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Abstract: We examine the implementation of reduced-form allocation rules in mechanism design problems. To handle the problem, we adopt a conic approach which uses a lift-and-project method to construct a projection cone and find its finite generators. This results in a set of implementable reduced forms for implementation. We then characterize projection cones for several typical mechanism design problems including single-item auctions, bilateral trade, compromise, and multiple-item auctions with group capacity constraints. We find that the implementation condition in general has a linear characterization by a class of sign functions, which is larger and richer than the well-known class of characteristic functions found by Border. These results admit meaningful economic interpretations.

Keywords: Implementation, Reduced-form rules, Auction, Bilateral trade, Mechanism design, Total unimodularity.

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1 Introduction

We consider the following implementation problem in the context of mechanism design: Under what conditions can a system of interim expected allocation rules (or reduced forms) be generated by some physically feasible allocation rule? Concerning the problem, several important results have been obtained. Regarding the classic auction design model of Myerson (1981), Maskin and Riley (1984) were the first to highlight the implementation problem and Matthews (1984) proposed an intuitive necessary condition and conjectured that it is also sufficient. Border (1991) proved the conjecture through a geometric approach. The result is now well-known as Border's theorem. Che et al. (2013) developed a network flow approach to the implementation problem in multi-unit auctions with homogeneous units and ceiling and floor constraints on agents. They derived the characterization of reduced-form auctions by constructing an appropriate network and using the existence conditions from the network flow literature for a feasible flow in the constructed network.

Alternative characterizations of Border's theorem have been proposed. Border (2007), Mierendorff (2011), and Che et al. (2018) provided characterizations for single-item auctions with asymmetric buyers. Hart and Reny (2015) presented a characterization by a majorization condition. Goeree and Kushnir (2016) extended the geometric approach to mechanism design problems with interdependent values. Gershkov et al. (2013) examined the equivalence of Bayesian and dominant strategy implementation and observed that some reduced form implementation problem can be viewed as a so-called "marginal problem" in probability theory studied by Strassen (1965) and Gutmann et al. (1991).

In general mechanism design problems, a social alternative may influence more than one player. For example in the bilateral trade problem of Myerson and Sattterthwaite (1983), the sale of an item gives the seller a revenue and the buyer a utility value. In other problems, each player may be interested in more than one social alternative. For instance, in the auctions with externalities of Jehiel et al. (1999), selling a nuclear weapon to one buyer could impose a negative externality on a losing buyer. In the two-player three-alternative compromise problem of Börgers and Postl (2009), two players have opposite preferences over three alternatives. Moving from the compromise outcome to one player's best alternative will make the other player worse off. In this paper, we investigate the implementation of reduced-form allocations in a large class of problems covering several well-known problems.

Our approach to the implementation is rooted in polyhedral combinatorics, a branch of discrete mathematics. In polyhedral combinatorics, most problems tend to have several alternative formulations, some of which are easier to handle than others. The lift-and-project procedure provides a connection between different formulations (Balas, 2001). By lifting, the combinatorial object of interest is first formulated by a linear system in some higher-dimensional space. Then by projecting away these additional variables, we obtain the linear system of interest. As the projection relies on a projection cone, we also refer to this procedure as a conic approach. This approach has become a powerful method to study linear characterizations for combinatorial objects (Balas and Pulleyblank, 1983; Balas, 2001). In mechanism design literature, Vohra (2013) first observed that the lift-and-project method can be used to study reduced-form auctions. He suggested this method without offering an explicit characterization. In this paper, we will further explore this conic approach in a variety of general environments and characterize the implementability condition.

We first introduce the lift-and-project procedure to a general implementation problem. The procedure starts with a linear system containing both feasible allocation rules and reduced forms in a higher-dimensional space. By projecting away the variables of allocation rules, we obtain a linear system of feasible reduced forms. A projection cone is introduced and we prove that the implementability condition is obtained by finding its finite generators. To obtain an explicit characterization of finite generators, we study geometric and combinatorial properties of the projection cone and its conic constraint matrix. With these properties, we characterize the projection cone and finite generators in the single-unit auction problem of Myerson (1981), the bilateral trade problem of Myerson and Satterthwaite (1983), the compromise problem of Börgers and Postl (2009), and the multi-unit auction of Che et al. (2013). Underlying the characterization of each implementation of these problems is *total unimodularity*, a fundamental feature of a class of well-behaved matrices extensively studied in discrete and combinatorial mathematics; see Schrijiver (1986, 2004) and Nemhauser and Wolsey (1988). These results suggest that the conic approach provides a general method for investigating different classes of implementation problems. Studying the projection cones of different problems allows us to classify different kinds of mechanism design problems by the underlying structure of reduced form implementation.

While Border's theorem (1991) suggests a linear characterization of the implementability condition by the coefficients from $\{0, +1\}$ (or by the characteristic functions of individual type subsets), by the conic approach we find that for a general implementation problem, the implementability condition contains the coefficients from $\{0, \pm 1\}$, which is strictly larger than the class found in Border (1991) and Che et al. (2013). This class of coefficients can be represented by the sign functions of individual type subsets. To see an economic interpretation of this result, we compare a single-unit auction with a bilateral trade. In an auction problem, an increase in the expected probability of winning of one buyer will decrease the expected probability of winning of another buyer. "+1" in the implementability condition reflects competition among the players. In a bargaining problem, an increase in the expected probability of trade for the seller also raises the expected probability of trade for the buyer. "-1" in the implementability condition represents cooperation between the players.

Our results also give new insights into the network flow approach in Che et al. (2013). To deal with a multi-unit auction with group capacity constraints, Che et al. (2013) formulated their problem as a digraph network flow problem. By a generalized max-flow min-cut theorem, the cut condition is necessary and sufficient for a feasible flow, and hence it also gives rise to the implementability condition. Unlike other models, in their model all goods are assumed to be perfectly divisible. We need to recast their model in an appropriate way to fit our framework. In each of these problems, the constraint matrix is totally unimodular, which allows the construction of a digraph incidence matrix and a flow formulation. The conic approach provides an alternative way for examining the problem.

The rest of this paper is organized as follows. Section 2 introduces an implementation problem. Section 3 investigates a necessary condition for the implementability. Section 4 presents the conic approach and the main theorems. Section 5 provides characterizations of projection cones in several typical design problems. Section 6 discusses the model of Che et al. (2013) which needs to be treated separately due to its special nature. Section 7 concludes.

2 The Problem

The implementation problem under consideration consists of several players who choose from multiple social alternatives. Let N be a finite set of players, $n := |N| \geq 2$. Let a_0 be a status quo and $A = \{a_1, ..., a_m\}, m \geq 1$, be a finite set of social alternatives other than a_0, and let $A \cup \{a_0\}$ be the set of all social alternatives. For each player $i \in N$, let $A_i \subseteq A$ be a set of player-interested social alternatives (possibly empty). Denote $|A_i| = m_i$. Every player $i \in N$ is associated with a finite set T_i of types with $|T_i| = \tau_i$. We define $T = \prod_{i \in N} T_i$ and $T_{-i} = \prod_{j \neq i} T_j$. Denote $|T| = \tau$. Let λ be a probability measure on T, as the common prior of players. Assume $\lambda(t) > 0$ for each $t \in T$ and $\sum_{t \in T} \lambda(t) = 1$. For each $i \in N$, let

λ_i be player i's marginal probabilities and write $\lambda_i(t_i)$ for each $t_i \in T_i$. W.l.o.g, we assume λ is statistically independent, i.e. $\lambda(t) = \prod_{i\in N} \lambda_i(t_i)$. Denote $\lambda_{-i}(t_{-i}) = \prod_{j\neq i} \lambda_j(t_j)$ for $t_{-i} \in T_{-i}$. In this paper we focus on the case where both social alternatives and types are discrete and finite, as this case is very common and practical in real life applications.

An **implementation environment** is given by

$$S = (N, A \cup \{a_0\}, (A_i)_{i\in N}, (T_i)_{i\in N}, \lambda).$$

Let $\mathscr{S}$ be the set of all implementation environments. We say $S \in \mathscr{S}$ is of **full-alternative** if $m_i = m$ for all $i \in N$, and S is of **partial-alternative** if $m_i < m$ for some $i \in N$. Let us first briefly review the following basic implementation environments but discuss them and other models in detail later:

(1) Myerson (1981): In his celebrated auction model, a seller (player 0) wishes to sell an indivisible object to a group of buyers $\{1, ..., m\}$. The set of alternatives $\{a_0, a_1, ..., a_m\}$ contains all potential winners of the object (including the seller). When there are no allocative externalities among the buyers, we have for each buyer i, $A_i = \{a_i\}$ and $A_0 = \{\emptyset\}$.

(2) Myerson and Satterthwaite (1983): In their well-known bilateral trade problem, seller 0 has one indivisible object for sale to buyer 1. There are two alternatives, a_0 and a_1, which correspond to no trade and trade, respectively. As trade incurs a cost for the seller and yields a value for the buyer, we have $A_0 = A_1 = \{a_1\}$.

(3) Bögers and Postl (2009): In a two-person compromise problem, there are two players 1 and 2 and three alternatives, a_0, a_1, and a_2. The players have opposite preferences: each alternative a_i is the best alternative for player i and the worst alternative for player j, while alternative a_0 is a compromise. As selecting alternative i imposes a negative externality on player j, we have $A_1 = A_2 = \{a_1, a_2\}$.

The auction problem in Myerson (1981) is an example of partial-alternative problems. The bilateral trade and the compromise are examples of full-alternative problems.

Let $S \in \mathscr{S}$. An (ex post) feasible allocation rule assigns to each type profile a lottery over social alternatives. Formally, an **ex post allocation rule** $q : A \times T \to \mathbb{R}$ **is feasible** if

$$q \geq 0 \text{ and } \sum_{a\in A} q(a, t) \leq 1, \text{ for all } t \in T. \tag{2.1}$$

The component $q(a, t)$ denotes the probability that alternative $a \in A$ is chosen by the society given type profile $t \in T$. Denote $d := m \times \tau$ and q is a vector in $\mathbb{R}^d$.

