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Triage in Kidney Exchange

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Abstract

This paper studies a kidney exchange problem in which a planner sorts patients into "priority groups" based on, e.g., the severity of their conditions. The planner may choose to allow cyclic exchanges, chains, altruistically unbalanced exchanges and desensitization. It presents a computational method for finding a new class of matchings that give priority to patients in higher priority groups, called "priority group matchings". These matchings are always Pareto efficient no matter how patients are sorted into priority groups or how the kidney exchange program is designed by the planner. Priority group matchings generalize several classes of matchings, where different classes correspond to different ways of prioritizing patients. This includes maximum matchings and (half-compatibility) priority matchings.

Keywords: Kidney exchange, triage, priority matching, priority group, threshold matching, desensitization.
JEL Classification: C78, D47.

1 Introduction

Patients in need of a healthy kidney may eventually reach a point where they are too sick for transplantation to be a viable option. As a result, thousands of people die every year waiting for a compatible kidney (Organ Procurement and Transplantation Network, 2019). The need for a healthy kidney is consequently more urgent for some patients than for others. A situation could arise where someone must decide whether to help one patient in urgent need of a kidney transplant or to help several patients with less urgent conditions.

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Triage is the process of prioritizing patients based on how severe their conditions are and allocating scarce health care resources accordingly. This paper introduces triage in the context of kidney exchange and provides a computational method for finding an efficient matching no matter how patients are prioritized. This new class of efficient matchings is shown to generalize several well-studied classes of matchings in the sense that different classes of matchings correspond to different approaches to triage.

The term 'triage' has its origins in the treatment of wounded soldiers and the chief surgeon of Napoleon's Imperial Guard, Baron Dominique-Jean Larrey, is believed to have been the first to employ a formal triage system (Iserson and Moskop, 2007). Larrey stated that *"[t]hose who are dangerously wounded should receive the first attention, without regard to rank or distinction. They who are injured in a less degree may wait until their brethren in arms, who are badly mutilated, have been operated on and dressed, otherwise the latter would not survive many hours; rarely, until the succeeding day"* (Larrey, 1812).

Since then, patients have been sorted into categories based on how urgently they need treatment along with other considerations in a number of different settings. Throughout this paper, patients sorted into a certain category will be referred to as a *priority group*.

In the US military, patients are sorted into four priority groups (Department of the Army, 1997). The first ("immediate") contains patients that require immediate resuscitative treatment. The second ("delayed") contains patients whose treatment can be delayed *"without unduly compromising the likelihood of a successful outcome."* If resource constraints are binding, patients in the second priority group must wait until all patients in the first priority group have been dealt with.[1] The third ("minimal") contains the patients with the least urgent conditions. The fourth priority group ("expectant") is given the lowest priority and contains patients with injuries that are so extensive that their survival is unlikely even if they were the only patient and medical resources were utilized optimally.

In emergency healthcare, patients are often sorted into priority groups in a similar manner. For example, the Australasian National Triage Scale adopted in Australia and New Zealand sorts patients into five priority groups based on whether they need treatment within 0, 10, 30, 60 or 120 minutes (Richardson, 1998).

Triage considerations may also arise in the context of *kidney exchange*. A patient suffering from kidney disease may have a friend or relative who is willing but unable to donate a kidney due to blood group or tissue type incompatibilities. If there are several patients in the same predicament, they may be able to exchange donors with one another so that some or all of them receive kidney transplants. A *kidney exchange program* can be thought of as a clearing house, the purpose of which is to facilitate kidney exchanges by assigning patients to donors. A set of selected assignments is called a *matching*. A typical kidney exchange program will select several matchings every year. For example, in the Czech Republic, France, the Netherlands, Portugal, the United Kingdom and Switzerland, matching algorithms are used to find matchings on a quarterly basis (Biró et al., 2018a).

The rules governing a kidney exchange program are made by a *planner*. Triage may be relevant to the planner due to the existence of a trade-off between maximizing the number

[1] In this sense, patients are prioritized lexicographically with respect to priority group membership.

of transplants and prioritizing patients with particularly severe conditions each time a matching algorithm is run. The unmatched patients remain on dialysis and may receive a healthy kidney at some point in the future when a new matching is selected. The less severe an unmatched patient's condition is, the more likely she is to receive a kidney transplant in the future. For example, highly HLA-sensitized patients are particularly difficult to find compatible donors for. They are consequently less likely to receive kidney transplants in the future if they remain unmatched after the matching algorithm is run. Giving priority to patients with severe conditions may consequently improve the number of transplants in the long run, even though it could reduce the number of transplants in the short run. Triage is consequently a potentially welfare improving tool.

Despite this, such trade-offs are generally not considered in the context of kidney exchange. Most kidney exchange programs do prioritize patients based on a variety of different criteria. For example, the kidney exchange programs in Belgium, Italy, the Netherlands, Poland, Portugal, Spain, Sweden and the United Kingdom all assign higher weight in their respective matching algorithms to patients that are difficult to match, such as highly HLA-sensitized patients. However, any concerns of this kind are strictly subordinate to the primary goal of maximizing the number of transplants (Biró et al., 2018b). That is, priorities are only used to break ties when there are multiple matchings that result in the same number of transplants.

While highly HLA-sensitized patients are more likely to die while waiting for a kidney transplant, patients are typically not explicitly prioritized based on how urgently they need kidney transplants *within* kidney exchange programs. However, patients on waiting lists for kidney transplants from cadaveric donors often are (Costa et al., 2007; Celebi et al., 2015; Assfalg et al., 2016). In principle, the model presented in this paper allows patients to be prioritized in accordance with any criteria. For example, in Belgium, Italy, the Netherlands, Portugal and Spain, patients are also given priority based on how long they have been on dialysis waiting for a transplant (Biró et al., 2018b).

Whether or not there is a trade-off between maximizing the number of transplants and prioritizing patients with more severe conditions depends on the design of the kidney exchange program. Roth et al. (2004) were the first to take a mechanism design approach to the kidney exchange problem. They develop a solution based on the notion of top trading cycles introduced by Shapley and Scarf (1974). Since a donor's promise is not legally binding, all transplants in a kidney exchange are often required to be simultaneous (Biró et al., 2017), although Ausubel and Morrill (2014) show that the simultaneity requirement can be relaxed somewhat without losing donor participation incentives. Such requirements create a logistic problem that is larger the more numerous the patients involved in the exchange are. To address this, Roth et al. (2005a) show how a class of matchings called *priority matchings* can be found using a greedy algorithm in a setting where only *pairwise exchanges* (involving two patient-donor pairs) are permitted. In practice, many kidney exchange programs impose some restrictions on the number of patient-donor pairs that may participate in a single exchange. For example, only pairwise exchanges are allowed in the French program, while *cyclic exchanges* involving at most three patient-donor pairs are allowed in Poland, Poland, Italy, Spain and the United Kingdom. Kidney exchange programs that do not impose simultaneity restrictions are better able to facilitate larger cyclic exchanges. For example, the Czech program has carried out cyclic exchanges involving six and seven patient-donor pairs (Biró et al., 2018b). In this paper, the planner may impose any arbitrary restrictions on the types of exchanges that are permitted.

In addition to maximizing the number of transplants, priority matchings rank all patients according to some criterion and then prioritize patients with higher ranking. Roth et al. (2005a) show that there is no trade-off between prioritizing patients in accordance with some ranking (based on, e.g., HLA-sensitization) and maximizing the the number of transplants in settings where only pairwise exchanges are permitted. Andersson and Kratz (2019) show that this result still holds in a setting with more general patient preferences. However, it is well known that the number of transplants can be increased by allowing cyclic exchanges involving three patient-donor pairs. Saidman et al. (2006) quantify this improvement in a simulation study. While Roth et al. (2007) demonstrate that the benefits of increasing the permitted size of cyclic exchanges from three to four pairs are typically small, there may be large gains from long cyclic exchanges (and chains) if many patients are highly HLA-sensitized (Ashlagi et al., 2012).

Although cyclic exchanges may increase the number of transplants, Sönmez and Ünver (2014) point out that allowing such exchanges may give rise to the trade-off mentioned above, between maximizing the number of transplants and matching patients in accordance with their assigned priorities. This trade-off is typically avoided altogether by either restricting attention to pairwise exchanges and selecting priority matchings as in Roth et al. (2005a), Okumura (2014) and Sönmez and Ünver (2014) or by selecting *maximum matchings* (that simply maximize the number of transplants) and at most use priorities to break ties as in Roth et al. (2005b; 2007) and Saidman et al. (2006).

Since triage is a potentially welfare improving tool, this paper, by contrast, presents a flexible method for finding a class of matchings called *priority group matchings*, in which the planner may choose to prioritize patients with severe conditions even if it would reduce the number transplants. Priority group matchings are guaranteed to be Pareto efficient regardless of the planner's approach to triage. The planner prioritizes patients by sorting them into priority groups. First, the number of patients in the first priority group that receive kidney transplants is maximized. Then, the number of patients in the second priority group that receive kidney transplants is maximized, and so on.[2] Finally, recent advances in medicine make it possible for some patients to receive kidneys from blood group incompatible and sometimes even tissue type incompatible donors. This is known as *desensitization* and involves delays and additional treatments before and after transplantation in addition to the cost of the treatment.[3] All else equal, patients are assumed to prefer avoiding desensitization. Priority group matchings will, subject to the prioritization discussed above, minimize the use of desensitization. Andersson and Kratz (2019) show that if the use of desensitization is not minimized, the outcome may not be Pareto efficient. This intuition

[2]This is in line with the lexicographic application of triage discussed above.

[3]As long as the patient is expected to survive a few years, blood group incompatible transplantation using desensitization is still cheaper than letting the patient remain on dialysis (Thydén et al., 2012; Wennberg, 2010).

carries over to the more general setting in this paper.

The planner chooses the number and size of priority groups in accordance with her approach to triage. Priority group matchings are always Pareto efficient, but the outcome may differ depending on how patients are sorted into priority groups, what types of exchanges are allowed and whether desensitization is permitted. Furthermore, the planner may allow patients that never strictly benefit from a kidney exchange program to participate for altruistic reasons. For different approaches to triage, priority group matchings reduce to a number of different classes of matchings.

A simple approach to triage is to set a threshold in terms of some measure of the severity of a patient's condition. The patients above the threshold are deemed to have sufficiently severe conditions to be prioritized in accordance with some predetermined rule. For example, the planner could first maximize the number of patients above the threshold that receive transplants, then maximize the number of patients below the threshold that receive transplants and finally minimize the use of desensitization. In this case, priority group matchings reduce to a class of matchings called *egalitarian threshold matchings*. This approach shares many similarities with the MAXCARD-FAIR objective function studied by Dickerson and Sandholm (2014) and Dickerson et al. (2014). MAXCARD-FAIR amends an objective function selecting maximum matchings by scaling up the weight assigned to "marginalized" patients (such as highly HLA-sensitized patients or children) by some factor. In the special case when desensitization is not permitted, egalitarian threshold matchings coincide with the MAXCARD-FAIR solutions whenever the additional weight assigned to marginalized patients is sufficiently large. Dickerson et al. (2014) also consider a problem where the planner imposes a requirement that the share of highly HLA-sensitized patients receiving transplants be above some threshold.

