UNIVERSITY of York

Discussion Papers in Economics

No. 19/12

A Conic Approach to the Implementation of Reduced-Form Allocation Rules

Xu Lang and Zaifu Yang

Department of Economics and Related Studies
University of York
Heslington
York, YO10 5DD

Markovian Core, Indivisibility, and Successive Pareto-Improvements[1]

Satoru Fujishige[2] and Zaifu Yang[3]

Previous: June 22, 2018
This version: September 6, 2019

Abstract

We study a general barter market in which every agent is initially endowed with several inherently indivisible items and wishes to exchange with other agents. There is no medium of exchange like money. Agents have general preferences over bundles of items and may acquire several items. It is well-known that the core of such an economy is typically empty. We propose a new but more general notion of core called a Markovian core. A Markovian core allocation is individually rational, but Pareto-efficient and stable against any possible coalition deviation by comparison with their current assignments instead of their initial endowments. We show that the market has always a nonempty strict Markovian core through a decentralized Pareto-improvement process.

Keywords: Decentralized market, barter market, indivisibility, efficiency, stability, Markovian core.

JEL classification: C62, D72.

1 Introduction

The barter market of Shapley and Scarf (1974) stands out as one of the most fundamental models in microeconomic theory and game theory. Their described Gale's top trading cycle (TTC) procedure has found important applications in mechanism design, two-sided matching, kidney exchange, and school choice etc. In this market there are finitely many traders or agents each of whom owns initially an indivisible item e.g., a house. Every agent has preferences over all the houses but has no use for more than one item. There is no

[1]The first author acknowledges the support from a JSPS KAKENHI grant (19K11839).

[2]S. Fujishige, Research Institute for Mathematical Sciences, Kyoto University, Kyoto, Japan, `fujishig@kurims.kyoto-u.ac.jp`

[3]Z. Yang, Department of Economics and Related Studies, University of York, York YO10 5DD, UK; `zaifu.yang@york.ac.uk`.

money or other medium of exchange. Agents swap their houses with each other in order to obtain their favorite possible houses. Shapley and Scarf (1974) proved that this market is a balanced game and therefore has a nonempty core by a theorem of Scarf (1967) and both core allocation and a competitive equilibrium can be found by the TTC procedure. Since then, many remarkable properties of this model and of the procedure have been discovered.

Roth and Postlewaite (1977) demonstrated that if every agent has strict preferences, this market has a unique competitive equilibrium coinciding with the unique strict core allocation. Roth (1982) proved that the TTC procedure can induce every agent to behave honestly. Wako (1984) illustrated that the strict core can be a proper subset of the set of competitive equilibrium allocations. Ma (1994) proved that under strict preferences, a procedure is individually rational, Pareto-efficient, and strategy-proof if and only if it is the TTC procedure. Konishi et al. (2001) studied a generalization of the Shapley-Scarf model by allowing each agent to initially own, say, one house and one car. They demonstrated that many distinctive features of the Shapley-Scarf economy cannot carry over to this extended model. But they found that if there are only three agents and all agents have additively separable strict preferences, then the core of this market is not empty. Inoue (2005) examined an extension of Shapley-Scarf model and established that if the aggregate upper contour set of all agents is discretely convex, then the core is nonempty, and that especially if the upper contour set of every agent is M-convex, the core is not empty. Dogan et al. (2011) extended the Shapley-Scarf model by permitting couples of agents and proved the nonemptiness of the core under lexicographic preferences. Inoue (2014) established an equivalence between the strict core and the set of Walrasian equilibria for an economy with indivisibility and a large but finitely many agents.

In stark contrast to the markets with perfectly divisible goods which always have a nonempty core under general environments (see Scarf 1967 and Arrow and Hahn 1971), the core of markets with indivisibility more general than the Shapley-Scarf model can easily become empty. The difficulty is caused by indivisibility which is an extreme form of non-convexity. For instance, Shapley and Scarf (1974, pp. 32-34) themselves showed the nonexistence of core in an apparently natural market with three agents having symmetric holdings in a tract of nice houses. In their example, agent j owns houses j, j', and j'' for $j = 1, 2, 3$. Moulin (1995) pointed out that the core generally does not exist in the market where each agent owns a car and a house and views cars and houses as complements. Konishi et al. (2001) demonstrated that the core can be empty even in the class of economies with several identical items and agents consuming multiple units. In such economies there is no complementarity among the items. Inoue (2005, pp.102-103) gave a different type of example with an empty core.

