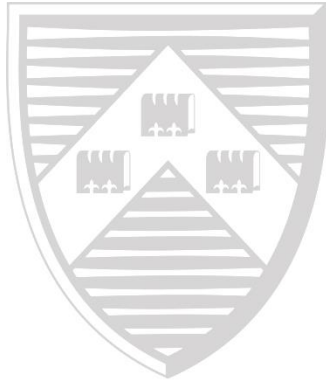


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Estimating the term structure with linear regressions:
Getting to the roots of the problem

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Abstract

Linear estimators of the affine term structure model are inconsistent since they cannot reproduce the factors used in estimation. This is a serious handicap empirically, giving a worse fit than the conventional ML estimator that ensures consistency. We show that a simple self-consistent estimator can be constructed using the eigenvalue decomposition of a regression estimator. The remaining parameters of the model follow analytically. The fit of this model is virtually indistinguishable from that of the ML estimator. We apply the method to estimate various models of U.S. Treasury yields and a joint model of the U.S. and German yield curves.

JEL classification: C13, G12.

Keywords: term structure, linear regression estimators, self-consistent model, estimation methods, two-country model.

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1 Introduction

This paper proposes a simple regression-based method for estimating the risk-neutral dynamics of the affine term structure model that is internally consistent in the sense that it can reproduce the factors used in estimation. We obtain this result using the eigenvalue decomposition of a linear estimator. The remaining parameters of the model follow analytically. Remarkably, the fit of this model is virtually indistinguishable from that of the maximum likelihood (*ML*) estimator.

The estimation of a no-arbitrage dynamic term structure model (*DTSM*) represents a challenging numerical problem, but in recent years significant progress has been made in tackling this (Joslin et al. (2011), Hamilton and Wu (2012), Adrian et al. (2013), Abrahams et al. (2015) and Diez de Los Rios (2015)). These new methods greatly facilitate the estimation of term structure models but have their own limitations. For instance, the method of Joslin et al. (2011) (henceforth *JSZ*) maximizes the likelihood of the cross-section of yields, but is slower than the new linear methods and requires the nature of the risk-neutral roots of the model to be specified (as real or complex and distinct or repeated). Moreover, the presence of multiple local optima means that good parameter starting values are necessary to increase the chances of finding the global optimum.

On the other hand, the linear estimators of Adrian et al. (2013) and Abrahams et al. (2015)¹ (henceforth *AACM*) yield instantaneous solutions that are unique, but as Joslin et al. (2013) point out, they contain redundant parameters. For example, with 3 factors, the risk-neutral response matrix that underpins the model of the cross section of yields, is defined by 3 characteristic roots (or eigenvalues), while the regression uses 9 parameters. Joslin et al. (2013) show that this means that the models are not self-consistent in the sense that they assume that the yield-based factors are observed without error, but the factors reconstructed from the yields fitted by the model differ from these observed values.

¹This working paper version is not to be confused with the published version (Abrahams et al., 2016), which eschews the regression approach and instead employs the Maximum Likelihood method, which is internally consistent.

This theoretical inconsistency may or may not be important in practice, but is just one potential consequence of over-fitting. As in any econometric model, eliminating redundant parameters should improve general performance in fitting the data, as well as forecasting and simulation exercises, reducing the risk of misleading inferences being drawn.

The constrained estimator (DLR_C) proposed by Diez de Los Rios (2015) (henceforth DLR), uses an iterative method to remove these redundant parameters from initial regression estimates (DLR_U) of the risk-neutral dynamics. Instead, we do this in a way that preserves the characteristic roots of the DLR_U risk-neutral response matrix and thus its dynamic characteristics. This matrix determines the factor loadings in the model of bond yields. We then exploit the sequential nature of the arbitrage-free parameter solution to find a closed form expression for the risk-neutral level parameter, conditional upon the response matrix. This determines the intercepts in the yield model.

By eliminating redundant parameters, our estimator markedly outperforms linear estimators that are not self-consistent. In practice this means that the fit of the latter needs to be improved by increasing the number of factors. Our procedure preserves the risk-neutral dynamics that are embedded in the characteristic roots, while iterative procedures that ensure consistency, like the one used by DLR , are unlikely to do this. His estimator does not fit a standard set of U.S. Treasury yield data and simulated data sets based on this benchmark data as well as ours does.

We show that any regression-based estimator of the risk-neutral dynamics (such as $AACM$ or the unconstrained DLR estimator, DLR_U), can easily be modified to give a self-consistent term structure model. This method is based on four key observations, of which two are well-established and two are novel. First, JSZ showed that ML estimates of the physical factor dynamics can be provided by a stand-alone vector autoregression (VAR) of the principal components of yields and second, that the factor covariance matrix provided by this VAR is consistent and very close to the ML estimate. These techniques are extensively used in the new linear regression literature. We take this further by noting that the risk-neutral roots of the self-consistent model can be estimated as the characteristic roots

of the risk-neutral response matrix of any linear regression model. Moreover, although raw regression estimates (such as DLR_U) are under-identified, these eigenvalues, which characterize the Q -dynamics, are estimated with an extremely high degree of precision. They are strongly rooted in the data, helping to explain why our approach works so remarkably well. Finally, we show that given these values, the remaining parameters follow analytically. This completes the specification of the risk-neutral dynamics, allowing the coefficients of the full cross-section of yields to be estimated using well-known affine recursion relationships.

Using a standard data set for U.S. Treasury yields, we find that applying our simple self-consistent (SSC) procedure to the unconstrained DLR estimator gives results that are virtually indistinguishable from those obtained by the JSZ ML estimator. Similarly, we find that using the characteristic roots of the risk-neutral response matrix from the DLR_U regressions as starting values for the root parameters in the JSZ ML algorithm, hardly any iteration is required to get the ML values, since these are so close (see Table 2).

We check the robustness of these findings using bootstrap simulations. The simulation exercise takes the ML estimates of the JSZ model as the ‘true’ values and uses these to generate 5,000 artificial yield data samples of the same length and character as the original U.S. Treasury yield data set. This exercise reveals that the $SSC(DLR_U)$ and ML methods both display negligible bias and that the former fits the simulated data almost as well as the latter. Remarkably, initiating the ML procedure using the DLR_U estimates is as effective as initiating it from the ‘true’ values used to generate the artificial data.

These exercises show how our new algorithm can estimate the Q -dynamics in the standard $VAR(1)$ $DTSM$, quickly and consistently. However, the recent literature has been focussed on generalizing and refining the specification of the \mathcal{P} -dynamics. These can be used to represent market expectations that, subtracted from the corresponding yields, gives residuals that can be interpreted as term premia. These new time-series approaches are based on ML methods and can be very time-intensive, but we show that they can be speeded up enormously using our regression-based approach, greatly extending the range of problems that can be handled.

We demonstrate this using a generalization of the standard $VAR(1)$ *DTSM* proposed by Joslin et al. (2013), who consider $VAR(p)$, $p \geq 1$, for the physical dynamics and $VAR(1)$ for the risk-neutral dynamics. For the three factor model, the Akaike (*AIC*) and Hannan-Quinn (*HQIC*) information criteria favour the $VAR(2)$, while the Bayesian information criterion (*BIC*), which is stricter, selects the $VAR(1)$. As usual, we find that many of the parameters of these time-series models are insignificant, leading to poorly-determined estimates of the term premium. So we apply the model selection procedure proposed by Joslin et al. (2014) to eliminate insignificant parameters and search over parameter combinations that is optimal in terms of these information criteria. For the $VAR(2)$ model we need to select a model from over two million (2^{21}) possible combinations. This would not be feasible using *ML*, but we can find this solution quickly using our regression-based framework. Imposing these restrictions reduces the variance of the term premium without significantly reducing the likelihood of the model.

Finally, we show how the *SSC* procedure can be extended to handle multi-market term structures with common factors and use this to model the U.S. and German government bond markets jointly. We represent the common factors as principal components from the joint yield covariance matrix and thus avoid the usual division into global and local factors, letting the data speak without imposing any such division. We find that six factors are sufficient to give a plausible fit to the term structure for both countries, in line with typical bid-ask spreads. The root-mean-square error of the *AACM* method is on average about double of that given by the *SSC*(DLR_U) procedure, which is very close to the fit of an *OLS* benchmark.

Our analysis raises doubts about the presence of pure global factors since we find that there is no reduction in the total number of factors needed to achieve a satisfactory result. For example, we need six factors in a joint model to obtain a similar fit to that of two separate country three factor models. We conclude that there is very little contemporaneous interaction between the U.S. and German markets that is not picked up by single country models. Nevertheless, we find that there are significant differences between the single country

and joint approaches in terms of the physical dynamics, which largely determine the way these models decompose yields into components representing interest rate expectations and the risk premium. Although, pooling the data for the U.S. and Germany seems to make little qualitative difference to the U.S. decomposition, it does give a more plausible decomposition of the German curve.

The paper is set out as follows. The next section sets out the theoretical model of the risk-neutral dynamics and the term structure and discusses the alternative *ML* and linear estimation approaches. Section 3 shows how a self-consistent model of the risk-neutral dynamics can be found using any linear estimator. Section 4 compares the performance of these self-consistent regression-based estimators with that of the *ML* approach. In Section 5 we apply our estimation method in conjunction with generalized physical dynamics of the factors and in Section 6 we apply our linear estimator to a two-country model and estimate jointly the term structure of the U.S. and German yields. Section 7 concludes with some observations on the the implications of these results for research on the term structure.

2 Estimating affine term structure models

Assume that the zero-coupon log bond prices, $p_{m,t}$, are affine in K observable factors \mathbf{q}_t :

$$-p_{m,t} = a_m + \mathbf{b}_m' \mathbf{q}_t. \quad (1)$$

We collect these in a vector \mathbf{p} of log prices with J maturities that can be collected in a vector $\mathbf{m} = [m_1, m_2, \dots, m_J]'$, where $-\mathbf{p}_t = \mathbf{a} + \mathbf{B}' \mathbf{q}_t$, $\mathbf{a} = [a_{m_1}, \dots, a_{m_J}]'$ and $\mathbf{B} = [\mathbf{b}_{m_1}, \dots, \mathbf{b}_{m_J}]$. Also assume that the factors follow a *VAR*(1) process under the risk-neutral measure:

$$\mathbf{q}_{t+1} = \boldsymbol{\mu}^{\mathcal{Q}} + \boldsymbol{\Phi}^{\mathcal{Q}} \mathbf{q}_t + \mathbf{u}_{t+1}^{\mathcal{Q}} \quad (2)$$

with $\mathbf{u}_t^{\mathcal{Q}} \sim i.i.d. N^{\mathcal{Q}}(\mathbf{0}, \boldsymbol{\Sigma})$.

The core of the estimation problem is to find the parameters of this process: $\boldsymbol{\mu}^{\mathcal{Q}}$ and $\boldsymbol{\Phi}^{\mathcal{Q}}$.

These are related to the price coefficients by the well-known recursions:

$$\mathbf{b}_m = \mathbf{b}_1 + \Phi^Q \mathbf{b}_{m-1}, \quad (3)$$

$$a_m = a_1 + a_{m-1} + \mathbf{b}_{m-1}' \boldsymbol{\mu}^Q - \frac{1}{2} \mathbf{b}_{m-1}' \Sigma \mathbf{b}_{m-1}. \quad (4)$$

2.1 Indirect Maximum Likelihood estimators of the risk-neutral dynamics

Until recently, the standard way of estimating the parameters of the risk-neutral dynamics was to embed them in the price coefficients using (3) and (4), substitute them back into (1) and, upon transformation to yields by $y_{m,t} = -p_{m,t}/m$, optimize the likelihood of observing the associated bond yields. Because yields do not exhibit an exact factor structure it is assumed that they are measured with error. Following Duffee (2011) and others we denote observed yields by \mathbf{y}_t^o and add the vector of pricing errors \mathbf{v}_t :

$$\begin{aligned} \mathbf{y}_t^o &= \hat{\mathbf{y}}_t + \mathbf{v}_t \\ &= \mathbf{a}_y + \mathbf{B}_y' \mathbf{q}_t + \mathbf{v}_t, \end{aligned} \quad (5)$$

where $\mathbf{a}_y \equiv \mathbf{M}^{-1} \mathbf{a}$, $\mathbf{B}_y' \equiv \mathbf{M}^{-1} \mathbf{B}'$, $\mathbf{v}_t \sim N(\mathbf{0}, \Sigma_v)$, and \mathbf{M} is a diagonal price-yield transformation matrix with maturities \mathbf{m} on the main diagonal. In particular, the short rate equation ($r_t \equiv -p_{1,t}$) becomes:

$$r_t^o = \delta_0 + \boldsymbol{\delta}_1' \mathbf{q}_t + v_{1,t}. \quad (6)$$

Since the measurement errors are not priced, they have the same distribution under both the physical and the risk-neutral measure: $\mathbf{v}_t \sim i.i.d. N(\mathbf{0}, \Sigma_v)$.

Hamilton and Wu (2012) note that an *OLS* regression of the form (5) provides a benchmark for these affine models, which ignores the no-arbitrage restrictions (3) and (4). They propose a minimum χ^2 estimator that minimizes the distance between the restricted and unrestricted estimates of \mathbf{a}_y and \mathbf{B}_y .

2.2 Direct regression-based estimators of the risk-neutral dynamics

In contrast, the regression-based approaches attempt to estimate the parameters of the risk-neutral dynamics $\boldsymbol{\mu}^Q$ and $\boldsymbol{\Phi}^Q$ directly using either the *OLS* regression estimates of \mathbf{a}_y and \mathbf{B}_y or those of excess *ex post* return regressions. As noted, these regression estimators cannot reproduce the pricing factors. It is usual to assume that these factors are the first K principal components (*PCs*) of the cross-section of yields:

$$\mathbf{q}_t \equiv \mathbf{W}'\mathbf{y}_t^o. \quad (7)$$

Substituting (5) into (7), we see that internal consistency requires:

$$\mathbf{W}'\mathbf{M}^{-1}\mathbf{a} = \mathbf{0}, \quad (8)$$

$$\mathbf{W}'\mathbf{M}^{-1}\mathbf{B}' = \mathbf{I}, \quad (9)$$

and $\mathbf{W}'\mathbf{v}_t = \mathbf{0}$. In other words, the factors reconstructed from the yield estimates $\mathbf{q}_t = \mathbf{W}'\hat{\mathbf{y}}_t$, must coincide with those constructed from the observed yields in (7).

This problem is a consequence of the basic problem that the regression models contain redundant parameters (Hamilton and Wu, 2012, Diez de Los Rios, 2015). The indirect ML estimators eliminate these using alternative (but equivalent) identification schemes. For example, *JSZ* scheme, which we use, is discussed in the next section. Conveniently, when the factors are *PCs*, the weights \mathbf{W} used in the construction of the factors \mathbf{q}_t are provided by the eigenvectors of the eigendecomposition of the yield covariance matrix and the *OLS* factor loadings are the transpose of this matrix, allowing us to write the Hamilton and Wu (2012) benchmark *OLS* regression (5) as: $\mathbf{y}_t^o = \mathbf{c} + \mathbf{W}\mathbf{q}_t + \hat{\mathbf{v}}_t$, where $\mathbf{W} \equiv \mathbf{B}'_y$, \mathbf{c} is a vector of constants and $\hat{\mathbf{v}}_t$ is a vector of *PC* or equivalently *OLS* residuals.