An alternative approach is to prioritize the patients above the threshold in accordance with the priority mechanism that selects priority matchings in Roth et al. (2005a). Then priority group matchings reduce to a new class of matchings called *threshold matchings*[4]. One result shows that a weakly higher number of transplants will be performed the higher the threshold is. If the threshold is high enough, then threshold matchings reduce to maximum matchings. If, on the other hand, the threshold is low enough, then threshold matchings reduce to a generalization of the half-compatibility priority matchings introduced by Andersson and Kratz (2019). These matchings are, in turn, a generalization of priority matchings. Recall the observation that the focus in the literature shifted from priority matchings to maximum matchings in models where more advanced exchanges were permitted. By the results above, this was not a shift from one class of matchings to a completely different class of matchings. Threshold matchings were selected in both cases and when more advanced exchanges were introduced, giving rise to a trade-off between maximizing the number of transplants and prioritizing patients with severe conditions, the threshold was simply raised so high that no patients qualified for preferential treatment.

Andersson and Kratz (2019) show that desensitization can improve the number of transplants significantly if implemented correctly. The suggested approach to desensitization is

[4]Not to be confused with the threshold mechanism studied by Ünver (2010).

translated into the more general setting of this paper. Each patients that could receive a transplant from her own blood group or tissue type incompatible donor is given an incentive to participate in kidney exchange by guaranteeing her a donor that is strictly more compatible or, depending on the patient's type, at least as compatible. While the same compatibility structure and a marginally more general preference domain is used here, the model in Andersson and Kratz (2019) restricts attention to pairwise exchanges. The method presented in this paper is, to the best of the author's knowledge, the only method flexible enough to simultaneously incorporate both desensitization and more advanced exchanges, like three-way cyclic exchanges and exchange chains. Chun et al. (2015) do allow cyclic exchanges in a setting with desensitization, but their model does not permit any restrictions on the length of cycles. By contrast, this paper allows the planner to impose any constraints on cycles to avoid infeasibly large cyclic exchanges. Another distinction is that Chun et al. (2015) assume that any immunological constraints can be overcome using desensitization.[5]

In addition to cyclic exchanges, the planner may choose to allow *exchange chains*. A person willing to donate a kidney to *any* unspecified patient for altruistic reasons can initiate a chain of non-simultaneous exchanges, resulting in a large number of patients receiving kidney transplants (Roth et al., 2006; Ashlagi et al., 2011; Anderson et al., 2015).

Moreover, the planner can increase the number of transplants by including compatible patient-donor pairs in the kidney exchange program (Roth et al., 2005b; Gentry et al., 2007). Such patients can receive kidneys from their own donors outside the kidney exchange program and their participation in the program is consequently either based on altruism or compulsion. Exchanges involving compatible pairs are called *altruistically unbalanced* exchanges. Sönmez and Ünver (2014) study Pareto efficient matchings in a setting where some patient-donor pairs are compatible and some are incompatible. They show that the same number of patients are matched at all Pareto efficient matchings and that when selecting a Pareto efficient matching, the decision of which compatible pairs to match can be separated from the decision of which incompatible pairs to match. The number of compatible patient-donor pairs that are matched within a kidney exchange program can therefore be minimized (to reduce altruistic participation) without adversely affecting any incompatible pairs.

In this paper, altruistically unbalanced exchanges are made possible by letting a patient either be *regular* or *altruistic*. A regular patient only accepts being assigned a different patient's donor if she is made strictly better off. An altruistic patient, on the other hand, will accept being assigned any patient's donor that is at least as good as her own donor. An

[5]Transplantation across the HLA (tissue type) barrier is associated with significantly lower graft and patient survival rates (Haririan et al., 2009), as well as other adverse effects. Montgomery et al. (2011) write that *"[a]lthough it is clear that desensitization increases transplantation rates and reduces the waiting time among patients with HLA sensitization, the inferior outcomes that have been reported to date raise the question of whether these patients would be better served by waiting for a compatible organ."* Furthermore, transplantation across the blood group barrier is only deemed safe if desensitization can keep the patient's titer values below some threshold for a period of time before the transplantation (Sönmez et al., 2018).

altruistic patient will therefore accept being included in a kidney exchange program even if she is already fully compatible with her own donor. Priority group matchings correspond to the class of matchings considered by Sönmez and Ünver (2014) when only pairwise exchanges are allowed, desensitization is disallowed and all patients are altruistic.

For any rules governing a kidney exchange program, the planner weakly prefers all priority group matchings to all other matchings. Priority group matchings can be found by constructing an objective function representing the planner's preferences and solving the corresponding optimization problem. This problem is a special case of the multidimensional knapsack problem and can be solved using integer programming techniques.

While this provides a flexible method for finding Pareto efficient matchings for any approach to triage the planner may have, it does not seek to find an optimal approach to triage. Investigating the optimal approach to triage would require answering several difficult ethical questions that are sidestepped here by treating the planner's approach to triage as exogenous. Furthermore, the optimal approach to triage would have to be analyzed in a fully dynamic model, as in Akbarpour et al. (2017).

Akbarpour et al. (2017) investigate how to maximize the number of matched agents in a general dynamic matching setting where agents arrive and depart from the market over time. The planner seeks to match as many agents as possible and has limited information about agent departure times. This information can be taken into account to give priority to agents whose departure is imminent. They provide bounds on the loss in terms of departed (unmatched) patients for different matching algorithms. A matching algorithm that waits and matches agents when they are about to depart outperforms an algorithm that immediately matches agents that enter the market (if possible). These results apply to kidney exchange settings where only pairwise exchanges are allowed.

Ünver (2010) studies a dynamic kidney exchange model that differs from the model in Akbarpour et al. (2017) in the sense that patients remain in the program until they are matched. In contrast with the results in Akbarpour (2017), Ünver (2010) shows that, when only pairwise exchanges are permitted, it is never necessary to sacrifice some currently feasible exchange in order to enable a larger number of future exchanges. A kidney exchange program that realizes exchanges as soon as they become feasible is therefore dynamically efficient. However, this result ceases to hold when cyclic exchange are introduced.

The remainder of the paper is structured as follows. Section 2 introduces the model. Section 3 contains the main results and discusses the general properties of priority group matchings. Sections 4, 5 and 6 focus on threshold matchings, (half-compatibility) priority matchings and maximum matchings respectively. Section 7 provides some concluding remarks. Finally, Appendix A contains all proofs.

2 Model

This section is divided into two subsections. The first subsection deals with the compatibility of different patients and donors. The second subsection introduces matchings and priority groups.

2.1 Compatibility

Let $N = \{1, \dots, n\}$ be a set of $n \geq 2$ patients, where each patient $i \in N$ has a living donor d_i. Let D be a (possibly empty) set of *altruistic donors* with no associated patient in N, where $D = \{n+1, \dots, n+|D|\}$ whenever $|D| \geq 1$. For ease of notation, let $j \coloneqq d_j$ for all $j \in D$. Unlike the donors belonging to patient-donor pairs, the donors in D are willing to donate a kidney without a guarantee that some particular patient receive a kidney. Each donor, altruistic or not, is either *fully compatible*, *half-compatible* or *incompatible* with a given patient. A patient can always receive a kidney from a fully compatible donor and never from an incompatible donor. A donor is half-compatible if transplantation is only possible using desensitization. Let C be an $n \times (n+|D|)$ compatibility matrix. The ijth entry of C is given by the following.

$$c_{i,j} = \begin{cases} 2 & \text{if patient } i \text{ is fully compatible with donor } d_j, \\ 1 & \text{if patient } i \text{ is half-compatible with donor } d_j, \\ 0 & \text{if patient } i \text{ is incompatible with donor } d_j. \end{cases}$$

Note that patients and donors are never half-compatible if desensitization is not permitted. In such cases, $c_{ij} \neq 1$ for all $i, j \in N \cup D$. Patients are either *regular* or *altruistic*. A *regular* patient prefers to receive a kidney from her own donor but will accept a kidney from a different donor if it means receiving a kidney with a higher degree of compatibility. If she is incompatible with her own donor, she will accept a kidney from any compatible donor. If she is half-compatible, she will accept a kidney from any fully compatible donor (and thereby avoid desensitization). An *altruistic* patient, on the other hand, will accept a kidney from a different donor as long as the donor is at least as compatible as her own donor. Contrary to a regular patient, the altruistic patient will agree to participate in kidney exchanges even if it does not strictly benefit her. For instance, a patient with a fully compatible donor has no incentive beyond altruism to participate in a kidney exchange program, since she is guaranteed a fully compatible kidney either way. Altruistic participation is known to increase the total number of transplants (Roth et al., 2005b; Sönmez and Ünver, 2014). Regular patients are gathered in the set I_R and altruistic patients are gathered in the set I_A, where $I_R \cup I_A = N$ and $I_R \cap I_A = \emptyset$. Since no patients can receive kidney transplants from incompatible donors, all patients with incompatible donors must be regular patients. Note that no regular patient with a fully compatible donor will participate in the kidney exchange program. I_R consequently contains no patients with fully compatible donors.

Each compatibility matrix can (though not without loss of information in settings with desensitization) be represented by a directed compatibility graph $G = (N \cup D, E)$, where E is the arc set. There is an arc ij from some patient i to some patient or altruistic donor j if i would accept receiving a kidney from patient j's donor d_j or altruistic donor j. The set of arcs from some patient i to other patients and altruistic donors is therefore determined by the compatibility matrix and whether i is a regular or altruistic patient.

$$\text{For } i \in I_R, \quad ij \in E \iff \begin{cases} i = j \text{ and } c_{i,i} = 1, \text{or} \\ j \in (N \cup D) \setminus \{i\} \text{ and } c_{i,j} > c_{i,i}. \end{cases}$$
$$\text{For } i \in I_A, \quad ij \in E \iff c_{i,j} \geq c_{i,i}.$$
$$\text{For } i \in D, \quad ij \notin E \text{ for all } j \in N \cup D.$$

This means that there is a loop ii at a regular patient i if she is half-compatible with her own donor d_i and there is a loop ii at an altruistic patient if she is not incompatible with her own donor d_i. There is an arc ij from a regular patient i to some other patient j if i is strictly "more compatible" with donor d_j than with her own donor d_i. There is also an arc ij from an altruistic patient i to some other patient j if i is at least as compatible with d_j as she is with her own donor d_i. Finally, there are no arcs from altruistic donors since they already have two healthy kidneys.

