M$ such that the expected total revenue obtained from selling items is maximized, based on the information of the value function v, the arrival density function λ and the distribution function F.

Note that the model described above is a generalization of the models studied by Gershkov and Moldovanu (2009, 2010). They assume that every arriving agent can buy at most one item and that all items are commonly ranked.

Throughout the paper we assume that the value range of random variable B is a subinterval I of the nonnegative real numbers, B is a continuous variable with a differentiable distribution F within the interior of I and the density function f is strictly positive and differentiable, where $f(x) = F'(x)$ for every x in the interior of I. It is clear from this assumption that the cumulative distribution function F has an inverse function F^{-1} on $(0, 1)$ and $F^{-1}(y) \in I$ for every $y \in (0, 1)$.

3 Dynamic Revenue Maximization

We will first discuss direct mechanisms. In such mechanisms, upon arrival every buyer will report his type to the designer who determines the rule of how to assign which bundle to whom at what price and at what time. Following Gershkov and Moldovanu (2009), we focus on the class of fairly practical nonrandomized and Markovian allocation policies. In such policies, at any time t and for any possible type of agent arriving at time t, the nonrandomized allocation rule to be used depends only on the arrival time t, the reported type of the arriving agent and the family of items available at time t. We further require such policies to be interim individually rational in the sense that no buyer should pay more than the utility he could obtain from the allocation given by the allocation rule. As pointed out in Gershkov and Moldovanu (2009), these policies can be intuitively interpreted as indirect mechanisms in which the designer specifies a time-dependent menu of prices, one for each bundle of items, and arriving buyers choose from the menu. In the following we will examine such policies without repeating the words of nonrandomized and Markovian.

At any time $t \in [0, T]$ and for any subset $M \subseteq N$ of items available at time t, a *pricing policy or strategy* specifies a price denoted by $P_V(S, s)$ for every bundle $S \subseteq V$ at any time s in the remaining time interval $[t, T]$ and for any bundle $V \subseteq M$ available at time s. This policy is a vector of prices with $P_V(S, s)$ as components and is a function of bundles and time. It will be denoted by $P_M(t)$ or P_M. We restrict attention to pricing strategies p_M for which for any nonempty $S \subseteq V \subseteq M$ the corresponding price function $p_V(S, s)$ for $s \in [t, T]$ is nonnegative and Lebesgue integrable with $P_V(\emptyset, s) = 0$.

With respect to the given value function $v : 2^N \to \mathbb{R}_+$, any time $t \in [0, T]$, any bundle $M \subseteq N$ available at time t and an associated pricing policy p_M, we represent the set of those types of agents who buy bundle $S \subseteq M$ upon arrival time $s \in [t, T]$ by

$$A_{(M,s)}(S) = \{b \in [0, \infty) \mid \text{Agent of type } b \text{ who arrives at time } s \text{ buys bundle } S\}.$$

To determine the set $A_{(M,s)}(S)$, a tie-breaking rule should be specified and applied by all agents. For example, the following rule can be used by all arriving agents. In case of a tie in maximizing his utility from choosing among several bundles, an arriving agent should first avoid any bundle containing item 1 if possible, and then avoid any bundle containing item 2 if possible, etc.

The following result is reminiscent of and extends Proposition 1 of Gershkov and Moldovanu (2009) from their unit-demand model to the current multi-item demand setting and shows that every implementable nonrandomized and Markovian allocation policy must be based on a partition of all the types of the arriving agents.

Lemma 1 *If all arriving agents follow the same tie-breaking rule as specified above, then the sets $A_{(M,s)}(S)$ are well defined for every $S \subseteq M$ and form a disjoint cover of all possible types of the arriving agents. Moreover, for every $S \subseteq M$ the set $A_{(M,s)}(S)$ is a (possibly empty) subinterval of all possible types of the arriving agents.*

Proof. Assuming that the tie-breaking rule is applied consistently it follows directly that any type $b \geq 0$ belongs to exactly one of the sets $A_{(M,s)}(S)$, $S \subseteq M$. Thus these sets form a disjoint cover of all possible types of the arriving agents, i.e., a set of nonnegative real numbers. Moreover, for any two different subsets S_1 and S_2 of M it follows that the difference in utility

$$u^j_{(M,s)}(S_1) - u^j_{(M,s)}(S_2) = b(v(S_1) - v(S_2)) - (p_M(S_1, s) - p_M(S_2, s))$$

depends linearly on the type b of an arriving agent. From this one can easily see that there do not exist nonnegative numbers $b_1 < b_2 < b_3$ such that $b_1, b_3 \in A_{(M,s)}(S_1)$ while $b_2 \in A_{(M,s)}(S_2)$. Therefore $A_{(M,s)}(S)$ must be a subset of types, i.e., a subinterval of nonnegative real numbers for every $S \subseteq M$. ■

Corollary 2 *If all arriving agents apply the same tie-break rule as stated in the previous lemma, then for every $S \subseteq M$ and at any time $s \in [t, T]$ the set $\beta(A_{(M,s)}(S))$ is an event in the probability space for B and the probability $g_{(M,s,p_M(s))}(S, t)$ that an agent arriving at time s will buy the bundle S is given by*

$$g_{(M,s,p_M(s))}(S, s) = P(\beta(A_{(M,s)}(S))).$$

Definition 3 *Let $R(M, t, p_M)$ denote the expected revenue obtained by selling items during time interval $[t, T]$ if at time t the set M of items is available for selling and p_M is the price strategy set by the seller.*

Note that a given price strategy p_M also sets the price strategies p_V for all nonempty $V \subseteq M$. Thus if at some future moment $s \in [t, T]$ the subset $V \subseteq M$ of items is available for selling then a potential buyer arriving at time s will buy with probability $g_{(V,s,p_V(s))}(S, s)$ the subset $S \subseteq V$ of available items and all relevant probabilities $g_{(V,s,p_V(s))}(S,s)$ are determined by price strategy p_M. We will use $s \in [t,T]$ auxiliary variables defined by

$$H_t(s) := \int_t^s \lambda(z)(1 - g_{(M,z,p_M(z))}(\emptyset))dz. \tag{1}$$

Remark 4 It follows from the definition that $H_t(s)$ is the expected number of buyers arriving between current time t and future time s who would buy at least one item from the available set M. Note that $H_t(s)$ depends on the arrival rate function $\lambda(z)$ and the price strategy p_M set by the seller, but that this is suppressed in notation. It follows from the assumption that potential buyers arrive according to a Poisson process that the number of buyers arriving in time interval (t,s) who would buy at least one item is Poisson distributed. Hence $\exp(-H_t(s))$ is the probability that no item is sold in time interval $[t,s]$. In other words $\exp(-H_t(s))$ is the probability that all items which are available at current time t are still available for selling at time s.

By conditioning on all the subsets $S \subseteq M$ that a potential buyer arriving at time s can buy it follows that $R(M,t,p_M)$ satisfies the following dynamic recursion.

$$R(M,t,p_M) = \int_t^T \sum_{S \subseteq M, |S| \geq 1} (p_M(S,s) + R(M \setminus S, s, p_{M \setminus S}))\lambda(s) g_{(M,s,p_M(s))}(S,s) e^{-H_t(s)} ds, \tag{2}$$

where $R(\emptyset, s, p_\emptyset) = 0$ for all s.

This recursive formula for $R(M,t,p_M)$ for price strategy p_M can be extended to a recursive formula for the maximal expected reward which is defined as follows.

Definition 5 *Let $R(M,t)$ denote the maximal expected revenue obtained by selling items during time interval $[t,T]$ if at time t the set M of items is available for selling.*

Tt follows that for all $M \subseteq N$ and $t \in [0,T]$ we have that

$$R(M,t) = \max_{p_M} R(M,t,p_M). \tag{3}$$

Moreover, from (2) the following result which is a type of Bellman equation follows for the maximal expected revenue $R(M,t)$.