The weights \mathbf{W} can thus be used to estimate $\mathbf{B}' = \mathbf{M}\mathbf{W}$ and hence the risk-neutral parameters using the Hamilton and Wu (2012) or *DLR* approaches. For example, the *DLR*

cross-section regression estimator DLR_U of the risk-neutral feedback matrix $\hat{\Phi}^Q$ in (3):

$$\hat{\Phi}^Q = \left(\sum_{m=m_2}^{m_J} \mathbf{b}_{m-1} \mathbf{b}'_{m-1} \right)^{-1} \left(\sum_{m=m_2}^{m_J} \mathbf{b}_{m-1} (\mathbf{b}'_m - \mathbf{b}'_1) \right), \quad (10)$$

(his Eq.(9)). Our self-consistent estimator focuses on this matrix.

The model is completed by specifying the physical dynamics of the factors. In this paper we first assume that they follow a $VAR(1)$ process:

$$\mathbf{q}_{t+1} = \boldsymbol{\mu} + \Phi \mathbf{q}_t + \mathbf{u}_{t+1}^P \quad (11)$$

with $\mathbf{u}_t^P \sim i.i.d.N^P(\mathbf{0}, \Sigma)$, but it is straightforward to make an extension so that they can include other, unspanned variables (Joslin et al., 2014) or include multiple lags (Joslin et al., 2013), as we do in Section 5.

3 The simple self-consistent term structure estimator

In this section, we show how the redundant parameters of any linear regression model (such as $AACM$ or the unconstrained DLR estimates outlined in the previous section) can be eliminated using the standard eigenvalue decomposition.

3.1 Structural restrictions and the JSZ identification scheme

A convenient identification scheme was proposed by JSZ . They assume that the underlying latent factors follow the risk-neutral dynamics:

$$\mathbf{x}_t = \boldsymbol{\mu}_x^Q + \Phi_x^Q \mathbf{x}_{t-1} + \mathbf{u}_{x,t}^Q, \quad (12)$$

where Φ_x^Q has an ordered Jordan form determined by K parameters only and $\mathbf{u}_{x,t}^Q \sim i.i.d.N^Q(\mathbf{0}, \Sigma_x)$. The level parameter can be identified by specifying jointly $\boldsymbol{\mu}_x^Q = (\mu_\infty^Q, 0, \dots, 0)'$

and $\delta_{\mathbf{x},0} = 0$.² The theoretical log bond prices are:

$$-\mathbf{p}_t = \mathbf{a}_{\mathbf{x}} + \mathbf{B}'_{\mathbf{x}} \mathbf{x}_t, \quad (13)$$

where $\mathbf{a}_{\mathbf{x}}$ and $\mathbf{B}_{\mathbf{x}}$ follow from recursion systems analogous to (3) and (4):

$$\mathbf{b}_{\mathbf{x},m} = \mathbf{b}_{\mathbf{x},1} + \Phi_{\mathbf{x}}^{\mathcal{Q}'} \mathbf{b}_{\mathbf{x},m-1}, \quad (14)$$

$$a_{\mathbf{x},m} = a_{\mathbf{x},1} + a_{\mathbf{x},m-1} + \mathbf{b}'_{\mathbf{x},m-1} \boldsymbol{\mu}_{\mathbf{x}}^{\mathcal{Q}} - \frac{1}{2} \mathbf{b}'_{\mathbf{x},m-1} \boldsymbol{\Sigma}_{\mathbf{x}} \mathbf{b}_{\mathbf{x},m-1}. \quad (15)$$

The observed yields can be expressed as:

$$\mathbf{y}_t^o = \mathbf{a}_{\mathbf{y},\mathbf{x}} + \mathbf{B}'_{\mathbf{y},\mathbf{x}} \mathbf{x}_t + \mathbf{v}_t, \quad (16)$$

where $\mathbf{a}_{\mathbf{y},\mathbf{x}} = \mathbf{M}^{-1} \mathbf{a}_{\mathbf{x}}$ and $\mathbf{B}'_{\mathbf{y},\mathbf{x}} = \mathbf{M}^{-1} \mathbf{B}'_{\mathbf{x}}$, and the short rate is given by:

$$r_t = \delta_{\mathbf{x},0} + \boldsymbol{\delta}'_{\mathbf{x},1} \mathbf{x}_t + v_{1,t}, \quad (17)$$

where $\delta_{\mathbf{x},0} = a_{\mathbf{x},1}$ and $\boldsymbol{\delta}_{\mathbf{x},1} = \mathbf{b}_{\mathbf{x},1}$.

In a single-market setting (e.g. when modelling nominal government bonds in one country), $\boldsymbol{\delta}_{\mathbf{x},1}$ is not identified under the *JSZ* parametrization and thus can be normalized to an arbitrary vector (usually a K -dimensional vector of ones, $\boldsymbol{\delta}_{\mathbf{x},1} = \mathbf{1}$). In a joint model of two or more markets with common factors, however, $\boldsymbol{\delta}_{\mathbf{x},1}$ needs to be estimated, since in general the latent factor loadings for one country affect the observable factor loading of the others.

Substituting (13) into (7) and assuming that $\mathbf{W}' \mathbf{v}_t = \mathbf{0}$ allows us to switch between the latent and observable factors using:

$$\mathbf{x}_t = -(\mathbf{W}' \mathbf{B}'_{\mathbf{y},\mathbf{x}})^{-1} \mathbf{W}' \mathbf{a}_{\mathbf{y},\mathbf{x}} + (\mathbf{W}' \mathbf{B}'_{\mathbf{y},\mathbf{x}})^{-1} \mathbf{q}_t. \quad (18)$$

²Alternatively, if the factors are stationary under the risk-neutral measure (the most persistent factor has the multiplicity one with the autoregressive coefficient $|\phi_1| < 1$), the level parameter can be identified by setting $\delta_{\mathbf{x},0} = r_{\infty}^{\mathcal{Q}}$ and $\boldsymbol{\mu}_{\mathbf{x}} = \mathbf{0}$. These two identification schemes are equivalent in this case and the parameters are related by $r_{\infty}^{\mathcal{Q}} = \mu_{\infty}^{\mathcal{Q}} / (1 - \phi_1)$.

This implies that the relationship between the coefficients in the dynamics of the observable factors (2) and latent factors (12) is:

$$\mathbf{\Phi}^{\mathcal{Q}} = (\mathbf{W}'\mathbf{B}'_{y,x})\mathbf{\Phi}_x^{\mathcal{Q}}(\mathbf{W}'\mathbf{B}'_{y,x})^{-1}, \quad (19)$$

$$\boldsymbol{\mu}^{\mathcal{Q}} = (\mathbf{W}'\mathbf{B}'_{y,x})\boldsymbol{\mu}_x^{\mathcal{Q}} + (\mathbf{I} - \mathbf{\Phi}^{\mathcal{Q}})\mathbf{W}'\mathbf{a}_{y,x}, \quad (20)$$

and

$$\boldsymbol{\Sigma}_x = (\mathbf{W}'\mathbf{B}'_{y,x})^{-1}\boldsymbol{\Sigma}(\mathbf{W}'\mathbf{B}'_{y,x})^{-1'}. \quad (21)$$

JSZ note that the parameters $\{\boldsymbol{\mu}, \mathbf{\Phi}\}$ describing the \mathcal{P} -dynamics of the observable factors (11) are unrestricted and can be estimated by *OLS* regression. This regression also provides a consistent (although not efficient) estimate of the $\boldsymbol{\Sigma}$ matrix, which is involved in determination of the affine coefficients a_m . Given these parameters, we are concerned in this paper with finding self-consistent estimates of the remaining parameters, which determine the risk-neutral dynamics of the factors.

3.2 The simple self-consistent (SSC) estimator

Any approach that returns unconstrained estimates of the risk-neutral dynamics uses $K \times K$ free parameters in the response $\mathbf{\Phi}^{\mathcal{Q}}$ matrix instead of the K parameters in $\mathbf{\Phi}_x^{\mathcal{Q}}$ and K parameters in $\boldsymbol{\mu}^{\mathcal{Q}}$ instead of the single level parameter in $\boldsymbol{\mu}_x^{\mathcal{Q}}$, and thus results in a model that is over-parameterized and therefore internally inconsistent. However, the factor structure of the $\mathbf{\Phi}^{\mathcal{Q}}$ matrix (19) suggests that the underlying roots of the model can be found from the eigenvalue decomposition of a regression-based $\hat{\mathbf{\Phi}}^{\mathcal{Q}}$.³ We can put these roots in the real Jordan form $\hat{\mathbf{\Phi}}_x^{\mathcal{Q}}$ and replace the regression-based response $\mathbf{\Phi}^{\mathcal{Q}}$ matrix by its self-consistent counterpart corresponding to (19): $\tilde{\mathbf{\Phi}}^{\mathcal{Q}} = (\mathbf{W}'\mathbf{B}'_{y,x})\hat{\mathbf{\Phi}}_x^{\mathcal{Q}}(\mathbf{W}'\mathbf{B}'_{y,x})^{-1}$, where the loadings $\mathbf{B}_{y,x}$ follow from (14) and (16). Substituting the latent factors (18) into (16), which represents yields in terms of the latent factors, and equating this with (5), which expresses them in

³One of the advantages of this approach is that we do not need to test different root configurations (i.e. whether they are real/complex, distinct/repeated).

terms of the observed factors, allows us to find its loadings and intercepts:

$$\mathbf{B}'_{\mathbf{y}} = \mathbf{B}'_{\mathbf{y},\mathbf{x}}(\mathbf{W}'\mathbf{B}'_{\mathbf{y},\mathbf{x}})^{-1} \quad (22)$$

$$\mathbf{a}_{\mathbf{y}} = \mathbf{H}\mathbf{a}_{\mathbf{y},\mathbf{x}}, \quad (23)$$

where \mathbf{H} is the idempotent matrix:

$$\mathbf{H} = \mathbf{I} - \mathbf{B}'_{\mathbf{y},\mathbf{x}}(\mathbf{W}'\mathbf{B}'_{\mathbf{y},\mathbf{x}})^{-1}\mathbf{W}' \quad (24)$$

and \mathbf{I} is the identity matrix. Also, integrating the intercept recursion in (15) and substituting $\boldsymbol{\mu}_{\mathbf{x}}^{\mathcal{Q}} = (\mu_{\infty}^{\mathcal{Q}}, 0, \dots, 0)'$ gives the closed form:

$$\mathbf{a}_{\mathbf{y},\mathbf{x}} = \mu_{\infty}^{\mathcal{Q}}\mathbf{c}_0 - \mathbf{c}_1, \quad (25)$$

where the elements of \mathbf{c}_0 and \mathbf{c}_1 are:

$$c_{0,m} = \frac{1}{m} \sum_{j=1}^m b_{\mathbf{x},1,j-1}, \quad c_{1,m} = \frac{1}{2m} \sum_{j=1}^m \mathbf{b}'_{\mathbf{x},j-1} \boldsymbol{\Sigma}_{\mathbf{x}} \mathbf{b}_{\mathbf{x},j-1}. \quad (26)$$

The consistent estimate of the level parameter is given by substituting (25) into (23) and equating this vector with the unrestricted yield intercepts: $\tilde{\mathbf{a}}_{\mathbf{y}} = \bar{\mathbf{y}}^o - \mathbf{B}'_{\mathbf{y}}\bar{\mathbf{q}}$, where $\bar{\mathbf{y}}^o$ and $\bar{\mathbf{q}}$ denote vectors of sample means of yields and the observable factors, respectively, and $\mathbf{B}_{\mathbf{y}}$ are the arbitrage-free slope coefficients (22). We can then solve for the level parameter as:⁴

$$\hat{\mu}_{\infty}^{\mathcal{Q}} = (\mathbf{c}'_0 \mathbf{H}' \mathbf{H} \mathbf{c}_0)^{-1} \mathbf{c}'_0 \mathbf{H}' (\tilde{\mathbf{a}}_{\mathbf{y}} + \mathbf{H} \mathbf{c}_1). \quad (27)$$

The terms on the right hand side of (27) are evaluated conditional on the estimated roots.

To summarize, our simple self-consistent estimator, $SSC(DLR_U)$, is obtained as follows. The vector of parameters to estimate is $\boldsymbol{\Theta} = \{\mu_{\infty}^{\mathcal{Q}}, \boldsymbol{\Phi}_{\mathbf{x}}^{\mathcal{Q}}, \boldsymbol{\mu}, \boldsymbol{\Phi}, \boldsymbol{\Sigma}, \boldsymbol{\Sigma}_{\mathbf{v}}\}$. We follow *JSZ*, *AACM* and others in estimating the \mathcal{P} -dynamics of the observable factors $(\boldsymbol{\mu}, \boldsymbol{\Phi}, \boldsymbol{\Sigma})$ in (11) by *OLS*.

⁴The formula for $r_{\infty}^{\mathcal{Q}}$ is identical, with \mathbf{c}_0 equal to a vector of ones, $\mathbf{c}_0 = \mathbf{1}$.

We obtain Φ^Q using (10). The roots of the model are the eigenvalues of Φ^Q , which form Φ_x^Q . Given the estimates of Σ and Φ_x^Q , we compute the loadings \mathbf{B} using (22) and find the level parameter using (27). Similarly, our simple self-consistent *AACM* estimator, $SSC(AACM)$, is obtained by finding Φ^Q from excess return regressions.

4 Initial empirical applications

This section reports the results of using the basic *VAR(1) DTSM* to represent a standard data set for the U.S. Treasury market. Later sections look at extensions of the model stemming from the recent literature on the physical dynamics and international bond markets.

4.1 The U.S. Treasury market

4.1.1 Data

To compare the performance of this new approach with others, we first employ these on a standard monthly data set for the U.S. Treasury market, one used for example in Goliński and Spencer (2017). This is similar to the data set used by Adrian et al. (2013) and comprises eight base yields with maturities 1 month, 1, 2, 3, 5, 7, 10 and 15 years for the period January 1983 to December 2015, which followed the Volker experiment. The annual maturities come from the well-known data set constructed by Gurkaynak et al. (2007) using the Svensson (1994) parametric method, which are published by the Federal Reserve Board.⁵ The short rate data comes from the Fama Treasury bills files available from the Center of Research in Security Prices. For more detailed description of the data, see Goliński and Spencer (2017).

4.1.2 The performance of different estimators

Table 1 shows the performance of different estimators, measured in terms of the root-mean-square error (*RMSE*) for each yield and the average across yields for 3, 4 and 5 factors.

⁵Available at: <http://www.federalreserve.gov/pubs/feds/2006/200628/200628abs.html>.

This is a natural measure of performance used for example by Adrian et al. (2013), even though their estimates are based on return regressions, perhaps making *RMSEs* for returns more relevant. The Hamilton and Wu (2012) regression model (5) and *ML* estimate of the *JSZ* model (reported respectively as *OLS* and *JSZ* for short) constitute benchmarks for other models. The *OLS* regression is the best linear unbiased estimator. The *ML* routine searches numerically for the parameters (including Σ) that maximize the likelihood function subject to the no-arbitrage constraints, providing the upper bound for the no-arbitrage approaches. These two routines optimize the likelihood function for these eight base yields directly, while the other methods that we consider estimate the price coefficients indirectly using estimates of the parameters of the risk-neutral dynamics for different sets of maturities. The information set used in these estimators is larger than in *JSZ*, since it includes both the eight base maturities and eight adjacent maturities. Finally, these other approaches satisfy the restriction that the Σ parameters appearing in the cross-section (2) and the time series dynamics (11) are the same by using the estimates from the latter, while the *ML* routine uses these parameters to optimize the fit of both dimensions jointly. However, all methods considered in the table use the same observable factors, which are the first K principal components of our eight base yields.

In our comparison we include the linear estimators that do not impose self-consistency restrictions, namely the *AACM* and *DLR_U*, as well as estimators that do, like the estimator proposed by Diez de Los Rios (2015), denoted *DLR_C*, adapted to our selection of yield maturities.⁶ As noted in the introduction, the *DLR* routine imposes the self-consistency constraints using iterative methods. However, as opposed to the original *DLR* estimator, we use the *OLS* version of the *DLR* method, since we found that his *GLS* estimator diverges when the number of factors is larger than three. For three factors, the *DLR* routine results obtained by constrained *OLS* and constrained *GLS* are similar. As input for our *SSC* method we then use the estimates of Φ^Q from the *AACM* and *DLR_U* estimators to get *SSC(AACM)* and *SSC(DLR_U)*, respectively.