2.2 Matchings and priority groups

Assignments of kidneys to patients are represented by cycles and chains. A *cycle* of length $q \geq 2$, or a *q-cycle*, is an ordered list of unique patients $(i_1, i_2, \ldots, i_q)$ such that $i_j i_{j+1} \in E$ for all $j \in \{1, 2, \ldots, q\}$ taken modulo q, and $i_1 \leq i_j$ for all $j \in \{1, 2, \ldots, q\}$. In other words, an ordered list of patients is a cycle if there is an arc from each patient to the next patient in the list and an arc from the last patient to the first patient in the list. The requirement that $i_1 \leq i_j$ for all $j \in \{1, 2, \ldots, q\}$ prevents copies of cycles by imposing a unique indexing of the patients in a cycle. A 1-cycle is a loop, $ii \in E$. Note that a cycle never involves a donor in D. An ordered list of unique patients and a donor $(a_1, a_2, \ldots, a_q)$ is called a *chain* if $a_q \in D$ and for all $j \in \{1, \ldots, q-1\}$, $a_j a_{j+1} \in E$. Patients that appear in a cycle or chain are said to *participate* or *be involved* in the cycle or chain.

Let $\mathcal{K}$ contain all cycles and chains in G and let $K = \{k_1, \ldots, k_{|K|}\}$ be a subset of $\mathcal{K}$. K can be interpreted as the set of all permitted cycles and chains. For example, it could contain all cycles involving no more than three patients. K is always assumed to be non-empty to avoid trivial situations where no transplants are possible. A matching is defined as a subset μ of K. Matchings specify which transplants are to be carried out. If i is the direct predecessor of j in a cycle k, this is denoted by $i \to_k j$. At $\mu \subseteq K$, patient i is assigned donor d_j if $i \to_k j$ for some $k \in \mu$. If a 2-cycle is selected, this is referred to as a *pairwise exchange*. If a cycle involving three or more patients is selected, this is referred to as a *cyclic exchange*.

Patient i is said to be *matched* at μ if $i \in k$ for some cycle or chain $k \in \mu$. Let $N^*(\mu) \subseteq N$ denote the set of patients that are matched at μ, i.e., the set of patients receiving transplants. A matching μ is *feasible* if for all $i \in N \cup D$, $i \in k$ for at most one cycle or chain $k \in \mu$. The feasibility constraint ensures that no patient receives more than one kidney and no donor donates more than one kidney. Let $\mathcal{M}$ denote the set of feasible matchings.

There is a *planner* designing a kidney exchange program. The planner decides which types of exchanges are permitted. For example, the planner must decide whether desensitization and cyclic exchanges are allowed. Furthermore, the planner sorts patients into *priority groups*. Whenever the sorting is based on the severity of a patient's condition, it corresponds to a form of triage. Formally, N is partitioned into $m \in \{1, \ldots, n\}$ priority groups $N_1, \ldots, N_m$. This means that each patient belongs to exactly one priority group. N_1 is called the first priority group, N_2 is called the second priority group and so on. The patients in the first and mth priority groups could be the patients with the most severe and least severe conditions, respectively. Let $N_t^*(\mu) := N^*(\mu) \cap N_t$ be the set of patients in N_t that are matched at μ and let $N_0 := \emptyset$ for notational convenience. It is assumed that $m, n, t \in \mathbb{Z}$ throughout the paper.

The planner has preferences over feasible matchings, which can be represented by a binary *preference relation* R. For any $\mu, \mu' \in \mathcal{M}$, $\mu R \mu'$ if and only if the planner considers μ to be at least as good as μ'.

3 Priority group matchings

The patients may be sorted into priority groups in accordance with the planner's preferences. That is, the planner may prefer to assign donors to patients belonging to higher priority groups. This section introduces *priority group preferences*, $\succsim$, that take priority groups into account.[6] All else equal, planners with priority group preferences prefer to minimize the use of desensitization. Let $B(\mu)$ be the number of patients that are assigned fully compatible donors at μ and thereby avoid desensitization. Priority group preferences are defined formally below, where $t \in \{1, \ldots, m\}$ and $\tau \in \{0, \ldots, m-1\}$. Throughout the paper, it is assumed that $\tau \in \mathbb{Z}$.

$$\mu \succ \mu' \iff \begin{cases} |N_t^*(\mu)| > |N_t^*(\mu')| \text{ for some } t \leq m \text{ and } |N_\tau^*(\mu)| = |N_\tau^*(\mu')| \text{ for all } \tau < t, \\ |N_t^*(\mu)| = |N_t^*(\mu')| \text{ for all } t \leq m \text{ and } B(\mu) > B(\mu'), \end{cases}$$

$$\mu \sim \mu' \iff |N_t^*(\mu)| = |N_t^*(\mu')| \text{ for all } t \leq m \text{ and } B(\mu) = B(\mu').$$

A planner with priority group preferences is primarily concerned with matching as many patients in the first priority group as possible. The planner's secondary concern is to match as many patients in the second priority group as possible, and so on. If the same number of patients receive transplants in each priority group at two different matchings, then the planner prefers whichever matching assigns more patients fully compatible donors. A matching $\mu \in \mathcal{M}$ is a *priority group matching* if $\mu \succsim \mu'$ for all $\mu' \in \mathcal{M}$. That is, a planner with priority group preferences weakly prefers priority group matchings to all other feasible matchings. Priority group matchings are gathered in the set $\mathcal{M}^*$.

The results in this section demonstrate that priority group matchings can be found by constructing an objective function that represents priority group preferences and solving an optimization problem. Some new notation is needed to formulate this optimization

[6] Strict preference and indifference are denoted by $\succ$ and $\sim$ respectively.

problem. The input of the objective function will be a vector representation of a matching. Any matching $\mu \subseteq K$ can be represented by a unique vector[7] $x \in \{0,1\}^{|K|}$, where the jth element in x is 1 if $k_j \in \mu$ and 0 otherwise. For each $\mu \subseteq K$, define $x(\mu) \in \{0,1\}^{|K|}$ such that for all $j \in \{1,\ldots,|K|\}$, the jth element in $x(\mu)$ is 1 if and only if $k_j \in \mu$. That is, $x(\mu)$ is the unique vector representing μ and each 1 in $x(\mu)$ corresponds to some cycle or chain in μ.

Cycles and chains are assigned different weights in the objective function depending on the patients that participate in them and whether or not they are assigned fully compatible donors. Let δ_1 be some positive real number and recursively define $\delta_t := n^{-2}\delta_{t-1}$. Let $\gamma : N \to \{\delta_1,\ldots,\delta_m\}$ be an injective function such that for each $t \in \{1,\ldots,m\}$, $\gamma(i) = \delta_t$ for all $i \in N_t$. The function γ assigns the same weight to all patients belonging to the same priority group and higher weight to patients belonging to higher priority groups. In addition to the weight assigned to patients participating in a cycle or chain, patients will also be assigned the weight δ_{m+1} whenever they are assigned fully compatible donors. In other words, each patient belonging to the priority group N_t will be assigned the weight δ_t if assigned a half-compatible donor and the weight $\delta_t + \delta_{m+1}$ if assigned a fully compatible donor in the chain or cycle in question. This means that a patient may be assigned different weights in different chains and cycles. Formally, for all $k_j \in K$, define $v(i,k_j) = \delta_{m+1}$ if $i \to_{k_j} i'$ for some $i' \in N \cup D$ such that $c_{i,i'} = 2$ and $v(i,k_j) = 0$ otherwise. The weight, f_j, assigned to a chain or cycle $k_j \in \mu$ is simply the sum of the individual weights of the patients participating in k_j. Let $f \in \mathbb{R}_+^{|K|}$ be a vector with jth element[8]

$$f_j := \sum_{i \in k_j \cap N} (\gamma(i) + v(i,k_j)).$$

Summing over $k_j \cap N$ rather than k_j ensures that only patients receiving kidney transplants are assigned positive weight and not altruistic donors. A function $V : \{0,1\}^{|K|} \to \mathbb{R}$ that assigns a real value to each $x \in \{0,1\}^{|K|}$ is called an *objective function*. Since each $x \in \{0,1\}^{|K|}$ represents some (potentially infeasible) matching, an objective function assigns a value to each matching $\mu \subseteq K$. Priority group matchings can be found by constructing an objective function that *represents* priority group preferenses.

Definition 1. An objective function $V : \{0,1\}^{|K|} \to \mathbb{R}$ represents a preference relation R if for all $\mu, \mu' \subseteq K$, $V(x(\mu)) \geq V(x(\mu'))$ if and only if $\mu R \mu'$.

In other words, if a matching μ is weakly preferred to some μ' by a preference relation R, then any objective function V representing R will assign a weakly higher value to the vector representation of μ than to the vector representation of μ'. Consider an objective function U defined by $U(x) := f^T x$. For each matching $\mu \subseteq K$, $U(x(\mu))$ assigns the weight f_j to each cycle or chain $k_j \in \mu$ and the weight 0 to each cycle or chain $k_j \in K \setminus \mu$. Theorem 1 states that this objective function represents priority group preferences.

[7] All vectors are column vectors unless explicitly transposed. The transpose of a vector x is denoted by x^T.

[8] $\mathbb{R}_+$ denotes the set of non-negative real numbers and $\mathbb{R}_{++}$ denotes the set of positive real numbers. $\mathbb{Z}_+$ is defined analogously.

Theorem 1. U represents $\succsim$.

This representation result is not immediately useful for finding priority group matchings. If the objective function $U(x)$ is maximized over all vectors $x \in \{0,1\}$, the solution will typically not represent a *feasible* matching. For this reason, a number of linear inequality constraints are imposed to ensure that the matching represented by the solution to the resulting constrained optimization problem always be feasible.

First, let A be an $(n + |D|) \times |K|$ matrix, where row i corresponds to a patient or altruistic donor $i \in N \cup D$ and column j corresponds to a cycle or chain $k_j \in K$. The ijth entry in A is 1 if $i \in N \cup D$ participates in k_j and 0 otherwise. Note that the ith entry in the vector $Ax(\mu)$ is the number of cycles and chains that patient or altruistic donor i participates in at matching μ. Since no patient or altruistic donor may participate in more than one cycle or chain, a vector $x \in \{0,1\}^{|K|}$ corresponds to a feasible matching if and only if $Ax \leq \mathbf{1}$, where $\mathbf{1} \coloneqq \{1\}^{n+|D|}$. In other words, the integral points in the $|K|$-polyhedron $\{x \in \mathbb{R}_+^{|K|} \mid Ax \leq \mathbf{1}\}$ correspond to the set of feasible matchings, $\mathcal{M} = \{\mu \subseteq K \mid Ax(\mu) \leq \mathbf{1}\}$. Consider the resulting constrained optimization problem.

$$\begin{aligned} \max_{x \in \{0,1\}^{|K|}} \quad & U(x) \\ \text{subject to} \quad & Ax \leq \mathbf{1} \end{aligned}$$

This optimization problem will henceforth be referred to as *the planner's maximization problem*. It is a special case of the multi-dimensional knapsack problem and can be solved using integer programming techniques. In settings where there is only one priority group and desensitization is not permitted, it reduces to the integer programming formulation of the problem of how to maximize the number of transplants in Roth et al. (2007), which is sometimes referred to as the cycle formulation (Constantino et al., 2013).