The notion of core cannot be used as a solution to many markets where agents initially possess more than one item and may acquire several items and for which the core is generally not guaranteed to be nonempty. It is therefore necessary to find an appropriate alternative solution to such markets. On the one hand, as core is one of the most important and most used solution concepts in the context of competition and cooperation, it will be important for any proposed solution to maintain some basic properties of the core such as individual rationality, Pareto-efficiency, and stability. On the other hand, any proposed solution has to be general enough to cover a variety of environments. In this paper we aim to establish such a solution. Briefly speaking, we investigate a general barter market in which each agent is initially endowed with several inherently indivisible items and wishes to exchange with other agents. There is no medium of exchange like money. Agents have general preferences over bundles of items and may acquire several items. Taking incomplete information and behavioural aspects of human decision-making into account, we propose a new solution concept of core called a Markovian core. Every Markovian core allocation is individually rational with respect to their initial endowments, but Pareto-efficient and stable against any coalition deviation by comparison with their current assignments instead of their initial endowments. We prove that the market has always a nonempty strict Markovian core through a decentralized Pareto-improvement process. We also show that a random decentralized process converges almost surely to a strict Markovian core allocation.

The rest is organized as follows. Section 2 presents the model. Section 3 establishes the main results and Section 4 concludes.

2 The Model

Consider a general barter market in which there are m agents and n different types of indivisible commodities. Let $M = \{1, 2, \cdots, m\}$ denote the set of agents. Each agent $j \in M$ is initially endowed with a non-zero bundle $\omega^j \in \mathbb{Z}^n_+$ of indivisible items and has a general preference relation $\succeq_j$ on a family of feasible bundles $F^j \subseteq \mathbb{Z}^n_+$ containing his initial endowment ω^j, where $\mathbb{Z}^n_+$ represents the space of indivisible commodities, i.e., the collection of all nonnegative n-dimensional integer vectors. There is no medium of exchange like money in the economy. All agents try to swap their items in order to improve their welfare.

Given the market, an allocation is a redistribution $X = (x^i \mid i \in M)$ of all items among all agents such that $\sum_{j\in M} x^j = \sum_{j\in M} \omega^j$ and $x^j \in F^j$ for every $j \in M$. At allocation X, agent j receives bundle x^j.

The conventional solution to this type of problem is the notion of core. It is defined to be a set of redistributions of all items among all agents that cannot be profitably blocked by any coalition of agents by comparing with their initial endowments. A coalition is a nonempty subset of the set M.

Definition 1 *An allocation $(x^j \mid j \in M)$ is a core allocation if there do not exist a coalition S and an allocation $(x^j \mid j \in S)$ with $\sum_{j\in S} x^j = \sum_{j\in S} \omega^j$ such that $x^j \succ_j \omega^j$ and $x^j \in F^j$ for all $j \in S$. An allocation $(x^j \mid j \in M)$ is a strict core allocation if there do not exist a coalition S and an allocation $(x^j \mid j \in S)$ with $\sum_{j\in S} x^j = \sum_{j\in S} \omega^j$ such that $x^j \succeq_j \omega^j$ and $x^j \in F^j$ for all $j \in S$ with at least one strict inequality.*

Shapley and Scarf (1974) considered a market in which each agent initially owns one item like one house, has preferences over all items but has no use for more than one item. They showed that this market has a nonempty core.

Because core is not guaranteed to be nonempty, it cannot be used as a solution to many markets where agents initially possess more than one item and may acquire more than one item, and it is therefore necessary and useful to find an appropriate alternative solution to such markets. It will be important for such a solution to possess at least individual rationality, Pareto-efficiency, and stability among many desirable properties. The Markovian core to be introduced has these desirable properties. From the classical notion of core (Definition 1), we understand that every agent has to be totally rational, infinitely patient and fully capable of cognitive thinking and computing, and to have a whole and clear picture of the entire market and that every member in every possible blocking coalition always compares with his initial endowment. Unlike this classical solution, the proposed solution has to relax some of these assumptions by taking both incomplete information and behavioural factors of human being into account. We may imagine an economic milieu in which people initially own several items and wish to exchange with each other. At the beginning they may not know everyone except their neighbors and friends but they gradually get to know each other as time goes on. They may be myopic or not patient enough but just want to grab every opportunity to improve their current position. In this process agents haggle with each other and exchange with each other as long as improvement can be made. Transactions take place all the time as long as people find it profitable to exchange. In the process agents act rationally but not necessarily optimally. When trade happens, ownership will automatically change hands. Trade terminates until none have incentive to trade any further. This final state of the market is the solution which we seek to establish.

Definition 2 *An allocation $(x^j \mid j \in M)$ is strongly blocked by a coalition S if there exists $(y^j \mid j \in S)$ such that $\sum_{j\in S} y^j = \sum_{j\in S} x^j$ and $y^j \succ_j x^j$ and $y^j \in F^j$ for all $j \in S$. An allocation $(x^j \mid j \in M)$ is blocked by a coalition S if there exists $(y^j \mid j \in S)$ such that $\sum_{j\in S} y^j = \sum_{j\in S} x^j$ and $y^j \succeq_j x^j$ and $y^j \in F^j$ for all $j \in S$ with at least one strict inequality.*

An allocation $(x^j \mid j \in M)$ is *individually rational* if, for every agent $j \in M$, x^j is at least as good as his initial endowment ω^j.