⁶We are very grateful to Antonio Diez de los Rios for providing us with his code.

Finally, in the comparison we use the constrained estimator DLR_C as input to get $SSC(DLR_C)$. Since the DLR_C estimator is already self-consistent, this estimator returns the same estimate of the Φ^Q matrix, and thus the \mathbf{B}_y factor loadings. The difference between these two methods comes solely from the method of estimation of μ ; while the DLR method simultaneously uses all risk-neutral parameters to fit the model to its OLS counterpart, the SSC estimator obtains the level parameter using (27) *conditional* on the Φ^Q matrix.

Table 1 indicates that three factors are sufficient to give a plausible fit to the term structure, in line with typical bid-ask spreads in the U.S. Treasury market. However, a four or perhaps a five factor model is needed to get the performance of the JSZ model close to that of the unrestricted OLS model.

The next two results are for the $AACM$ and DLR_U regressions, which do not impose the self-consistency condition. The $AACM$ algorithm gives the worst fit of all approaches for the 3 factor model, with an average $RMSE$ exceeding 12 basis points. The table shows that it requires 4 (or perhaps 5) factors to get a plausible fit, but even then it is noticeably worse than the OLS and JSZ benchmarks. The average $RMSE$ for the DLR_U method for the 3 factor model is about 8.5 basis points, which is much lower than for the $AACM$ routines. The fit obtained by the DLR_U method for the 5 factor model is, however, slightly worse than for the $AACM$ method. This is mainly due to the poor fit apparent for the 1-year yield.

Surprisingly, the performance of the constrained linear estimator proposed by DLR is always visibly worse than the JSZ benchmark. This is particularly glaring for the model with 4 and 5 factors, where the $RMSEs$ exceed those of the JSZ benchmark by a several basis points.

The last three entries in the table, $SSC(AACM)$, $SSC(DLR_C)$ and $SSC(DLR_U)$, compare the fit of the simple self-consistent estimation methods based on the roots of the estimates of the Φ^Q matrix obtained by different methods. Imposing the self-consistency restrictions in this way always improves the fit. For instance, $SSC(AACM)$ with 3 factors reduces the average $RMSE$ from over 12 basis points to under 8 basis points. The absolute

improvement in fit for a larger number of factors is smaller but is still considerable: e.g. for the 5 factor model the *RMSE* of the *AACM* and *DLR_U* estimators are reduced from about 1.6 – 2.0 basis points to under 1 basis point. Remarkably, the *SSC(DLR_U)* estimator is very close to that obtained by the *JSZ* method. This is impressive given the fact that the key parameters are estimated by simple linear regression, using data for a selection of *DLR_U* regressions rather than by optimizing the fit for the eight base yields, using all the model parameters. This reflects the observation made in the introduction that the estimates of the roots of $\Phi_{\mathbf{x}}^Q$ provided by the basic *DLR_U* regressions are very close to those of the *ML* estimates. Finally, it is worth noting that applying our *SSC* method to the *DLR_C* gives considerably better fit to the data, than the *DLR_C* routine, which demonstrates the gain from estimating the level parameter upon conditioning on the fitted loadings.

4.1.3 Parameter estimates

While practitioners are likely to be interested in the ability of a method to fit the cross-section of bond yields, a researcher is more likely to be interested in drawing statistical inferences about the risk-neutral parameters. In this respect, direct regression-based estimates provide a useful check upon inferences drawn indirectly from the parameters embedded in bond yields. Table 2 reports the *ML* estimates of μ_{∞}^Q and the roots of $\Phi_{\mathbf{x}}^Q$ in the *JSZ* model for 3, 4 and 5 factors. We checked different root configurations and report those with the highest likelihood value. Specifically, the roots for the 3 factor models are all real, while those for the 4 and 5 factor models include a complex pair. Table 2 also reports the structural parameters implied by the *DLR_C* routine, as well as *SSC(AACM)* and *SSC(DLR_U)* methods.

It is striking how close the roots of the *DLR_U* estimator always are to the *ML* estimates. The roots obtained from the *SSC(AACM)* estimator are generally some distance away from the *ML* values. Most problematic for the *SSC(AACM)* method is the estimate of the least persistent root, which is always much too small. This is most apparent for the 4 and 5 factor models, where the estimates of the least persistent real root is negative, while the *ML*

estimate is about 0.6.

Interestingly, while the direct regression methods, *AACM* and *DLR_U*, return the same roots as *JSZ*, i.e. real roots of $\Phi_{\mathbf{x}}^Q$ for the 3 factor model and complex roots for 4 and 5 factors, for the 4 factor model the roots obtained by the *DLR_C* method are all real. This demonstrates the likely ‘distortion’ caused by the iterative procedure of imposing nonlinear self-consistency constraints that can drive the roots away from those in the linear regression estimates (*DLR_U*). Also, for the specification with 4 and 5 factors, the 2 largest roots obtained by the *DLR_C* method are furthest away from the *ML* estimates. The next section analyses the robustness of these findings.

The standard errors reported in this table come from the asymptotic variance matrix of the parameters found using the delta method (see e.g. Greene, 2002, Theorem D.21A). Specifically, the risk-neutral parameters $\Psi = (\mu_{\infty}^Q, \lambda')'$, where λ are the risk-neutral roots, are calculated from

$$\sqrt{T} \left(\hat{\Psi} - \Psi \right) \xrightarrow{d} N(\mathbf{0}, \mathbf{\Gamma} \mathbf{\Omega} \mathbf{\Gamma}'), \quad (28)$$

where $\mathbf{\Omega}$ is the variance matrix of the reduced form parameters \mathbf{a}_y and \mathbf{B}_y in (5) for both the set of J yields with our basic maturities and J yields with subsequent maturities, and $\mathbf{\Gamma}$ is the Jacobian matrix of partial derivatives calculated numerically. Since Σ has no effect on the roots and only a second-order effect on the level parameter, we assume that it is known with certainty in these calculations, having checked that it does not materially affect the results.

As we would expect, the efficient *JSZ* maximum likelihood procedure generates the smallest standard errors. The self-consistent linear regression methods based on yields, *DLR_C* and *SSC(DLR_U)* have slightly larger standard errors, but they are generally of the same order of magnitude as those returned by the maximum likelihood algorithm. In particular, the very low standard errors of the latter indicate that although the raw regression estimates are under-identified, their eigenvalues are strongly rooted in the data, helping to explain why our approach works as well as it does. On the other hand, the standard errors from the *SSC(AACM)* routine are an order of magnitude larger, making some of them

statistically insignificant. This is particularly evident for the level parameter, which is poorly identified from the excess return regressions. This provides further evidence that the model parameters obtained from excess returns should be treated with caution.

4.1.4 Robustness

To examine the robustness of these results, we conducted a bootstrap simulation exercise that takes the *ML* estimates of the *JSZ* model in Table 2 as the ‘true’ values and uses these to generate 5,000 artificial data samples of the same length and character as the original data set.

In the first stage we use the estimates of the \mathcal{P} -dynamics in (11) to simulate the time series of the principal components.⁷ This gives the ‘true factors’ for each sample. We next use each set of factors to generate a cross-section of yields using (5) with \mathbf{a}_y^* and \mathbf{B}_y^* obtained from the recursions (3-4) using the ‘true’ *JSZ* parameters. Then we add the measurement errors, obtained by randomly drawing the joint residuals \mathbf{v}_t^* from the ‘true’ *JSZ* model estimates using the circular stationary bootstrap proposed by Politis and Romano (1994).

We then present a hypothetical researcher with each data sample. Importantly they only see the cross-section of yields. They do not know what the true factors are, but have to back these out of the cross-section using principal components and the *JSZ* assumption that weighting up the ‘true’ residuals \mathbf{v}_t^* using the sample-dependent weights \mathbf{W} gives $\mathbf{W}'\mathbf{v}_t^* = \mathbf{0}$. Since this may not be true in these samples, setting the simulations up in this way allows us to test the validity of this assumption by checking the parameter estimates for the measurement error bias discussed in Section 3.1. Finally, our researcher uses these estimated factors to estimate the term structure model using the *ML*, *DLR_C* and *SSC(DLR_U)* methods. Table 3 reports the bias and root-mean-square error (*RMSE*) of each method. It shows that the *SSC(DLR_U)* and *ML* estimates display negligible bias and that their *RMSEs* are

⁷Specifically, starting from the value \mathbf{q}_1 in the first period of the original sample, we use (11) to project this forward in time over a period of 896 months given the values of the previous period and adding forecast errors \mathbf{u}_{t+1}^P that are randomly selected (with replacement) from the original sample of forecast errors. We dispose of the first 500 generated observations, leaving a set of 396 observations that match the size and character of the historical sample.

close. However, the $SSC(AACM)$ estimator always has a much larger $RMSE$ than the $SSC(DLR_U)$. It also exhibits large negative bias in the least persistent real root, consistent with our observation in Section 4.1.3 regarding the point estimate of this parameter.

The largest bias and $RMSEs$, however, are obtained by the DLR_C estimator, which are generally an order of magnitude bigger than for other methods. Particularly notable are the large biases for the 2 largest roots for the 4 and 5 factor models, which, again, is consistent with the comparison of the point estimates in Table 2.

As mentioned in the introduction, one of advantages of the SSC method is that it does not require the researcher to specify whether the roots of the model are real or complex, distinct or repeated. This could potentially inhibit the simulation exercise, since the estimated roots might not match the ‘true’ roots. We found, however, that the SSC methods estimated the same roots as ‘true’ roots in over 99% of simulations (when calculating the bias and $RMSE$ we rejected the remaining results when the estimated roots where ‘incorrect’). Also, the DLR_C estimator returned the correct roots in over 99% cases for the model with 3 and 5 factors, but for the model with 4 factors it found the correct roots in only about 70% of simulations.

4.1.5 Linear regression estimates as starting values in ML estimation

So far we have considered the DLR_U and $SSC(DLR_U)$ estimators as stand-alone estimation methods. However, it is well known that the likelihood function exhibits multiple local maxima and it is standard practice to use a range of starting values to identify these, making it likely (though not certain) that they include the global optimum. As suggested in the introduction, a researcher interested in the ML estimates could view the roots from the linear DLR_U method (i.e. the SSC_{DLR_U} estimates) as potentially useful starting values. To investigate this option, we used these estimates as one of the sets of parameter starting values for the ML routine in the bootstrap simulations. As another set we used the ‘true’ values which were used to simulate the artificial data. Finally, to represent the usual prac-

tice, we use 5 sets of random starting values.⁸ The parameters reported in Table 2 are those that gave the maximum for the optimized likelihood out of these seven sets. How likely is it that any particular starting value specification returns this best fit? Although there can be no guarantee that this represents the true global optimum, this should indicate the ability of different specifications to distinguish the global from other local optima.

As Hamilton and Wu (2012) observed, we find that parametrization in terms of the level parameter r_{∞}^Q frequently leads the numerical routine to stall. They showed that this problem is alleviated by parametrization in terms of μ_{∞}^Q , but we go a step further with our *ML* algorithm by concentrating the level parameter out from the likelihood function, as shown in Section 3.2. Thus, we only estimate the K roots of the Φ^Q matrix and the $K(K+1)/2$ parameters of Σ . We find that this algorithm is very robust for the model with 3 factors, always converging very close to the same maximum, irrespective of the choice of starting values. The ability of different specifications to find the largest local maximum for the 4 and 5 factor models is reported in Figure 1. This shows that with 4 and 5 factors, initiating the search from the roots of the DLR_U estimator as the starting values gives the best fit about as often as initiating the search from the ‘true’ values of the parameters. In both cases the convergence rate is about 99%. This contrasts starkly with the performance of random values. When using a single set of random starting values, the algorithm converges to the best fit in about 40% and 61% of simulations for the 4 and 5 factor models, respectively. Even if 5 sets of different starting values are used, this is found in only 91% of simulations (see Figure 8(a)). Interestingly, the performance of random starting values for the 5 factor model appears to be better than for the 4 factor model (see Figure 8(b)).

⁸Specifically, we randomly draw the starting values for the real roots (and the real part of the complex roots) from the range $(1 - 0.1K, 1)$, where K is the number of factors, and the imaginary part of the complex roots is drawn from the uniform distribution in the range $(0, 0.1)$. The starting values for Σ again come from the *VAR* model of the physical dynamics.

4.2 Discussion

It is perhaps not surprising to find that imposing self-consistency on regression estimators by removing redundant parameters improves their performance. But it is a surprise to find that in practice the constrained DLR estimator does not perform as well as our SSC approach. This is reflected by the $RMSEs$ of DLR_C , $SSC(DLR_U)$ and $SSC(DLR_C)$ in Table 1 and the parameter biases and $RMSEs$ shown in Table 3.

The DLR_C method is based on a linear approximation of the highly nonlinear self-consistency constraints. Indeed, several iterations are typically necessary to reach a self-consistent solution, which indicates that the non-linearities involved are important. In other words, the convergence gradient is not optimal in the ML sense and in the iteration process the roots tend to move away from their OLS and ML counterparts. On the other hand, the roots recovered by the SSC method preserve the characteristics of the OLS risk-neutral dynamics. As originally noted by Hamilton and Wu (2014), the OLS slope coefficients are numerically very close to the no-arbitrage values estimated by their χ^2 minimization procedure (and by implication ML), so there is little efficiency loss in using our SSC approach instead. In contrast, Hamilton and Wu (2014) noted that the no-arbitrage constraints on the yield intercepts were strongly rejected in their data set.

Importantly, the SSC method estimates the level parameter conditional on the risk-neutral roots, which is similar to concentrating the parameters out of the likelihood in ML procedures. On the other hand, the DLR_C estimator constrains all parameters simultaneously, which means that, due to the iteration process, the risk-neutral level parameter μ_∞^Q may not be estimated efficiently. The importance of this point can be seen by comparing $RMSEs$ of DLR_C and $SSC(DLR_C)$ in Table 1. Since both estimators use the same roots, they have the same factor loadings \mathbf{B}_y , and therefore the difference in performance between them is solely due to the superior SSC estimate of the level parameter.

5 Modelling the physical dynamics and the term premium⁹

Thus far, we have focussed on our new algorithms for estimating the \mathcal{Q} -dynamics quickly and consistently. However, the recent literature has been focussed on the \mathcal{P} -dynamics, which are used to represent market expectations. These are especially important from a policy perspective because subtracting these expectations from the corresponding yields gives residuals that can be interpreted as term premiums. Unfortunately, because the physical dynamics are estimated from the time-series dynamics they are subject to prediction errors that are an order of magnitude larger than the measurement errors found in the cross-section, as noted by Dai and Singleton (2000), Cochrane and Piazzesi (2008) and others. They usually contain many insignificant parameters and may also be subject to small sample bias. The new time series methods have been focussed on ways of dealing with these problems. However, they can be very time-intensive when used in conjugation with ML estimates of the risk-neutral dynamics. Because our algorithms solve almost instantly for the \mathcal{Q} -dynamics, they complement these innovations nicely.