Proposition 6 *For any set $M \subseteq N$ of available items and time $t \in [0,T]$ we have the following dynamic recursion for the maximal expected revenue $R(M,t)$.*

$$R(M,t) = \max_{p_M} \int_t^T \sum_{S \subseteq M, |S| \geq 1} (p_M(S,s) + R(M \setminus S, s, p_{M \setminus S}))\lambda(s) g_{(M,s,p_M(s))}(S,s) e^{-H_t(s)} ds,$$

with $R(\emptyset, t) = 0$ for all $t \in [0,T]$.

4 Main Results

We now define an auxiliary variable which helps to simplify expressions and obtain more insight on the optimal solution.

Definition 7 *Let $T > 0$, $0 < t < T$ and the arrival rate function $\lambda(s), s \in (0,T)$ be given. Then we define the auxiliary variable*

$$Q(t) = \int_t^T \lambda(s)ds,$$

the expected number of agents which will arrive in the remaining time interval $(t,T]$.

The main result of this paper will be Theorem 9 which shows that the optimal solution for any general positive and Lebesgue integrable arrival function $\lambda(s)$ can be obtained from a canonical problem with a simple constant arrival rate $\lambda(s) = 1$. Let $p_m^*(l,t)$ denote the optimal price for l items at time t if $m \geq l$ items are available at time t. Then Theorem 9 implies that for any $t \in [0,T]$ the maximal expected revenue $R(k,t)$ obtained by selling items during the remaining time interval $[t,T]$ depends only on $Q(t)$ apart from the k valuations $v(l), l = 1,2,\dots,k$. Moreover, all the corresponding optimal prices $p_m^*(l,s)$ for $1 \leq l \leq m \leq k$ and $s \in [t,T]$ which the seller should set to obtain the maximal expected revenue $R(k,t)$ depend only on $Q(s)$ apart from the valuations $v(l), l = 1,2,\dots,m$. To prove Theorem 9, we need to introduce the following lemma, which is mathematically interesting on its own.

Lemma 8 (Integration Transformation Lemma) *Let $\lambda_1(s)$ and $\lambda_2(s)$ be strictly positive Lebesgue integrable functions on intervals $[t_1,T]$ and $[t_2,T]$, respectively. For $t_1 \leq x \leq T$ let $\mu(x) = \int_x^T \lambda_1(s)ds$ and for $t_2 \leq x \leq T$ let $\nu(x) = \int_x^T \lambda_2(s)ds$. Suppose that $f(x)$ and $g(y)$ are Lebesgue integrable functions on $[t_1,T]$ and $[t_2,T]$ respectively such that $f(x) = g(y)$ if $\mu(x) = \nu(y)$. Then if*

$$\mu(x_0) = \nu(y_0) \text{ and } \mu(x_1) = \nu(y_1) \text{ with } t_1 \leq x_0 \leq x_1 \leq T \text{ and } t_2 \leq y_0 \leq y_1 \leq T,$$

we have

$$\int_{x_0}^{x_1} \lambda_1(x)f(x)dx = \int_{y_0}^{y_1} \lambda_2(y)g(y)dy.$$

Proof. The functions μ and ν are strictly decreasing differentiable functions with $\mu'(x) = -\lambda_1(x)$ and $\nu'(x) = -\lambda_2(x)$ for $x \in [t_1,T]$ respectively $x \in [t_2,T]$. Hence the inverse functions $m := \mu^{-1}$ and $n := \nu^{-1}$ are well defined for $s \in [0,\mu(t_1)]$ respectively $s \in [0,\nu(t_2)]$. It follows that both functions m and n are positive, strictly decreasing and differentiable. Moreover, we have that $m'(s) = \frac{1}{\mu'(m(s))} = -\frac{1}{\lambda_1(m(s))}$ and $n'(s) = \frac{1}{\nu'(n(s))} = -\frac{1}{\lambda_2(n(s))}$. If $\nu(t_2) \geq \mu(t_1)$ let $D = [t_1,T]$ and if $\nu(t_2) < \mu(t_1)$ let $D = [m(\nu(t_2)),T]$. For $s \in D$ define $w(s)$ by the composition $w = n \circ \mu$. Then w is nonnegative, strictly increasing and differentiable on D and for $x \in D$ we have that $n'(\mu(x)) = -\frac{1}{\lambda_2(w(x))}$. Hence

$$w'(x) = n'(\mu(x))\mu'(x) = \frac{-\mu'(x)}{\lambda_2(w(x))} = \frac{\lambda_1(x)}{\lambda_2(w(x))} \text{ for } x \in D.$$

Let $t_1 \leq x_0 \leq x_1 \leq T$ and $t_2 \leq y_0 \leq y_1 \leq T$ be such that $w(x_0) = y_0$ and $w(x_1) = y_1$. Then it follows that $x_0 \in D$ with $\mu(x_0) = n^{-1}(w(x_0)) = n^{-1}(y_0) = \nu(y_0)$ and thus $f(x_0) = g(y_0)$. Similarly it follows that $x_1 \in D$ and $f(x_1) = g(y_1)$. Hence we have for all $x_0 \leq x \leq x_1$ that $x \in D$ and thus

$$\lambda_1(x) = \lambda_2(w(x))w'(x). \tag{4}$$

Moreover, for all $x \in D$ we have $w(x) = y$ if and only if $\mu(x) = \nu(y)$. Hence by the assumption on $f(x)$ and $g(y)$ it follows that

$$f(x) = g(w(x)) \text{ for } x \in D. \tag{5}$$

Combining (4) and (5) and substituting $y = w(x)$ for $x \in D$ it follows that

$$\int_{x_0}^{x_1} \lambda_1(x)f(x)dx = \int_{x_0}^{x_1} \lambda_2(w(x))g(w(x))w'(x)dx = \int_{y_0}^{y_1} \lambda_2(y)g(y)dy.$$

■

We are now ready to establish the following major result of the paper from which other results will be derived.

Theorem 9 (Revenue and Pricing Strategy Equivalence Theorem) *Let value function v be given for set of items M and let distribution function F for the type of arriving agents be given. Let $R_1(M, t, p_M^1)$ and $R_2(M, t, p_M^2)$ be the expected revenues, respectively, from selling items from M in time interval $[t, T]$ obtained by using pricing strategies p_M^1 and p_M^2 if potential buyers arrive according to Poisson processes with arrival rates $\lambda_1(s)$ and $\lambda_2(s)$ which are strictly positive and Lebesgue integrable on intervals $I_1 = [t_1, T]$ with $t_1 \leq t$ and $I_2 = [t_2, T]$ with $t_2 \leq t$. For $x \in I_1$ let $\mu(x) = \int_x^T \lambda_1(s)ds$ and for $x \in I_2$ let $\nu(x) = \int_x^T \lambda_2(s)ds$. If $\nu(t_2) \geq \mu(t_1)$ let $D = I_1$ and if $\nu(t_2) < \mu(t_1)$ let $D = [\mu^{-1}(\nu(t_2)), T]$. For $s \in D$ define $w(s)$ by the composition $w = \nu^{-1} \circ \mu$. Suppose that pricing strategies p_M^1 and p_M^2 satisfy the following equality*

$$p_M^2(S, w(s)) = p_M^1(S, s) \text{ for all nonempty } S \subseteq M \text{ and for all } s \in D. \tag{6}$$

Then we have for all $t \in D$ that

$$R_1(M, t, p_M^1) = R_2(M, w(t), p_M^2). \tag{7}$$

Proof. Recall that for $i = 1, 2$ the expected revenue $R_i(M, t, p^i_M)$ is given by

$$R_i(M,t,p^i_M) = \int_t^T \sum_{S\subseteq M, |S|\geq 1} (p^i_M(S,s) + R_i(M\setminus S, s, p^i_{M\setminus S}))\lambda_i(s) g_{(M,s,p^i_M)}(S) e^{-H^i_t(s)} ds, \quad (8)$$

where $H^i_t(s) = \int_t^s \lambda_i(z)(1 - g_{(M,z,p^i_M)}(\emptyset))dz$ and $R_i(\emptyset, t, p^i_M) = 0$ for all $t \leq T$ and price strategies p^i_M.