Definition 3 *An individually rational allocation is a Markovian core allocation if it cannot be strongly blocked by any coalition. An individually rational allocation is a strict Markovian core allocation if it cannot be blocked by any coalition.*

By definition, a Markovian core allocation is individually rational with respect to the initial endowment of every agent. This property is shared with the classical notion of core. The current definition of blocking coalitions with at least two members is similar to that in the classical core (Definition 1) but differs from the latter in one major aspect that the current definition requires agents in each blocking coalition to compare the proposed assignments (y^j) with their current assignments (x^j) instead of their initial endowments (ω^j). Our Markovian core allocation is Pareto-efficient and immune from any possible coalition deviation by comparing with every coalition member's current assignment and captures some behavioural aspects of human decision-making. That we use the term of Markovian core is to try to reflect behind the model a process of decision-making in which agents make their decision rationally but not necessarily optimally at every time based solely on their present states.

3 The Results

An allocation $(x^j \mid j \in M)$ is a Pareto-improvement of an allocation $(y^j \mid j \in M)$ if $x^j \succeq_j y^j$ for all $j \in M$ with at least one strict inequality. An allocation $(x^j \mid j \in M)$ is a strong Pareto-improvement of an allocation $(y^j \mid j \in M)$ if $x^j \succ_j y^j$ for all $j \in M$. An allocation is strongly Pareto-efficient if it has no Pareto-improvement. An allocation is Pareto-efficient if it has no strong Pareto-improvement. We first consider the case of complete information in which every agent knows everything about the market.

Theorem 1 *The market has a nonempty strict Markovian core. Every strict Markovian core allocation is strongly Pareto-efficient.*

Proof. We prove this by constructing a finite number of successive Pareto-Improvements. Imagine that the market opens at day 0 with the initial market state $X^0 = (\omega^j \mid j \in M)$. We also denote it by $X^0 = (x^{0,j} \mid j \in M)$. If X^0 is not a strict Markovian core allocation, then there must exist a blocking coalition S^1 with $(y^j \mid j \in S^1)$ against X^0. Then we $(x^j \mid j \in M)$ obtain a new market state $X^1 = (x^{1,j} \mid j \in M)$ on day 1 with $x^{1,j} = y^j$ for all $j \in S^1$ and $x^{1,j} = \omega^j$ for all $j \in M \setminus S^1$. Clearly, X^1 is a new allocation at which none is worse than his initial state and at least one is strictly better off. So X^1 is a Pareto-Improvement of X^0. Again, if X^1 is not a strict Markovian core allocation, there must exist a blocking coalition S^2 with $(y^j \mid j \in S^2)$ against X^1. We have a new allocation X^2 on day 2 being a Pareto-Improvement of X^1. We repeat this process until there is no blocking coalition anymore. This is a monotonic process and must stop in finite time as the number of allocations is finite. Therefore the market must have a nonempty strict Markovian core. By definition, every strict Markovian core allocation is strongly Pareto-efficient. $\square$

Secondly, we consider the case of incomplete information in which agents do not have a clear and complete picture of the whole market but are somehow well-informed in a sense that as long as there will be opportunities for some coalition of agents to improve themselves, this coalition can grasp such opportunities with a positive probability. To be precise, the market opens at day 0 with every agent coming with his initial endowment. Trade takes place everyday $t = 0, 1, \cdots$, as long as agents can find opportunities to exchange and improve themselves. As agents do not have a complete knowledge of the market, we can only impose a mild condition upon the economy that on each day t, every blocking coalition should occur with a positive probability.

Theorem 2 *Assume that the market starts at day* 0 *with every agent* $j \in M$ *endowed with* ω^j *and that on each day* $t = 0, 1, \cdots$, *every blocking coalition happens with a positive probability. Then the market will converge almost surely to a strict Markovian core allocation.*