An ex post feasible allocation rule q induces an interim allocation rule $Q = (Q_i)_{i\in N}$ where $Q_i : A_i \times T_i \to \mathbb{R}$ represents player i's interim expected allocation given his type.

For each $i \in N$, $t_i \in T_i$, and $a_i \in A_i$, we have

$$Q_i(a_i, t_i) := \sum_{t_{-i} \in T_{-i}} q(a_i, t)\lambda_{-i}(t_{-i}). \tag{2.2}$$

We then say Q is the **reduced form** of q and q **implements** Q. Denote $l := \sum_{i\in N} m_i \times \tau_i$ and Q is a vector in $\mathbb{R}^l$.

Conversely, one could begin with an arbitrary interim allocation rule and ask whether it can be implemented by an ex post feasible allocation rule.

DEFINITION 1: An interim allocation rule $Q = (Q_i)_{i\in N}$ is **implementable** if there exists an ex post allocation rule q such that (q, Q) satisfies both (2.1) and (2.2).

Let $\mathcal{A}$ denote the set of feasible allocation rules given by (2.1) and let $\mathcal{Q}$ denote the set of implementable interim allocation rules. For a given environment S, $\mathcal{A}$ is known while $\mathcal{Q}$ is unknown. It is easy to see that $\mathcal{Q}$ is a polyhedron in $\mathbb{R}^l$ and it can be described in half-space representation:

$$\mathcal{Q} = \{Q \in \mathbb{R}^l : MQ \leq b\},$$

for some $M \in \mathbb{R}^{k\times l}$ and $b \in \mathbb{R}^k$. The **implementation problem** S is to find a system of (M, b) that defines $\mathcal{Q}$.

3 A Necessary Condition

The difficulty in obtaining a characterization for the set $\mathcal{Q}$ is that there is no combinatorial property for the set, and hence characterizing its defining linear inequalities is extremely difficult. A key to circumvent this problem is a change of variables for both q and Q.

For every q and Q, we define x and y by

$$x(a, t) := q(a, t)\lambda(t), \tag{3.1}$$

$$y_i(a_i, t_i) := Q_i(a_i, t_i)\lambda_i(t_i). \tag{3.2}$$

The vector x is an ex post allocation rule q weighted by the prior probabilities and the vector y is an interim allocation rule Q weighted by the marginal probabilities. Intuitively, x and y are ex ante variables for q and Q.

Assume $\lambda(t) > 0$ for all $t \in T$, the feasibility for q in (2.1) is reformulated by the feasibility for x:

$$x \geq 0 \text{ and } \sum_{a\in A} x(a, t) \leq \lambda(t), \text{ for all } t \in T. \tag{3.3}$$

The equations for Q implemented by q in (2.2) can be reformulated by: for each $i \in N$, $t_i \in T_i$, and $a_i \in A_i$,

$$y_i(a_i, t_i) = \sum_{t_{-i} \in T_{-i}} x(a_i, t). \tag{3.4}$$

It is clear that Q is implemented by q if and only if y is implemented by x, or (x, y) satisfies (3.3) and (3.4).

We define B an $l \times d$ matrix that takes each x to its reduced form y given by (3.4). Let $\mathcal{A}^* \subset \mathbb{R}^d$ denote the set of feasible weighted allocation rules given by (3.3). Let $\mathcal{Q}^* \subset \mathbb{R}^l$ denote the set of implementable weighted interim allocation rules by

$$\mathcal{Q}^* = \{Bx : x \in \mathcal{A}^*\}. \tag{3.5}$$

The following Proposition 1 is based on a half-space characterization for the implementability, using the geometric approach in Border (1991). As the set of implementable weighted reduced forms is a convex and compact subset of a Euclidean space, a standard separation argument implies that this set is the intersection of *all* supporting half-spaces containing it. Hence, a necessary condition is that this set is contained in the intersection of *some* finitely many supporting half-spaces containing it.

PROPOSITION 1: *Let $S \in \mathscr{S}$. If $y \in \mathbb{R}^l$ is implementable, then*

$$f^\top y \le \sup\{f^\top Bx : x \in \mathcal{A}^*\}, \ \textit{for all } f \in \{0, \pm 1\}^l. \tag{H}$$

Proof. Notice that $\mathcal{Q}^* \subset \mathbb{R}^l$ is a polyhedron since $\mathcal{A}^*$ is a polyhedron and B is linear. A standard hyperplane separation theorem implies that $\mathcal{Q}^*$ is the intersection of all supporting half-spaces containing it. Hence it is contained in the intersection of a subset of these supporting half-spaces whose normal vectors lie in $\{0, \pm 1\}^l$. □

As we will show in Section 5, condition (H) in Proposition 1 is not only necessary but also sufficient for several implementation problems, including Myerson (1981) and Myerson and Satterthwaite (1983).

The necessary condition in Proposition 1 allows for a more concrete expression. For each $i \in N$, $E_i \subseteq T_i$, define its characteristic function by $\chi_{E_i}(t_i) = 1$ if $t_i \in E_i$ and 0 otherwise. For each $E_i^+, E_i^- \subseteq T_i$, and $E_i^+ \cap E_i^- = \emptyset$, define a sign function

$$f_{E_i^+, E_i^-} = \chi_{E_i^+} - \chi_{E_i^-}, \tag{3.6}$$

that is, the difference of two characteristic functions for two non-overlapping type subsets of player i. Let $E^+_{i,a_i}, E^-_{i,a_i} \subseteq T_i$, $E^+_{i,a_i} \cap E^-_{i,a_i} = \emptyset$, for all $a_i \in A_i$, and define $f_{\boldsymbol{E}} = (f_{E^+_{i,a_i},E^-_{i,a_i}})_{i\in N, a_i \in A_i}$. Then, the necessary condition for the implementability requires the linear inequalities in (H) to hold for all $f_{\boldsymbol{E}}$ in this class.

To illustrate condition (H), we consider a problem of allocating complementary goods. A seller has two homogeneous indivisible goods for sale and there are two buyers, who may value two goods as complementary. The seller can either sell both goods to one buyer, or split two goods between the buyers, or withhold both goods. Miralles (2012) considered a similar problem but assumed a buyer to have additive valuations for the goods. It can be seen that if valuations are non-additive, Border's condition cannot be applied separately to each good. In this case, condition (H) is necessary for the implementability.

Example 1: Let S be such that $N = \{1, 2\}$, $A \cup \{a_0\} = \{a_0, a_1, a_2, a_3\}$, $A_1 = \{a_1, a_3\}$, $A_2 = \{a_2, a_3\}$. If $y \in \mathbb{R}^l_+$ is implementable, then for all $T^0_i, T^+_i, T^-_i \subseteq T_i$, $T^+_i \cap T^-_i = \emptyset$, $i \in N$,

$$\begin{aligned}\sum_{i\in N}\sum_{t_i\in T^0_i} y_i(a_i, t_i) &+ \sum_{i\in N}\sum_{t_i\in T^+_i} y_i(a_3, t_i) - \sum_{i\in N}\sum_{t_i\in T^-_i} y_i(a_3, t_i) \\ &\le \lambda(E_0) + \lambda(E_1) + \lambda(E_2) - \lambda\left(E_0 \bigcap (E_1 \bigcup E_2)\right), \end{aligned} \tag{3.7}$$

where

$$E_0 = \bigcup_{i\in N} (T^0_i \times T_{-i}) \text{ and } E_i = T^+_i \times (T^-_{-i})^c, i \in N.$$

As we will show in the Appendix 1, this multi-item auction problem nests a two-buyer single-item auction in Myerson (1981) and a bilateral trade of Myerson and Satterthwaite (1983). The implementability condition in this larger problem requires more linear inequalities than pooling all inequalities from the smaller nested problems. The models of Myerson (1981) and Myerson and Satterthwaite (1983) will be discussed in detail in Section 5.

4 A Conic Approach

In polyhedral combinatorics, a very useful method is lifting-and-projection (Balas, 2001; Vohra, 2013). A combinatorial object of interest is first described using a linear system in some higher-dimensional space, or lifting. One is then interested in the linear system obtained by projecting away these additional variables. Because the projection needs to construct a projection cone, we call this procedure *a conic approach.*

For our problem, our goal is to obtain a linear system of interim allocation rules. To achieve this, the lift-and-project procedure starts with a linear system in terms of both ex post and interim allocation rules. By projecting away the variables of ex post allocation rules, we obtain a linear system of interim allocation rules. In this section, we use this idea to obtain a conic characterization for the implementability (Theorem 1) and a necessary condition in conic form (Proposition 2). We also provide a condition on an implementation problem such that the necessary condition in Proposition 1 or 2 is necessary and sufficient for the implementability (Theorem 2).

4.1 Roadmap

Before introducing the conic approach, we need to recall several basic mathematical concepts. See Schrijver (1986) and Nemhauser and Wolsey (1988) for further details. Let $x_1, ..., x_k \in \mathbb{R}^q$ be given vectors. A linear combination $\alpha_1 x_1 + ... + \alpha_k x_k$ is **conic** if $\alpha_1, ..., \alpha_k \geq 0$. The convex cone generated by a subset $X \subseteq \mathbb{R}^q$ is the set of all conic combinations of the elements from X.

$$\text{cone}(X) = \{\alpha_1 x_1 + ... + \alpha_k x_k : x_1, ..., x_k \in X, \alpha_1, ..., \alpha_k \geq 0\}. \tag{4.1}$$

A convex cone P is **finitely generated** if there exists a finite set $\hat{P} \subset P$ such that $P = \text{cone}(\hat{P})$. A generating set $\hat{P}$ is **minimal** if there exists no $\hat{P}' \subset \hat{P}$ such that P is finitely generated by $\hat{P}'$. A convex cone P is **polyhedral** if $P = \{x \in \mathbb{R}^q : Mx \leq 0\}$ for some $M \in \mathbb{R}^{p \times q}$. By the Weyl-Minkowski theorem, P is polyhedral if and only if P is finitely generated.