By (6), (8) and applying Lemma 8 we will now prove by induction that $R_2(M, w(t), p^2_M) = R_1(M, t, p^1_M)$ for all $t \in D$. First consider the case that there is only $|M| = 1$ item to be sold. Then (8) simplifies for $i = 1, 2$ to

$$R_i(M,t,p^i_M) = \int_t^T (p^i_M(M,s)\lambda_i(s) g_{(M,s,p^i_M)}(M) e^{-H^i_t(s)} ds, \quad (9)$$

where $H^i_t(s) = \int_t^s \lambda_i(z)(1 - g_{(M,z,p^i_M)}(\emptyset))dz$.

From $t \in D$ it follows that $z \in D$ for all $z \in [t, T]$. Hence for all $z \in [t, T]$ we have by (6) that $p^2_M(M, w(z)) = p^1_M(M, z)$. Hence $g_{(M,z,p^1_M)}(\emptyset) = g_{(M,w(z),p^2_M)}(\emptyset)$ for all $z \in [t, T]$. Then by applying Lemma 8 with $x = z$, $y = w(z)$, $f(x) = 1 - g_{(M,x,p^1_M)}(\emptyset)$ and $g(y) = (1 - g_{(M,y,p^1_M)}(\emptyset))$ it follows that $H^1_t(s) = H^2_{w(t)}(w(s))$ for all $t \in D$ and $s \in [t, T]$. Next we apply Lemma 8 again, but now for $x = s$, $y = w(s)$, $f(x) = p^1_M(M, x) g_{(M,x,p^1_M)}(M) e^{-H^1_t(x)}$ and $g(y) = p^2_M(M, y) g_{(M,y,p^2_M)}(M) e^{-H^2_{w(t)}(y)}$. From this we obtain that

$$R_1(M,t,p^1_M) = \int_t^T (p^1_M(M,s)\lambda_1(s) g_{(M,s,p^1_M)}(M) e^{-H^1_t(s)} ds =$$

$$\int_{w(t)}^T (p^2_M(M,w(s))\lambda_2(w(s)) g_{(M,w(s),p^2_M)}(M) e^{-H^2_{w(t)}(w(s))} dw(s) = R_2(M, w(t), p^2_M).$$

Hence (7) holds in case $|M| = 1$.

To complete the prove by induction consider the case that $|M| > 1$ and assume that (7) holds for all proper nonempty subsets of M. By this assumption we have for all $S \subseteq M$ with $|S| \geq 1$ and all $s \in [t, T]$ that $R_1(M \setminus S, s, p^1_{M\setminus S}) = R_2(M \setminus S, w(s), p^2_{M\setminus S})$. Now we apply Lemma 8 again with $x = s$, $y = w(s)$,

$$f(x) = \sum_{S\subseteq M, |S|\geq 1} (p^1_M(S,x) + R_1(M\setminus S, x, p^1_{M\setminus S})) g_{(M,x,p^1_M)}(S) e^{-H^1_t(x)}$$

and

$$g(y) = \sum_{S\subseteq M, |S|\geq 1} (p^2_M(S,y) + R_2(M\setminus S, y, p^2_{M\setminus S})) g_{(M,y,p^2_M)}(S) e^{-H^2_{w(t)}(y)}.$$

Then $y = w(x)$ implies that $f(x) = g(y)$ since the values of all corresponding terms coincide. Thus Lemma 8 is applicable which completes the proof by induction. ■

Corollary 10 *Let value function v be given for set of items M and let distribution function F for the type of arriving agents be given. Let $R_1(M,t)$ and $R_2(M,t)$ be the maximal expected revenues, respectively, from selling items from M in time interval $[t,T]$ if potential buyers arrive according to Poisson processes with arrival rates $\lambda_1(s)$ and $\lambda_2(s)$ which are strictly positive and Lebesgue integrable on intervals $I_1 = [t_1,T]$ with $t_1 \leq t$ and $I_2 = [t_2,T]$ with $t_2 \leq t$. For $x \in I_1$ let $\mu(x) = \int_x^T \lambda_1(s)ds$ and for $x \in I_2$ let $\nu(x) = \int_x^T \lambda_2(s)ds$. If $\nu(t_2) \geq \mu(t_1)$ let $D = I_1$ and if $\nu(t_2) < \mu(t_1)$ let $D = [\mu^{-1}(\nu(t_2)),T]$. For $s \in D$ define $w(s)$ by the composition $w = \nu^{-1} \circ \mu$. Then we have for all $t \in D$ that*

$$R_1(M,t) = R_2(M,w(t)). \tag{10}$$

Proof. By Theorem 9 there exists for any price strategy p_M^1 and $t \in D$ a corresponding price strategy p_M^2 such that $R_1(M,t,p_M^1) = R_2(M,w(t),p_M^2)$. Hence $R_1(M,t) \leq R_2(M,w(t))$. Vice versa let $U = w(D)$ be the image of D under the invertible function w where on domain U we have that $w^{-1} = \mu^{-1} \circ \nu$. By applying Theorem 9 and reversing the roles of arrival rate functions $\lambda_1(s)$ and $\lambda_2(s)$ it follows that for any price strategy p_M^2 there exists for all $u \in U$ a corresponding price strategy p_M^1 such that $R_2(M,u,p_M^2) = R_2(M,w^{-1}(u),p_M^1)$. In particular for $u = w(t)$ it follows that for any price strategy p_M^2 there exists a price strategy p_M^1 such that $R_2(M,w(t),p_M^2) = R_1(M,t,p_M^1)$. Hence we also have that $R_2(M,w(t)) \leq R_1(M,t,p_M^1)$ and thus $R_1(M,t) = R_2(M,w(t))$. ■

Corollary 11 *Let p_M^1 be an optimal pricing strategy for the canonical problem with a constant arrival rate $\lambda_1(s) = 1$ for all $s \in I_1 = [t_1,T]$. Then any problem with a positive Lebesgue integrable Poisson arrival rate $\lambda_2(s)$ on some interval $I_2 = [t_2,T]$ can be reduced to the canonical problem on interval $I_1 = [t_1,T]$ with $t_1 \leq w^{-1}(t_2)$, where $w^{-1}(s) = T - \int_s^T \lambda_2(x)dx$ for any $s \in U = w(D)$ described in the proof of the previous corollary. For any nonempty $S \subseteq V \subseteq M$ and at any time $t \in I_2$, the optimal price $p_V^2(S,t)$ is given by*

$$p_V^2(S,t) = p_V^1(S,w^{-1}(t))$$

where V is the set of items available at time t. Moreover, for all $t \in I_2$ and $V \subseteq M$, the maximal expected revenue $R_2(V,t)$ for selling items from V during the remaining time interval $[t,T]$ is given by

$$R_2(V,t) = R_1(V,w^{-1}(t)).$$

We have now shown that under some mild assumption (Lebesgue integrable arrival rate function) any problem is equivalent to a problem with constant arrival rate $\lambda_1(s) = 1$. Therefore the optimal price at time t for a given subset of items and a given set of available items should (apart from static input such as the valuations of all relevant subsets) depend only on one dynamic factor which is determined by the current time t. From the proof of

Theorem 9 it is clear that this dynamic factor is $Q(t) = \int_t^T \lambda(s)ds$, the expected number of buyers which will arrive in remaining time interval $[t, T]$. In summary we have obtained the following result.

Corollary 12 *Suppose that a set $V \subseteq M$ of items is available for sale during a remaining time interval $[t, T]$ under the assumptions we have made. Then all the optimal prices and maximal expected revenue $R(V, t)$ depend apart from valuations $v(S)$ for $S \subseteq V$ and distribution function $F(x)$ only on $Q(t)$, the expected number of agent arrivals in the remaining time interval in which items can be sold.*

5 The Case Study of Identical Items

In this section we consider the case where there are n identical items and every buyer may acquire several items.[4] This is a special case of the general model studied in the previous sections. Let N denote the collection of identical items. Then for the value function $v : 2^N \to \mathbb{R}$ we have $v(S) = v(T)$ for any two bundles S and T with $|S| = |T|$. Thus if $|S| = k$, we may write $v(k)$ instead of $v(S)$. From this it follows that for every subset $M \subseteq N$ the value $R(M, t)$ should only depend on the cardinality $|M|$ of M and t and the price of a subset of available items should only depend on the cardinality of the subset. Therefore we may specify the earlier definitions for the case of identical items as follows.