Theorem 2 states that a matching μ is a priority group matching if and only if its vector representation $x(\mu)$ is a solution to the planner's maximization problem. This means that priority group matchings can be found by solving the integer programming problem above.

Theorem 2. For any $\mu \in \mathcal{M}$, $\mu \in \mathcal{M}^*$ if and only if

$$x(\mu) \in \arg\max_{Ax \leq \mathbf{1}} U(x).$$

Priority group matchings are always guaranteed to be Pareto efficient. Pareto efficiency is defined in terms of patient preferences. Both patient preferences and Pareto efficiency are defined formally in Appendix A. Informally, patients have preferences over donors. They prefer fully compatible donors to half-compatible donors and half-compatible donors to incompatible donors. Furthermore, regular patients differ from altruistic patients in that they, all else equal, have a preference for their own donors. A matching $\mu \in \mathcal{M}$ is *Pareto efficient* if there exists no other feasible matching that is weakly preferred to μ by all patients and strictly preferred by at least one patient.

Proposition 1. Priority group matchings are Pareto efficient.

If the planner only permits pairwise exchanges, then all Pareto efficient matchings match the same number of patients (Roth et al., 2005a). This result no longer holds whenever cyclic exchanges and chains are permitted. To see this, consider the following example.

Example 1. Let $N = \{1, 2, 3, 4\}$ and $K = \{(1, 2), (2, 3, 4)\}$. Then there are two feasible matchings, $\mu = \{(1, 2)\}$ and $\mu' = \{(2, 3, 4)\}$. Both are Pareto efficient, even though two patients are matched at μ and three patients are matched at μ'.

Priority group matchings, which constitute a subset of the Pareto efficient matchings, do, however, match the same number of patients.

Proposition 2. $|N^*(\mu)| = |N^*(\mu')|$ for all $\mu, \mu' \in \mathcal{M}^*$.

Note that the method described in this section is relatively flexible. Is desensitization used within the kidney exchange program? What types of exchanges are possible? Are exchange chains allowed? Can compatible patient-donor pairs participate? Should the total number of transplants be maximized, or should patients with severe conditions be prioritized? In what way should they be prioritized? The method is agnostic about all these questions and depending on the answers, the selected priority group matchings may belong to a variety of different classes of matchings. The following sections investigate this point further.

4 Threshold matchings

Recall Example 1 in Section 3. There was a choice between a matching μ, in which Patients 1 and 2 receive transplants, and a matching μ', in which Patients 2, 3 and 4 receive transplants. A planner who is only concerned with maximizing the number of kidney transplants would clearly find μ' preferable to μ. However, matching algorithms are typically run more than once in kidney exchange programs. The unmatched patients may still have a chance to receive a healthy kidney at some point in the future when the algorithm is run again. This makes the notion of triage important since the planner may wish to prioritize patients that are less likely to receive kidneys in subsequent runs. For example, the patients could be approaching the point when they are too sick for transplantation, or perhaps they are heavily HLA-sensitized. If Patient 1 urgently needs a healthy kidney, the planner may prefer μ in the example above even though it results in fewer transplants than μ'.

For any approach the planner may take on such triage problems, a Pareto efficient priority group matching can be found by sorting the patients into priority groups and solving the planner's maximization problem, by Theorem 2 and Proposition 1. However, even if there are only 25 patients, there are more than four quintillion ways to sort them into priority groups. The planner may therefore want some rule to follow when deciding how to prioritize patients in any given situation. This section introduces a simple rule for sorting patients into priority groups such that the the solutions to the planner's maximization problem correspond to a class of matchings called *threshold matchings*.

The idea is to set some threshold in terms of some measure of the severity of a patient's condition and then prioritize the patients above the threshold. It is a simple sorting rule in the sense that the planner only needs to decide how severe a condition must be for a patient to receive preferential treatment and set the threshold accordingly. Threshold matchings generalize both maximum matchings and a class of matchings introduced by Roth et al. (2005a) called *priority matchings*. These subclasses are studied in further detail in Sections 5 and 6.

Consider a permutation of N called a *priority order*. The priority order ranks patients by the severity of their conditions. It is given by the bijective function $\pi : N \to \{1, \ldots, n\}$, where a lower value indicates higher priority. That is, if $\pi(i) = 1$, then i is the top priority patient. Patients above the threshold will be prioritized in accordance with a process corresponding to the *priority mechanism* introduced by Roth et al. (2005a). Such patients are called *prioritized* patients. Let m be the number of patients above the threshold plus one. In other words, there are $m - 1$ patients above the threshold. Let $\mathcal{E}_0^m := \mathcal{M}$ and let the set of threshold matchings, $\mathcal{M}_T^m$, be recursively defined by the following.

$$\text{For } t \in (0, m),\ \ \mathcal{E}_t^m = \begin{cases} \{\mu \in \mathcal{E}_{t-1}^m \mid \pi^{-1}(t) \in N^*(\mu)\} & \text{if } \pi^{-1}(t) \in N^*(\mu) \text{ for some } \mu \in \mathcal{E}_{t-1}^m \\ \mathcal{E}_{t-1}^m & \text{otherwise} \end{cases}$$

$$\text{For } t \in [m, n],\ \ \mathcal{E}_t^m = \operatorname*{arg\,max}_{\mu \in \mathcal{E}_{m-1}^m} |N^*(\mu)|$$

$$\mathcal{M}_T^m = \operatorname*{arg\,max}_{\mu \in \mathcal{E}_n^m} B(\mu)$$

The first condition corresponds to the priority mechanism in Roth et al. (2005a). The second condition means that the number of patients below the threshold receiving kidney transplants is maximized subject to the constraint that the patients above the threshold are prioritized in accordance with the first statement. That is, first the patients above the threshold are prioritized in accordance with a priority mechanism, then as many patients below the threshold as possible are matched. The third condition simply states that if there are multiple matchings satisfying the first and second conditions, the matchings that maximize the number of patients assigned fully compatible donors should be maximized to minimize the use of desensitization.

Suppose that the planner sorts the patients into m priority groups where each of the top $m-1$ priority patients belongs to her own priority group and all the remaining patients belong to the lowest priority group. Then threshold matchings can be found by solving the planner's maximization problem.

Formally, for some number of priority groups $m \in \{1, \ldots, n\}$, $|N_m| = n - (m - 1)$ and whenever $m \geq 2$, $N_t = \{\pi^{-1}(t)\}$ for all $t \in \{1, \ldots, m-1\}$. That is, each patient i above the threshold belongs to her own priority group $N_{\pi(i)}$, while each patient below the threshold is not prioritized and gathered in the priority group N_m. Whenever patients are sorted into priority groups in this way, Proposition 3 states that the set of priority group matchings reduces to the set of threshold matchings.

Proposition 3. Consider some priority order π and some $m \in \{1, \ldots, n\}$. Let $|N_m| = n - (m-1)$ and if $m \geq 2$, let $N_t = \{\pi^{-1}(t)\}$ for all $t \in \{1, \ldots, m-1\}$. Then $\mathcal{M}^* = \mathcal{M}_T^m$.

This not only implies that threshold matchings share the general properties of priority group matchings discussed in Section 3, such as Pareto efficiency. It also means that, by Theorem 2, threshold matchings can be found by solving the planner's maximization problem.

Note that the planner may choose a threshold which is so high that no patients qualify for preferential treatment. This is equivalent to setting $m = 1$. Then all patients belong to the same priority group and no patients are prioritized. If all patients belong to the same priority group, threshold matchings will maximize the total number of transplants without regard to priorities. With a lower threshold, there may be some patients above the threshold. Since prioritizing these patients may reduce the number of transplants, there is a relation between the choice of threshold and the number of patients receiving kidney transplants. Proposition 4 states that the more patients are prioritized, the (weakly) lower the number of transplants.

Proposition 4. Consider some priority order π. For any $m \in \{1, \ldots, n\}$, any $m' \in \{1, \ldots, m\}$, any $\mu \in \mathcal{M}_T^m$ and any $\mu' \in \mathcal{M}_T^{m'}$, $|N^*(\mu')| \geq |N^*(\mu)|$.

In other words, the higher the threshold, the (weakly) larger the number of transplants. A donor is a scarce resource that may be used more or less efficiently in terms of the number of patients receiving kidney transplants, as a single donor may be a part of a cyclic exchange involving as many patients as the planner permits. The lower the threshold, the higher the number of prioritized patients that can potentially prevent longer cyclic exchanges by being assigned donors participating in these exchanges.

There is a clear analogy to triage in war situations. The treatment of some wounded soldiers may consume more of the available resources than others. If the doctor simply wants to treat as many patients as possible, he will focus on those whose treatments are cheap in terms of resources. However, the doctor may wish to carry out more costly treatments of patients with particularly urgent conditions. This would reduce the total number of treated patients at a given point in time. How severe the condition should be to warrant special treatment corresponds to the doctor's decision when setting the threshold. The lower the requirements, the larger the number of patients that can potentially reduce the number of treated patients by consuming a large share of the available resources.

Dickerson et al. (2012) also discuss donors as resources that can be used more or less efficiently, but take a different approach that is not based on the notion of triage. They point out that some altruistic donors have higher potential usefulness than others (such as donors with blood group O) and suggest that, in dynamic settings, it is sometimes beneficial to save altruistic donors with high potential until they can be used to form a long chain rather than using them right away in some exchange involving a smaller number of patients.

The way threshold matchings prioritize patients above the threshold corresponds to a generalization of the priority mechanism introduced by Roth et al. (2005a). This is not the only way patients above the threshold could be prioritized. An interesting question is whether the reduction in the number of transplanted patients caused by prioritizing some patients would be lower if all patients above the threshold were assigned equal weight rather than being weighted in accordance with a priority mechanism. In general, the answer is no. To see this, let $N_\Gamma \subseteq N$ be the set of prioritized patients and let the set of *egalitarian threshold matchings* $\mathcal{M}_E$ be defined by the following.

$$\Gamma = \arg\max_{\mu \in \mathcal{M}} |N^*_\Gamma(\mu)|$$
$$\mathcal{M}_\Gamma = \arg\max_{\mu \in \Gamma} |N^*(\mu)|$$
$$\mathcal{M}_E = \arg\max_{\mu \in \mathcal{M}_\Gamma} B(\mu)$$

That is, egalitarian threshold matchings prioritize the patients in N_Γ by first maximizing the number of matched patients in N_Γ and then maximizing the number of matched patients in $N \setminus N_\Gamma$. Finally, $\mathcal{M}_E$ consists of the matchings in $\mathcal{M}_\Gamma$ that minimize the use of desensitization. Proposition 5 states that the set of priority group matchings reduces to the set of egalitarian threshold matchings whenever $m = 2$ and $N_\Gamma = N_1$.