We present a general idea of the conic approach. Let $X \subset \mathbb{R}^d$ be a given polyhedron and let $L \in \mathbb{R}^{l \times d}$ be a given matrix. Let Y denote the image set of X under L. Our problem is to characterize the linear inequalities defining Y. To obtain a linear characterization, we introduce a lifted polyhedron $Z = \{(x, y) \in \mathbb{R}^{d+l} : x \in X, y = Lx\}$. The orthogonal projection of Z onto y-space is Y. In order to project away x-variables, we can use theory of alternatives to construct a polyhedral cone P such that a set of finite generators of P gives rise to the linear inequalities defining Y. In this way, finding the linear inequalities that define Y amounts to finding finite generators of P.

4.2 Two Matrices

We now apply the conic approach to an implementation problem $S \in \mathscr{S}$. To proceed, we reformulate the problem in matrix form. We focus on two matrices that are important for our analysis.

Recall that B is an $l \times d$ matrix that represents $x \mapsto y$ in (3.4). B is an incidence matrix where each row is indexed by (i, a_i, t_i),[4] each column is indexed by (a, t), and the entry in row (i, a_i, t_i) and column (a', t') is 1 if $a_i = a'$ and $t_i = t'_i$ and 0 otherwise. That is,

$$B = \left(\begin{array}{cccccccccccc} 1 & \cdots & 1 & & & & & & & & & \\ & & & 1 & \cdots & 1 & & & & & & \\ & & & & & & 1 & \cdots & 1 & & & \\ 1 & & & 1 & & & & \cdots & & & & \\ & & & & & & & & & 1 & \cdots & 1 \\ & \cdots & & & & & & & & & \cdots & \end{array} \right)_{l \times d}.$$

Observe that column (a, t) contains k $1s$, $k \geq 1$, only if alternative a is interested by exactly k players. For illustration, suppose there are only two players i and j. If $a \in A_i \cap A_j$, then column (a, t) contains two 1s, one in row (i, a, t_i) and one in row (j, a, t_j). If $a \in A_i \setminus A_j$, then column (a, t) contains one 1 in row (i, a, t_i).

The equations in (3.4) are rewritten by

$$y = Bx. \tag{4.2}$$

We define C a $\tau \times d$ matrix that represents $x \mapsto \sum_{a \in A} x(a, \cdot)$ in (3.3). Then C is an incidence matrix where each row is indexed by t, each column is indexed by (a, t), and the entry in row t and column (a', t') is 1 if $t = t'$ and 0 otherwise. Note that C is the concatenation of m copies of $\tau \times \tau$ identity matrix given by

$$C = \begin{pmatrix} 1 & & & 1 & & & & 1 & & \\ & \ddots & & & \ddots & & \cdots & & \ddots & \\ & & 1 & & & 1 & & & & 1 \end{pmatrix}_{\tau \times d}.$$

The inequalities in (3.3) are rewritten by

$$x \geq 0 \text{ and } Cx \leq \lambda. \tag{4.3}$$

[4]Here (i, a_i, t_i) should be listed according to the lexicographic order. For instance, if $A_1 = \{a_{i_1}, a_{i_2}\}$ with $i_1 < i_2$ and $T_1 = \{t_{j_1}, t_{j_2}\}$ with $j_1 < j_2$, then we list $(1, a_1, t_1)$ as $(1, a_{i_1}, t_{j_1})$, $(1, a_{i_1}, t_{j_2})$, $(1, a_{i_2}, t_{j_1})$, and $(1, a_{i_2}, t_{j_2})$. The other indices follow the same rule.

The **lifted polyhedron** of feasible allocation rule-reduced forms is given by

$$\mathcal{Z} = \{\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) \in \mathbb{R}^{d+l} : Cx \leq \lambda, x \geq 0, y = Bx\}. \tag{4.4}$$

By construction, the projection of $\mathcal{Z}$ onto the subspace of y-variables is given by

$$\mathcal{Q}^* = \{y \in \mathbb{R}^l : \text{ there exists } x \in \mathbb{R}^d \text{ such that } \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) \in \mathcal{Z}\}. \tag{4.5}$$

We wish to obtain a linear system whose solution set is $\mathcal{Q}^*$.

4.3 Theorems

For $S \in \mathscr{S}$, define the **projection cone** by

$$\mathcal{C} = \{\left(\begin{smallmatrix} f \\ g \end{smallmatrix}\right) \in \mathbb{R}^{l+\tau} : -B^\top f + C^\top g \geq 0, g \geq 0\}. \tag{4.6}$$

THEOREM 1: *(Conic Representation) Let $S \in \mathscr{S}$ and let $\hat{\mathcal{C}}$ be any set of finite generators of the projection cone $\mathcal{C}$. $y \in \mathbb{R}^l$ is implementable if and only if*

$$f^\top y \leq g^\top \lambda, \text{ for all } \left(\begin{smallmatrix} f \\ g \end{smallmatrix}\right) \in \hat{\mathcal{C}}. \tag{4.7}$$

Proof. (Only If) Suppose $(f, g) \in \mathcal{C}$ and $y \in \mathcal{Q}^*$. Let $(x, y) \in \mathcal{Z}$. Then $-f^\top B + g^\top C \geq 0$, $g \geq 0$, $y = Bx$, $Cx \leq \lambda$, and $x \geq 0$ imply $f^\top y \leq f^\top Bx - g^\top Cx + g^\top \lambda \leq g^\top \lambda$. Hence y satisfies (4.7). (If) Suppose $y \notin \mathcal{Q}^*$. There exists no $x \geq 0$ such that $Bx = y$ and $Cx \leq \lambda$. By Farka's Lemma (see Schrijver 1986, p.89), there exists $\left(\begin{smallmatrix} f \\ g \end{smallmatrix}\right) \in \mathbb{R}^{l+\tau}$ such that $-B^\top f + C^\top g \geq 0$, $g \geq 0$, and $-y^\top f + \lambda^\top g < 0$. But then $\left(\begin{smallmatrix} f \\ g \end{smallmatrix}\right) \in \mathcal{C}$. There must be $\left(\begin{smallmatrix} f \\ g \end{smallmatrix}\right) \in \hat{\mathcal{C}}$ such that $-y^\top f + \lambda^\top g < 0$, contradicting the assumption that y satisfies (4.7). $\square$

Notice that the condition given by (4.7) may contain redundant inequalities, even when the latter come only from the elements of $\hat{\mathcal{C}}$. That is, a generator of $\mathcal{C}$ does not necessarily give rise to a defining inequality of $\mathcal{Q}^*$.

By Theorem 1, to find the implementability condition, we only need to find a set of finite generators of the projection cone $\mathcal{C}$. Motzkin et al. (1953) introduced a simple and useful algorithm for finding a set of finite generators of a general polyhedral cone; see also Fukuda and Prodon (1995).

To obtain a clear-cut characterization of a set of finite generators, we investigate the structure of the projection cone. Rewrite the projection cone $\mathcal{C}$ and its conic constraint matrix M^* by

$$\mathcal{C} = \left\{ \begin{pmatrix} f \\ g \end{pmatrix} : M^* \begin{pmatrix} f \\ g \end{pmatrix} \geq 0 \right\}, \quad M^* = \begin{pmatrix} -B^\top & C^\top \\ O & I \end{pmatrix}. \tag{4.8}$$

A matrix is **combinatorial** if it contains only entries from $\{0, \pm 1\}$. Similarly, a vector is combinatorial if each coordinate is -1, 0, or $+1$. A matrix M is **totally unimodular** (TUM) if every nonsingular square submatrix has determinant equal to either -1 or $+1$. Note that a TUM matrix is combinatorial but a combinatorial matrix is not necessarily TUM.

The following Proposition 2 is a corollary of Theorem 1. As the intersection of combinatorial vectors and the projection cone is always a subset of some system of finite generators, the system of linear inequalities given by these vectors is a necessary condition for the implementability. Theorem 2 below provides a condition on an implementation problem such that we can obtain an explicit characterization of finite generators and the necessary condition in Proposition 1 or 2 is also sufficient for the implementability.

Proposition 2: *Let $S \in \mathscr{S}$. If $y \in \mathbb{R}^l$ is implementable, then*

$$f^\top y \leq g^\top \lambda, \quad \textit{for all} \quad \left(\begin{smallmatrix} f \\ g \end{smallmatrix}\right) \in \mathcal{C} \cap \{0, \pm 1\}^l \times \{0, +1\}^\tau. \tag{C}$$

Theorem 2: *(A sufficient condition) Let $S \in \mathscr{S}$. If $\mathcal{C}$ is finitely generated by a set of combinatorial vectors, then $y \in \mathbb{R}^l$ is implementable if and only if (i) condition (C) or (ii) condition (H) holds.*

Proof. Define $\hat{\mathcal{C}}^* = \mathcal{C} \cap \{0, \pm 1\}^l \times \{0, +1\}^\tau$. (i) The only if part is implied by Proposition 2. Conversely, if $\mathcal{C}$ is finitely generated by a set of combinatorial vectors $\hat{\mathcal{C}}$, then $\hat{\mathcal{C}} \subseteq \hat{\mathcal{C}}^*$ and $\hat{\mathcal{C}}^*$ is also a set of finite generators. By Theorem 1, the claim holds. (ii) The only if part is implied by Proposition 1. We prove that the if part holds. Suppose y satisfies condition (H) and let $\hat{\mathcal{C}}^*$ be defined as above. By Theorem 1, $\sup\{f^\top \hat{y} : \hat{y} \in \mathcal{Q}^*\} \leq g^\top \lambda$ for all $\left(\begin{smallmatrix} f \\ g \end{smallmatrix}\right) \in \hat{\mathcal{C}}^*$. Note that $\{f \in \mathbb{R}^l :$ there exists $g \in \mathbb{R}^\tau$ such that $\left(\begin{smallmatrix} f \\ g \end{smallmatrix}\right) \in \hat{\mathcal{C}}^*\} \subset \{0, \pm 1\}^l$. By condition (H) , it implies $f^\top y \leq g^\top \lambda$ for all $\left(\begin{smallmatrix} f \\ g \end{smallmatrix}\right) \in \hat{\mathcal{C}}^*$. By Theorem 1, y is implementable. $\square$

An advantage of the conic representation (C) over the half-space representation (H) is that condition (C) does not need to compute the value of the supporting function at each normal vector and it is easy to check $\mathcal{C} \cap \{0, \pm 1\}^l \times \{0, +1\}^\tau$.