Definition 13 *In case of identical items we denote for any positive integer k with $R(k, t)$ the maximal expected revenue for the seller if at time t there are exactly k identical items available for sales. Moreover, the price strategy set by the seller is then shortly denoted by p_k and $p_k(t) = (p_k(l, t), l \in \{1, 2, \ldots, k\})$ is the k-dimensional current price vector, where l is the number of items to be bought by an buyer. Moreover, we define $A_{(k,t)}(l) = \{b \in [0, \infty)$: if at time t an agent of type b would arrive then this agent would choose to buy exactly l items of the k available identical items$\}$.*

Corollary 14 *For current time $t \in [0, T]$ and $1 \leq k \leq |N|$ the current price vector set by the seller is $p_k(t) = (p_k(l, t), 1 \leq l \leq k)$. Then for every $l \leq k$ we have that $B^{-1}(A_{(k,t)}(l)) := \{\omega \in \Omega : B(\omega) \in A_{(k,t)}(l)\}$ is an event in the probability space for B. Hence for every $l \leq k$ the probability $g_{(k,p_k(t))}(l, t)$ that an agent arriving at time t will buy exactly l items of k available identical items is well defined and given by*

$$g_{(k,p_k(t))}(l, t) = P(B^{-1}(A_{(k,t)}(l))).$$

[4]Although it is a relatively simple and reasonable case, it has not been studied in the literature.

For identical items the dynamic recursion given in Proposition 6 simplifies to the following.

Proposition 15 *For given number k of available identical items and time $t \in [0,T]$ we have the following dynamic recursion equation for the maximal expected revenue $R(k,t)$.*

$$R(k,t) = \max_{p_k} \int_t^T \sum_{1 \le l \le k} (p_k(l,s) + R(k-l,s))\lambda(s) g_{(k,p_k(s))}(l,s) e^{-H_t(s)} ds$$

, where $R(0,t) = 0$ for all $t \in [0,T]$.

In the following two subsections we will first deal with the basic case of $k = 1$ only one item available at time $t \in [0,T]$ and then move to the general case of $k \ge 2$ multiple items available at time t.

5.1 Solving the dynamic recursive equation for the basic case

We explore methods to solve the dynamic recursion of Proposition 15 in an analytical or numerical way. We start with the simplest case in which only $k = 1$ item is available for sale at time $t \in [0,T]$. Since the price vector $p_1(t)$ the seller can set at time t consist of only one component we denote this price shortly with $p(1,t)$ to simplify notation. Note that in this $k = 1$ case the expression for the auxiliary variable $H_t(s)$ simplifies to

$$H_t(s) := \int_t^s \lambda(z)(1 - g_{(1,p(1,z))}(0,z))dz = \int_t^s \lambda(z) g_{(1,p(1,z))}(1,z)dz.$$

Hence from Proposition 15 we obtain for $k = 1$ that

$$R(1,t) = \max_{p_1} \int_t^T p(1,s)\lambda(s) g_{(1,p(1,s))}(1,s) e^{-H_t(s)} ds. \tag{11}$$

Since a potential buyer j arriving at time s will buy the item if and only if $B_j v(1) > p(1,s)$ it follows that

$$g_{(1,s,p(1,s))}(1) = 1 - F(\frac{p(1,s)}{v(1)}). \tag{12}$$

The case $k = 1$ is examined in Gershkov and Moldovanu (2009) where the maximal revenue $R(1,t)$ as given by (11) is rewritten as $R(1,t) = \max_{p_1} \int_t^T L(s, H_t(s), H_t'(s))ds$ by expressing the price function $p(1,s)$ and the probabilities $g_{(1,p(1,s))}(l,s)$ in terms of $H_t(s)$ and its derivative $H_t'(s)$. Then according to the variational principle of Euler-Lagrange this function $L(s, H_t(s), H_t'(s)$ satisfies the following differential equation (13):

$$\frac{d}{ds}\frac{\delta L}{\delta H_t'(s)}(s, H_t(s), H_t'(s)) = \frac{\delta L}{\delta H_t(s)}(s, H_t(s), H_t'(s)). \tag{13}$$

In Gershkov and Moldovanu (2009) this differential equation (13) is used to solve (11). We will now summarize some implications of their results for the $k = 1$ case in the current paper. Therefore we consider now the case that $\lambda(t) = \lambda$ for all t and we put $y(1,t) := \frac{p(1,t)}{v(1)}$. Then it follows that the revenue maximizing price $p(1,t)$ should be such that the corresponding function $y(1,t)$ satisfies the differential equation

$$y(1,t) = \frac{1 - F(y(1,t))}{f(y(1,t))} + \lambda \int_t^T \frac{[1 - F(y(1,s))]^2}{f(y(1,s))} ds. \tag{14}$$

Let $y^*(1,t)$, $t \in [0,T]$ be the corresponding optimal solution satisfying (14). Then for every $t \in [0,T]$ it holds that

$$R(1,t) = v(1)\lambda \int_t^T \frac{[1 - F(y^*(1,s))]^2}{f(y^*(1,s))} ds. \tag{15}$$

By results (14) and (15) we will see that the analysis of Gershkov and Moldovanu (2009) can be used here. Next we obtain the solution of differential equation (14) and a corresponding optimal price function $p^*(1,t)$ and maximal expected revenue $R(1,t)$ for an explicit example.

Example 16 *Assume without loss of generality that* $v(1) = 1$ *and thus* $p(1,t) = y(1,t)$. *Moreover, let* $F(x) = x$ *for* $x \in [0,1]$ *which implies that* B *is uniformly distributed on* $[0,1]$. *Let the Poisson arrival rate be homogeneous,* $\lambda(t) = \lambda$ *for all* $t \in [0,T]$.
Put $z(t) = y^*(1,t)$ *for* $t \in [0,T]$. *Then from (14) it follows that* $2z(t) = 1 + \lambda \int_t^T [1 - z(s)]^2 ds$ *which implies that* $z(T) = \frac{1}{2}$ *and* $2\frac{dz(t)}{dt} = -\lambda(1 - z(t))^2$. *This ordinary differential equation with boundary condition* $z(T) = \frac{1}{2}$ *has unique solution*

$$z(t) = \frac{\lambda(T-t) + 2}{\lambda(T-t) + 4} \text{ for } t \in [0,T].$$

This implies that for all $v(1) > 0$ *the optimal price function* $p^*(1,t)$ *is given by*

$$p^*(1,t) = v(1)\frac{\lambda(T-t) + 2}{\lambda(T-t) + 4} \text{ for } t \in [0,T]. \tag{16}$$

Moreover, it follows that the maximal expected revenue function $R(1,t)$ *is given by*

$$R(1,t) = v(1)\frac{\lambda(T-t)}{\lambda(T-t) + 4} \text{ for } t \in [0,T]. \tag{17}$$

In Figure 1 the graphs of the optimal price function $p^*(1,t)$ *and maximal expected revenue function* $R(1,t)$ *are plotted in case* $\lambda = 2$ *and* $T = 10$.