5.1 Generalizing the dynamic structure

Thus far, we have assumed that the dynamics are $VAR(1)$ under both the risk-neutral and the physical measures, which is the most common $DTSM$ specification in this literature. However, recently some papers, have considered generalizing the factor dynamics. For instance, Abbritti et al. (2016) proposed $VARFIMA$ real world dynamics, while Goliński and Spencer (2017) considered $ARFIMA$ risk-neutral factor dynamics. Both of these papers assumed a linear specification of the price of risk, which implies non-standard dynamics for the corresponding equivalent measure through the no-arbitrage condition. Le et al. (2010) considered a model with $VAR(1)$ dynamics under the risk-neutral measure but with a non-

⁹We are grateful to two referees of this journal who urged us to investigate more general dynamic models and recently developed methods for improving the performance of the physical dynamics.

linear specification of the price of risk, which also implies non-standard \mathcal{P} -dynamics. There is also an immense literature that examines the time series properties of interest rates outside the no-arbitrage framework.

These models are computationally difficult and lie well beyond the linear-in-variables framework considered here. However, given the evidence suggesting that the $VAR(1)$ dynamics are too simple, we investigate a linear generalization proposed by Joslin, Le and Singleton (2014), who consider $VAR(p)$, $p > 1$, for the physical dynamics and $VAR(1)$ for the risk-neutral dynamics. The advantage of this framework is that the dynamics are explicit under both probability measures, while the price of risk specification remains implicit and linear-in-variables. This means that the decomposition of a yield into an expectation and risk premium remains just as straightforward as it is in the basic $VAR(1)$ model. In this section we employ standard statistical selection criteria to find the best specification of the time series dynamics. We show that our linear estimation methods allow restrictions within this generalized setting to be tested much more quickly than ML does.

Thus, we now consider a model defined by (5) and (6), but replace the specification of the \mathcal{P} -dynamics (11) by:

$$\mathbf{q}_{t+1} = \boldsymbol{\mu} + \sum_{j=1}^p \boldsymbol{\Phi}_j \mathbf{q}_{t-j+1} + \mathbf{u}_{t+1}^{\mathcal{P}}. \quad (29)$$

We restrict our attention to the three factor model, $K = 3$, since a higher number of factors poses significant risk of over-fitting the model.

We consider the $VAR(p)$ model with up to $p = 12$ lags. We start with the likelihood ratio test (LR_c in Table 4) for each $VAR(p)$, making the adjustment for small sample bias in the likelihood ratio¹⁰ proposed by Sims (1980). This statistic tests the significance of the last lag (i.e. $VAR(p-1)$ under H_0 vs. $VAR(p)$ under H_A). The results for the first six lags are reported in Panel A of Table 4. The LR test strongly rejects the null of $p = 1$ against the alternative $p = 2$. Panel B of Table 4 reports (the negative of) the information criteria

¹⁰See also Hamilton (1994), ch.11.

for each lag. The Akaike (*AIC*) and Hannan-Quinn (*HQIC*) information criteria favour the $VAR(2)$, while the Bayesian information criterion (*BIC*), which is stricter, selects the $VAR(1)$. The unrestricted parameters of the $VAR(1)$ process for the factors are reported in Panel A of Table 5, while Panel A of Table 6 reports the estimates for the $VAR(2)$ model.

5.2 Selecting the best model of the physical dynamics

As we have noted, many of the parameters of the price of risk and the physical dynamics are insignificant, making estimates of the term premium poorly determined. Moreover, the physical dynamics are less persistent, possibly due to the well-known small sample bias. Researchers have recently been trying to improve these estimates by imposing various parameter restrictions. We can exploit our rapid estimation procedure to implement these for models that would otherwise be too time consuming to consider.

We first apply the model selection procedure proposed by Joslin et al. (2014). Following Bauer (2018) we write the parameters of the \mathcal{P} -dynamics in (11) in terms of those of the \mathcal{Q} -dynamics in the $VAR(1)$ model (2) using:

$$\boldsymbol{\mu} = \boldsymbol{\mu}^{\mathcal{Q}} + \mathbf{l}_0, \quad \boldsymbol{\Phi} = \boldsymbol{\Phi}^{\mathcal{Q}} + \mathbf{L}_1, \quad (30)$$

and testing zero restrictions on the risk-adjustment parameters \mathbf{l}_0 and \mathbf{L}_1 . Similarly, in the $VAR(2)$ model, we test the significance of \mathbf{l}_0 , \mathbf{L}_1 and the extra lag parameters ($\boldsymbol{\Phi}_2 = \mathbf{L}_2$). Since the zero price-of-risk restrictions effectively push the parameters of the \mathcal{P} -dynamics towards those of the \mathcal{Q} -dynamics, which are more strongly determined, one would expect that the restricted models would be more persistent (Cochrane and Piazzesi (2008)).

Thus, for the optimal $VAR(p)$ model of the \mathcal{P} -dynamics selected by the information criteria, we consider all possible combinations of the zero restrictions on the price of risk parameters. Specifically, we estimate 2^{12} different model specifications for $VAR(1)$ selected by the *BIC* and 2^{21} specifications for $VAR(2)$ using *AIC* and *HQIC*.

Importantly, the restricted \mathcal{P} -dynamics can be estimated by the method of restricted

least squares, which fits neatly into our linear-in-variables framework. Moreover the roots of the risk neutral response matrix Φ^Q do not change as this model of the \mathcal{P} -dynamics shifts (i.e. they remain as reported in Table 2).

However, as in the basic $VAR(1)$ model, the level parameter introduces a slight complication: our solution for Σ and hence μ^Q is no longer sequential as it is with the unconstrained OLS model of the \mathcal{P} -dynamics in earlier sections. We need to find the constrained \mathcal{P} -dynamics and μ^Q simultaneously, which we do using a recursive estimation strategy. Specifically, we start with the estimate of Σ from the unconstrained OLS model, use this to find μ^Q and hence the constrained \mathcal{P} -dynamics. We repeat this procedure until the norm of the Σ matrix is smaller than 10^{-6} . This procedure is similar in spirit to that proposed by Diez de Los Rios (2015). However, in practice because Σ has only a second-order effect on the \mathbf{a} coefficients, we find that the cross-sectional fit is virtually unaffected by the specification of the \mathcal{P} -dynamics and typically convergence is achieved in a single iteration.

In marked contrast, the selection procedure for the full ML estimates requires the whole model of the cross-section to be re-estimated for each combination of the restrictions. That is because as Joslin et al. (2014) note, when there are restrictions across the model of the \mathcal{Q} - and \mathcal{P} -dynamics as in (30), the latter can no longer be estimated separately. We find that this puts up the time required for ML estimation from around a second for the unrestricted model to between 8 and 20 seconds for the restricted model. Using this procedure on a similar macro-finance model, which generalizes the \mathcal{P} -dynamics by introducing growth and inflation variables, Joslin et al. (2014) manage to select a model from 2^{19} possible combinations. This is unlikely to be feasible for our 3 factor $VAR(2)$ model, for which there are over two million combinations, four times as many as in their problem. However, using our linear method we are able to analyse this number of combinations in just 7 – 8 hours using a single desktop computer.

We use our regression approach¹¹ to select the best $VAR(1)$ and $VAR(2)$ models using

¹¹The information criteria select the optimal model based on the value of the likelihood function at the maximum. As explained in Joslin et al. (2011), the model likelihood can be decomposed into a part coming from the cross-sectional fit and another from the physical dynamics of the factors. In order to avoid the risk of distorting the likelihood value by using our sequential linear estimation method for finding the cross-

information criteria and report the ML estimates of the resulting models in Tables 5 and 6. For the $VAR(1)$ dynamics, the BIC selects the model with nine restrictions, leaving only the first column of the Φ_1^Q matrix unconstrained. For the $VAR(2)$ dynamics, the AIC criterion selects a model with six restrictions. The $HQIC$, which is stricter, selects the same six restrictions as the AIC plus three additional restrictions. As expected, the restricted models are more persistent than the unrestricted ones, as indicated by the largest eigenvalue of the \mathcal{P} -dynamics, reported in the last column of Tables 5 and 6.

5.3 Directly restricted degree of persistence

To deal directly with the problem of small sample bias in the \mathcal{P} -dynamics Bauer et al. (2012) propose a simulation-based correction. We apply this technique to the $VAR(2)$ process, to compare with the results of the price of risk restrictions shown in Panels A to C of Table 6. In our data, the constraint in the adjustment procedure that keeps the process stationary is binding and the shrinkage method is necessary to keep the eigenvalues of the system smaller than unity. The estimates are reported in Panel D of Table 6. The highest eigenvalue that results is 0.9999, which is practically a unit root.

Finally, we apply the restriction proposed by Joslin et al. (2011), that the highest eigenvalue under the physical measure is equal to the value (of 0.9976) under the risk-neutral measure, shown in Panel A of Table 2. We implement this adjustment using linear reduced rank regression. Since the process under the risk-neutral measure is more persistent than under the physical measure, this method is also designed to reduce the small sample autoregressive bias. We call this model $Eig(Q)$ and report the estimates in Panel E of Table 6.

sectional part of the likelihood function, the cross-sectional part of the likelihood is obtained for each model from the benchmark OLS regression (5).

5.4 Evaluating the term premium

For each specification of the \mathcal{P} -dynamics, we calculate the term premium, defined as the difference between the fitted yield and the average forecast of the 1-month rate for a given maturity. We focus on the 10-year term premium, which is the common benchmark in the literature. The diagonal entries in Table 7 show the standard deviation of each estimate of the term premium. The largest variation (1.07%) is displayed by the term premium from the model with unrestricted $VAR(2)$ \mathcal{P} -dynamics, while the $VAR(1)$ BIC model, which has the most heavily restricted dynamics, exhibits the smallest variation (0.19%). The cross-correlations between these six estimates are reported in the off-diagonal entries of Table 7.

Figure 2 plots the estimates of the different term premia. The grey areas in the plot denote the NBER recession periods. Clearly, the differences between different term premium estimates can be very large at times. Consistent with the results of Table 7, the term premium from the unrestricted $VAR(2)$ model takes the most extreme values, while the BIC term premium is the smoothest in our sample.

The decomposition of the yield into a market expectation and a term premium can be used to analyse the behaviour of the U.S. Treasury market during key monetary policy episodes. For example, this has been used extensively to analyse the well-known ‘conundrum’ identified originally by Alan Greenspan, when in 2004 and 2005 short term policy rates rose but the 10 year forward rate continued to fall (see e.g. Backus and Wright, 2007). The term premia in these models exhibit counter-cyclical behaviour, increasing during recessions and declining between recessions. So they suggest that the premium would decline as short rates increase. However, the response in the basic $VAR(1)$ BIC model is hard to discern from the chart, as is the response in the $VAR(2)$ $HQIC$. These models suggest that the conundrum occurred because long term rate expectations fell or were flat. However, the premium in the more flexible $VAR(2)$ AIC , BRW and $Eig(\mathcal{Q})$ models fell back markedly over this period. Like the recent macro-finance term structure literature (Joslin et al., 2011), this suggests more plausibly, that the conundrum occurred because the premium fell, offsetting an increase in

long term rate expectations. These results underline the importance of developing richer models of dynamic adjustment in term structure models carefully.

6 The U.S. Treasury and German Bund markets

Encouraged by the results for the U.S. Treasury market, we then examined the performance of the *SSC* approach for other countries like Germany, and then go on to look at a joint two-country model. *AACM* have already used their regression approach to model the U.S. Treasury nominal bond and *TIPS* markets jointly. In a similar spirit we use the *SSC(DLR_U)* estimator to estimate a two-country German-U.S. model, which should offer an interesting test. Specifically, we look at U.S. Treasury bonds and German bunds from the perspective of a U.S. dollar based investor and use the dollar risk-neutral probability measure for pricing. The common factor model algebra is set out in Appendix A.

For this two-country application we used data for the period January 1987 to December 2015, which was dictated by the availability of data for the German market. For the U.S. the annual maturities again come from Gurkaynak et al. (2007). For Germany, we use estimates published by the Bundesbank, also estimated using the Svensson method.¹² In the absence of 15 year maturity data for Germany, we use maturities 1 month, 1, 2, 3, 4, 5, 7, and 10 years for both countries. In the absence of the equivalent Fama-Bliss estimates of German short rates we use the 1-month Euribor rate and, for consistency, we also use the 1-month Libor to complete the U.S. data set. These data are shown in Figure 3.

Table 8 reports basic summary statistics for these data. The first principal component for each country is depicted in Figure 4. These are similar, exhibiting a strong downward trend, but there is some suggestion that the U.S. leads Germany. Figure 5 shows the factor loadings for the two countries, which are remarkably close, indicating that a common factor model might be appropriate.

We begin by estimating separate models for the two countries. Table 9 shows the results

¹²Available at: http://www.bundesbank.de/Navigation/EN/Statistics/Money_and_capital_markets/Interest_rates_and_yields/Term_structure_of_interest_rates/term_structure_of_interest_rates.html.

of this exercise in terms of model fit. Once again, the improvement in fit of the $SSC(DLR_U)$ estimator over $AACM$ is impressive. To conserve space, we do not report other methods, such as DLR_U or SSC estimator based on $AACM$ estimate of the Φ^Q matrix, but they are available upon request. The ranking of the results is consistent with those reported in Section 4.1: the $SSC(DLR_U)$ method greatly improves upon the estimators that do not impose the self-consistency restrictions and gives the best fit among the methods based on linear regression. Generally, as reported in Table 9 the $SSC(DLR_U)$ estimator gives a close fit to the unrestricted OLS regressions for any number of factors. The fit for Germany is better than for the U.S. For the models with 3 factors the average $RMSE$ for the U.S. obtained by the $SSC(DLR_U)$ method is 1 basis point larger for the U.S. yields and 0.2 basis points for Germany, relative to the OLS fit. For the 5 factor model the fit of the $SSC(DLR_U)$ is virtually the same as the OLS benchmark. The $RMSE$ results for the $AACM$ method are generally much higher, often more than double that of the OLS .

Table 10 reports the cumulative percentage of the variance in the U.S. and German yields explained by the PCs extracted (a) separately from the single country yield covariance matrices and (b) from the joint covariance matrix of yields. This provides a *prima facie* indication of the performance of an unrestricted OLS common factor model. These results do not suggest that common factors are present since to achieve a similar fit, the common factor model needs as many factors as the total number used in the two separate country models. For instance, if we assume that the yields of each country are appropriately fitted by the 3 first PCs extracted from each country's data (which explain about 99.97% of their variance), then we need to use 6 PCs extracted from the joint data set to achieve the same performance.

The results for the common factor models are shown in Table 11. The technical details extending the $SSC(DLR_U)$ estimator to multi-market framework are relegated to Appendix A. For Germany, the fit of the 6 factor joint model is similar to that for the 3 factor single-country model shown in Table 9, although the fit of the 6 factor joint model for the U.S. is slightly better than that obtained for the single country model. Importantly, consistent with

the single-country results, the $SSC(DLR_U)$ estimator performs very well, giving $RMSEs$ generally very close to those of the OLS benchmark. On the other hand, the $AACM$ estimator (and DLR_U regressions in unreported results), which do not impose self-consistency of the model, perform much worse with $RMSEs$ often more than double those given by the OLS estimator

These results support the argument that the $SSC(DLR_U)$ estimator offers a computationally efficient alternative to the full ML method used by JSZ . It provides a similar fit to the data, which is not achieved by the routines that do not impose the self-consistency restrictions. In the remainder of this section we look at the relationship between the single and two-country models and the view that the U.S. and German markets are spanned by common factors.

The factors in the common and single country models exhibit an interesting pattern. The first common factor is depicted in Figure 4 and is clearly a compromise between the level factors in the two single country models. We find that the second common factor is highly correlated with the difference between the level factors in the two single country models. This relationship is shown in the top panel of Figure 6. Furthermore, we find that regressing the level factors for the single country models on the first two common factors gives an almost perfect fit (not reported). We find similar pair-wise factor-rotation relationships between the slope and curvature factors. Specifically, the third common factor is a slope factor and is a compromise between the second factors in the single country models, while the fourth common factor reflects the difference between them, as shown by the middle panel in Figure 6. Similarly, the fifth common factor is a compromise between the third (curvature) factors in the single country models and the sixth common factor reflects the difference between them, as shown by the final panel in Figure 6.