5 Characterization of Projection Cones

By Theorem 2, if the projection cone of a problem is finitely generated by combinatorial vectors, then the implementability condition is given by condition (C) or (H). This sufficient condition in Theorem 2 is, however, not directly stated on a problem's primitives. In this section, we provide a useful condition on a problem's primitives for the sufficient condition in Theorem 2 to hold: the constraint matrix of the projection cone is totally unimodular (Lemmas 1 and 2). This condition holds for many problems, including Myerson (1981), Myerson and Satterthwaite (1983), and Börgers and Postl (2009). We then use this condition to characterize the projection cones of these problems. We also obtain an alternative characterization of Border's theorem.

5.1 Basics

Let $P = \{x \in \mathbb{R}^q : Mx \leq 0\}$ be a polyhedral cone. Let

$$\text{lin}(P) = \{x \in \mathbb{R}^q : Mx = 0\}. \tag{5.1}$$

We call $\text{lin}(P)$ the **lineality space** of P. Let $s \geq 0$ denote the dimension of the linearity space. If $\text{lin}(P)$ contains only the zero vector, then P is a **pointed** cone. Otherwise, we say P is non-pointed. Geometrically, a pointed cone contains the zero vector as its unique vertex while a non-pointed cone contains the linearity space as an s-dimensional subspace.

Let I be the index set of the inequalities in P. Let $I^=(x)$ be the index set in I for which the corresponding inequalities hold as equations (or active constraints) at $x \in P$. We say $J \subset I$ is a **maximal proper subset** of I, if $J = I^=(x)$ for some $x \in P$ and there exists no $z \in P$ such that $J \subset I^=(z) \subset I$. Let $\mathcal{J}$ be all maximal proper subsets J of I. For each $J \in \mathcal{J}$, let $r(J)$ consist of all $x \in P$ such that $J = I^=(x)$. The **extreme elements** of P are the members of $r(J)$, for any $J \in \mathcal{J}$. For a pointed cone P, the sets $r(J)$ each consist of all positive multiples of a single member of P, i.e. $\{\alpha x : \alpha > 0\}$ for some $x \in P$. These sets are called the **extreme rays** of P.

For a polyhedral cone P, we can find a minimal set of (finite) generators $\hat{P}$ by the following procedure:

- Let $P^=$ be a basis of $\text{lin}(P)$.
- Let P^+ consist of one member of $r(J)$ for each $J \in \mathcal{J}$.

- Every member of P can be expressed as a linear combination of members of $P^=$ plus a nonnegative linear combination of members of P^+. Let $\hat{P} = P^= \cup -P^= \cup P^+$.

For a pointed cone P, the set of extreme rays provides a unique (up to positive scaling) minimal set of generators, i.e. P is the conical hull of its extreme rays. However, if P is non-pointed, a minimal set of generators is not unique: if $w \in P^=$ and $x \in r(J)$ for some $J \in \mathcal{J}$, then $w + x \in r(J)$. For a non-pointed cone, there are different ways of construction or decomposition, and each will find different sets of generators. We will mainly use one construction by writing a non-pointed cone into a union of finitely many pointed cones (Lemma 2). We will also discuss a canonical decomposition of a non-pointed cone into the Minkowski sum of the linearity space and its projection onto the orthogonal complement in the Appendix 3.

Before stating our main lemmas, we provide a useful rank condition that gives the rank of the submatrix defined by active constraints at an extreme element of a cone. Let M_J be the corresponding submatrix for $J = I^=(x)$, and let $\mathrm{rk} M_J$ denote the rank of M_J. Then $\mathrm{rk} M_J = q - s - 1$ if $x \in r(J)$ for some $J \in \mathcal{J}$. In particular, for a pointed cone, $\mathrm{rk} M_J = q - 1$ if x is on an extreme ray $r(J)$, i.e. an extreme ray is an intersection of $q - 1$ linearly independent active constraints.

5.2 Lemmas

The following lemma 1 gives a sufficient condition for a pointed cone being generated by combinatorial vectors. The lemma is developed from and slightly more general than Lemma 3.1 in Hoffman (1976), where the result there is proved for a certain class of pointed cones.

Lemma 1: *Let $P = \{x \in \mathbb{R}^q : Mx \leq 0\}$ be a polyhedral cone and M totally unimodular. If P is pointed, then every extreme ray contains a combinatorial vector.*

Proof. Let P be pointed. Assume that $z \in P$ is on an extreme ray $r(J)$, $J \in \mathcal{J}$. Since $\mathrm{rk} M_J = q - 1$, there is a submatrix M' with $q - 1$ linearly independent rows in M_J. Since M is TUM, M' is also TUM. We need to show that if M' with columns $M'_1, ..., M'_q$ has rank $q - 1$, then $M'z = 0$ implies that all nonzero coordinates of z are either α or $-\alpha$, for some $\alpha > 0$. The idea is based on Lemma 3.1 of Hoffman (1976). Note that for any $z_j \neq 0$,

$$M'_j = \sum_{i=1, i \neq j}^{q} \frac{z_i}{z_j} M'_i, \tag{5.2}$$

and M'_i, $i \neq j$ are linearly independent. The linear system has a unique solution $(\frac{z_i}{z_j})$, $i \neq j$.

Since M' is TUM, each $\frac{z_i}{z_j}$ is integer. As this argument applies to every nonzero entry in z, for any nonzero $z_j, z_{j'}$ and $j \neq j'$, $\frac{z_{j'}}{z_j}$ and $\frac{z_j}{z_{j'}}$ are integers and hence $|z_j| = |z_{j'}| = \alpha$ for some $\alpha > 0$. Therefore every extreme ray of P contains a combinatorial vector. $\square$

The following lemma 2 obtains a result similar to Lemma 1 for a non-pointed cone. The idea is to write a non-pointed cone into a union of finitely many pointed cones. To do this properly, we take the intersection of the cone and each closed orthant (i.e. $x \geq 0$), which turns out to be a cone and is pointed. Lemma 1 then applies.

LEMMA 2: *Let $P = \{x \in \mathbb{R}^q : Mx \leq 0\}$ be a polyhedral cone and M totally unimodular. If P is non-pointed, then P is finitely generated by a set of combinatorial vectors.*

Proof. Let P be non-pointed. We write P as a union of finitely many pointed cones. Notice that the Euclidean space $\mathbb{R}^q$ is a union of $k := 2^q$ closed orthants O_i, $i = 1, ..., k$. Rewrite $O_i = \{x \in \mathbb{R}^q : E_i x \leq 0\}$ with $E_i \in \{0, \pm 1\}^{q \times q}$ a diagonal matrix. For each i, $M_i := \binom{M}{E_i}$ and $P_i := P \bigcap O_i = \{x \in \mathbb{R}^q : M_i x \leq 0\}$, with P_i possibly empty. Then $P = \bigcup_{i=1}^k P_i$.

We need to show that (i) P_i is a pointed cone and M_i is TUM, for each i, and hence by Lemma 1, P_i is finitely generated by combinatorial vectors $r_{i1}, ..., r_{i\eta_i}$. (ii) If for each i, $P_i = \text{cone}(\{r_{i1}, ..., r_{i\eta_i}\})$, then $P = \text{cone}(\bigcup_{i=1}^k \{r_{i1}, ..., r_{i\eta_i}\})$. Combining (i) and (ii) completes the proof. To verify claim (i), notice that by construction, each P_i is a pointed cone. Since M is TUM and M_i is obtained from M by adding rows with at most one non-zero entries, M_i is also TUM. To see claim (ii), suppose $x \in P$. Then $x \in P_i$ for at least one i. Hence $x \in \text{cone}(\bigcup_{i=1}^k \{r_{i1}, ..., r_{i\eta_i}\})$ and $P \subseteq \text{cone}(\bigcup_{i=1}^k \{r_{i1}, ..., r_{i\eta_i}\})$. Conversely, $\text{cone}(\bigcup_{i=1}^k \{r_{i1}, ..., r_{i\eta_i}\}) \subseteq P$ as P is a cone and each $r_{ij} \in P_i \subset P$. $\square$

It is worth pointing out that while total unimodularity is sufficient for a cone generated by combinatorial vectors, it is not necessary. For example, the constraint matrix of the following cone is not totally unimodular, but its extreme rays are combinatorial.

$$P = \left\{x \in \mathbb{R}^2 : x_1 + x_2 \leq 0, -x_1 + x_2 \leq 0\right\}. \tag{5.3}$$

The above two lemmas will be used in the following sections.

5.3 Auction

We first discuss an application to the classic auction design problem of Myerson (1981). The following Lemma 3 shows that the projection cone in his problem satisfies both pointedness and total unimodularity.

Lemma 3: *Let $S \in \mathscr{S}$ in Myerson (1981). Then (i) $\mathcal{C}$ is pointed. (ii) M^* is totally unimodular.*

Proof. (i) $M^* \binom{f}{g} = 0$ implies that for the last τ rows of M^*, we have $(O, I)\binom{f}{g} = 0$ and hence $g = 0$. But then in the first d rows of M^*, for any row indexed by (a_i, t), $-f_i(a_i, t_i) + g(t_i, t_{-i}) = 0$ implies $f_i(a_i, t_i) = 0$ and hence $f = 0$.

(ii) As TUM is preserved by transposing or by deleting columns with at most one nonzero entries, we only need to show that $\binom{B}{C}$ is TUM. Notice that either B or C has one nonzero entry in each column. The rows in $\binom{B}{C}$ can be partitioned into two classes, R_1 for the rows in B and R_2 for rows in C, such that for each column, the first 1 is in R_1 and the second 1 is in R_2. By the Heller-Tompkins theorem in the Appendix 2 (see also Schrijver 1986, p.276), $\binom{B}{C}$ is TUM. □

By Lemma 3, Lemma 1 and Theorem 2, condition (C) gives the implementability condition in Myerson (1981). By further removing some redundant inequalities, the following proposition provides a conic characterization of a well-known result of Border (1991).