Example 17 *In this example we continue the case of Example 16, except that the arrival rate* $\lambda(t)$ *is no longer a constant but time dependent. Namely let* $\lambda_2(t) = t^2 + t + 1$ *for*

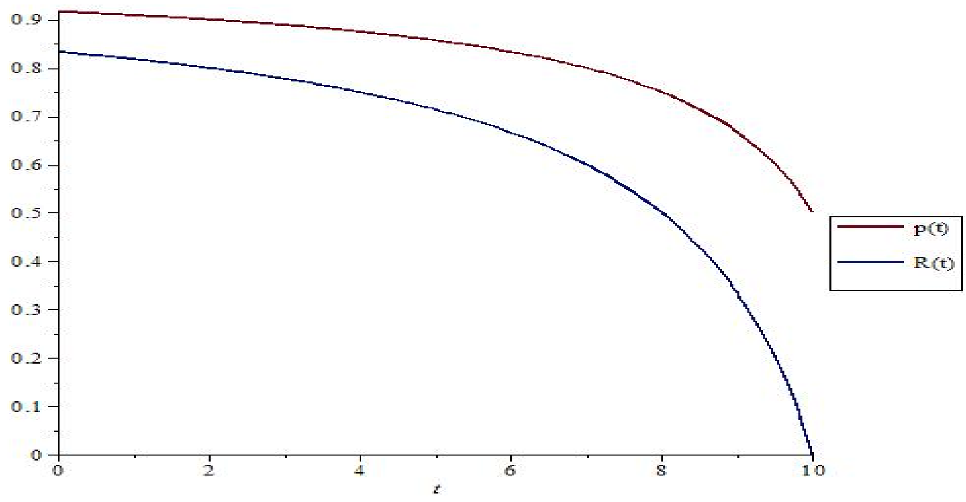

Figure 1: Optimal price and maximal revenue

$t \in [0, T]$ with $T = 2$. Moreover, assume that $v(1) = 1$. We now apply the results from Section 4 (in particular Corollary 11) to (16) and (17) to directly obtain (without solving differential equations) the optimal price function $p_2^*(t)$ and maximal expected revenue function $R_2(t)$ for the current example in which the Poisson arrival rate is non-homogeneous.

Following the notation from Corollary 11 we have in this example that

$$w^{-1}(s) = T - \int_s^T \lambda_2(x)dx = 2 - \int_s^2 (x^2 + x + 1)dx = \frac{1}{3}s^3 + \frac{1}{2}s^2 + s - \frac{14}{3}.$$

Moreover, by Corollary 11 we have that the optimal price function $p_2^*(t)$ for arrival rate function $\lambda_2(t)$ is obtained by $p_2^*(t) = p^*(w^{-1}(t))$ where $p^*(t) = p^*(1, t)$ is the optimal price function given by (16). Hence

$$p_2^*(t) = p^*(\frac{1}{3}t^3 + \frac{1}{2}t^2 + t - \frac{14}{3}) = \frac{52 - 2t^3 - 3t^2 - 6t}{64 - 2t^3 - 3t^2 - 6t} \text{ for } t \in [0, 2].$$

Similarly the corresponding maximal expected revenue $R_2(t)$ is obtained by $R_2(t) = R(w^{-1}(t))$ where $R(t) = R(1, t)$ is the maximal expected revenue function given by (17). Hence

$$R_2(t) = R(\frac{1}{3}t^3 + \frac{1}{2}t^2 + t - \frac{14}{3}) = \frac{40 - 2t^3 - 3t^2 - 6t}{64 - 2t^3 - 3t^2 - 6t} \text{ for } t \in [0, 2].$$

In Figure 2 the graphs of the optimal price function $p_2^*(t)$ and maximal expected revenue function $R_2(t)$ are plotted for this example with an inhomogeneous arrival rate function and deadline $T = 2$.
It is easily verified that at the deadline $T = 2$ it holds that $p_2^*(T) = \frac{1}{2}$ and $R_2(T) = 0$. These boundary conditions are satisfied as in Example 16.

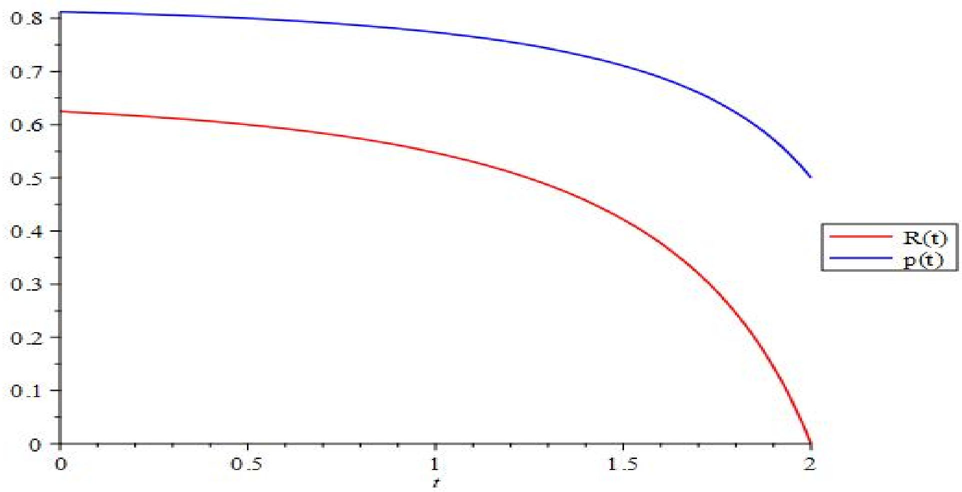

Figure 2: Optimal price and maximal revenue

Remark 18 *Recall the definition of $Q(t)$ from Definition 7. In Example 16 we have $Q(t) = \lambda(T - t) = \lambda(10 - t)$ for $t \in [0, 10]$ and in Example 17 we have $Q(t) = \int_t^2 \lambda_2(s)ds = \frac{20}{3} - \frac{1}{3}t^3 - \frac{1}{2}t^2 - t$ for $t \in [0, 2]$. From this it can easily be verified that in both examples the optimal price function $p^*(1, t)$ satisfy for all $t \in [0, T]$ the equality*

$$p^*(1,t) = v(1)\frac{Q(t)+2}{Q(t)+4}. \tag{18}$$

Moreover, in both examples the maximal expected revenue $R(1, t)$ satisfy for all $t \in [0, T]$ the equality

$$R(1,t) = v(1)\frac{Q(t)}{Q(t)+4}. \tag{19}$$

It is no coincidence that formulas (18) and (19) apply to both examples since these examples have in common that only one item is available for selling and the customer type distribution function $F(x)$ is the same. It is clear that in both examples the optimal price $p^*(1, t)$ and the maximal expected revenue $R(1, t)$ apart from being proportionate to $v(1)$ only depend on the auxiliary variable $Q(t)$ as introduced in Definition 7.

5.2 The case of $k > 1$ identical items

In this section we consider the case that at some time $t \in [0, T)$ the seller has $k > 1$ identical items for sale.[5] As before the value of items will expire if they have not been sold before

[5]Although this case in which there are multiple identical items and each agent may acquire several items is different from the one studied by Gershkov and Moldovanu (2009) (see also our Section 6) in which there are multiple heterogeneous commonly ranked items and each agent demands at most one item, we can still apply some of their techniques and results to the current case.

time T which is given. To obtain optimal prices in such cases we describe a method in case $k = 2$, but this method can be generalized to the case $k > 2$. Suppose for now $k = 2$ Then the objective is to set for all $s \in [t, T]$ prices $p_1(s)$ for buying one item and a bundle price $p_2(s)$ for buying both items such that the expected revenue $R(2, t)$ is maximized. We note that if at time $t' \in [t, T]$ an arriving agent given the current prices $p_1(t')$ and $p_2(t')$ chooses to buy one item for price $p_1(t')$ that then in the remaining time interval $[t', T]$ the price $p^*(1, s) = v(1)y^*(1, s)$ of the one remaining item is given by the optimal solution of (14). Then it follows from (15) that the expected revenue of the remaining item is $R(1, t') = v(1)\lambda \int_{t'}^{T} \frac{[1-F(y^*(1,s))]^2}{f(y^*(1,s))} ds$. Thus under the condition that at some time t' one of the two items is sold for price $p_1(t')$ the total expected revenue for the seller will be

$$p_1(t') + R(1, t') = p_1(t') + v(1)\lambda \int_{t'}^{T} \frac{[1 - F(y^*(1, s))]^2}{f(y^*(1, s))} ds.$$

Moreover, it is obvious that if at some time $t' < T$ an arriving agent chooses to buy both items for the bundle price $p_2(t')$ which the seller has set at time t' that then the (expected) revenue for the seller is just $p_2(t')$. In general according to Proposition 15 for any $t \in [0, T]$ the maximal expected revenue $R(2, t)$ satisfies the dynamic recursion equation

$$\begin{aligned} R(2,t) &= \max \int_t^T [(p_1(s) + R(1, s))g_{(2,p_1(s),p_2(s))}(1) \\ &\quad + p_2(s)g_{(2,p_1(s),p_2(s))}(2)]\lambda(s)e^{-\int_t^s \lambda(z)(1-g_{(2,p_1(z),p_2(z))}(0))dz} ds, \end{aligned} \tag{20}$$

where $g_{(2,p_1(s),p_2(s))}(l)$ is defined as the probability that an agent arriving at time s when two items are available will buy $l \in \{0, 1, 2\}$ items given current set prices $p_1(s)$ for a single item and $p_2(s)$ for the bundle of both items.