These pair-wise factor-rotation relationships are reflected in the factor loadings for the two countries in the 6 common factor model shown in Figure 7. The continuous lines show the loadings on the odd-numbered factors, which are interpreted as common level (black), slope (blue) and curvature (red) factor loadings respectively. These clearly resemble the

respective loadings in the single country models shown in Figure 5. The loadings on the associated even numbered common factors are shown with dashed lines. Consistent with the view that these allow for relative effects missed by the associated common factors, the signs are reversed for the U.S. and Germany.

Thus, we find that if we do not restrict the loadings but just allow the data to speak, these data tell us that the joint model really just mimics the U.S. and German single country models, giving a similar fit. Figure 4 shows that the level factor in the joint model picks up the down-trend in the single country factors, but misses out important relative effects - indeed there is a suggestion that the U.S. level factor leads the German level factor. Clearly, the second joint factor is needed to allow for these differences, with similar effects in the slope and curvature factors. We conclude that most of the contemporaneous interaction between the U.S. and Germany can be spanned equally well by single-country models.

Nevertheless, a paper by Meldrum et al. (2016) using these data finds strong evidence of *unspanned* (i.e. delayed) spillovers between the U.S. and Germany. This finds that overseas unspanned factors - constructed from the components of overseas yields that are uncorrelated with domestic yields - have significant explanatory power for subsequent domestic bond returns. Reflecting this, we find that there are significant differences between the single country and joint approaches in terms of the \mathcal{P} -dynamics and hence the risk premia (which reflect differences between the risk-neutral and \mathcal{P} -dynamics). Figure 8 shows how the model decomposes the 10-year yield into components representing 10-year interest rate expectations and a risk premium. This decomposition largely depends upon the \mathcal{P} -dynamics represented by the time series VAR in (11), and in particular its maximum eigenvalue which governs their persistence. The VAR is used to forecast the interest rates and hence compute the yields that would be observed if the expectations hypothesis held and investors were risk-neutral. Subtracting this estimate from the model-fitted yield for any maturity then gives the standard estimate of the risk premium. This is shown in Figure 8 for both countries and for both single and common factor models. Pooling the data for the two countries seems to make little qualitative difference to the U.S. decomposition. However, it does give a more

plausible decomposition of the German curve. This is explained by the difference in the persistence of the \mathcal{P} -dynamics and hence the risk-neutral expectations.

Table 8 shows that the maximum eigenvalue for the common factor VAR is straddled by those of the two single country $VARs$. The maximum eigenvalue in our German-only VAR model is 0.9992, which is very close to a unit root. This makes interest rates highly persistent, suggesting that the negative rates seen recently will last, making the risk-neutral expectations component of the 10-year yield negative. Thus, the German-only model attributes most of the fall in the 10-year yield since 2012 to falling interest rate expectations, leaving the risk premium relatively stable. In contrast, the joint model suggests that German rates will normalize more quickly, attributing the recent fall in the 10-year yield largely to the fall in the risk premium, with falling interest rate expectations playing a secondary role.

7 Conclusion

Until the advent of the recent regression-based estimators, estimates of the parameters μ^Q and Φ^Q of the risk-neutral dynamics were obtained by embedding them in the bond pricing coefficients and then using numerical methods to find values that optimized the fit of these equations. However, the new linear regression methods allow us to estimate this structure by *OLS*. Although these estimators are not internally consistent, we can estimate the response matrix Φ^Q by *OLS* and use its characteristic roots to obtain a consistent response matrix for the *JSZ* canonical form. The level parameter then follows directly from the affine bond pricing recursions and the remaining risk-neutral parameters are determined by the internal consistency conditions. Finally, the affine recursions determine the bond yields.

It is reassuring to find that our parameter estimates are very close to global *ML* estimates of the risk-neutral parameters embedded in bond yields, and that the bias revealed by our bootstrap simulations is negligibly small. It is clear that the risk-neutral dynamics are strongly rooted in the data, allowing both approaches to be used to draw statistical inferences about the risk-neutral parameters and, in the case of our estimator, to fit the term structure

of bond yields by extrapolation.

Our algorithms ensure consistency by eliminating redundant parameters from the risk-neutral dynamics embedded in the cross-section of yields. Recent innovations in handling the time-series part of the *DTSM* work by eliminating insignificant parameters from the \mathcal{P} -dynamics and using other constraints to improve precision. However, they are used in conjunction with *ML* methods for estimating the risk-neutral dynamics, which are very time intensive when used in this way. Because our algorithms solve almost instantly for the \mathcal{Q} -dynamics, they complement these innovations nicely.

Our linear estimator offers the researcher a useful addition to the term structure toolkit. As the exercises reported here demonstrate, its speed gives it a clear edge in model selection and simulation exercises involving a large number of models or replications. *ML* estimation remains an option for estimating the final model if warranted. Reflecting this, regression methods have been extensively employed by central banks and other policy-makers (Bauer and Rudebusch, 2014), despite their lack of consistency and efficiency. Our new estimator avoids this trade-off and allows both academic researchers and practitioners to obtain results with speed, consistency and efficiency.

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8 APPENDIX A

Appendix A The two-country model

Let $\mathbf{p}_t^\$$ and \mathbf{p}_t^ϵ represent vectors of US and German bond prices in the local currency, the dollar or Euro respectively. We follow the international bond market literature in assuming that these prices are both lognormally distributed and, similar to (1), affine in a set of common set of K factors \mathbf{q}_t . Collecting bond prices for the two countries in one ($J = J^\$ + J^\epsilon$) vector:

$$-\mathbf{p}_t = - \begin{bmatrix} \mathbf{p}_t^\$ \\ \mathbf{p}_t^\epsilon \end{bmatrix} = \begin{bmatrix} \mathbf{a}^\$ \\ \mathbf{a}^\epsilon \end{bmatrix} + \begin{bmatrix} \mathbf{B}^{\$'} \\ \mathbf{B}^{\epsilon'} \end{bmatrix} \mathbf{q}_t. \quad (\text{A-1})$$

As before we assume that portfolios of bond yields represented by common principal components:

$$\mathbf{q}_t = -\mathbf{W}'\mathbf{M}^{-1}\mathbf{p}_t,$$

are noiseless, where \mathbf{M} is a diagonal matrix composed of the respective maturities of the U.S. and German bonds, $\mathbf{m} = [\mathbf{m}^\$, \mathbf{m}^\epsilon]'$. Thus, we can estimate the roots of the risk-neutral dynamics from the eigenvalues of the $\hat{\Phi}^\mathcal{Q}$ matrix estimated from (10).

In a multi-market setting we need to estimate the common factor parameters $\{\Phi_x^\mathcal{Q}, \mu, \Phi, \Sigma\}$ and $\{r_\infty^\mathcal{Q}, \delta_{x,1}, \Sigma_v\}$ specifically for each market. The loading recursions (14) are common to all markets, but $\mathbf{b}_{x,1}$ needs to be estimated for each one. To estimate these, we propose the following strategy. First, from (22) we can write the relation between $\delta_{x,1}$ and δ_1 for each market:

$$\begin{aligned} \delta'_{x,1} &= \delta'_1(\mathbf{W}'\mathbf{B}'_{y,x}) \\ &= \delta'_1(\mathbf{W}'\mathbf{B}'_{y,x})\hat{\Phi}_x(\mathbf{W}'\mathbf{B}'_{y,x})^{-1}(\mathbf{W}'\mathbf{B}'_{y,x})\hat{\Phi}_x^{-1} \\ &= \delta'_1\hat{\Phi}(\mathbf{W}'\mathbf{B}'_{y,x})\hat{\Phi}_x^{-1}. \end{aligned} \quad (\text{A-2})$$

Although the matrix Φ and the coefficients \mathbf{B}_x in a multi-country setting depend on the short rate coefficients $\delta_{x,1}$, we can substitute the eigendecomposition of the empirical estimate of $(\hat{\Phi} = \hat{\mathbf{L}}\hat{\Phi}_x\hat{\mathbf{L}}^{-1})$ and use the fact that $\hat{\mathbf{L}} \simeq (\mathbf{W}'\mathbf{B}'_{y,x})$ to obtain the estimated short rate loadings as:

$$\delta_{x,1} = \hat{\mathbf{L}}'\delta_1 \quad (\text{A-3})$$

using the *OLS* estimate of δ_1 for each market as in (6).

The level parameters for a multi-market model follow in a similar manner as for the one-market model. Since there are two countries, the level of the term structures might be different and thus a more natural and intuitive parametrization of the level is in terms of r_∞^Q rather than μ_∞^Q ; since e.g. level of the term structure in each country is generally different, we need multiple level parameters, and is it more natural to interpret multiple r_∞^Q rather than multiple μ_∞^Q . Recall that in this parametrization scheme we need to assume that there is no unit root under the risk-neutral measure. Although this assumption might seem restrictive, in the model estimation it is hardly an issue, since the probability of estimating the one of the risk-neutral roots of the model of exactly 1 in a linear regression is virtually zero. Having calculated the loading parameters $\mathbf{B}_x = [\mathbf{B}_x^{\$}, \mathbf{B}_x^{\text{€}}]$, we can find \mathbf{H} and $\mathbf{c}_1 = [\mathbf{c}_1^{\$}, \mathbf{c}_1^{\text{€}}]'$ from (24) and (26), respectively. The vector of level parameters, $\mathbf{r}_\infty^Q = [r_\infty^{Q\$}, r_\infty^{Q\text{€}}]'$, can be estimated by:

$$\hat{r}_\infty^Q = (\mathbf{C}_0'\mathbf{H}'\mathbf{H}\mathbf{C}_0)^{-1} \mathbf{C}_0'\mathbf{H}'(\tilde{\mathbf{a}}_y + \mathbf{H}\mathbf{c}_1), \quad (\text{A-4})$$

where \mathbf{C}_0 is a $J \times 2$ block diagonal matrix:

$$\mathbf{C}_0 = \begin{bmatrix} \iota_{J^{\$}} & \mathbf{0} \\ \mathbf{0} & \iota_{J^{\text{€}}} \end{bmatrix}$$

and $\tilde{\mathbf{a}}_y = [\tilde{\mathbf{a}}_y^{\$}, \tilde{\mathbf{a}}_y^{\text{€}}]'$ is a vector of unrestricted yield intercepts given the arbitrage-free slope coefficients given by $\tilde{\mathbf{a}}_y = \bar{\mathbf{y}}^o - \mathbf{B}_y'\bar{\mathbf{q}}$.

Tables

yields:	Root-mean-square error (bp)								
	1m	1y	2y	3y	5y	7y	10y	15y	Av.RMSE
Panel A: $K = 3$									
<i>OLS</i>	2.5	11.2	3.0	6.6	8.4	5.8	4.0	9.9	6.43
<i>JSZ</i>	2.6	12.0	3.7	7.4	8.5	6.1	5.6	10.2	7.01
<i>AACM</i>	2.5	20.4	6.4	7.9	14.4	15.2	10.6	20.2	12.20
<i>DLR_U</i>	2.5	17.6	7.0	7.3	8.9	7.5	7.6	10.0	8.55
<i>DLR_C</i>	2.6	11.7	4.9	7.3	9.4	8.3	6.5	13.6	8.05
<i>SSC(AACM)</i>	2.7	13.1	4.3	8.3	8.8	6.7	6.4	11.0	7.65
<i>SSC(DLR_C)</i>	2.6	12.4	3.9	7.7	8.7	6.1	5.9	10.1	7.17
<i>SSC(DLR_U)</i>	2.6	12.4	4.0	7.7	8.7	6.1	5.8	10.1	7.16
Panel B: $K = 4$									
<i>OLS</i>	0.3	3.1	3.0	3.1	1.4	3.4	3.4	4.0	2.71
<i>JSZ</i>	0.3	3.1	3.0	3.1	1.5	3.4	3.4	4.1	2.72
<i>AACM</i>	0.3	5.0	3.8	3.5	1.6	3.8	4.0	4.2	3.28
<i>DLR_U</i>	0.3	4.5	3.3	3.2	1.5	3.4	3.6	4.1	2.99
<i>DLR_C</i>	0.5	6.6	5.1	7.5	1.6	7.9	9.9	11.1	6.29
<i>SSC(AACM)</i>	0.3	3.4	3.3	3.2	1.7	3.5	3.6	4.2	2.89
<i>SSC(DLR_C)</i>	0.6	4.3	3.7	4.5	1.5	4.8	5.7	6.4	3.90
<i>SSC(DLR_U)</i>	0.3	3.1	3.0	3.1	1.5	3.4	3.4	4.1	2.72
Panel C: $K = 5$									
<i>OLS</i>	0.1	0.8	1.3	0.3	1.3	0.4	1.7	0.8	0.82
<i>JSZ</i>	0.1	0.8	1.4	0.3	1.3	0.5	1.7	0.8	0.86
<i>AACM</i>	0.1	3.2	2.6	1.4	1.9	0.8	2.2	1.0	1.64
<i>DLR_U</i>	0.1	6.5	2.8	1.6	1.5	0.5	1.7	0.9	1.96
<i>DLR_C</i>	0.3	3.6	5.1	2.2	6.3	3.3	9.8	4.3	4.34
<i>SSC(AACM)</i>	0.1	0.9	1.6	0.4	1.5	0.5	1.9	0.9	0.97
<i>SSC(DLR_C)</i>	0.2	2.1	3.2	1.2	3.7	1.6	5.2	2.3	2.42
<i>SSC(DLR_U)</i>	0.1	0.8	1.5	0.6	1.4	0.6	1.7	0.8	0.92

Table 1. Goodness of fit for the Treasury yields, as measured by the root-mean-square error of the measurement errors. The model is estimated with 3, 4 and 5 principal components of yields. The reported values are in basis points. The sample period is January 1983 to December 2015.

	$\mu_{\infty}^Q \times 100$	λ_1	λ_2	λ_3	λ_4	$(\lambda_{real} \pm i\lambda_{imag})$	
Panel A: $K = 3$							
JSZ	0.0403 0.0005	0.9971 0.0001	0.9714 0.0006	0.7537 0.0098			
$SSC(AACM)$	0.0434 0.2871	0.9958 0.0409	0.9779 0.0654	0.1657 0.2986			
$SSC(DLR_U)$	0.0338 0.0019	0.9976 0.0003	0.9686 0.0021	0.7862 0.0322			
DLR_C	0.0391 0.0016	0.9975 0.0004	0.9677 0.0026	0.7711 0.0301			
Panel B: $K = 4$							
JSZ	0.0143 0.0004	0.9998 0.0001	0.5930 0.0250			0.9664 0.0008	0.0194 0.0008
$SSC(AACM)$	0.0204 0.0631	0.9991 0.0089	-0.0666 0.2562			0.9656 0.0434	0.0221 0.0563
$SSC(DLR_U)$	0.0136 0.0015	0.9998 0.0002	0.6103 0.0630			0.9655 0.0020	0.0188 0.0019
DLR_C	0.0199 0.0089	0.9958 0.0040	0.9791 0.0153	0.9472 0.0217	0.6163 0.0660		
Panel B: $K = 5$							
JSZ	0.0129 0.0003	0.9998 0.0001	0.9621 0.0052	0.6020 0.0186		0.9728 0.0019	0.0184 0.0011
$SSC(AACM)$	-0.0006 0.0241	1.0012 0.0020	0.9741 0.0932	-0.0656 0.2446		0.9625 0.0286	0.0233 0.0692
$SSC(DLR_U)$	0.0125 0.0022	1.0003 0.0003	0.9528 0.0066	0.7100 0.0395		0.9789 0.0010	0.0211 0.0018
DLR_C	0.0092 0.0010	1.0041 0.0007	0.8691 0.0181	0.7249 0.0332		0.9878 0.0011	0.0282 0.0008

Table 2. Estimates of the parameters of risk-neutral dynamics obtained by different methods for 3, 4 and 5 factor JSZ models.