Proposition 5. Let $m = 2$ and set $N_\Gamma = N_1$. Then $\mathcal{M}^* = \mathcal{M}_E$.

Note that whenever $N_\Gamma = \{i \in N \mid \pi(i) < m\}$ for some $m \in \{1, \dots, n\}$, the set of prioritized patients is still the same as under $\mathcal{M}_T^m$. The only difference is that the prioritized patients are given equal priority rather than being ranked internally. The following example demonstrates that this could even reduce the total number of transplants.

Example 2. Let $N = \{1, 2, 3, 4, 5, 6\}$, $N_\Gamma = \{1, 2, 3, 4\}$, $K = \{(1, 2, 5, 6), (2, 3, 4)\}$ and $\pi(i) = i$ for all $i \in N$. Note that matchings in $\mathcal{M}_E$ and $\mathcal{M}_T^5$ prioritize the same set of patients, N_Γ. In this case $\{(2, 3, 4)\}$ is the unique matching in $\mathcal{M}_E$, resulting in a lower number of transplants than the unique matching in $\mathcal{M}_T^5$, $\{(1, 2, 5, 6)\}$.

All else equal, egalitarian threshold matchings will always match a higher number of *prioritized* patients than threshold matchings. However, as the example above showed, the total number of transplants may be lower. They may still be preferable whenever the goal is to match as many patients in urgent need of a healthy kidney as possible.

Egalitarian threshold matchings are similar to the MaxCard-Fair solutions studied by Dickerson and Sandholm (2014) and Dickerson et al. (2014). They construct a MaxCard-Fair objective function by taking an objective function finding maximum matchings and then scaling up the weights assigned to "marginalized" patients (e.g., patients in N_Γ) by a factor of $(1 + \beta)$. For sufficiently large values of β, the set of MaxCard-Fair solutions coincides with $\mathcal{M}_\Gamma$. Furthermore, in the special case when the planner disallows transplantation over the blood group barrier, the set of MaxCard-Fair solutions is identical to the set of egalitarian threshold matchings.

Proposition 6. Let $\beta > n - |N_\Gamma| - 1$. Then $\mathcal{M}_\Gamma$ is the set of *MaxCard-Fair* solutions. If the planner disallows desensitization, then the egalitarian threshold matchings and the *MaxCard-Fair* solutions coincide.

5 Priority matchings

Threshold matchings generalize various types of *priority matchings*. For example, the priority matchings originally defined in Roth et al (2005a), the half-compatibility priority matchings introduced in Andersson and Kratz (2019) and matchings permitting altruistically unbalanced exchanges as in Sönmez and Ünver (2014) can be found by solving the planner's maximization problem.

5.1 Priority matchings

Priority matchings were first introduced by Roth et al. (2005a) in a setting where only pairwise exchanges are permitted, desensitization is disallowed and there are no altruistic donors or patients. Priority matchings are defined in terms of a selection process called a *priority mechanism*. The priority mechanism first selects the set of matchings at which the top priority patient is matched (if non-empty), then it selects the subset of that set at which the second priority patient is matched (if non-empty) and so on. The set of matchings remaining when the process has terminated is the set of priority matchings. Formally, let $\mathcal{E}_0 := \mathcal{M}$ and for each $t \in \{1, \ldots, n\}$, let $\mathcal{E}_t \subseteq \mathcal{E}_{t-1}$ be recursively defined by

$$\mathcal{E}_t = \begin{cases} \{\mu \in \mathcal{E}_{t-1} \mid \pi^{-1}(t) \in N^*(\mu)\} \text{ if } \pi^{-1}(t) \in N^*(\mu) \text{ for some } \mu \in \mathcal{E}_{t-1}, \\ \mathcal{E}_{t-1} \text{ otherwise.} \end{cases}$$

$\mathcal{E}_n$ is the set of priority matchings.[9] Proposition 7 states that whenever the planner does not allow desensitization and each patient is sorted into her own priority group, the set of priority group matchings reduces to the set of priority matchings.

Proposition 7. Suppose that the planner disallows desensitization. Consider some priority order π and partition N such that $N_t = \{\pi^{-1}(t)\}$ for all $t \in \{1, \ldots, n\}$. Then $\mathcal{M}^* = \mathcal{E}_n$.

An equivalent interpretation of this result is that whenever desensitization is not permitted and the planner selects a low enough threshold, the set of threshold matchings reduces to the set of priority matchings. To see this, note that for a low enough threshold, $m = n$ if, as in Section 4, $m - 1$ is the number of prioritized patients. Whenever $c_{i,j} \neq 1$ for all $i, j \in N$ and $m = n$, the definition of the set of threshold matchings $\mathcal{M}_T^m$ is identical to the definition of the set of priority matchings $\mathcal{E}_n$. Therefore, an immediate implication of Proposition 4 is that abandoning priority matchings in favor of threshold matchings will weakly increase the number of transplants.

Corollary 1. For any threshold, threshold matchings match a weakly larger number of patients than priority matchings.

[9] Priority matchings can also be defined in terms of a class of preference relations as in Okumura (2014). Andersson and Kratz (2019) show that these definitions are equivalent.

Corollary 1 continues to hold for the other forms of priority matchings discussed in the following two subsections. Note that this definition of priority matchings is more general than the one found in Roth et al. (2005a), since it does not rule out chains, cyclic exchanges or the use of desensitization. Whenever the planner only allows pairwise exchanges and desensitization is not allowed, the set of priority matchings defined in this section reduces to the set of priority matchings introduced by Roth et al. (2005a).

Proposition 2 states that the number of matched patients is the same for all priority group matchings. Still, the set of matched patients may differ. To see this, consider the following example.

Example 3. Let $N = \{1, 2, 3\}, N_1 = \{1\}, N_2 = \{2, 3\}$ and $K = \{(1, 2), (1, 3)\}$. Then both $\mu = \{(1, 2)\}$ and $\mu' = \{(1, 3)\}$ are priority group matchings, even though $N^*(\mu) \neq N^*(\mu')$.

However, Proposition 8 shows that the set of matched patients will always be the same for all priority matchings.[10]

Proposition 8. For any priority order π, $N^*(\mu) = N^*(\mu')$ for all $\mu, \mu' \in \mathcal{E}_n$.

Proposition 8 holds for any subset of priority matchings, including the set of half-compatibility priority matchings discussed in the next subsection.

5.2 Half-compatibility priority matchings

Andersson and Kratz (2019) introduce a setting where some patients can receive kidneys from blood group incompatible donors using desensitization. Only pairwise kidney exchanges are considered and there are no altruistic donors or patients. Whenever desensitization is permitted, priority matchings are no longer Pareto efficient. For this reason, they introduce a subset $\mathcal{M}_B$ of priority matchings called *half-compatibility priority matchings*. Half-compatibility priority matchings take the distinction between half-compatible and fully compatible donors into account and are guaranteed to be Pareto efficient. Formally, let $\mathcal{E}_0 := \mathcal{M}$ and let $\mathcal{M}_B$ be recursively defined by the following conditions.

$$\text{For } t \in \{1, \ldots, n\}, \;\; \mathcal{E}_t = \begin{cases} \{\mu \in \mathcal{E}_{t-1} \mid \pi^{-1}(t) \in N^*(\mu)\} & \text{if } \pi^{-1}(t) \in N^*(\mu) \text{ for some } \mu \in \mathcal{E}_{t-1}, \\ \mathcal{E}_{t-1} & \text{otherwise.} \end{cases}$$

$$\mathcal{M}_B = \arg\max_{\mu \in \mathcal{E}_n} B(\mu)$$

That is, half-compatibility priority matchings are those priority matchings that maximize the number of patients assigned fully compatible donors. This definition is more general than the one found in Andersson and Kratz (2019) since altruistic participation, chains and cyclic exchanges are allowed. Proposition 9 shows that the set of priority group matchings reduces to the set of half-compatibility priority matchings whenever each patient is assigned a priority group in accordance with the priority order.

[10] This relies on the assumption that the priority order is a permutation of the set of patients. In Okumura (2014), a priority order is simply defined as a function $\pi : N \to \mathbb{R}_{++}$. In such settings, Proposition 8 no longer holds unless the restriction that each patient be assigned a *unique* priority is imposed.

Proposition 9. Consider some priority order π and partition N such that $N_t = \{\pi^{-1}(t)\}$ for all $t \in \{1, \ldots, n\}$. Then $\mathcal{M}^* = \mathcal{M}_B$.

In settings where only pairwise exchanges are permitted and there are no altruistic donors or patients, $\mathcal{M}_B$ reduces to the set of half-compatibility priority matchings considered in Andersson and Kratz (2019). Note that their observation that whenever the planner permits blood group incompatible transplantation using desensitization, half-compatibility priority matchings are (in contrast with priority matchings) guaranteed to be Pareto efficient continues to hold when cyclic exchanges, chains, altruistic donors and altruistic patients are permitted, by Proposition 1.

5.3 Altruistically unbalanced exchange

Some authors, such as Roth et al. (2005b) and Sönmez and Ünver (2014) have advocated the inclusion of compatible patient-donor pairs in exchange programs where only pairwise exchanges are permitted. In their settings, the planner does not allow desensitization. This means that there are no half-compatible patient-donor pairs. A patient is either incompatible or fully compatible with her own donor. A patient with a fully compatible donor has no incentive beyond altruism to participate in a kidney exchange program, since she could receive a kidney from her own donor without participating or using desensitization. However, Roth et al. (2005b) and Sönmez and Ünver (2014) show that the number of transplants can be increased by including compatible patient-donor pairs in a kidney exchange program selecting priority matchings.

Kidney exchange programs including all compatible patient-donor pairs can be studied in the present model by letting all patients with fully compatible or half-compatible donors be altruistic patients. If the planner allows desensitization, then i is never matched to some $j \in N$ such that $c_{i,j} = 1$. This is to ensure that no patients are made strictly worse off by participating.

The matchings considered in Roth et al. (2005b) and Sönmez and Ünver (2014) are called *altruistically unbalanced priority matchings* and are gathered in the set $\mathcal{M}_U$. It follows immediately from Proposition 7 that these matchings correspond to solutions to the planner's maximization problem in the setting studied in their papers.

Corollary 2. Consider some priority order π and suppose that the planner disallows desensitization and that for each $i \in N$, $i \in I_A$ if and only if $c_{i,i} = 2$. Let $D = \emptyset$, let K be the set of all 2-cycles and partition N such that $N_t = \{\pi^{-1}(t)\}$ for all $t \in \{1, \ldots, n\}$. Then $\mathcal{M}^* = \mathcal{M}_U$.

6 Maximum matchings

The preceding sections demonstrated that prioritizing some patients, e.g. patients in urgent need of a healthy kidney, may reduce the total number of transplants. Some planners may therefore wish to abandon triage and simply focus on maximizing the number of patients receiving kidney transplants, as in Roth et al. (2005b) and Saidman et al. (2006).