Next we investigate obtaining optimal price functions $p_1^*(s)$, $p_2^*(s)$ for which the dynamic recursion (20) is satisfied. We recall that in Gershkov and Moldovanu (2009) the variational principle of Euler-Lagrange was utilized. We have seen that in case $k = 1$ an auxiliary function $H_t(s)$ can be defined such that the maximal revenue $R(1, t)$ as given by (11) is rewritten as $R(1, t) = \max \int_t^T L(s, H_t(s), H_t'(s))ds$. Then according to the variational principle the solution of the recursion satisfies

$$\frac{d}{ds}\frac{\delta L}{\delta H_t'(s)}(s, H_t(s), H_t'(s)) = \frac{\delta L}{\delta H_t(s)}(s, H_t(s), H_t'(s)). \tag{21}$$

In Gershkov and Moldovanu (2009) it is shown that from the differential equation (21) and the boundary condition $R(1, T) = 0$ the function $H_t(s)$ can be obtained and subsequently also the optimal price function $p_1^*(s)$ and the corresponding maximal expected revenue $R(1, s)$ for all $s \in [t, T]$ resulting in (14) and (15).

Our approach is to generalize this by defining in case of $k = 2$ identical items two functions $H_t(s)$ and $G_t(s)$ such that both the price of single item $p_1(s)$ and the bundle price $p_2(s)$ for both items can be expressed in terms of $H_t(s)$, $G_t(s)$ and their derivatives $H_t'(s)$ and $G_t'(s)$. Moreover, the probabilities $g_{(2,p_1(s),p_2(s))}(l)$ for $l \in 0, 1, 2$ should then also be expressed in terms of these functions. In this way the maximal revenue $R(2,t)$ given by (20) will be expressed by

$$R(2,t) = \max \int_t^T L(s, H_t(s), H_t'(s), G_t(s), G_t'(s))ds. \tag{22}$$

Then according to the variational principle of Euler-Lagrange the solution of (20) satisfies the following system of differential equations

$$\begin{array}{lcl} \frac{d}{ds}\frac{\delta L}{\delta H_t'(s)}(s, H_t(s), H_t'(s), G_t(s), G_t'(s)) & = & \frac{\delta L}{\delta H_t(s)}(s, H_t(s), H_t'(s), G_t(s), G_t'(s)) \\ \frac{d}{ds}\frac{\delta L}{\delta G_t'(s)}(s, H_t(s), H_t'(s), G_t(s), G_t'(s)) & = & \frac{\delta L}{\delta G_t(s)}(s, H_t(s), H_t'(s), G_t(s), G_t'(s)) \end{array}. \tag{23}$$

Next this system (23) together with appropriate boundary conditions following from the maximal revenue of a single item (see 15) should determine the solution (at least numerical if the distribution function $F(x)$ for the buyers is explicitly given) of $H_t(s)$ and $G_t(s)$. Then also the optimal price functions $p_1^*(s)$, $p_2^*(s)$ and the maximal revenue $R(2,s)$ for all $s \in [t,T]$ can be (numerically) obtained from that.

To express the maximal revenue $R(2,t)$ given by (20) in the Euler-Lagrange format (22) for all $s \in [t,T]$ auxiliary variables $a(s)$ and $b(s)$ are defined as follows.

Definition 19 *For $s \in [t,T]$ let the prices $p_1(s) \geq 0$ and $p_2(s) \geq 0$ for respectively a single item and the bundle of both items be given and assume that $0 < v(1) < v(2)$. Then we define*

$$a(s) := \min(\frac{p_1(s)}{v(1)}, \frac{p_2(s)}{v(2)}) \text{ and } b(s) := \max(\frac{p_2(s) - p_1(s)}{v(2) - v(1)}, \frac{p_2(s)}{v(2)}).$$

According to the following lemma we have to distinguish (only) two different cases for $a(s)$ and $b(s)$.

Lemma 20 *Assuming that $0 < v(1) < v(2)$ it holds that*

$$a(s) = \begin{cases} \frac{p_2(s)}{v(2)} = b(s) & \text{if } \frac{p_2(s)}{v(2)} \leq \frac{p_1(s)}{v(1)} \\ \frac{p_1(s)}{v(1)} \leq \frac{p_2(s)-p_1(s)}{v(2)-v(1)} = b(s) & \text{if } \frac{p_2(s)}{v(2)} \geq \frac{p_1(s)}{v(1)} \end{cases}. \tag{24}$$

Proof. Since $0 < v(1) < v(2)$ it follows that

$$\frac{p_1(s)}{v(1)} \leq \frac{p_2(s)}{v(2)} \Leftrightarrow p_1(s)v(2) \leq p_2(s)v(1) \Leftrightarrow -p_1(s)v(2) \geq -p_2(s)v(1)$$

$$\Leftrightarrow v(2)(p_2(s) - p_1(s)) \geq p_2(s)(v(2) - v(1)) \Leftrightarrow \frac{p_2(s) - p_1(s)}{v(2) - v(1)} \geq \frac{p_2(s)}{v(2)}.$$

Hence $a(s) = \frac{p_1(s)}{v(1)}$ if and only if $b(s) = \frac{p_2(s)-p_1(s)}{v(2)-v(1)}$ and vice versa $a(s) = \frac{p_2(s)}{v(2)}$ if and only if $b(s) = \frac{p_2(s)}{v(2)}$. ■

Next we have the following result.

Proposition 21 *For all $s \in [t, T]$ it holds that $0 \leq a(s) \leq b(s)$. Moreover, if an agent arrives at some time $s \in [t, T]$ for which the prices are $p_1(s)$ and $p_2(s)$ for a single respectively bundle of both items then the probabilities $g_{(2,p_1(s),p_2(s))}(l)$ that the agent will buy $l \in \{0, 1, 2\}$ items are determined by*

$$\begin{aligned} g_{(2,p_1(s),p_2(s))}(0) &= F(a(s)) \\ g_{(2,p_1(s),p_2(s))}(1) &= F(b(s)) - F(a(s)) \\ g_{(2,p_1(s),p_2(s))}(2) &= 1 - F(b(s)) \end{aligned} \tag{25}$$

where $F(x)$ with $F(0) = 0$ is the cumulative distribution function of the type B of arriving agents.

Proof. Since $p_1(s) \geq 0$ and $p_2(s) \geq 0$ it holds for both cases described in Lemma 20 that $0 \leq a(s) \leq \frac{p_2(s)}{v(2)} \leq b(s)$. Also recall the tiebreak rule we have assumed in Section 3 from which it follows that if for some arriving agent j it holds that $u_j(m) = u_j(n)$ with $0 \leq m < n \leq 2$ (i.e. the expected utilities of buying m or n identical items are equal for the considered agent) that then the agent will prefer to buy m items above buying n items. Let x denote the realization of type B of an arbitrary agent j arriving at time s. We have to distinguish three cases for the type x of the arriving agent.

First consider the case that $x \leq a(s)$. Then for this agent the utility of buying one item is $u_j(1) = x(v(1)) - p_1(s) \leq a(s)v(1) - p_1(s) \leq \frac{p_1(s)}{v(1)}v(1) - p_1(s) \leq 0$ and similarly the utility of buying both items is $u_j(2) = x(v(2)) - p_2(s) \leq a(s)v(2) - p_2(s) \leq \frac{p_2(s)}{v(2)}v(2) - p_2(s) = 0$. Thus $u_j(1) \leq 0$ and $u_j(2) \leq 0$ and it follows that the arriving agent j will choose to not buy any item if $x \leq a(s)$.