Proposition 3: *Let $S \in \mathscr{S}$ in Myerson (1981). Then $y \in \mathbb{R}^l$ is implementable if and only if $y \geq 0$ and $f^\top y \leq g^\top \lambda$, for all $\binom{f}{g} \in \mathcal{C} \cap \{0, +1\}^{l+\tau}$.*

Proof. We will show that if $z = (f, g)$ is a combinatorial vector on an extreme ray $r(J)$ and if for some i, a_i, t_i, $f_i(a_i, t_i) = -1$, then all the other coordinates in z must be 0. Note that for all $t_{-i} \in T_{-i}$, $g(t_i, t_{-i}) \geq 0$ implies $-f_i(a_i, t_i) + g(t_i, t_{-i}) > 0$. Then in M^*, the rows indexed by (a_i, t_i, t_{-i}) are not in J. Define $\tilde{z} = (\tilde{f}, g)$ by $\tilde{f}_i(a_i, t_i) = 0$ and $\tilde{f} = f$ otherwise. Then $M^*_J \tilde{z} = 0$, i.e., replacing z by $\tilde{z}$ does not affect the equalities for the rows in J but it may produce new equalities for the rows (a_i, t_i, t_{-i}), i.e. $J \subseteq \tilde{J} = I^=(\tilde{z})$. Since J is maximal, either $\tilde{J} = J$ or $\tilde{J} = I$. In the first case, $\tilde{z} = \alpha z$ for some $\alpha > 0$, yielding a contradiction. Hence, $\tilde{J} = I$ and $M^* \tilde{z} = 0$. Since P is pointed, $\tilde{z} = 0$. This implies $z = (0, ..., 0, -1, 0, ..., 0)$, which further implies $y_i(a_i, t_i) \geq 0$. This completes the proof. □

To facilitate a better understanding of the above discussion we provide an illustrative example.

Example 2: (Myerson, 1981) Let $N = \{0, 1, 2\}$, $A \cup \{a_0\} = \{a_0, a_1, a_2\}$, $A_0 = \{\emptyset\}$, $A_1 = \{a_1\}$, $A_2 = \{a_2\}$, $T_1 = \{t_{11}, t_{12}\}, T_2 = \{t_{21}, t_{22}\}$. The constraint matrix of the projection cone is given by

$$
M^{*\top} = \begin{pmatrix}
-1 & -1 & & & & & & & & & & \\
 & & -1 & -1 & & & & & & & & \\
 & & & & -1 & & -1 & & & & & \\
 & & & & & -1 & & -1 & & & & \\
1 & & & & 1 & & & & 1 & & & \\
 & 1 & & & & 1 & & & & 1 & & \\
 & & 1 & & & & 1 & & & & 1 & \\
 & & & 1 & & & & 1 & & & & 1
\end{pmatrix}.
$$

$M^* \left(\begin{smallmatrix} f \\ g \end{smallmatrix} \right) = 0$ implies $f = g = 0$ and $\text{lin}(\mathcal{C}) = \{0\}$. We find that $\mathcal{C}$ is generated by a set of combinatorial vectors:

$$
\begin{aligned}
\mathcal{C}^+ = \{&(-1,0,0,0,0,0,0,0), && (0,-1,0,0,0,0,0,0),\\
&(0,0,-1,0,0,0,0,0), && (0,0,0,-1,0,0,0,0),\\
&(1,0,1,0,1,1,1,0), && (0,1,1,0,1,0,1,1),\\
&(1,0,0,1,1,1,0,1), && (0,1,0,1,0,1,1,1),\\
&(1,1,1,1,1,1,1,1)\}.
\end{aligned}
$$

Denote the index set of the columns of $M^{*\top}$ by $I = \{1, ..., 12\}$. For each $x \in \mathcal{C}^+$, the corresponding maximal proper subset $J = I^=(x)$ satisfies $\text{rk}M_J^{*\top} = 7$ and $J = I \setminus K$, where K is set as $\{1,2\}$, $\{3,4\}$, $\{5,7\}$, $\{6,8\}$, $\{3,6,9,10,11\}$, $\{1,8,9,11,12\}$, $\{4,5,9,10,12\}$, $\{2,7,10,11,12\}$, $\{9,10,11,12\}$. $\mathcal{C}^+$ corresponds to the extreme rays of $\mathcal{C}$ and is the unique minimal set of generators (up to positive scaling).

5.4 Bilateral Trade

In their classic paper Myerson and Satterthwaite (1983) considered a bilateral trade problem. Different from Myerson (1981), the seller now has a private cost and trade will influence the payoffs of both the seller and the buyer. We will characterize the projection cone and the implementability condition for this problem. Lemma 4 proves that the constraint matrix of the projection cone is TUM. In contrast to the auction problem of Myerson (1981) whose projection cone is *pointed* (Lemma 3), Lemma 4 shows that the projection cone of a bilateral trade is *non-pointed.*

LEMMA 4: *Let $S \in \mathscr{S}$ in Myerson and Satterthwaite (1983). Then (i) $\mathcal{C}$ is non-pointed. (ii) M^* is totally unimodular.*

Proof. (i) We prove that $\text{lin}(\mathcal{C}) \neq \{0\}$ by showing $\text{rk}M^* < l + \tau$. Since $\text{rk}M^* = \text{rk}B^\top + \tau$, we only need to show that $\text{rk}B^\top < l = \tau_0 + \tau_1$. That is, the columns of $B^\top$, denoted by

$\beta(i, a_1, t_i)$, are linearly dependent. First notice that for any reduced form y implemented by x,

$$\sum_{t \in T} x(a_1, t) = \sum_{t_0 \in T_0} y_0(a_1, t_0) = \sum_{t_1 \in T_1} y_1(a_1, t_1). \tag{5.4}$$

Let $c_{0,a_1,t_0} = 1$ for all $t_0 \in T_0$ and $c_{1,a_1,t_1} = -1$ for all $t_1 \in T_1$, we find a nontrivial linear combination $c = (c_{i,a_1,t_i})$ of the columns of $B^\top$ satisfying

$$\sum_{i \in \{0,1\}} \sum_{t_i \in T_i} c_{i,a_1,t_i} \beta(i, a_1, t_i) = 0. \tag{5.5}$$

(ii) We only need to verify that $\left(\begin{smallmatrix} B \\ C \end{smallmatrix}\right)$ is TUM. Since each row in C has only one nonzero entry and TUM is preserved by deleting a row with at most one nonzero entry, these rows can be deleted. It remains to show that B is TUM. Notice that the rows of B can be partitioned into two classes, R_0 for seller 0 and R_1 for buyer 1, such that for each column, the first 1 is in R_0 and the second 1 is in R_1. By the Heller-Tompkins theorem, B is TUM. $\square$

We combine Lemma 4, Lemma 2 and Theorem 2 to obtain a conic characterization of the implementability condition in the bilateral trade.

PROPOSITION 4: *Let $S \in \mathscr{S}$ in Myerson and Satterthwaite (1983). $y \in \mathbb{R}^l$ is implementable if and only if $f^\top y \leq g^\top \lambda$, for all $\left(\begin{smallmatrix} f \\ g \end{smallmatrix}\right) \in \mathcal{C} \cap \{0, \pm 1\}^l \times \{0, +1\}^\tau$.*

In a totally different context, Strassen (1965) and Gutmann et al. (1991) studied a so-called marginal problem in probability theory which is in essence identical to the implementation problem in a general bilateral trade model. Gershkov et al. (2013) were the first to reveal this link.

We provide some economic interpretation of the implementability conditions, by comparing an auction with a bilateral trade. In Border's theorem, only coefficients "+1" appear in the linear inequalities. In contrast, the implementability condition for a bilateral trade problem requires coefficients "−1" in the linear inequalities. The interpretation for this result is intuitive: In Myerson (1981), selling to buyer 1 with a higher expected probability tightens the probabilistic budget for selling to buyer 2. The buyers are competing for the expected probabilities of winning. In Myerson and Satterthwaite (1983), however, selecting trade with a higher expected probability for the seller relaxes the probabilistic budget that trade can be selected for the buyer. The players have common interests at this alternative.

Example 3: (Myerson and Satterthwaite, 1983) Let $N = \{0,1\}$, $A \cup \{a_0\} = \{a_0, a_1\}$, $A_0 = A_1 = \{a_1\}$, $T_0 = \{t_{01}, t_{02}\}, T_1 = \{t_{11}, t_{12}\}$. The constraint matrix of the projection cone is given by

$$M^{*\top} = \begin{pmatrix} -1 & -1 & & & & & & \\ & & -1 & -1 & & & & \\ -1 & & -1 & & & & & \\ & -1 & & -1 & & & & \\ 1 & & & & 1 & & & \\ & 1 & & & & 1 & & \\ & & 1 & & & & 1 & \\ & & & 1 & & & & 1 \end{pmatrix}.$$

Then $M^* \binom{f}{g} = 0$ implies $f_0(a_1, t_{01}) = f_0(a_1, t_{02})$, $f_1(a_1, t_{11}) = f_1(a_1, t_{12})$, $f_0(a_1, t_{01}) = -f_1(a_1, t_{11})$, $g = 0$. We find the 1-dimensional linearity space $\mathrm{lin}(\mathcal{C}) = \{\alpha(1, 1, -1, -1, 0, 0, 0, 0) : \alpha \in \mathbb{R}\}$. $\mathcal{C}$ is generated by a set of combinatorial vectors:

$$\begin{aligned} \mathcal{C}^= \cup -\mathcal{C}^= = \{&(1,1,-1,-1,0,0,0,0), && (-1,-1,1,1,0,0,0,0)\}, \\ &\textit{and} \\ \mathcal{C}^+ = \{&(-1,0,0,0,0,0,0,0), && (0,-1,0,0,0,0,0,0), \\ &(0,0,-1,0,0,0,0,0), && (0,0,0,-1,0,0,0,0), \\ &(1,0,0,0,1,1,0,0), && (0,1,0,0,0,0,1,1), \\ &(0,0,1,0,1,0,1,0), && (0,0,0,1,0,1,0,1), \\ &(1,0,-1,0,0,1,0,0), && (1,0,0,-1,1,0,0,0), \\ &(0,1,-1,0,0,0,0,1), && (0,1,0,-1,0,0,1,0)\}. \end{aligned}$$

Let $I = \{1, ..., 8\}$ be the index set of the columns of $M^{*\top}$. For each $x \in \mathcal{C}^+$, the corresponding maximal proper subset $J = I^=(x)$ satisfies $\mathrm{rk} M_J^{*\top} = 6$ and $J = I \setminus K$, where K is set as $\{1,2\}$, $\{3,4\}$, $\{1,3\}$, $\{2,4\}$, $\{5,6\}$, $\{7,8\}$, $\{5,7\}$, $\{6,8\}$, $\{3,6\}$, $\{4,5\}$, $\{1,8\}$, $\{2,7\}$. Since $\mathcal{C}$ is non-pointed, a minimal set of generators is not unique (up to positive scaling).