Proof. We prove this by constructing a finite random process of successive Pareto-Improvements. Imagine that the market opens at day 0 with the initial market state $X^0 = (\omega^j \mid j \in M)$. We also denote it by $X^0 = (x^{0,j} \mid j \in M)$. If X^0 is not a strict Markovian core allocation, then there must exist a blocking coalition against X^0. By hypothesis we may take S^1 as a realized blocking coalition with $(y^j \mid j \in S^1)$ against X^0 with a positive probability. Then we $(x^j \mid j \in M)$ obtain a new market state $X^1 = (x^{1,j} \mid j \in M)$ on day 1 with $x^{1,j} = y^j$ for all $j \in S^1$ and $x^{1,j} = \omega^j$ for all $j \in M \setminus S^1$. Clearly, X^1 is a new allocation at which none is worse than his initial state and at least one is strictly better off. So X^1 is a Pareto-Improvement of X^0. Again, if X^1 is not a strict Markovian core allocation, there must exist a realized blocking coalition S^2 with $(y^j \mid j \in S^2)$ against X^1 with a positive probability. We have a new allocation X^2 on day 2 being a Pareto-Improvement of X^1. We repeat this process until there is no blocking coalition anymore. This is a monotonic process and must converge almost surely to a strict Markovian core allocation in finite time. More precisely, because of the finite number of market states and the assumption of the theorem, there exists a positive ϵ such that every blocking coalition happens with probability greater than a fixed common $\epsilon > 0$. Hence the probability of the infinite sequence of no occurrence of blocking coalitions is equal to zero since $(1-\epsilon)^n$ converges to zero as n goes to infinity. So a blocking coalition occurs at a finite time instance t with probability one, and then after t this process repeats as many times as possible till we reach a strict Markovian core allocation with probability one. $\square$

The interested reader may refer to Chen et al. (2016), Fujishige and Yang (2017), Kojima and Ünver (2008), and Roth and Vande Vate (1990) for different random market processes.

4 Conclusion

In this paper we have introduced a general barter market with many heterogeneous indivisible items. Every agent is initially endowed with multiple items and may acquire several items. There is no medium of exchange like money. Agents have general preferences over bundles of items. This model reduces to the well-known model of Shapley and Scarf (1974) and has a nonempty core when every agent is initially endowed with one item and has no use for more than one item. However, it is well-known that the core of such an economy is typically empty when every agent is allowed to acquire more than one item. We have proposed a new and appealing solution to this general resource allocation problem called a Markovian core, generalizing the classical solution of core. A Markovian core allocation is individually rational, but Pareto-efficient and immune from any possible coalition deviation by comparison with their current assignments instead of their initial endowments. This solution has relaxed the requirement of full rationality of the classical notion of core by capturing some behavioral aspects of human decision making such as myopic behavior, impatience, and bounded rationality. We have shown that the market has always a nonempty strict Markovian core through a decentralized Pareto-improvement process, and that a random decentralized process converges almost surely to a strict Markovian core allocation.

References

[1] Arrow, K.J., and F. Hahn (1971): *General Competitive Analysis*, Holden-Day, San Francisco, CA.

[2] Chen, B., Fujishige, S., and Z. Yang (2016): "Random decentralized market processes for stable job matchings with competitive salaries," *Journal of Economic Theory,* 165, 25–36.

[3] Dogan, O., Laffond, G., and J. Laine (2011): "The core of Shapley-Scarf markets with couples," *Journal of Mathematical Economics*, 47, 60–67.

[4] Fujishige, S., and Z. Yang (2017): "On a spontaneous decentralized market process," *Journal of Mechanism and Institution Design*, 2, 1–37.

[5] Inoue, T. (2008): "Indivisible commodities and the nonemptiness of the weak core," *Journal of Mathematical Economics*, 44, 96–111.

[6] Inoue, T. (2014): "Indivisible commodities and an equivalence on the strong core," *Journal of Mathematical Economics*, 54, 22–35.

[7] Kojima, F., and M. Ünver (2008): "Random paths to pairwise stability in many-to-many matching problems: A study on market equilibration," *International Journal of Game Theory,* 36, 473–488.

[8] Konishi, H., Quint, T., and J. Wako (2001): "On the Shapley–Scarf economy: The case of multiple types of indivisible goods," *Journal of Mathematical Economics*, 35, 1–15.

[9] J. Ma (1994): "Strategy-proofness and the strict core in a market with indivisibilities," *International Journal of Game Theory,* 23, 75–83.

[10] H. Moulin (1995): *Cooperative Microeconomics*, Princeton University Press, Princeton, NJ.

[11] A. Roth (1982): "Incentive compatibility in a market with indivisibilities," *Economics Letter*, 9, 127–132.

[12] Roth, A., and A. Postlewaite (1977): "Weak versus strong domination in a market with indivisible goods," *Journal of Mathematical Economics*, 4, 131–137.

[13] Roth, A., and J.H. Vande Vate (1990): "Random paths to stability in two-sided matching," *Econometrica,* 58, 1475–1480.

[14] H. Scarf (1967): "The core of an n-person game," *Econometrica,* 35, 50–69.

[15] Shapley, L., and H. Scarf (1974): "On cores and indivisibilities," *Journal of Mathematical Economics*, 1, 23–37.

[16] J. Wako (1984): "A note on the strong core of a market with indivisible goods," *Journal of Mathematical Economics*, 13, 189–194.