Second consider the case that $a(s) < x \leq b(s)$. Then it follows from Lemma 20 that $x > a(s) = \frac{p_1(s)}{v(1)}$ and $x \leq b(s) = \frac{p_2(s)-p_1(s)}{v(2)-v(1)}$. Hence $u_j(1) = x(v(1)) - p_1(s) > a(s)v(1) - p_1(s) = 0$ and

$$\begin{aligned} u_j(2) - u_j(1) &= x(v(2) - v(1)) - p_2(s) + p_1(s) \\ &\leq b(s)(v(2) - v(1)) - p_2(s) + p_1(s) \\ &= \tfrac{p_2(s)-p_1(s)}{v(2)-v(1)}(v(2) - v(1)) - p_2(s) + p_1(s) \\ &= 0 \end{aligned}$$

Thus $u_j(1) \geq u_j(2)$ and $u_j(1) > 0$ and it follows that the arriving agent j chooses to buy one item for price $p_1(s) \geq 0$.

Finally consider the case that $a(s) \leq b(s) < x$. Then $u_j(2) = x(v(2)) - p_2(s) > \frac{p_2(s)}{v(2)} v(2) - p_2(s) = 0$ and

$$\begin{aligned} u_j(2) - u_j(1) &= x(v(2) - v(1)) - p_2(s) + p_1(s) \\ &> b(s)(v(2) - v(1)) - p_2(s) + p_1(s) \\ &\geq \tfrac{p_2(s) - p_1(s)}{v(2) - v(1)}(v(2) - v(1)) - p_2(s) + p_1(s) \\ &= 0 \end{aligned}$$

Thus $u_j(2) > u_j(1)$ and $u_j(2) > 0$ and it follows that the arriving agent j chooses to buy the bundle of both item for price $p_2(s) \geq 0$.

Now we have determined the choice of an arriving agent in all possible cases for the type x of an arriving agent. We conclude that an arriving agent buys zero items if and only if the type $x \leq a(s)$ from which it follows that $g_{(2,p_1(s),p_2(s))}(0) = F(a(s))$. We also conclude that an arriving agent of type x buys one item for price $p_1(s)$ if and only if $a(s) < x \leq b(s)$ from which it follows that $g_{(2,p_1(s),p_2(s))}(1) = F(b(s)) - F(a(s))$. Finally we conclude that an arriving agent of type x buys both item for bundle price $p_2(s)$ if and only if $x > b(s)$ from which it follows that $g_{(2,p_1(s),p_2(s))}(2) = 1 - F(b(s))$. ■

Next the auxiliary variables $a(s)$ and $b(s)$ are used to define the two functions $H_t(s)$ and $G_t(s)$ such that revenue maximization problem for two identical items can be expressed in the Euler-Lagrange form (22).

Definition 22 *Assuming that $0 < v(1) < v(2)$ and let $a(s)$ and $b(s)$ defined by Definition 19 we define for $t \in [0, T]$ the functions*

$$H_t(s) := \int_t^s \lambda(z)(1 - F(a(z)))dz \text{ for } s \in [t, T],$$

$$G_t(s) := \int_t^s \lambda(z)(F(b(z)) - F(a(z)))dz \text{ for } s \in [t, T].$$

From this definition is obvious that both $H_t(s)$ and $G_t(s)$ are non-decreasing differentiable functions on the interval $[t, T]$. Moreover,

$$H_t'(s) = \lambda(s)(1 - F(a(s))) \text{ for } s \in [t, T], \tag{26}$$

$$G_t'(s) = \lambda(s)(F(b(s)) - F(a(s))) \text{ for } s \in [t, T]. \tag{27}$$

Then it follows by combining (25),(26) and (27) that

$$\begin{aligned} g_{(2,p_1(s),p_2(s))}(0) &= 1 - \tfrac{H_t'(s)}{\lambda(s)} \\ g_{(2,p_1(s),p_2(s))}(1) &= \tfrac{G_t'(s)}{\lambda(s)} \\ g_{(2,p_1(s),p_2(s))}(2) &= \tfrac{H_t'(s) - G_t'(s)}{\lambda(s)} \end{aligned} \tag{28}$$

Remark 23 Note that given the two price functions for one and both items for any $s > t$ the number of agents arriving in (t, s) who will buy at least one item is a Poisson distributed random variable. Moreover, from the definition of $H_t(s)$ it is easily seen that $H_t(s)$ is the expected number of agents arriving in (t, s) who will buy at least one item. Therefore it follows that for all $s > t$,

$$e^{-H_t(s)}$$

is the probability that at time s both identical items are still unsold given the price functions set by the seller.

For the maximal revenue $R(2, t)$ from selling two identical items it follows from (20), Definition 22 and (28) that

$$R(2,t) = \max \int_t^T [(p_1(s) + R(1,s))G_t'(s) + p_2(s)(H_t'(s) - G_t'(s))]e^{-H_t(s)}ds. \tag{29}$$

Equation (29) is transformed into an appropriate Euler-Lagrange functional equation as (22) if the prices $p_1(s)$ and $p_2(s)$ are expressed in terms of $H_t'(s)$ and $G_t'(s)$. However, it should be noted that the two cases we have established in Lemma 20 will give different expressions of $p_1(s)$ and $p_2(s)$ in terms of $H_t'(s)$ and $G_t'(s)$ and therefore also different systems of differential equations will result from the variational principle of Euler-Lagrange.

5.3 Rewriting price functions

We continue the $k = 2$ case where $p_1(s)$ and $p_2(s)$ should be expressed in terms of $H_t'(s)$ and $G_t'(s)$ and recall Lemma 20 for the two cases which have to be distinguished.

We first consider the case that $a(s) = b(s) = \frac{p_2(s)}{v(2)}$ which is easier to analyze than the other case. Namely in this case it follows immediately that $G_t'(s) = 0$ and then $p_1(s)$ is eliminated from (29) determining $R(2, t)$. So, in this case we only have to rewrite $p_2(s)$ and this is easily done by noting that $p_2(s) = v(2)a(s) = v(2)F^{-1}(1 - \frac{H_t'(s)}{\lambda(s)})$. The maximal expected revenue under the restriction that the prices $p_1(s)$ and $p_2(s)$ are such that this first case is valid will be denoted by $R_1(2, t)$ and we have obtained the following result.

Lemma 24 *If for $s \in [t, T]$ the prices $p_1(s)$ and $p_2(s)$ are such that $a(s) = b(s) = \frac{p_2(s)}{v(2)}$ then the maximal expected revenue $R_1(2, t)$ satisfies the Euler-Lagrange functional equation*

$$R_1(2,t) = v(2) \max \int_t^T F^{-1}(1 - \frac{H_t'(s)}{\lambda(s)})H_t'(s)e^{-H_t(s)}ds, \tag{30}$$

where $H_t(s)$ is defined by Definition 22.

We note that (30) is in fact exactly the same Euler-Lagrange functional equation as was encountered for revenue maximization for one item (see the proof of Claim 1 in Gershkov and Moldovanu (2009)) except that the value $v(1)$ of the single item is now replaced with $v(2)$, the price of the bundle of two identical items. By this fact it is straightforward to obtain exact solutions for the optimal price functions in this case.

Now consider the more difficult case of rewriting the price functions when $\frac{p_2(s)}{v(2)} \geq \frac{p_1(s)}{v(1)}$ which according to Lemma 20 implies that $a(s) = \frac{p_1(s)}{v(1)}$ and $b(s) = \frac{p_2(s)-p_1(s)}{v(2)-v(1)}$. Then it follows that

$$p_1(s) = v(1)a(s) = v(1)F^{-1}(1 - \frac{H'(s)}{\lambda(s)}) \tag{31}$$

and

$$\begin{aligned} p_2(s) &= p_1(s) + (v(2) - v(1))b(s) \\ &= v(1)F^{-1}(1 - \tfrac{H'(s)}{\lambda(s)}) + (v(2) - v(1))F^{-1}(1 - \tfrac{H'(s)-G'(s)}{\lambda(s)}) \end{aligned} \tag{32}$$

The expressions (31) and (32) for the prices can be substituted in (29) to obtain an Euler-Lagrange functional equation in proper form. It seems untractable to obtain an exact solution of this functional equation, but a numerical solution of the corresponding system of differential equations can be obtained if all parameters are given.