5.5 Compromise

Börgers and Postl (2009) studied a compromise problem with two players and three alternatives and without money. The players have opposite preferences: the best alternative to one player is the worst alternative to the other player. More complicated than the bilateral trade, each player here can be influenced by more than one alternative. We will examine the projection cone in this setting by Lemma 5. We find that the projection cone is non-pointed, similar to a bilateral trade. Characterizing the combinatorial properties of the projection cone is more difficult. We prove that for 2×2 type sets, the constraint matrix of the projection cone is TUM.

LEMMA 5: *Let $S \in \mathscr{S}$ in Börgers and Postl (2009). Then (i) $\mathcal{C}$ is non-pointed. (ii) M^* is totally unimodular if $\tau_1 = \tau_2 = 2$.*

Proof. (i) We will show that $\operatorname{lin}(\mathcal{C}) \neq \{0\}$. As shown in Lemma 4, for each $a = a_1, a_2$, the columns of $B^\top$, $\beta(i, a, t_i)$, are linearly dependent. Hence $\mathrm{rk}B^\top < l$ and $\mathrm{rk}M^* < l + \tau$. (ii) For $\tau_1 = \tau_2 = 2$, let

$$B = \begin{pmatrix} 1 & 1 & & & & & & \\ & & 1 & 1 & & & & \\ & & & & 1 & 1 & & \\ & & & & & & 1 & 1 \\ 1 & & 1 & & & & & \\ & 1 & & 1 & & & & \\ & & & & 1 & & 1 & \\ & & & & & 1 & & 1 \end{pmatrix}, C = \begin{pmatrix} 1 & & & & 1 & & & \\ & 1 & & & & 1 & & \\ & & 1 & & & & 1 & \\ & & & 1 & & & & 1 \end{pmatrix}.$$

Denote the index set of the columns of $\left(\begin{smallmatrix} B \\ C \end{smallmatrix}\right)$ by $I = \{1, ..., 12\}$. Partition I into $\Omega_1 = \{1, 4, 6, 7\}$ and $\Omega_2 = \{2, 3, 5, 8\}$. For any subset $\Omega \subseteq I$, let

$$\Omega_1' = \Omega \bigcap \Omega_1 \text{ and } \Omega_2' = \Omega \bigcap \Omega_2.$$

It can be seen that the two 1s in each row either belong to the different sets Ω_1' and Ω_2', or at least one of the two 1s belongs to neither of Ω_1' and Ω_2'. By the Ghouila-Houri theorem in the Appendix 2 (see also Schrijver 1986, p.269), $\left(\begin{smallmatrix} B \\ C \end{smallmatrix}\right)$ is TUM. $\square$

In the above we show that when the type set of each player contains only two elements, the constraint matrix of the projection cone is TUM. However, this will be no longer true when type sets become large. We will leave the discussion to the Appendix 2.

6 Capacity Constraints

In contrast to the models discussed in the previous section which typically consider the problem of allocating an indivisible good, Che et al. (2013) investigated a problem of allocating a perfectly divisible good via reduced-form auction. In their model there are several units of the divisible good to be distributed among a group N of players. Every subset of players can receive at most a certain amount of the good but must also receive at least a certain amount of the good. Each player has a finite set of types. The entire type set is denoted by T.

An ex post allocation rule $q : N \times T \to \mathbb{R}$ assigns $q_i(t)$ units of the good to player i at type profile $t \in T$. For a constraint set $G \subseteq N$, the players in G will receive at least $L(G, t)$ units of the good and at most $U(G, t)$ units of the good. Note that we allow the bounds to be type-contingent. Let $\mathcal{G} \subseteq 2^N \setminus \{\emptyset\}$ be a collection of constraint sets, we call $\mathcal{G}$ a constraint structure. We say an ex post allocation rule q is feasible if for each $G \in \mathcal{G}$ and $t \in T$,

$$L(G, t) \leq \sum_{i \in G} q_i(t) \leq U(G, t). \tag{6.1}$$

An ex post feasible allocation rule q induces an interim allocation rule $Q = (Q_i)_{i \in N}$ where $Q_i : T_i \to \mathbb{R}$ represents player i's interim expected assigned units given his type. For each $i \in N$ and $t_i \in T_i$,

$$Q_i(t_i) := \sum_{t_{-i} \in T_{-i}} q_i(t)\lambda_{-i}(t_{-i}). \tag{6.2}$$

By change of variables, we define x and y by $x_i(t) := q_i(t)\lambda(t)$, $y_i(t_i) := Q_i(t_i)\lambda_i(t_i)$, and define $U^\lambda(G, t) := U(G, t)\lambda(t)$ and L^λ similarly. Then, conditions (6.1) and (6.2) can be rewritten by

$$L^\lambda(G, t) \leq \sum_{i \in G} x_i(t) \leq U^\lambda(G, t), \tag{6.3}$$

$$y_i(t_i) = \sum_{t_{-i} \in T_{-i}} x_i(t). \tag{6.4}$$

We define matrices B and C as follows. Denote $d := n \times \tau$ and $l := \sum_{i \in N} \tau_i$. Matrix B is an $l \times d$ matrix that represents $x \mapsto y$ in (6.4). Then B is an incidence matrix where each row is indexed by (i, t_i), each column is indexed by (i, t), and the entry in row (i, t_i) and column (i', t') is 1 if $i = i'$ and $t_i = t'_i$ and 0 otherwise. Matrix C is a $(|\mathcal{G}| \times \tau) \times d$ matrix that represents $x \mapsto \sum_{i \in G} x_i$ in (6.3). Then C is an incidence matrix where each row is indexed by (G, t), each column is indexed by (i, t), and the entry in row (G, t) and column (i', t') is 1 if $i' \in G$ and $t' = t$ and 0 otherwise.

The lifted polyhedron of feasible allocation rule-reduced forms is given by

$$\mathcal{Z} = \{\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) \in \mathbb{R}^{d+l} : L^\lambda \leq Cx \leq U^\lambda, y = Bx\}. \tag{6.5}$$

We obtain a counterpart of Theorem 1 below.

PROPOSITION 5: *Let S be in Che et al. (2013) and let $\hat{\mathcal{C}}$ be any finite set of generators of the projection cone*

$$\mathcal{C} = \{\begin{pmatrix} f \\ g \\ h \end{pmatrix} \in \mathbb{R}^{l+2|\mathcal{G}|\times\tau} : -B^\top f + C^\top g - C^\top h = 0, g \geq 0, h \geq 0\}. \tag{6.6}$$

Then $y \in \mathbb{R}^l$ is implementable if and only if

$$f^\top y \leq g^\top U^\lambda - h^\top L^\lambda, \text{ for all } \begin{pmatrix} f \\ g \\ h \end{pmatrix} \in \hat{\mathcal{C}}. \tag{6.7}$$

COROLLARY 1: *Let S and S' be two problems in Che et al. (2013), with quotas (L, U) and (L', U'). Then $\mathcal{C} = \mathcal{C}'$, and the implementability conditions for S and S' are the same except replacing (L, U) with (L', U') in (6.7).*

Corollary 1 suggests that the projection cone of a problem does not depend on the quotas (L, U), which implies that the same sets of finite generators arise if two problems differ only in quotas. Hence the conic approach is, by construction, a "universal implementation" defined by Che et al. (2013).

For a non-universal implementation where (L, U) further satisfies paramodularity, Che et al.'s network flow approach obtained a tighter characterization (their Theorem 3) and provided three examples to show that their characterization no longer holds if (L, U) are not paramodular. Corollary 1 provides additional insights to theirs. To see this, notice that the conic approach provides a complete set of finite generators. As (L, U) satisfies paramodularity, some linear inequalities given by some generators are redundant. As (L, U) fails paramodular, these inequalities are needed for a characterization.

To characterize the projection cone, we consider an important family of constraint structures for capacities. Let X be a finite ground set. A constraint structure $\mathcal{F} \subseteq 2^X \setminus \{\emptyset\}$ is a **laminar** (or a hierarchy or a tree family) if for all $F, F' \in \mathcal{F}$,

$$F \subset F', \text{ or } F' \subset F, \text{ or } F' \cap F = \emptyset. \tag{6.8}$$

Note that given a ground set X and a laminar $\mathcal{F}$, we can represent $\mathcal{F}$ using a 0-1 matrix M as follows. Define a row for each member of $\mathcal{F}$ and a column for each element of X. Set $M_{ij} = 1$ if the set corresponding to row i contains $j \in X$. We say a 0-1 matrix that arises in this way a laminar matrix.

The following lemma, due to Edmonds (1970), will be used for our purpose. The reader may refer to Theorem 41.11 in Schrijver (2004) for a proof.

LEMMA 6: *Let $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2$ be the union of two laminar families of subsets of a ground set X. Let M be the $\mathcal{F} \times X$ incidence matrix. Then M is totally unimodular.*

The conic constraint matrix M^* is given by

$$M^* = \begin{pmatrix} -B^\top & C^\top & -C^\top \\ & I & \\ & & I \end{pmatrix}. \tag{6.9}$$

We use Lemma 6 to characterize the projection cone in Che et al. (2013).