		$\mu_{\infty}^Q \times 100$	λ_1	λ_2	λ_3	λ_4	λ_{imag}	\pm	xi
Panel A: $K = 3$									
JSZ	Bias	0.2521	-0.0001	0.0023	-0.0015				
	RMSE	0.2634	0.0003	0.0028	0.0244				
$SSC(AACM)$	Bias	0.6978	-0.0012	0.0034	-0.0747				
	RMSE	0.7329	0.0016	0.0050	0.0822				
$SSC(DLR_U)$	Bias	0.6094	0.0001	-0.0012	0.0133				
	RMSE	0.6341	0.0004	0.0022	0.0371				
DLR_C	Bias	0.5710	0.0002	-0.0022	0.0095				
	RMSE	0.5944	0.0007	0.0036	0.0353				
Panel B: $K = 4$									
JSZ	Bias	0.2480	0.0002	0.0064			0.0014	-0.0027	
	RMSE	0.2563	0.0003	0.1057			0.0022	0.0040	
$SSC(AACM)$	Bias	0.5469	0.0000	-0.1084			0.0005	0.0013	
	RMSE	0.5696	0.0005	0.1204			0.0034	0.0037	
$SSC(DLR_U)$	Bias	0.5561	0.0000	0.0010			-0.0005	-0.0006	
	RMSE	0.5805	0.0002	0.0562			0.0022	0.0027	
DLR_C	Bias	0.8418	-0.0077	-0.0193			0.0031	-0.0024	
	RMSE	6.1602	0.0240	0.0674			0.0082	0.0077	
Panel B: $K = 5$									
JSZ	Bias	0.4522	0.0004	-0.0011	0.0291		-0.0001	-0.0031	
	RMSE	0.4748	0.0007	0.0098	0.0346		0.0039	0.0051	
$SSC(AACM)$	Bias	0.7212	-0.0016	0.0092	-0.1070		-0.0055	0.0049	
	RMSE	0.7701	0.0318	0.0170	0.1229		0.0498	0.0082	
$SSC(DLR_U)$	Bias	0.6570	0.0002	-0.0051	0.0441		0.0025	0.0014	
	RMSE	0.6864	0.0004	0.0099	0.0548		0.0035	0.0028	
DLR_C	Bias	0.4024	0.0033	-0.0569	0.0744		0.0127	0.0070	
	RMSE	0.4258	0.0035	0.0584	0.0800		0.0128	0.0071	

Table 3. Bias and RMSE calculated for different estimation methods for 3, 4 and 5 factor JSZ model based on 5,000 simulation bootstrap assuming the JSZ estimates are the ‘true’ values. The simulated model for 3 factors was based on 3 real roots, while for 4 and 5 factors included a complex root.

Lags (k):	1	2	3	4	5	6
Panel A: LR test						
LR_c (p - value)	0.0000	0.0000	0.4067	0.1024	0.4903	0.6492
Panel B: Information Criteria						
$-BIC$	32.7973	32.7744	32.6677	32.5754	32.4667	32.3538
$-AIC$	32.8899	32.9596	32.9455	32.9458	32.9296	32.9094
$-HQIC$	32.8531	32.8861	32.8353	32.7988	32.7460	32.6890

Table 4. VAR lag selection for factor dynamics under the physical measure. Panel A shows the p -values of the likelihood ratio test for the significance of the k -th lag. The table shows the LR_c statistic, which adjusts for the small sample bias as proposed by Sims (1980). In Panel B we report the negative of Bayesian, Akaike and Hannan-Quinn information criteria. The bold font denotes the highest value for each criterion. The sample period is January 1983 to December 2015.

μ	Φ_1			\mathcal{P} -eigenvalues
Panel A: VAR(1) (unrestricted)				
0.0023 0.0014	0.9936 0.0048	0.0008 0.0295	-0.2055 0.0880	0.9933
-0.0019 0.0007	-0.0014 0.0024	0.9483 0.0147	0.3683 0.0439	0.9678
0.0030 0.0007	0.0009 0.0021	0.0186 0.0133	0.6191 0.0397	0.6000
Panel B: Var(1) BIC (9 restrictions)				
-0.0014 0.0004	1.0043 0.0026	0.0617 0.0075	-0.2112 0.0398	0.9995
-0.0008 0.0002	-0.0075 0.0016	0.9754 0.0047	0.2944 0.0295	0.9709
0.0013 0.0001	0.0071 0.0012	-0.0013 0.0030	0.7342 0.0066	0.7436

Table 5. MLE Estimates for different $VAR(1)$ specifications of the physical dynamics in the DTSM. Panel A shows the unrestricted estimates of the $VAR(1)$ model. Panel B shows the estimates of the model selected by the BIC model. The bold font denotes restricted parameters. The sample period is January 1983 to December 2015.

μ	Φ_1			Φ_2			\mathcal{P} -eigenvalues
Panel A: VAR(2) (unrestricted)							
0.0012 0.0015	1.1334 0.0524	-0.1155 0.1264	-0.3231 0.1358	-0.1395 0.0523	0.1067 0.1247	0.2585 0.1467	0.9938
-0.0017 0.0007	-0.0295 0.0264	0.9811 0.0637	0.4140 0.0684	0.0284 0.0264	-0.0294 0.0628	-0.0789 0.0739	0.9658
0.0030 0.0007	-0.0640 0.0235	0.1607 0.0567	0.6285 0.0609	0.0652 0.0234	-0.1460 0.0559	-0.0113 0.0658	0.6033
Panel B: VAR(2) AIC (6 restrictions)							
-0.0014 0.0004	1.1388 0.0478	0.0201 0.0248	-0.2110 0.0399	-0.1400 0.0477	0 0.0000	0.2449 0.0758	0.9995
-0.0023 0.0007	-0.0170 0.0018	0.9755 0.0047	0.4259 0.0509	0.0169 0.0029	-0.0192 0.0152	-0.0650 0.0473	0.9695
0.0031 0.0006	-0.0688 0.0186	0.1593 0.0371	0.6153 0.0395	0.0679 0.0186	-0.1458 0.0376	0 0.0000	0.6015
Panel C: VAR(2) HQIC (9 restrictions)							
-0.0014 0.0004	1.1312 0.0475	0.0616 0.0075	-0.2109 0.0399	-0.1361 0.0476	0 0.0000	0.1933 0.0632	0.9963
-0.0028 0.0006	-0.0170 0.0018	0.9755 0.0047	0.3769 0.0424	0.0162 0.0030	0 0.0000	0 0.0000	0.9801
0.0033 0.0006	-0.0674 0.0187	0.1426 0.0345	0.6157 0.0393	0.0685 0.0187	-0.1367 0.0344	0 0.0000	0.6098
Panel D: VAR(2) BRW							
0.0004 0.0015	1.1291 0.0520	-0.1098 0.1255	-0.3279 0.1349	-0.1291 0.0519	0.1022 0.1239	0.2545 0.1457	0.9999
-0.0020 0.0007	-0.0300 0.0262	0.9909 0.0631	0.4149 0.0678	0.0297 0.0261	-0.0301 0.0623	-0.0872 0.0733	0.9778
0.0029 0.0007	-0.0653 0.0233	0.1612 0.0562	0.6363 0.0604	0.0657 0.0233	-0.1435 0.0554	-0.0026 0.0652	0.6170
Panel E: VAR(2) Eig(\mathcal{Q}) (1 restriction)							
0.0007 0.0015	1.1360 0.0520	-0.1160 0.1253	-0.3223 0.1347	-0.1388 0.0519	0.1071 0.1237	0.2617 0.1455	0.9971
-0.0018 0.0007	-0.0292 0.0262	0.9811 0.0631	0.4141 0.0678	0.0284 0.0261	-0.0294 0.0623	-0.0784 0.0733	0.9658
0.0031 0.0007	-0.0643 0.0233	0.1608 0.0562	0.6284 0.0603	0.0652 0.0232	-0.1461 0.0554	-0.0117 0.0652	0.6027

Table 6. MLE Estimates for different $VAR(2)$ specifications of the physical dynamics in the DTSM. Panel A shows the unrestricted estimates of the $VAR(2)$ model. Panel B and Panel C report the estimates of the models selected by AIC and $HQIC$, respectively. Panel D reports the $VAR(2)$ estimates with the Bauer et al. (2012). Panel E shows the estimates for a model that restricts the largest eigenvalue of the physical dynamics to be the same as the one for the risk-neutral dynamics shown in Panel A of Table 2. The bold font denotes restricted parameters. The sample period is January 1983 to December 2015.

	(1)	(2)	(3)	(4)	(5)	(6)	(7)
(1) VAR(1) unrestr.	1.0372						
(2) VAR(2) unrestr.	0.9903	1.0658					
(3) VAR(1) BIC	0.2943	0.1658	0.1905				
(4) VAR(2) AIC	0.8387	0.7701	0.7274	0.5650			
(5) VAR(2) HQIC	0.8298	0.8959	-0.2765	0.4461	0.5152		
(6) VAR(2) BRW	0.7267	0.6345	0.8626	0.9703	0.2426	0.9395	
(7) VAR(2) Eig(\mathcal{Q})	0.9670	0.9294	0.5147	0.9469	0.6783	0.8749	0.8590

Table 7. The standard deviation of the term premium estimates is on the diagonal. The lower-triangular part of the matrix shows the correlations between particular term premium estimates. The sample period is January 1983 to December 2015.

	joint	U.S. only	German only
Max(eig $\{\Phi^P\}$)	0.9945	0.9918	0.9992
ρ_1	0.9925	0.9915	0.9923
ρ_{12}	0.8746	0.8497	0.8672
ρ_{24}	0.7403	0.6676	0.7195
ADF	-0.0026	-0.0034	-0.0023
$KPSS$	15.5124	14.6489	14.0887

Table 8. The table shows time series (physical) properties of the principal components of the joint model and U.S. and German only yields. The first row shows the maximum eigenvalue of the feedback matrix under the physical measure for the joint and individual models. In the following three rows the table reports the first, 12th and 24th order autocorrelation of the first principal component. The last two rows present the ADF test and $KPSS$ test for unit root and stationarity, respectively with one lag. The 10% critical value for the ADF test with no intercept and no time trend amounts to -2.5657 . The 1% critical value for the $KPSS$ test amounts to 0.739 .

Model	Country	Root-mean-square error (bp)								Av.RMSE
		1m	1y	2y	3y	4y	5y	7y	10y	
Panel A: $K = 2$										
OLS	U.S.	24.6	12.5	14.4	12.3	8.4	4.5	7.8	18.0	12.81
	Germany	25.2	12.4	14.9	12.4	8.0	3.6	7.4	17.4	12.66
$AACM$	U.S.	24.6	21.4	18.6	19.9	19.4	16.5	10.2	33.4	20.50
	Germany	25.2	15.6	19.0	22.3	23.5	21.9	13.1	31.1	21.45
$SSC(DLR_U)$	U.S.	26.6	17.5	15.1	12.3	9.2	7.0	8.7	19.6	14.50
	Germany	26.6	15.1	15.9	12.5	8.3	5.7	8.5	17.8	13.80
Panel B: $K = 3$										
OLS	U.S.	3.0	7.5	2.6	3.0	4.2	4.1	1.8	6.1	4.03
	Germany	2.4	7.1	2.0	2.9	3.4	3.0	1.3	4.6	3.34
$AACM$	U.S.	3.0	14.3	5.7	4.6	6.6	7.3	5.3	7.4	6.77
	Germany	2.4	8.4	3.6	3.3	3.8	3.7	2.7	5.1	4.14
$SSC(DLR_U)$	U.S.	3.6	10.2	4.0	4.4	4.6	4.4	2.6	6.5	5.03
	Germany	2.5	7.4	2.4	3.1	3.4	3.1	1.6	4.6	3.51
Panel C: $K = 4$										
OLS	U.S.	0.3	2.0	2.1	1.4	0.5	1.4	1.8	1.5	1.38
	Germany	0.1	1.4	2.0	1.0	0.6	1.4	1.3	1.5	1.18
$AACM$	U.S.	0.3	6.4	6.5	6.6	4.8	2.8	2.6	1.7	3.97
	Germany	0.1	5.1	2.8	1.7	1.3	1.7	1.5	1.6	1.98
$SSC(DLR_U)$	U.S.	0.4	2.6	3.2	1.5	1.2	1.8	2.1	1.7	1.81
	Germany	0.1	1.5	2.2	1.1	0.7	1.5	1.3	1.6	1.26
Panel D: $K = 5$										
OLS	U.S.	0.0	0.2	0.6	0.3	0.4	0.1	0.7	0.3	0.33
	Germany	0.0	0.1	0.4	0.3	0.3	0.1	0.5	0.2	0.25
$AACM$	U.S.	0.0	6.4	4.2	2.6	1.7	1.3	1.6	1.2	2.37
	Germany	0.0	4.0	1.7	1.1	0.9	0.7	0.6	0.3	1.16
$SSC(DLR_U)$	U.S.	0.0	0.3	0.7	0.4	0.5	0.1	0.7	0.3	0.38
	Germany	0.0	0.1	0.5	0.3	0.3	0.1	0.5	0.3	0.26

Table 9. Estimator performance in terms of fit for the two separate U.S. and German markets, as measured by the root-mean-square error of the measurement noise. The model is estimated separately for the U.S. and for Germany with 2, 3, 4 and 5 principal components of yields. The reported values are in basis points. The sample period is January 1987 to December 2015.

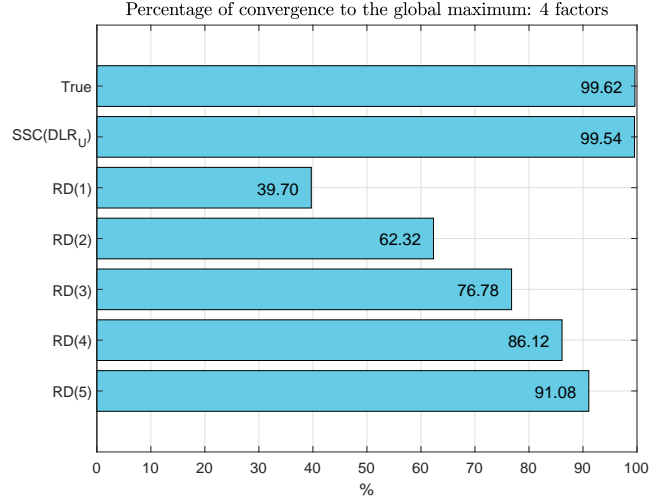
#	Separate models		Joint model	
	U.S.	Germany	U.S.	Germany
1	97.1893	97.2250	89.9414	89.7814
2	99.6709	99.6657	97.8825	97.8953
3	99.9679	99.9766	99.2972	99.2636
4	99.9962	99.9972	99.7138	99.7115
5	99.9998	99.9998	99.8848	99.8983
6	100.0000	100.0000	99.9722	99.9784
7	100.0000	100.0000	99.9928	99.9857
8	100.0000	100.0000	99.9967	99.9974

Table 10. The explanatory power of single country and global factors. The table shows the percentage of the variance of the U.S. and German yields explained by single-country and two-country principal components. The sample period is June 1986 to December 2015.