6 The Unit Demand Case of Heterogeneous Items

In this section we discuss the case of n heterogeneous items in which every buyer demands at most one item, i.e., the unit demand case. All agents rank these items in the same way in a sense that these items can be ordered by their qualities $0 \leq q_n \leq q_{n-1} \leq \cdots \leq q_1$. These qualities are shared by all agents. However, each agent j has her private information, her type x_j. So the value of item with quality q_i to agent j will be $x_j q_i$. Here q_i and x_j correspond to $v(\{i\})$ and B_j, respectively, as previously described in Section 2. This case has been extensively studied in the literature and is a special case of the general model studied in the current paper.

Gershkov and Moldovanu (2009) have studied this case in detail. We will briefly review their model and some of their basic results. In their model, potential buyers arrive according to an inhomogeneous Poisson process until a given deadline as described in our current paper. Items for sale can be heterogeneous having different qualities q_i as described above. Each arriving buyer j declares his type x_j at the moment of arrival. Depending on this type at most one of the available items at the moment will be assigned to the buyer, who will pay the price of the item. In general, a buyer of higher type will obtain an item of higher quality. Because of the nature of unit demand, it is possible to derive an explicit pricing formula as shown by Gershkov and Moldovanu (2009).

Following the notation in Gershkov and Moldovanu (2009), the set of items available at t is denoted by Π_t and the cardinality of Π_t by k_t. Let $Q_t : [0,\infty) \times 2^{\Pi_0} \to \Pi_0 \cup \emptyset$ denote a non-randomized Markovian allocation policy for time t and $P_t : [0,\infty) \times 2^{\Pi_0} \to \mathbb{R}$ the associated payment rule. Recall that by ordering items to quality q_i is the i-th highest quality of the k_t items in the set Π_t.

Proposition 1 in Gershkov and Moldovanu (2009) shows that a nonrandomized Markovian allocation policy is implementable if and only if it is based on a partition of the agents' type space. In other words a price is set for each item and an arriving agent can choose (at most) one item for the set price. It follows that if the type of an agent is in a certain interval, then the agent will choose the item corresponding to that interval or the agent will not buy any item if his type is below the interval corresponding to the item of lowest quality. Moreover an interval with higher bounds corresponds to an item of higher quality and there are no gaps between the intervals.

Their Proposition 1 says that if the items are of different quality (i.e. $q_i \neq q_k$ for any $q_i, q_k \in \Pi_t$, $i \neq k$) that then there exist $k_t + 1$ functions $\infty = y_{0,\Pi_t}(t) \geq y_{1,\Pi_t}(t) \geq y_{2,\Pi_t}(t) \geq \ldots \geq y_{k_t,\Pi_t}(t) \geq 0$, such that the arriving agents type $x \in [y_{i,\Pi_t}(t), y_{i-1,\Pi_t}(t)) \Rightarrow Q_t(x,\Pi_t) = q_i$ and $x < y_{k_t,\Pi_t}(t) \Rightarrow Q_t(x,\Pi_t) = \emptyset$. In other words if $x \in [y_{i,\Pi_t}(t), y_{i-1,\Pi_t}(t)]$ then an agent of type x will be allocated the item of i-th highest quality. The associated payment scheme for the item of i-th highest quality at time t is given by

$$P_t(x,\Pi_t) = \sum_{m=i}^{k_t} (q_m - q_{m+1}) y_{m,\Pi_t}(t) + S(t)$$

where $S(t)$ is some allocation- and type-independent function.

Observe that this payment scheme has an intuitive interpretation: The payment for the item with the i-th highest quality represents the externality imposed by the agent of type x on k_t dummy agents with types $y_{1,\Pi_t}(t)$, $y_{2,\Pi_t}(t)$, $\ldots$, $y_{k_t,\Pi_t}(t)$ in the corresponding efficient allocation.

The functions $y_{i,\Pi_t}(t)$ are called *cutoff curves*. To describe a Markovian nonrandomized policy, we need for each subset of available items to give a number of cutoff curves equal to the cardinality of the subset. There will be $n2^{n-1}$ cutoff curves when there are n items available. A key result in Gershkov and Moldovanu (2009) states that the dynamic revenue maximizing policy can be described by only n cutoff curves. Moreover, these cutoff curves are independent of the qualities of the available items but depend on the number of available items. Acccording to Gershkov and Moldovanu (2009) these cutoff curves can be derived as follows. Let $R(1_k, t)$ denote the expected revenue at time t to be obtained in the unit-demand case from the optimal cutoff policy if k identical items of quality $q = 1$ are still available at that moment. Let 1_k denote a set of 1's of cardinality k. To simplify the equations we write in the remainder of this section simply $y_i(t)$ instead of $y_{i,\Pi_t}(t)$ for the cutoff curves.

Assume only one item of quality $q = 1$ is available. Then the optimal cutoff curve $y_1(t)$ is obtained from the differential equation

$$y_1(t) - \frac{1 - F(y_1(t))}{f(y_1(t))} = \lambda \int_t^T \frac{[1 - F(y_1(s))]^2}{f(y_1(s))} ds = R(1_1, t).$$

Notice that we have already seen this differential equation in (14) although the notation is slightly adjusted now for the unit-demand case. Namely in case there is only one item available the function $y(1, t)$ as used in (14) and the cutoff function $y_1(t)$ of the current section have the same meaning. After the functions $y_1(t)$ and $R(1_1, t)$ for one available item have been obtained the i-th optimal cutoff curve $y_i(t)$, $1 \leq i \leq k_t$ for the general unit-demand case with k_t available items, can be recursively obtained from the differential equation

$$y_i(t) - \frac{1 - F(y_i(t))}{f(y_i(t))} + R(1_{i-1}, t) = R(1_i, t),$$

with

$$R(1_k, t) = \lambda \int_t^T \frac{[1 - F(y_k(s))]^2}{f(y_k(s))} ds \text{ for } k = 1, 2, \ldots, k_t.$$

Since the virtual valuations $y_i(t) - \frac{1-F(y_i(t))}{f(y_i(t))}$ should be positive to sell items it follows from the differential equation that $R(1_i, t) > R(1_k, t)$ for any $i > k \geq 0$ and $t < T$. In case of n items of different quality the same cutoff curves as for identical items can be used to set optimal prices. Notice that the prices of remaining items increase after each sale. Namely, if one of k available items is sold, then the optimal allocation policy for the remaining $k - 1$ items will be based on the $k - 1$ pre-sale highest cutoff curves.

7 Conclusion

The literature has focused on an important case of multiple heterogenous commonly ranked items in which every agent consumes *at most one item*, i.e., the unit demand case. In this paper we have made a step toward the study of a more general, more practical, and equally important case of multiple heterogenous commonly ranked items in which every agent may acquire *several items*. Our model covers a wider class of practical revenue management problems arising from airlines, trains, buses, hotels, and many other leisure and travel industries. In our model, a number of heterogeneous indivisible items will be sold to randomly arriving buyers before a given deadline. Each buyer may consume more than one item. They arrive sequentially and randomly and have their own private valuations over those items. They rank all bundles of items in the same fashion but differ in values involving their types. Buyers make their decision on what to buy immediately on the moment of arrival. The seller tries to maximize her expected revenue over the given period of time by setting a price for each bundle of items at each time. We have formulated this as a dynamic, finite horizon, continuous-time incentive compatible revenue maximization problem under random and sequential demand and characterized optimal allocation policies and derived various properties.

We hope that the analysis developed here will prove to be a useful and necessary basis for the study of more practical allocation problems with random and sequential demand and private information and incentive. A challenging open problem is how to tackle the case when items are not commonly ranked.

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