Let $\mathcal{M}_{\max} := \arg\max_{\mu \in \mathcal{M}} |N^*(\mu)|$ be the set of *maximum matchings* and let $\mathcal{M}^{\varepsilon}_{\max} := \arg\max_{\mu \in \mathcal{M}_{\max}} B(\mu)$ be the set of *desensitization-minimizing maximum matchings*. In other words, maximum matchings maximize the number of transplants, while desensitization-minimizing maximum matchings are the subset of maximum matchings for which the number of patients receiving kidney transplants from half-compatible donors is minimized, thereby minimizing the use of desensitization. If the planner does not permit desensitization, maximum matchings and desensitization-minimizing maximum matchings coincide. When only pairwise exchanges are permitted, $\mathcal{M}_{\max}$ is a superset of the set of priority matchings (Roth et al., 2005a). Proposition 10 shows that the set of priority group matchings reduces to the set of desensitization-minimizing maximum matchings whenever there is only one priority group.

Proposition 10. Let $N_1 = N$. Then $\mathcal{M}^* = \mathcal{M}^{\varepsilon}_{\max}$.

By Proposition 1, desensitization-minimizing maximum matchings are Pareto efficient. It also follows immediately from Proposition 4 that abandoning desensitization-minimizing maximum matchings in favor of threshold matchings will weakly decrease the number of transplants.

Corollary 3. For any threshold, threshold matchings match weakly fewer patients than desensitization-minimizing maximum matchings.

7 Conclusion

This paper introduced a new class of matchings called priority group matchings and presented a computational method for finding such matchings. Priority group matchings constitute a Pareto efficient solution to the kidney exchange problem for any approach to triage that planner may adopt. A subclass called threshold matchings was also introduced. Threshold matchings give priority to patients above some threshold. The choice of threshold depends on the planner's approach to triage. A lower threshold results in (weakly) fewer transplants. Threshold matchings include a number of classes of matchings such as (half-compatibility) priority matchings and maximum matchings as special cases.

The notion of triage becomes relevant due to matching algorithms being run multiple times each year. Yet, time is typically not explicitly modelled in the kidney exchange literature, with some exceptions (Ünver, 2010; Dickerson et al., 2012; Ashlagi et al., 2013; Akbarpour et al., 2017). This temporal aspect is often only implicitly addressed by the focus on matchings that respect patient priorities. Modelling this kidney exchange problem as a dynamic optimization problem could potentially yield insights into the planner's optimal approach to triage.

The formulation of the planner's maximization problem is simple, yet relatively flexible. It would be straightforward to carry out a simulation study comparing variants of the various types of matchings that priority group matchings generalize. For instance, such a study could investigate the choice of threshold, the potential benefits of egalitarian threshold matchings relative to threshold matchings or the impact of allowing chains and cyclic exchanges in kidney exchange programs that permit the use of desensitization.

A Proofs

This appendix contains all proofs. The theorems and propositions are restated for convenience.

Theorem 1. U represents $\succsim$.

Proof. ($\Rightarrow$) It will first be shown that for all $\mu, \mu' \subseteq K$, $U(x(\mu)) \geq U(x(\mu'))$ only if $\mu \succsim \mu'$. To reach a contradiction, consider some $\mu, \mu' \subseteq K$ such that $U(x(\mu)) \geq U(x(\mu'))$ and $\mu' \succ \mu$. If $\mu' \succ \mu$, then either there exists some $t \in \{1, \dots, m\}$ such that $|N_t^*(\mu')| > |N_t^*(\mu)|$ and $|N_\tau^*(\mu')| = |N_\tau^*(\mu)|$ for all $\tau \in \{0, \dots, t-1\}$, or $|N_t^*(\mu)| = |N_t^*(\mu')|$ for all $t \in \{1, \dots, m\}$ and $B(\mu') > B(\mu)$. First, suppose there exists some $t \in \{1, \dots, m\}$ such that $|N_t^*(\mu')| > |N_t^*(\mu)|$ and $|N_\tau^*(\mu')| = |N_\tau^*(\mu)|$ for all $\tau \in \{0, \dots, t-1\}$. Then,

$$\begin{aligned}
U(x(\mu')) - U(x(\mu)) &= f^T x(\mu') - f^T x(\mu) \\
&= \sum_{k_j \in \mu'} \sum_{i \in k_j \cap N} (\gamma(i) + v(i, k_j)) - \sum_{k_j \in \mu} \sum_{i \in k_j \cap N} (\gamma(i) + v(i, k_j)) \qquad (1) \\
&\geq \delta_t - (n-1)\delta_{m+1} - \sum_{k_j \in \mu} \sum_{i \in k_j \cap \{i \in N_{t'} | t < t' \leq m\}} \delta_{t+1} \qquad (2) \\
&> \delta_t - n(\delta_{t+1} + \delta_{m+1}) \\
&> \delta_t - 2n\delta_{t+1} \\
&= \delta_t - \frac{2\delta_t}{n} \\
&\geq 0
\end{aligned}$$

The first inequality follows from the fact that expression (2) corresponds to the case when (A) some $i \in N_t$ is the only patient matched at μ' that does not belong to some N_τ where $\tau \in \{0, \dots, t-1\}$; (B) i is assigned to a half-compatible donor; (C) every patient except i is matched to a fully compatible donor at μ; (D) every patient with lower priority than i that is matched at μ belongs to N_{t+1}, thereby having as high priority as possible. Hence, (2) constitutes a lower bound on (1). The final inequality follows from that fact that $n \geq 2$. $U(x(\mu')) - U(x(\mu)) > 0$ contradicts $U(x(\mu)) \geq U(x(\mu'))$. Then it must be true that

$|N_t^*(\mu)| = |N_t^*(\mu')|$ for all $t \in \{1, \ldots, m\}$ and $B(\mu') > B(\mu)$. Thus,

$$\begin{aligned}
U(x(\mu')) - U(x(\mu)) &= f^T x(\mu') - f^T x(\mu) \\
&= \sum_{k_j \in \mu'} \sum_{i \in k_j \cap N} (\gamma(i) + v(i, k_j)) - \sum_{k_j \in \mu} \sum_{i \in k_j \cap N} (\gamma(i) + v(i, k_j)) \\
&= \sum_{t=1}^{m} (|N_t^*(\mu')| - |N_t^*(\mu)|)\delta_t + \sum_{k_j \in \mu'} \sum_{i \in k_j \cap N} v(i, k_j) - \sum_{k_j \in \mu} \sum_{i \in k_j \cap N} v(i, k_j) \\
&= \sum_{k_j \in \mu'} \sum_{i \in k_j \cap N} v(i, k_j) - \sum_{k_j \in \mu} \sum_{i \in k_j \cap N} v(i, k_j) \\
&= \delta_{m+1}(B(\mu') - B(\mu)) \\
&> 0
\end{aligned}$$

The final inequality follows from the fact that $\delta_{m+1} > 0$ and $B(\mu') > B(\mu)$. $U(x(\mu')) - U(x(\mu)) > 0$ contradicts $U(x(\mu)) \geq U(x(\mu'))$. Since the existence of some $\mu, \mu' \subseteq K$ for which $U(x(\mu)) \geq U(x(\mu'))$ and $\mu' \succ \mu$ leads to a contradiction, it must be the case that $U(x(\mu)) \geq U(x(\mu'))$ only if $\mu \succsim \mu'$ for all $\mu, \mu' \subseteq K$.

($\Leftarrow$) Next, it will be shown that for all $\mu, \mu' \subseteq K$, $U(x(\mu)) \geq U(x(\mu'))$ if $\mu \succsim \mu'$. To reach a contradiction, consider some $\mu, \mu' \subseteq K$ such that $\mu \succsim \mu'$ and $U(x(\mu')) > U(x(\mu))$. If $\mu \succsim \mu'$, then either there exists some $t \in \{1, \ldots, m\}$ such that $|N_t^*(\mu)| > |N_t^*(\mu')|$ and $|N_\tau^*(\mu)| = |N_\tau^*(\mu')|$ for all $\tau \in \{0, \ldots, t-1\}$, or $|N_t^*(\mu)| = |N_t^*(\mu')|$ for all $t \in \{1, \ldots, m\}$ and $B(\mu) \geq B(\mu')$. First, suppose that there exists some $t \in \{1, \ldots, m\}$ such that $|N_t^*(\mu)| > |N_t^*(\mu')|$ and $|N_\tau^*(\mu)| = |N_\tau^*(\mu')|$ for all $\tau \in \{0, \ldots, t-1\}$. Then by reversing the argument above, it is easy to see that this implies $U(x(\mu)) - U(x(\mu')) > 0$, which contradicts $U(x(\mu')) > U(x(\mu))$.

Hence, it must be the case that $|N_t^*(\mu)| = |N_t^*(\mu')|$ for all $t \in \{1, \ldots, m\}$ and $B(\mu) \geq B(\mu')$. As before, note that $U(x(\mu)) - U(x(\mu')) = \delta_{m+1}(B(\mu) - B(\mu'))$. Since $\delta_{m+1} > 0$ and $B(\mu) \geq B(\mu')$, it follows that $\delta_{m+1}(B(\mu) - B(\mu')) \geq 0$. Thus, $U(x(\mu)) - U(x(\mu')) \geq 0$, which contradicts $U(x(\mu')) > U(x(\mu))$. Since the existence of some $\mu, \mu' \subseteq K$ such that $\mu \succsim \mu'$ and $U(x(\mu')) > U(x(\mu))$ leads to a contradiction, $U(x(\mu)) \geq U(x(\mu'))$ if $\mu \succsim \mu'$.

To conclude, $U(x(\mu)) \geq U(x(\mu'))$ if and only if $\mu \succsim \mu'$ for all $\mu, \mu' \subseteq K$. By Definition 1, U represents $\succsim$. □

Theorem 2. For any $\mu \in \mathcal{M}$, $\mu \in \mathcal{M}^*$ if and only if

$$x(\mu) \in \arg\max_{Ax \leq \mathbf{1}} U(x).$$

Proof. Consider some $\mu \in \mathcal{M}^*$. By definition of $\mathcal{M}^*$, $\mu \succsim \mu'$ for all $\mu' \in \mathcal{M}$. Since U represents $\succsim$, $U(x(\mu)) \geq U(x(\mu'))$ for all $\mu' \in \mathcal{M}$. Since $\mu \in \mathcal{M}$ if and only if $Ax(\mu) \leq \mathbf{1}$ by definition, $x(\mu) \in \arg\max_{Ax \leq \mathbf{1}} U(x)$.