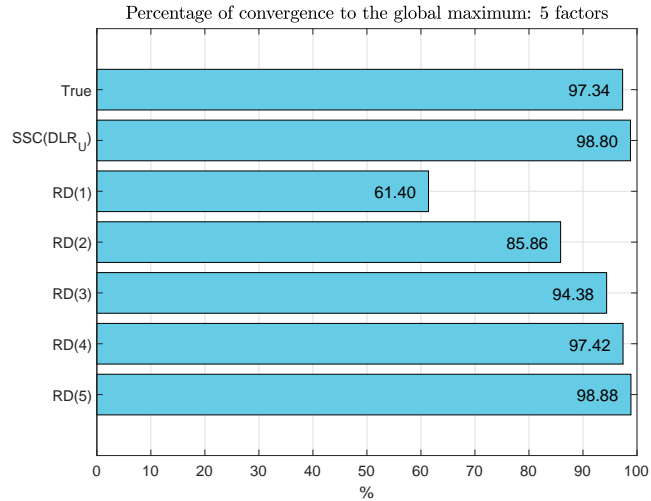
Model	Country	Root-mean-square error (bp)								Av.RMSE
		1m	1y	2y	3y	4y	5y	7y	10y	
Panel A: $K = 4$										
OLS	U.S.	22.4	11.9	12.9	10.7	7.0	3.7	8.6	18.1	11.90
	Germany	23.4	11.0	14.1	12.3	8.3	4.1	6.1	15.5	11.83
$AACM$	U.S.	22.4	24.8	22.7	22.4	20.1	15.9	13.3	35.9	22.16
	Germany	23.4	19.2	25.4	32.3	35.3	33.4	19.3	46.1	29.30
$SSC(DLR_U)$	U.S.	24.7	17.1	13.5	10.9	8.1	6.3	9.2	19.6	13.68
	Germany	24.9	15.5	15.4	12.5	9.1	6.8	7.5	16.6	13.55
Panel B: $K = 5$										
OLS	U.S.	11.7	9.8	8.4	6.8	4.8	3.6	6.1	11.6	7.85
	Germany	13.2	7.8	7.6	6.8	4.9	3.1	4.1	9.7	7.18
$AACM$	U.S.	11.7	14.4	11.8	12.4	12.1	9.6	7.2	17.7	12.09
	Germany	13.2	14.8	12.3	9.2	6.9	6.2	6.8	12.0	10.16
$SSC(DLR_U)$	U.S.	12.2	11.8	8.9	7.1	5.3	4.3	6.2	12.3	8.50
	Germany	13.4	9.3	7.8	7.4	5.8	4.0	4.5	10.9	7.89
Panel C: $K = 6$										
OLS	U.S.	2.9	7.1	2.3	2.9	3.8	3.5	1.6	5.7	3.73
	Germany	2.6	6.8	2.1	2.6	3.2	2.9	1.3	4.3	3.23
$AACM$	U.S.	2.9	14.6	7.6	4.3	8.2	9.9	6.3	11.4	8.16
	Germany	2.6	13.5	7.8	3.9	4.0	5.0	4.0	5.9	5.83
$SSC(DLR_U)$	U.S.	3.6	10.0	3.9	4.5	4.3	3.9	2.8	6.1	4.88
	Germany	2.7	7.6	2.6	3.1	3.3	3.0	1.7	4.3	3.54
Panel D: $K = 7$										
OLS	U.S.	1.2	3.2	2.1	1.6	1.7	1.9	1.6	2.6	2.00
	Germany	2.1	5.5	1.9	2.3	2.4	2.2	1.3	3.5	2.65
$AACM$	U.S.	1.2	10.6	6.8	5.5	4.3	3.5	2.3	3.2	4.66
	Germany	2.1	9.3	2.3	2.5	2.7	2.8	2.6	3.8	3.51
$SSC(DLR_U)$	U.S.	1.4	4.2	3.0	1.9	1.9	2.0	1.8	2.6	2.35
	Germany	2.4	6.3	2.1	2.7	2.6	2.3	1.6	3.8	2.97
Panel E: $K = 8$										
OLS	U.S.	0.3	1.8	2.0	1.3	0.5	1.4	1.6	1.5	1.28
	Germany	0.1	1.3	1.9	0.9	0.6	1.4	1.2	1.5	1.12
$AACM$	U.S.	0.3	8.5	6.8	6.5	4.6	2.9	2.1	1.9	4.20
	Germany	0.1	8.3	3.5	2.1	1.4	2.0	2.1	2.0	2.68
$SSC(DLR_U)$	U.S.	0.3	2.2	2.9	1.4	1.0	1.6	1.8	1.7	1.61
	Germany	0.1	1.5	2.3	1.0	0.9	1.5	1.3	1.6	1.26

Table 11. Estimator performance in terms of fit for the joint two-country model, as measured by the root-mean-square error of the measurement noise. The model is estimated with 4, 5, 6, 7 and 8 principal components of yields. The reported RMSE values are in basis points. The sample period is January 1987 to December 2015.

Figures



(a)



(b)

Figure 1. Finding the global maximum likelihood. The figure shows the percentage of simulations converging to the global maximum when initiating the numerical *MLE* routine from different starting values for the 4 and 5 factor model in Panel (a) and Panel (b), respectively. The starting values are the values used to generate the simulated yield samples ('True'), the roots of the Φ^Q matrix estimated by the DLR_U estimator, $SSC(DLR_U)$, and from 1 to 5 different random starting values ('RD'). The global maximum is assumed to be the optimum with the highest likelihood values found from any of these starting values.

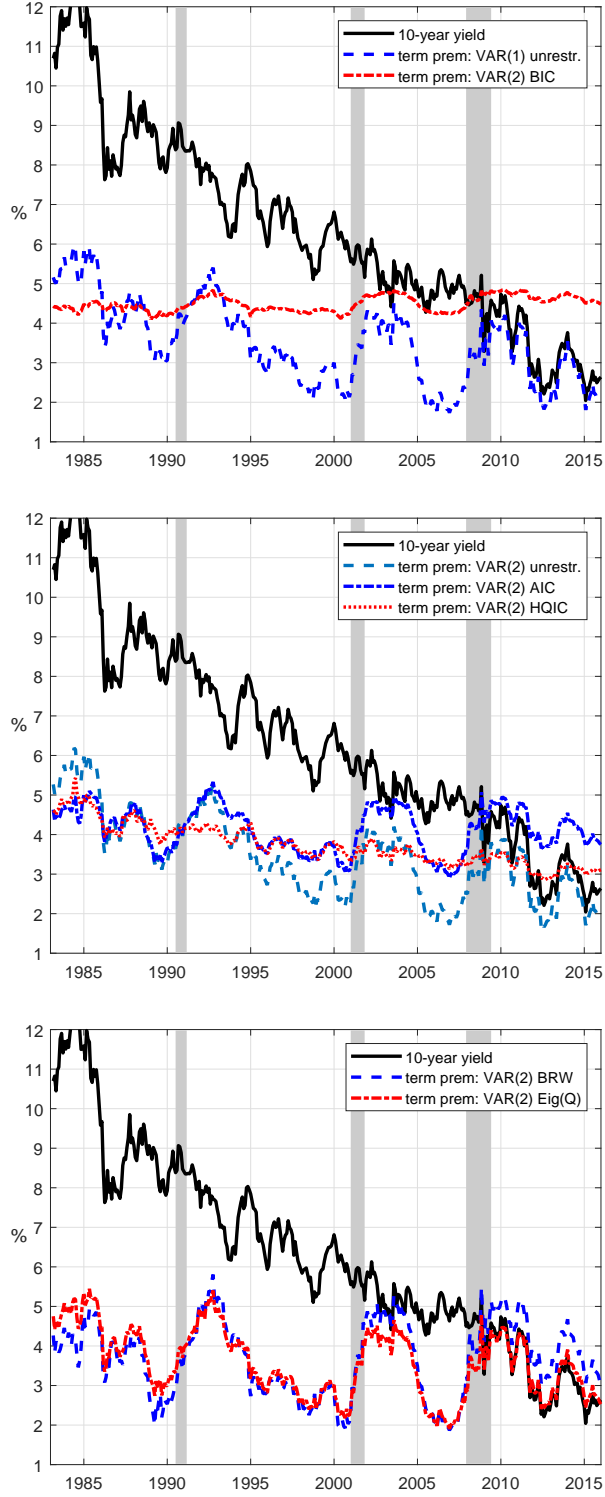


Figure 2. U.S. 10-year term premium. This figure shows 10-year yield and the 10-year term premium implied by different specifications of the \mathcal{P} -dynamics. The grey areas denote the NBER recession periods. The time series models are as specified in Tables 5 and 6.

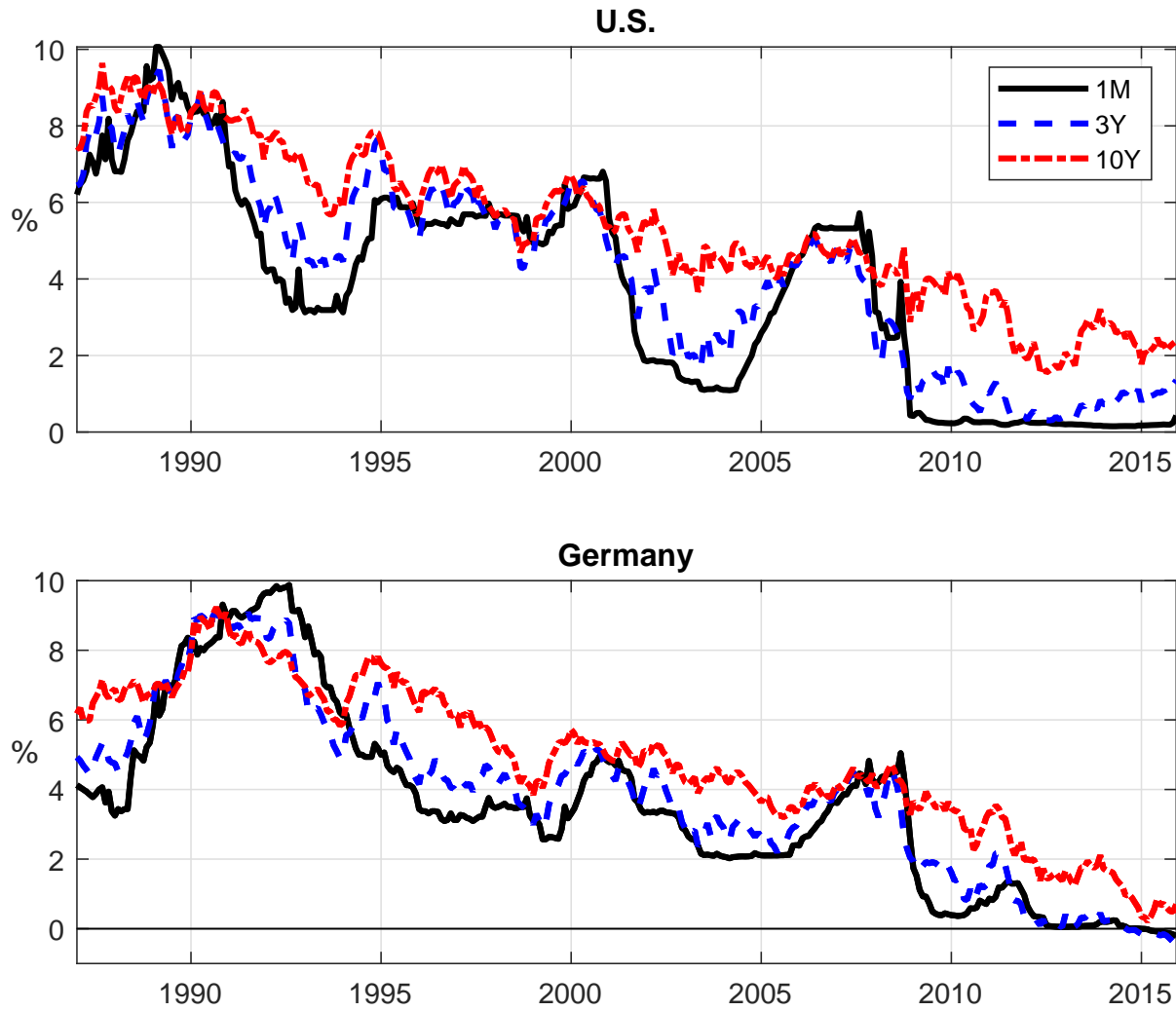


Figure 3. U.S. and German Treasury markets. This figure shows yield data for three of the maturities used in the two country model. For the U.S. the annual maturities come from Gurkaynak et al. (2007) estimated using the Svensson (1994) method. For Germany, we use estimates published by the Bundesbank, also estimated using the Svensson method.

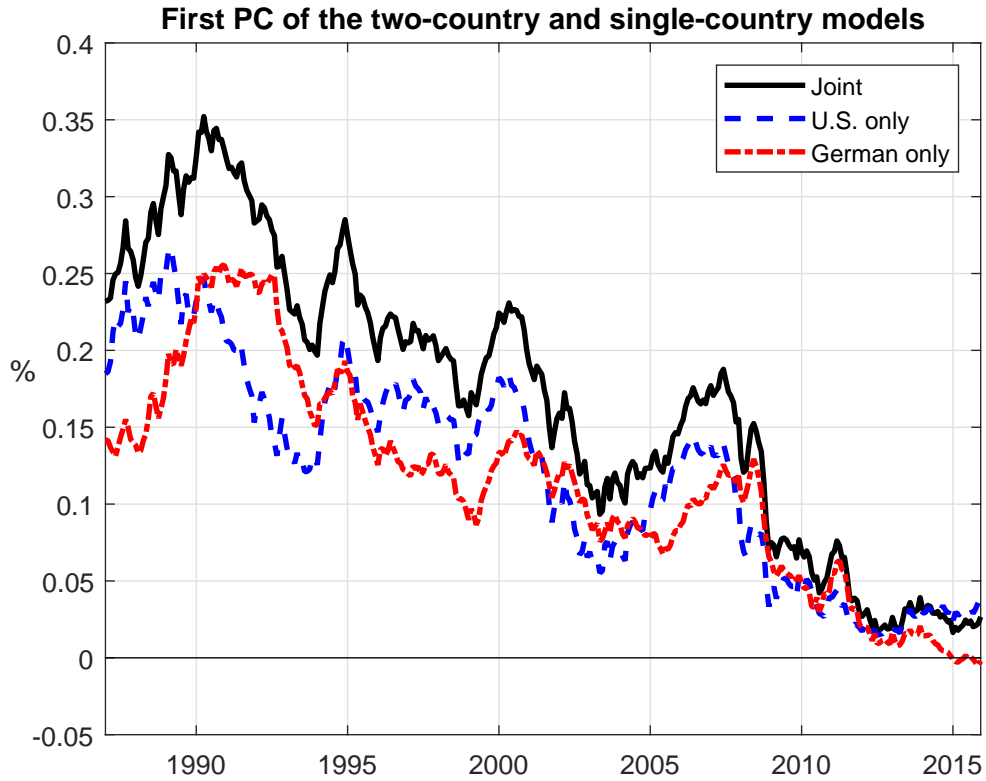


Figure 4. The levels of yields in the U.S. and German markets. This figure shows the first principal components for the two 3–factor U.S. and German models and for the joint 6–factor model. These are similar, exhibiting the strong downward trend in the level of interest rates in the two countries, but there is a tendency for the U.S. to lead the German market. The continuous black line shows the first common factor in the joint model and is a compromise between the level factors in the two single country models. Their differences are reflected in the second common factor, shown in top panel of Figure 6.