LEMMA 7: *Let S be in Che et al. (2013) and let $\mathcal{G}$ be a laminar. Then (i) $\mathcal{C}$ is pointed. (ii) M^* is totally unimodular.*

Proof. (i) $M^* \begin{pmatrix} f \\ g \\ h \end{pmatrix} = 0$ implies that for the last $2|\mathcal{G}| \times \tau$ rows of M^*, we have $\left(\begin{smallmatrix} O & I & O \\ O & O & I \end{smallmatrix}\right) \begin{pmatrix} f \\ g \\ h \end{pmatrix} = 0$, and hence $g = h = 0$. But then in the first d rows of M^*, for any row indexed by (i, t), $-f_i(t_i) + g(G, t) - h(G, t) = 0$ for $i \in G$ implies $f_i(t_i) = 0$ and hence $f = 0$.

(ii) Since TUM is preserved by deleting columns with at most one nonzero entry, or by deleting duplicated columns, or by transposing, we only need to show that $\left(\begin{smallmatrix} B \\ C \end{smallmatrix}\right)$ is TUM. We define the ground set $X = N \times T$ and show that B and C are the incidence matrices of two laminar families of subsets of X. Then, applying Lemma 6 completes the proof.

First note that B is the incidence matrix of $\mathcal{F}_1 \times X$, where $\mathcal{F}_1$ consists of the subsets $B(i, t_i) := \{(i', t') \in X : i' = i, t'_i = t_i\}$ indexed by the rows of B. We show that $\mathcal{F}_1$ is a laminar. To see this, notice that for all $(i, t_i) \neq (j, t_j)$, $B(i, t_i) \cap B(j, t_j) = \emptyset$. Since any two elements in $\mathcal{F}_1$ are disjoint, $\mathcal{F}_1$ is a laminar. Similarly, C is the incidence matrix of $\mathcal{F}_2 \times X$, where $\mathcal{F}_2$ consists of the subsets $C(G, t) := \{(i', t') \in X : i' \in G, t' = t\}$ indexed by the rows of C. We claim that $\mathcal{F}_2$ is a laminar. Consider any (G, t) and (G', t'). For all $t \neq t'$, $C(G, t) \cap C(G', t') = \emptyset$. For all $t = t'$, since $\mathcal{G}$ is a laminar, either $G \subset G'$ and $C(G, t) \subset C(G', t)$, or $G \cap G' = \emptyset$ and $C(G, t) \cap C(G', t) = \emptyset$. This proves that for every two elements of $\mathcal{F}_2$, either they are disjoint or one is a subset of another, $\mathcal{F}_2$ is a laminar. $\square$

PROPOSITION 6: *Let S be an allocation problem of Che et al. (2013) and let $\mathcal{G}$ be a laminar. Then $y \in \mathbb{R}^l$ is implementable if and only if*

$$f^\top y \leq g^\top U^\lambda - h^\top L^\lambda, \ \textit{for all} \ \begin{pmatrix} f \\ g \\ h \end{pmatrix} \in \mathcal{C} \cap \{0, \pm 1\}^l \times \{0, +1\}^{2|\mathcal{G}| \times \tau}. \tag{6.10}$$

We have considered reduced-form allocation of multiple units of one divisible good. In the remaining of this section, we will briefly discuss how to generalize our result to the case where there are several heterogeneous divisible goods and each good has multiple units, and every player may demand several units of every good.

Let N be a finite set of players and let O be a finite index set of heterogeneous divisible goods. Each player has a finite set of types. The entire type set is denoted by T. An ex post allocation rule $q : N \times O \times T \to \mathbb{R}$ assigns $q(i, a, t)$ units of the good $a \in O$ to player $i \in N$ at type profile $t \in T$. The total number of units for a set of player-good pairs $G \subseteq N \times O$ may be rationed. A typical constraint is that all the player-good pairs (i, a) in G will receive at least $L(G, t)$ units and at most $U(G, t)$ units in total, namely,

$$L(G,t) \leq \sum_{(i,a)\in G} q(i,a,t) \leq U(G,t). \tag{6.11}$$

Let $\mathcal{G} \subseteq 2^{N\times O} \setminus \{\emptyset\}$ be a constraint structure. It is clear that the environment reduces to Che et al. (2013) if we set $|O| = 1$, in which case $G \subseteq N$ is a subgroup of players.

An ex post feasible allocation rule q induces an interim allocation rule $Q = (Q_i)_{i\in N}$ where $Q_i : O \times T_i \to \mathbb{R}$ represents player i's interim expected allocation of units for the goods given his type. For each $i \in N$, $t_i \in T_i$, and $a \in O$,

$$Q_i(a, t_i) := \sum_{t_{-i}\in T_{-i}} q(i, a, t)\lambda_{-i}(t_{-i}). \tag{6.12}$$

For the above environment, we obtain the following generalization of Che et al. (2013). Since the proof is similar, we omit it.

Proposition 7: *Let S be an allocation problem with heterogeneous goods and multi-unit demand for each good, and let $\mathcal{G}$ be a laminar. Then the implementability condition similar to Proposition 6 holds.*

To illustrate Proposition 7, we provide an example. We assume each good has a supply more than the number of buyers. For each good, each buyer demands at most one unit,

$$0 \leq q(i, a, t) \leq 1. \tag{6.13}$$

Each buyer i may have a demand constraint for a subset of goods $O' \subseteq O$,

$$L(i, O', t) \leq \sum_{a \in O'} q(i, a, t) \leq U(i, O', t). \tag{6.14}$$

Also, a designated group of buyers $N' \subseteq N$ may face some lower and upper bounds on the total number of units assigned for all the goods,

$$L(N', O, t) \leq \sum_{i \in N'} \sum_{a \in O} q(i, a, t) \leq U(N', O, t). \tag{6.15}$$

7 Conclusion

In this paper, we studied a conic approach to the implementation of reduced-form allocation rules. Using a lift-and-project method, we provided a conic characterization for the implementability condition. We showed how this conic approach can be applied to various important problems including auctions, bilateral trade, compromise, and auctions with capacity constraints. In the literature there are two major methods: the geometric approach and the network flow approach. The geometric approach was first proposed by Border (1991) and developed further by Goeree and Kushnir (2016). The network flow approach was introduced by Che et al. (2013). Our conic approach is closely related to the lift-and-project method suggested by Vohra (2013), although he does not give any characterization. We demonstrate how the conic approach allows for a unified treatment of different classes of problems. For each problem, the main task is to verify the properties of its projection cone and the conic constraint matrix, i.e., totally unimodular matrices as well as other larger classes of matrices containing totally unimodular matrices. These matrices are well-studied mathematical objects and possess desirable properties. Analyzing such matrices also offers interesting criteria that can be used to classify different classes of economic problems.

The reduced form implementation has also computational implications. This has recently attracted attention of some researchers from computer science. Gopalan et al. (2015) discussed the computational complexity of Border's characterization. We leave the computational issue raised by the conic approach for future work.

Appendix 1

The two-buyer two-item allocation problem in Example 1 nests a two-buyer single-item auction problem in Myerson (1981) and a bilateral trade in Myerson and Satterthwaite (1983). If we eliminate some alternative(s) from the decision set, the problem reduces to the nested problems. Let $T_1^+ = T_1^- = T_2^+ = T_2^- = \emptyset$, condition (3.7) reduces to the implementability condition in the two-buyer single-item auction problem,

$$\sum_{i\in N}\sum_{t_i\in T_i^0} y_i(a_i,t_i) \leq \lambda\left(\bigcup_{i\in N}(T_i^0\times T_{-i})\right). \tag{A.1}$$

Let $T_1^0 = T_2^0 = \emptyset$, condition (3.7) reduces to the implementability condition in the bilateral trade problem,

$$\sum_{i\in N}\sum_{t_i\in T_i^+} y_i(a_3,t_i) - \sum_{i\in N}\sum_{t_i\in T_i^-} y_i(a_3,t_i) \leq \sum_{i\in N}\lambda\left(T_i^+\times(T_{-i}^-)^c\right). \tag{A.2}$$

It is easy to see that condition (3.7) is weakly tighter than conditions (A.1) and (A.2). This implies that the implementability condition in this larger problem requires more linear inequalities than pooling all inequalities from the smaller nested problems.

Proof for Example 1. We now derive condition (3.7). Let $f\in\{0,\pm1\}^l$. For each $t\in T$, the value of the supporting function in (H) is given by

$$\max_{q(\cdot,t)}\sum_{i\in N}\sum_{a\in A_i} f_i(a,t_i)q(a,t) = \max\{0, f_1(a_1,t_1), f_2(a_2,t_2), f_1(a_3,t_1)+f_2(a_3,t_2)\}. \tag{A.3}$$

First notice that we only need to consider f where $f_i(a_i,t_i)\geq 0$, for all $t_i\in T_i$, $i=1,2$. This is because for $y\geq 0$, replacing $f_i(a_i,t_i)=-1$ by 0 weakly increases the value of $f^\top y$ without changing the maximum value in (A.3), and hence it results in a tighter condition. Let $T_1^0, T_1^+, T_1^- \subseteq T_1, T_2^0, T_2^+, T_2^- \subseteq T_2$, where $T_1^+\bigcap T_1^- = \emptyset$, $T_2^+\bigcap T_2^- = \emptyset$, and let

$$f = (\chi_{T_1^0}, \chi_{T_2^0}, \chi_{T_1^+}-\chi_{T_1^-}, \chi_{T_2^+}-\chi_{T_2^-}). \tag{A.4}$$

For each $t\in T$, consider the point-wise maximization problem

$$\max_{q(\cdot,t)}\chi_{T_1^0}(t_1)q(a_1,t)+\chi_{T_2^0}(t_2)q(a_2,t)+(\chi_{T_1^+}(t_1)-\chi_{T_1^-}(t_1)+\chi_{T_2^+}(t_2)-\chi_{T_2^-}(t_2))q(a_3,t). \tag{A.5}$$

The solutions are provided in Table A.1. The first four columns partition the set of all type profiles into different subsets, e.g. $(1, 1, 1, 1)$ corresponds to $(T_1^0 \times T_2^0) \bigcap (T_1^+ \times T_2^+)$. The fifth column provides the point-wise solution and the last column is the point-wise maximum value of this problem. The extreme allocation rules are found by combining the point-wise solutions at all type profiles.

Case 1. $t \in E^* = T_1^+ \times T_2^+$. For such t, the probability weighted value is $2\lambda(t)$.