To show the opposite direction, consider some $x' \in \arg\max_{Ax \leq \mathbf{1}} U(x)$. Since $x' \in \{0,1\}^{|K|}$, $x' = x(\mu)$ for some $\mu \subseteq K$. $Ax(\mu) \leq \mathbf{1}$ implies that $\mu \in \mathcal{M}$ and $x(\mu) \in \arg\max_{Ax \leq \mathbf{1}} U(x)$ implies that $U(x(\mu)) \geq U(x(\mu'))$ for all $\mu' \in \mathcal{M}$. Since U represents $\succsim$, $\mu \succsim \mu'$ for all $\mu' \in \mathcal{M}$ and $\mu \in \mathcal{M}^*$. □

In order to prove Proposition 1, Pareto efficiency and patient preferences must first be defined formally. Denote the patient or altruistic donor that patient i is matched to at matching μ by $\mu(i)$. Consider two arbitrary matchings $\mu, \nu \in \mathcal{M}$ and let each patient $i \in N$ have preferences $\succsim_i$ over matchings defined by the following.

$$\text{For } i \in I_R, \quad \mu \succ_i \nu \iff \begin{cases} \mu(i) \neq i \text{ and } c_{i,\mu(i)} > c_{i,\nu(i)}, \\ \mu(i) = i \neq \nu(i) \text{ and } c_{i,\mu(i)} \geq c_{i,\nu(i)}, \end{cases}$$

$$\mu \sim_i \nu \iff \begin{cases} \mu(i) = \nu(i), \\ \mu(i), \nu(i) \in (N \cup D) \setminus \{i\} \text{ and } c_{i,\mu(i)} = c_{i,\nu(i)}. \end{cases}$$

$$\text{For } i \in I_A, \quad \mu \succ_i \nu \iff c_{i,\mu(i)} > c_{i,\nu(i)},$$

$$\mu \sim_i \nu \iff c_{i,\mu(i)} = c_{i,\nu(i)}.$$

That is, patients prefer fully compatible donors to half-compatible donors and half-compatible donors to incompatible donors. Patients in I_R differ from patients in I_A by, all else equal, having a preference for their own donors. A matching $\mu \in \mathcal{M}$ is *Pareto efficient* if there exists no matching $\nu \in \mathcal{M}$ such that $\nu \succsim_i \mu$ for all $i \in N$ and $\nu \succ_i \mu$ for some $i \in N$.

Proposition 1. Priority group matchings are Pareto efficient.

Proof. Consider some $\mu \in \mathcal{M}^*$ and assume that μ is not Pareto efficient to reach a contradiction. Then there exists some $\nu \in \mathcal{M}$ such that $\nu \succsim_i \mu$ for all $i \in N$ and $\nu \succ_i \mu$ for some $i \in N$. Since $\nu \succsim_i \mu$ for all $i \in N$, it follows that $N^*(\mu) \subseteq N^*(\nu)$. Then, either $N^*(\mu) \subset N^*(\nu)$ or $N^*(\mu) = N^*(\nu)$.

First, suppose that $N^*(\mu) \subset N^*(\nu)$. Let j be the highest priority patient in $N^*(\nu) \setminus N^*(\mu)$ and let N_t be the priority group for which $j \in N_t$. Since there is no $j' \in N^*(\nu) \setminus N^*(\mu)$ for whom $\pi(j') < \pi(j)$, it follows that $|N^*_\tau(\mu)| = |N^*_\tau(\nu)|$ for all $\tau \in \{0, \ldots, t-1\}$. Furthermore, since $N^*(\mu) \subset N^*(\nu)$ and $j \in N^*(\nu) \setminus N^*(\mu)$ it must be the case that $|N^*_t(\nu)| > |N^*_t(\mu)|$. Then, by the definition of $\succsim$, $\nu \succ \mu$. However, since $\mu \in \mathcal{M}^*$ and $\nu \in \mathcal{M}$, $\mu \succsim \nu$. This is a contradiction.

This implies that $N^*(\mu) = N^*(\nu)$. Since $\nu \succsim_i \mu$ for all $i \in N$, $c_{i,\nu(i)} \geq c_{i,\mu(i)}$ for all $i \in N$. Hence, $B(\nu) \geq B(\mu)$. Consider some patient j for whom $\nu \succ_j \mu$. Note that j is either a regular patient or an altruistic patient. That is, either $j \in I_A$ or $j \in I_R$.

Suppose that $j \in I_A$. Whereas a regular patient would prefer her own donor to a different donor with the same degree of compatibility, an altruistic patient would only prefer one donor over another if the degree of compatibility is higher. That is, $\nu \succ_j \mu$ implies that $c_{j,\nu(j)} > c_{j,\mu(j)}$ by the definition of the preference relation $\succsim_j$. Since both μ and ν are feasible, $c_{j,\mu(j)} \geq 1$ and $c_{j,\nu(j)} \geq 1$. It therefore follows from $c_{j,\nu(j)} > c_{j,\mu(j)}$ that $c_{j,\mu(j)} = 1$ and $c_{j,\nu(j)} = 2$. In conjunction with the observation that $c_{i,\nu(i)} \geq c_{i,\mu(i)}$ for all $i \in N$, this implies that $B(\nu) > B(\mu)$. The intuition is that since the donor assigned to some patient i at ν has a weakly higher degree of compatibility than the donor i is assigned at μ for all $i \in N$ and since patient j is assigned a fully compatible donor at ν but not at μ, the number of patients assigned fully compatible donors is strictly higher at ν than it is at μ. However, since $\mu \succsim \nu$ and $N^*(\mu) = N^*(\nu)$ by assumption, it must be the case that $B(\mu) \geq B(\nu)$. This is a contradiction.

It must therefore be the case that $j \in I_R$. First suppose that $\nu(j) \neq j$. This means that patient j's preference for ν over μ is not due to j being a regular patient with a preference for her own donor. That is, since $\nu(j) \neq j$ and $\nu \succ_j \mu$ it must be the case that $c_{j,\nu(j)} > c_{j,\mu(j)}$ by the definition of the preference relation $\succsim_j$. Then the situation is identical to the case when $j \in I_A$, resulting in the same contradiction.

Thus, $j \in I_R$ and $\nu(j) = j$. Since $c_{j,\mu(j)} \geq c_{j,\nu(j)}$, either $c_{j,\mu(j)} = c_{j,\nu(j)}$ or $c_{j,\nu(j)} > c_{j,\mu(j)}$. First consider the case when $c_{j,\mu(j)} = c_{j,\nu(j)}$. Since $\nu \succ_j \mu$ it must be the case that $\mu(j) \neq \nu(j)$. However, since $j \in I_R$, $\mu(j) \in (N \cup D) \setminus \{j\}$ and $c_{j,\mu(j)} \not> c_{j,j} = c_{j,\nu(j)}$ it follows from the definition of the arc set E that the arc $j\mu(j) \notin E$. This violates the assumption that μ is a matching. In other words, since j is a regular patient, she may only be assigned her own donor and donors with a strictly higher degree of compatibility. This contradicts the assumption that she is assigned a different donor with the same degree of compatibility as her own at μ.

Therefore, $N^*(\mu) = N^*(\nu)$, $j \in I_R$, $\nu(j) = j$ and $c_{j,\nu(j)} > c_{j,\mu(j)}$. In conjunction with the observation that $c_{i,\nu(i)} \geq c_{i,\mu(i)}$ for all $i \in N$, this, again, implies that $B(\nu) > B(\mu)$. However, since $\mu \succsim \nu$ and $N^*(\mu) = N^*(\nu)$ by assumption, it must be the case that $B(\mu) \geq B(\nu)$. This is a contradiction.

It has now been shown that whenever $\mu \in \mathcal{M}^*$, the existence of some $\nu \in \mathcal{M}$ such that $\nu \succsim_i \mu$ for all $i \in N$ and $\nu \succ_i \mu$ for some $i \in N$ will always result in a contradiction. This means that every priority group matching is Pareto efficient. □

Proposition 2. $|N^*(\mu)| = |N^*(\mu')|$ for all $\mu, \mu' \in \mathcal{M}^*$.

Proof. Consider some $\mu, \mu' \in \mathcal{M}^*$ and assume that $|N^*(\mu)| \neq |N^*(\mu')|$ to reach a contradiction. Let N_t be the highest priority group for which $|N_t^*(\mu)| \neq |N_t^*(\mu')|$. Without loss of generality, assume that $|N_t^*(\mu)| > |N_t^*(\mu')|$. Then $|N_t^*(\mu)| > |N_t^*(\mu')|$ for some $t \in \{1, \dots, m\}$ and $|N_\tau^*(\mu)| = |N_\tau^*(\mu')|$ for all $\tau \in \{0, \dots, t-1\}$. Thus, $\mu \succ \mu'$ by definition. This contradicts the assumption that $\mu' \in \mathcal{M}^*$. □

Proposition 3. Consider some priority order π and some $m \in \{1, \dots, n\}$. Let $|N_m| = n - (m-1)$ and if $m \geq 2$, let $N_t = \{\pi^{-1}(t)\}$ for all $t \in \{1, \dots, m-1\}$. Then $\mathcal{M}^* = \mathcal{M}_T^m$.

Proof. The idea of the proof is to show that both $\mathcal{M}_T^m \subseteq \mathcal{M}^*$ and $\mathcal{M}^* \subseteq \mathcal{M}_T^m$, which is only true whenever $\mathcal{M}^* = \mathcal{M}_T^m$.

It will first be shown that $\mathcal{M}_T^m \subseteq \mathcal{M}^*$. Consider some $\mu \in \mathcal{M}$ and assume that $\mu \in \mathcal{M}_T^m \setminus \mathcal{M}^*$ to reach a contradiction. Since $\mu \notin \mathcal{M}^*$, there must exist some $\mu' \in \mathcal{M}$ such that $\mu' \succ \mu$. Recall the definition of priority group preferences on page 10. As $\mu' \succ \mu$, it follows that either $|N_t^*(\mu)| = |N_t^*(\mu')|$ for all $t \in [1, m]$ and $B(\mu') > B(\mu)$, or $|N_t^*(\mu')| > |N_t^*(\mu)|$ for some $t \in \{1, \dots, m\}$ and $|N_\tau^*(\mu)| = |N_\tau^*(\mu')|$ for all $\tau \in \{0, \dots, t-1\}$.

First, suppose that $|N_t^*(\mu)| = |N_t^*(\mu')|$ for all $t \in \{1, \dots, m\}$ and $B(\mu') > B(\mu)$. Recall the definition of threshold matchings on page 14. If $m = 1$, then $\mathcal{E}_{m-1}^m = \mathcal{M}$ and $\mu' \in \mathcal{E}_{m-1}^m$. If $m \in \{2, \dots, n\}$, then since $N_t = \{\pi^{-1}(t)\}$ for all $t \in \{1, \dots, m-1\}$, $|N_t^*(\mu)| = |N_t^*(\mu')|$

for all $t \in \{1, \ldots, m\}$ implies that $\mu' \in \mathcal{E}_{m-1}^m$. That is, $\mu' \in \mathcal{E}_{m-1}^m$ for all $m \in \{1, \ldots, n\}$. Furthermore, $|N_m^*(\mu)| = |N_m^*(\mu')|$ implies that $\mu' \in \mathcal{E}_n^m$. Finally, $\mu' \in \mathcal{E}_n^m$ and $B(\mu') > B(\mu)$ contradict the assumption that $\mu \in \mathcal{M}_T^m$, since it requires that $\mu \in \arg\max_{\nu \in \mathcal{E}_n^m} B(\nu)$.