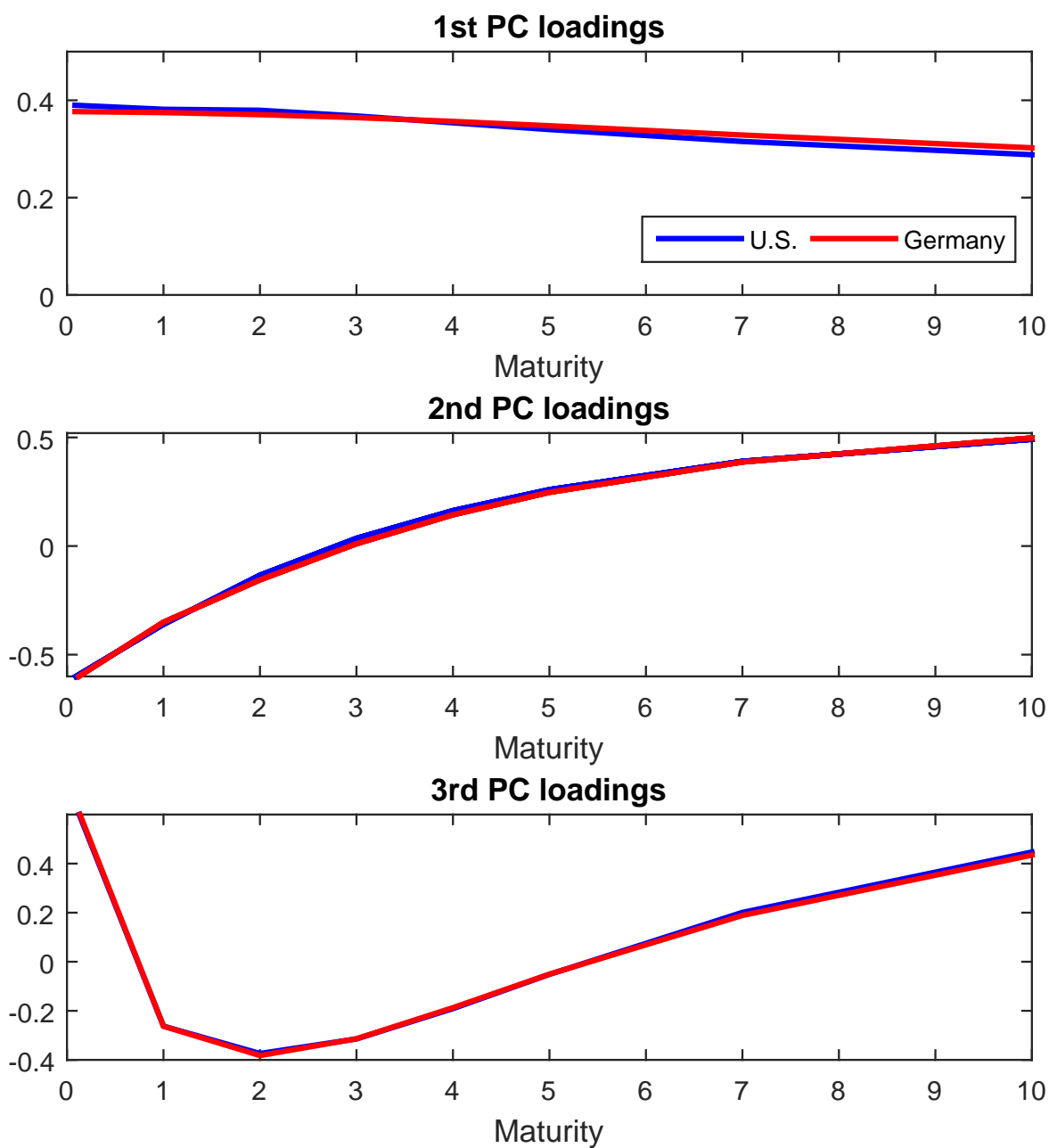


Figure 5. Factor loadings in the separate U.S. and German market models. This figure shows the factor loadings for the two separate 3-factor country models.

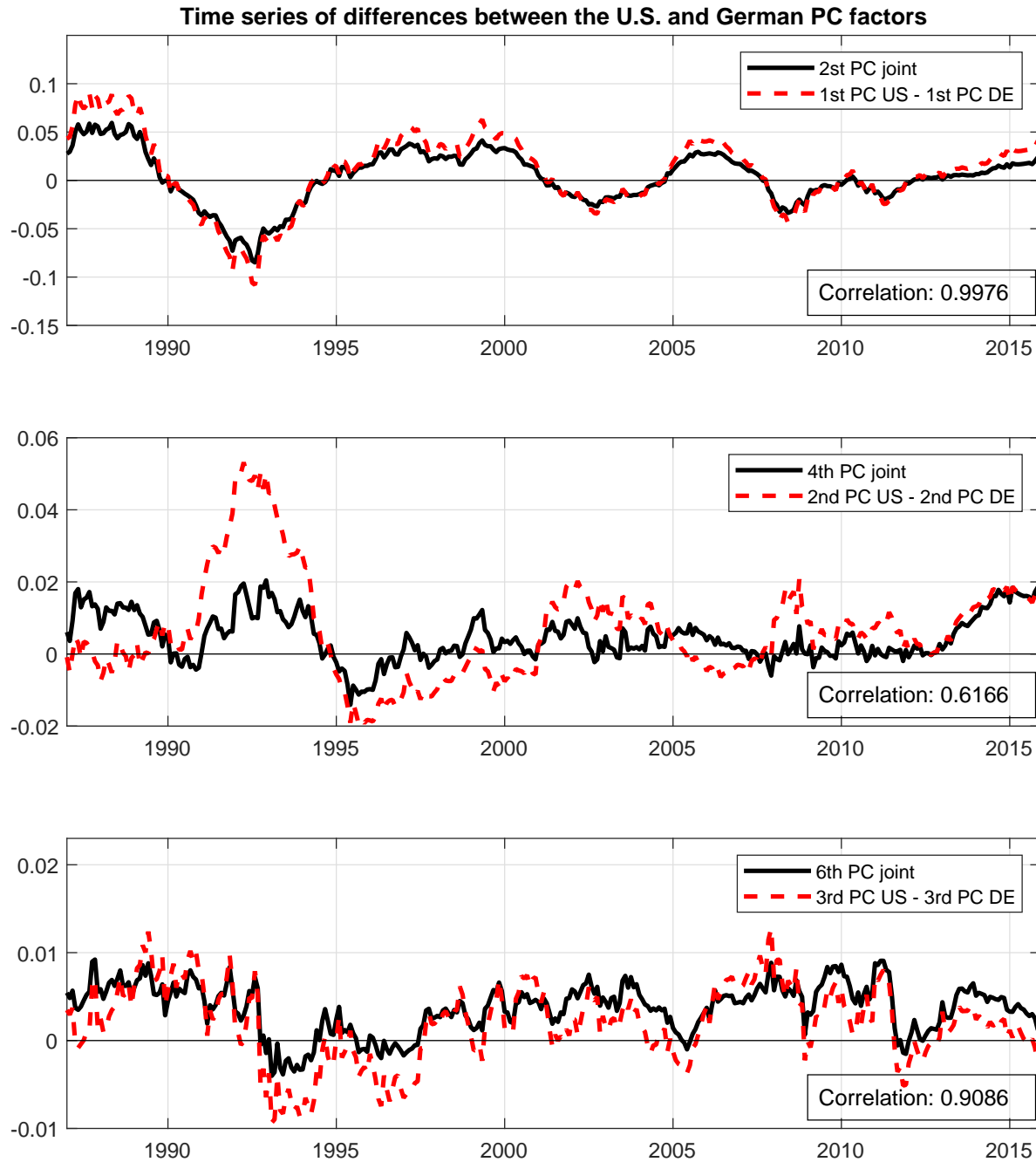


Figure 6. The relationships between the factors in the separate and joint U.S.-German models. This figure shows how the even-numbered common factors allow for relative effects missed by the associated odd-numbered common factors. So for example, the top panel shows how the second common factor is aligned with the difference between the level factors in the two single country models and thus complements the first common factor shown in Figure 4. Thus, we find that regressing the level factors in the single country models on the first two common factors gives an almost perfect fit. Similarly, the third and fourth (fifth and sixth, respectively) factors in the joint model explain most of the variance in the slope (curvature) factors in the single country models. The loadings of the yields on these factors are shown in Figure 7.

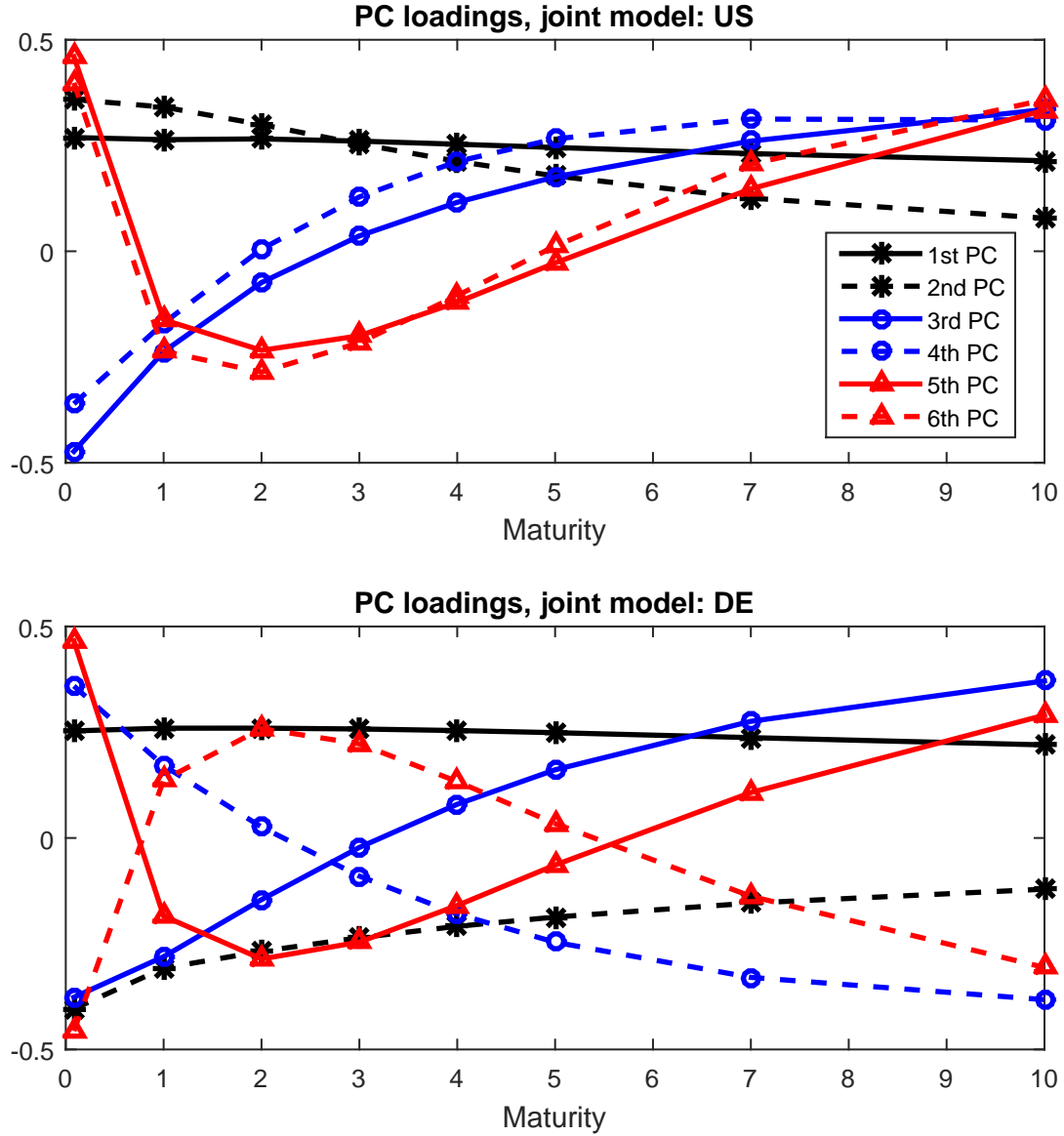


Figure 7. The relationships between the factor loadings in the separate and joint U.S.-German models. The pair-wise factor-rotation relationships shown in Figure 6 are reflected in the factor loadings for the two countries in the 6 common factor model shown in this figure. The continuous lines show the loadings on the odd-numbered factors, which are interpreted as common level (black), slope (blue) and curvature (red) factor loadings respectively. These clearly resemble the respective loadings in the single country models shown in Figure 5. The loadings on the associated even-numbered common factors are shown with dashed lines. Since these allow for relative level, slope and curvature effects, the signs are reversed for the U.S. and Germany.

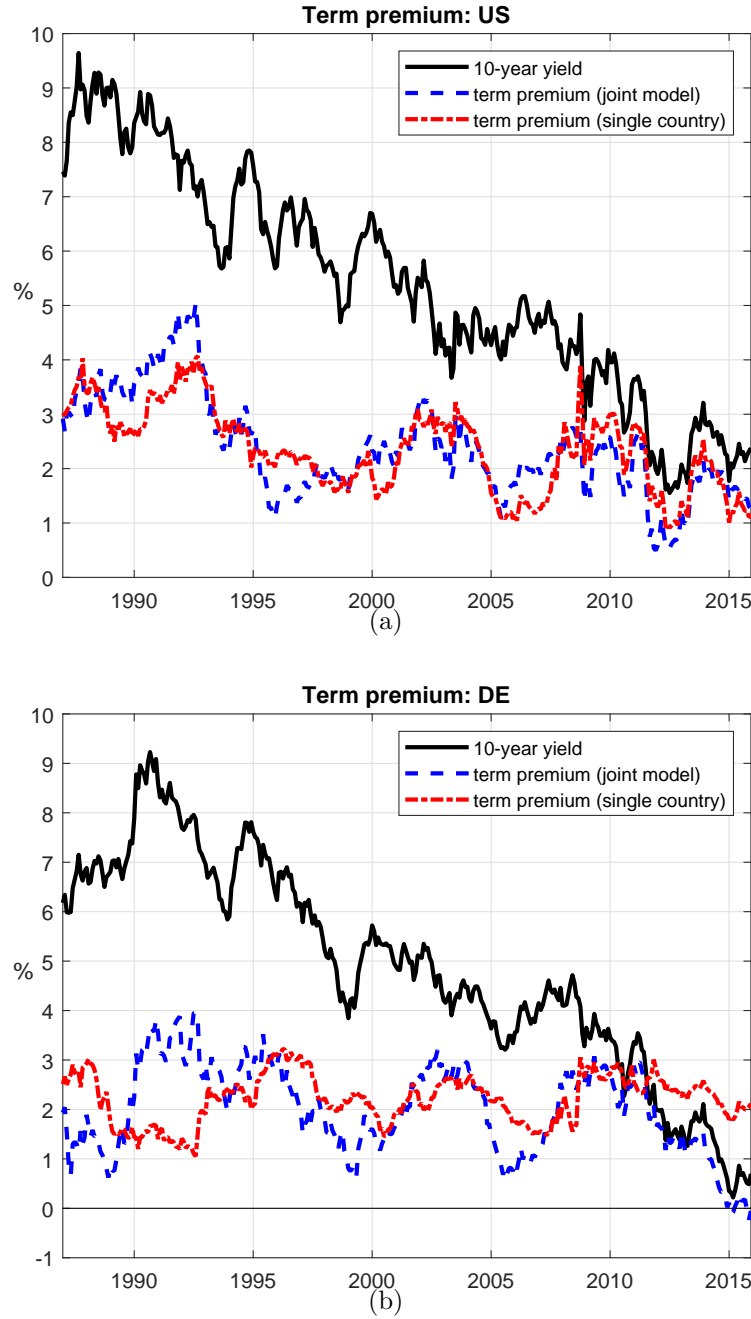


Figure 8. 10-year yield decomposition. This figure shows how these models decompose the 10-year yield into components representing 10-year interest rate expectations and a risk premium for the U.S. in Panel (a) and Germany in Panel (b). This decomposition largely depends upon the physical dynamics represented by the time series VAR , Eq.(11) which is used to forecast the interest rates and hence the yields that would be observed if investors were risk-neutral. Subtracting this estimate from the model-fitted yield for any maturity then gives the estimate of the risk premium.