Case 2. $t \in E^{**}$, the intersection of $(T_1^0)^c \times (T_2^0)^c$ and $T \setminus (T_1^+ \times (T_2^-)^c) \bigcup ((T_1^-)^c \times T_2^+)$. For such t, the probability weighted value is 0.

Case 3. $t \in T \setminus (E^* \bigcup E^{**})$. For such t, the probability weighted value is $\lambda(t)$.

Compared to $\lambda(T) = 1$, where each type profile counts once, the value of the supporting function in (H) counts $t \in E^*$ twice and $t \in E^{**}$ zero times. Hence, the value of the supporting function in (H) is given by $1 + \lambda(E^*) - \lambda(E^{**})$.

Let E_0, E_1, E_2 be defined in Example 1. Then $E^* = E_1 \bigcap E_2$ and $E^{**} = E_0^c \bigcap (E_1 \bigcup E_2)^c$. The value of the supporting function in (H) is given by

$$\begin{aligned} 1 + \lambda(E^*) - \lambda(E^{**}) =& \lambda(E^*) + \lambda(E_0 \bigcup E_1 \bigcup E_2) \\ =& \lambda(E^*) + \lambda(E_0) + \lambda(E_1 \bigcup E_2) - \lambda(E_0 \bigcap (E_1 \bigcup E_2)) \\ =& \lambda(E_1 \bigcap E_2) + \lambda(E_0) + \lambda(E_1 \bigcup E_2) - \lambda(E_0 \bigcap (E_1 \bigcup E_2)) \\ =& \lambda(E_0) + \lambda(E_1) + \lambda(E_2) - \lambda(E_0 \bigcap (E_1 \bigcup E_2)). \end{aligned} \tag{A.6}$$

It can be seen that

$$\lambda(E_0) + \lambda(E_1) + \lambda(E_2) - \lambda(E_0 \bigcap (E_1 \bigcup E_2)) \leq \lambda(E_0) + \lambda(E_1) + \lambda(E_2). \tag{A.7}$$

That is, the value of the supporting function in this two-buyer two-item allocation problem is weakly lower than the bound by simply summing up the values of supporting functions $\lambda(E_0)$ in the two-buyer single-item auction problem and $\lambda(E_1) + \lambda(E_2)$ in the bilateral trade problem.

$f_1(a_1,t_1)$	$f_1(a_3,t_1)$	$f_2(a_2,t_2)$	$f_2(a_3,t_2)$	$(q(a_1,t),q(a_2,t),q(a_3,t))=(q^1,q^2,q^3)$	Value
1	1	1	1	$q^3=1$	2
1	1	1	0	$q^1+q^2+q^3=1$	1
1	1	1	-1	$q^1+q^2=1$	1
1	1	0	1	$q^3=1$	2
1	1	0	0	$q^1+q^3=1$	1
1	1	0	-1	$q^1=1$	1
1	0	1	1	$q^1+q^2+q^3=1$	1
1	0	1	0	$q^1+q^2=1$	1
1	0	1	-1	$q^1+q^2=1$	1
1	0	0	1	$q^1+q^3=1$	1
1	0	0	0	$q^1=1$	1
1	0	0	-1	$q^1=1$	1
1	-1	1	1	$q^1+q^2=1$	1
1	-1	1	0	$q^1+q^2=1$	1
1	-1	1	-1	$q^1+q^2=1$	1
1	-1	0	1	$q^1=1$	1
1	-1	0	0	$q^1=1$	1
1	-1	0	-1	$q^1=1$	1
0	1	1	1	$q^3=1$	2
0	1	1	0	$q^2+q^3=1$	1
0	1	1	-1	$q^2=1$	1
0	1	0	1	$q^3=1$	2
0	1	0	0	$q^3=1$	1
0	1	0	-1	$0\le q^1+q^2+q^3\le 1$	0
0	0	1	1	$q^2+q^3=1$	1
0	0	1	0	$q^2=1$	1
0	0	1	-1	$q^2=1$	1
0	0	0	1	$q^3=1$	1
0	0	0	0	$0\le q^1+q^2+q^3\le 1$	0
0	0	0	-1	$0\le q^1+q^2\le 1$, $q^3=0$	0
0	-1	1	1	$q^2=1$	1
0	-1	1	0	$q^2=1$	1
0	-1	1	-1	$q^2=1$	1
0	-1	0	1	$0\le q^1+q^2+q^3\le 1$	0
0	-1	0	0	$0\le q^1+q^2\le 1$, $q^3=0$	0
0	-1	0	-1	$0\le q^1+q^2\le 1$, $q^3=0$	0

Table A.1

$\square$

Appendix 2

We will apply the following characterization results for TUM matrices. Lemma 9 is a sufficient condition for TUM and is a corollary of a characterization theorem in Lemma 10. The proofs of these results can be found in Schrijver (1986) and Nemhauser and Wosley (1988).

LEMMA 8: *Total unimodularity is preserved under the following operations:*
(i) Transpose.
(ii) Permute rows or columns.
(iii) Multiply a row or column by -1.
(iv) Add or delete a row or column with at most one non-zero entry.
(v) Duplicate or delete an identical row or column.

LEMMA 9: *(Heller-Tompkins, 1956) Let* M *be a* $0, \pm 1$ *matrix with exactly two non-zero entries in each column.* M *is totally unimodular if and only if the rows of* M *can be partitioned into* R_1 *and* R_2 *such that for each column, the following holds,*

i. If the two non-zeros have the same sign, then one corresponding row is in R_1 *and one is in* R_2.

ii. If the two non-zeros have opposite signs, then either the corresponding rows are in R_1*, or both in* R_2.

LEMMA 10: *(Ghouila-Houri, 1962) Let* M *be a* $p \times q$ *matrix.* M *is totally unimodular if and only if for every subset of columns* $\Omega \subseteq \{1, ..., q\}$*, there exists a partition* Ω_1*,* Ω_2 *of* Ω *such that*

$$\left| \sum_{j \in \Omega_1} m_{ij} - \sum_{j \in \Omega_2} m_{ij} \right| \leq 1 \textit{ for } i = 1, ..., p. \tag{A.8}$$

We now show that in the compromise problem discussed in Section 5.5, if $\tau_1 \geq 3$, or $\tau_2 \geq 3$, then the constraint matrix of the projection cone is not TUM. It will be useful to compare the proof of Lemma 5 with the following one. Denote B_{τ_1,τ_2} and C_{τ_1,τ_2} the matrices B and C parameterized by (τ_1, τ_2).

Proof. It is sufficient to show that for $\tau_1 = 2, \tau_2 = 3$, $\binom{B_{2,3}}{C_{2,3}}$ is not TUM, where

$$
\begin{pmatrix} B_{2,3} \\ C_{2,3} \end{pmatrix} =
\begin{pmatrix}
1 & 1 & 1 & & & & & & & & & \\
 & & & 1 & 1 & 1 & & & & & & \\
 & & & & & & 1 & 1 & 1 & & & \\
 & & & & & & & & & 1 & 1 & 1 \\
1 & & & 1 & & & & & & & & \\
 & 1 & & & 1 & & & & & & & \\
 & & 1 & & & 1 & & & & & & \\
 & & & & & & 1 & & & 1 & & \\
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 & & & & & & & & 1 & & & 1 \\
1 & & & & & & 1 & & & & & \\
 & 1 & & & & & & 1 & & & & \\
 & & 1 & & & & & & 1 & & & \\
 & & & 1 & & & & & & 1 & & \\
 & & & & 1 & & & & & & 1 & \\
 & & & & & 1 & & & & & & 1
\end{pmatrix}.
$$

This is because any $\begin{pmatrix} B_{\tau_1,\tau_2} \\ C_{\tau_1,\tau_2} \end{pmatrix}$ obtained from $\begin{pmatrix} B_{2,3} \\ C_{2,3} \end{pmatrix}$ by increasing either τ_1 or τ_2, will add new columns and rows. Hence the latter is a submatrix of the former and if the latter is not TUM, the former is also not TUM.

Denote the index set of the columns of $\begin{pmatrix} B_{2,3} \\ C_{2,3} \end{pmatrix}$ by $I = \{1, ..., 12\}$. By the Ghouila-Houri theorem, we only need to find a subset $\Omega \subseteq I$ such that no partition Ω_1, Ω_2 of Ω satisfying (A.8) exists.

Let $\Omega = \{1, 2, 3, 4, 6, 7, 8, 11, 12\}$. For any partition Ω_1, Ω_2 of Ω to satisfy (A.8), a chain of necessary conditions is given by: $1 \in \Omega_1 \longrightarrow 4 \in \Omega_2 \longrightarrow 6 \in \Omega_1$, and $1 \in \Omega_1 \longrightarrow 7 \in \Omega_2 \longrightarrow 8 \in \Omega_1 \longrightarrow 11 \in \Omega_2 \longrightarrow 12 \in \Omega_1$. But then $6 \in \Omega_1$ and $12 \in \Omega_1$ violates (A.8). Hence there exists no partition Ω_1, Ω_2 of Ω such that (A.8) holds. □

Appendix 3

We introduce an alternative characterization of finite generators of a cone by using a canonical decomposition. Formally, let P be a cone. Let $\text{lin}(P)^\perp$ be the orthogonal complement of the linearity space $\text{lin}(P)$, and let $P \cap \text{lin}(P)^\perp$ be the projection of P onto the subspace $\text{lin}(P)^\perp$, which is a cone and is pointed. Then, $P = \text{lin}(P) + P \cap \text{lin}(P)^\perp$. Here "+" represents the Minkowski sum of two sets of vectors. The canonical decomposition suggests that for a non-pointed cone P, the sets $r(J)$ each correspond to an extreme ray of $P \cap \text{lin}(P)^\perp$. Hence a minimal set of finite generators can be found by (i) a basis of $\text{lin}(P)$ and (ii) the set of extreme rays of $P \cap \text{lin}(P)^\perp$.

For the bilateral trade problem in Example 3, we have found a set of generators in combinatorial vectors. As the projection cone $\mathcal{C}$ is non-pointed, we can instead apply the canonical decomposition to $\mathcal{C}$ and hence find another system of finite generators, which contains non-combinatorial vectors.

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