Hence, $|N_t^*(\mu')| > |N_t^*(\mu)|$ for some $t \in \{1, \ldots, m\}$ and $|N_\tau^*(\mu)| = |N_\tau^*(\mu')|$ for all $\tau \in \{0, \ldots, t-1\}$. Since $t \in \{1, \ldots, m\}$ and thus $N_\tau = \{\pi^{-1}(\tau)\}$ for all $\tau \in \{0, t-1\}$, $|N_\tau^*(\mu)| = |N_\tau^*(\mu')|$ for all $\tau \in \{0, t-1\}$ implies that $\mu' \in \mathcal{E}_{t-1}^m$.

Suppose that $t \in \{1, \ldots, m-1\}$. Then $N_t = \{\pi^{-1}(t)\}$. In that case, $|N_t^*(\mu')| > |N_t^*(\mu)|$ implies that $\pi^{-1}(t) \in N^*(\mu') \setminus N^*(\mu)$. That is, if $t \in \{1, \ldots, m-1\}$, then priority group N_t contains only the unique patient with priority t. Since the number of patients in priority group N_t is higher at μ' than at μ, it must therefore be the case that the only patient in N_t, $\pi^{-1}(t)$, is matched at μ' but not at μ. Thus, since $\mu' \in \mathcal{E}_{t-1}^m$, $\mu \notin \mathcal{E}_t^m$ by the definition of $\mathcal{E}_t^m$. This contradicts the assumption that $\mu \in \mathcal{M}_T^m$, because $\mathcal{M}_T^m \subseteq \mathcal{E}_t^m$.

It must therefore be the case that $t = m$. Then $\mu' \in \mathcal{E}_{m-1}^m$. Since m is the number of priority groups and $|N_\tau^*(\mu)| = |N_\tau^*(\mu')|$ for all $\tau \in \{0, \ldots, m-1\}$, $|N_t^*(\mu')| > |N_t^*(\mu)|$ if and only if $|N^*(\mu')| > |N^*(\mu)|$. This contradicts the assumption that $\mu \in \mathcal{M}_T^m \subseteq \arg\max_{\nu \in \mathcal{E}_{m-1}^m} |N^*(\nu)|$. Hence, $\mathcal{M}_T^m \subseteq \mathcal{M}^*$.

Next, it will be shown that $\mathcal{M}^* \subseteq \mathcal{M}_T^m$. Consider some $\mu \in \mathcal{M}$ and assume that $\mu \in \mathcal{M}^* \setminus \mathcal{M}_T^m$ to reach a contradiction. Since $\mathcal{M}$ is assumed to be non-empty, $\mathcal{M}_T^m$ is guaranteed to be non-empty. Consider some $\mu' \in \mathcal{M}_T^m$. Note that $\mu \in \mathcal{E}_0^m \setminus \mathcal{M}_T^m$, as $\mathcal{E}_0^m$ is the set of all feasible matchings. Since $\mathcal{E}_t^m \subseteq \mathcal{E}_{t-1}^m$ for all $t \in \{1, \ldots, m\}$, it must therefore either be the case that $\mu \in \mathcal{E}_n^m \setminus \mathcal{M}_T^m$ or $\mu \in \mathcal{E}_{t'-1}^m \setminus \mathcal{E}_{t'}^m$ for some $t' \in \{1, \ldots, m\}$. Also note that $\mathcal{E}_t^m = \mathcal{E}_{t+1}^m$ for all $t \in \{m, \ldots, n-1\}$ whenever $m < n$.

First, suppose that $\mu \in \mathcal{E}_n^m \setminus \mathcal{M}_T^m$. In this case, $|N_t^*(\mu)| = |N_t^*(\mu')|$ for all $t \in \{1, \ldots, m\}$. Hence, $\mu' \in \mathcal{M}_T^m$ implies that $B(\mu') > B(\mu)$. This, in turn, implies that $\mu' \succ \mu$, contradicting the assumption that $\mu \in \mathcal{M}^*$.

This means thas $\mu \in \mathcal{E}_{t'-1}^m \setminus \mathcal{E}_{t'}^m$ for some $t' \in \{1, \ldots, m\}$. First suppose that $t' = m$ so that $\mu \in \mathcal{E}_{m-1}^m \setminus \mathcal{E}_m^m$. Then $|N_t^*(\mu)| = |N_t^*(\mu')|$ for all $t \in \{0, \ldots, m-1\}$. (Recall that $N_0 = \emptyset$ by definition.) Since m is the number of priority groups, $\mu \notin \mathcal{E}_m^m$ and $\mu' \in \mathcal{E}_m^m$ then imply that $|N_m^*(\mu')| > |N_m^*(\mu)|$ and $|N^*(\mu')| > |N^*(\mu)|$. Hence, $\mu' \succ \mu$, contradicting the assumption that $\mu \in \mathcal{M}^*$.

Thus, $t' \in \{1, \ldots, m-1\}$. Note that $m \geq 2$, since $m = 1$ requires that $t' = m$, which has been shown to result in a contradiction. Since $t' < m$, $N_\tau = \{\pi^{-1}(\tau)\}$ for all $\tau \in \{1, \ldots, t\}$. Then, by the definition of $\mathcal{E}_{t'}^m$ and the fact that $\mu, \mu' \in \mathcal{E}_{t'-1}^m$, $|N_\tau^*(\mu)| = |N_\tau^*(\mu')|$ for all $\tau \in \{0, \ldots, t'-1\}$. Since $\mu \in \mathcal{E}_{t'-1}^m \setminus \mathcal{E}_{t'}^m$ and $\mu' \in \mathcal{E}_{t'-1}^m$, $\pi^{-1}(t') \in N^*(\mu') \setminus N^*(\mu)$. Thus, $|N_{t'}^*(\mu')| > |N_{t'}^*(\mu)|$ as $|N_{t'}| = 1$. $|N_\tau^*(\mu)| = |N_\tau^*(\mu')|$ for all $\tau \in \{0, \ldots, t'-1\}$ and $|N_{t'}^*(\mu')| > |N_{t'}^*(\mu)|$ together imply that $\mu' \succ \mu$. This contradicts $\mu \in \mathcal{M}^*$.

Hence, there exists no $\mu \in \mathcal{M}^* \setminus \mathcal{M}_T^m$, implying that $\mathcal{M}^* \subseteq \mathcal{M}_T^m$ since $\mathcal{M}_T^m$ is non-empty. In conclusion, as both $\mathcal{M}_T^m \subseteq \mathcal{M}^*$ and $\mathcal{M}^* \subseteq \mathcal{M}_T^m$ are true, it must be the case that $\mathcal{M}^* = \mathcal{M}_T^m$. □

Proposition 4. Consider some priority order π. For any $m \in \{1, \ldots, n\}$, any $m' \in \{1, \ldots, m\}$, any $\mu \in \mathcal{M}_T^m$ and any $\mu' \in \mathcal{M}_T^{m'}$, $|N^*(\mu')| \geq |N^*(\mu)|$.

Proof. First, let $m \in \{2, \ldots, n\}$ and $m' \in \{1, \ldots, m-1\}$. Consider some $\mu \in \mathcal{M}_T^m$ and some $\mu' \in \mathcal{M}_T^{m'}$. First note that $\mathcal{E}_t^m = \mathcal{E}_t^{m'}$ for all $t \in \{1, \ldots, m'-1\}$. Consequently, $\mathcal{E}_{m'-1}^m = \mathcal{E}_{m'-1}^{m'}$. By definition, $|N^*(\mu')| \geq |N^*(\nu)|$ for all $\nu \in \mathcal{E}_{m'-1}^{m'} = \mathcal{E}_{m'-1}^m$. Furthermore, since $\mathcal{E}_\tau^m \subseteq \mathcal{E}_{m'-1}^m$ for all $\tau \in \{m'-1, \ldots, n\}$ and $\mathcal{M}_T^m \subseteq \mathcal{E}_n^m$, it follows that $|N^*(\mu')| \geq |N^*(\mu)|$.

Next, let $m \in \{1, \ldots, n\}$ and $m' = m$. Consider some $\mu \in \mathcal{M}_T^m$ and some $\mu' \in \mathcal{M}_T^{m'}$. Since $m = m'$, $\mathcal{M}_T^m = \mathcal{M}_T^{m'}$. By Proposition 3, there is a partitioning of N such that $\mathcal{M}_T^m = \mathcal{M}^*$. By Proposition 2, all priority group matchings match the same number of patients. Then $|N^*(\mu)| = |N^*(\mu')|$, as $\mu, \mu' \in \mathcal{M}^*$.

Hence, for any $\mu \in \mathcal{M}_T^m$ and any $\mu' \in \mathcal{M}_T^{m'}$, $|N^*(\mu')| \geq |N^*(\mu)$ both in the case when $m \in \{2, \ldots, n\}$, $m' \in \{1, \ldots, m-1\}$ and in the case when $m \in \{1, \ldots, n\}$ and $m' = m$. □

Proposition 5. Let $m = 2$ and set $N_\Gamma = N_1$. Then $M^* = M_E$.

Proof. It follows immediately from the definition of priority group preferences on page 10 that matchings in $\mathcal{M}_\Gamma^\varepsilon$ are preferred to all other feasible matchings in this setting. □

Proposition 6. Let $\beta > n - |N_\Gamma| - 1$. Then $\mathcal{M}_\Gamma$ is the set of *MAXCARD-FAIR* solutions. If the planner disallows desensitization, then the egalitarian threshold matchings and the *MAXCARD-FAIR* solutions coincide.

Proof. Dickerson and Sandholm (2014) assign each arc $e \in E$ some weight $w_e \in \mathbb{R}_{++}$. A cycle or chain c is defined in terms of the relevant arcs rather than (as in this paper) in terms of the participating patients. A matching is feasible if the cycles and chains it contains are disjoint. Consider an objective function u where for any $\mu \in \mathcal{M}$, $u(\mu) = \sum_{c \in \mu} \sum_{e \in c} w_e$. If each arc has the same weight, $w \in \mathbb{R}_{++}$, the solutions to the corresponding maximization problem will simply maximize the number of matched patients. Dickerson and Sandholm (2014) call this objective function MAXCARD. Let N_Γ be the set of "marginalized patients". They obtain the MAXCARD-FAIR objective function from the MAXCARD objective function by assigning each arc ij the same weight as before (w) whenever $i \in N \setminus N_\Gamma$ and increasing the weight to $(1+\beta)w$ whenever $i \in N_\Gamma$, where $\beta > 0$. Let $\beta > n - |N_\Gamma| - 1$. Then $(1+\beta)w > (n - |N_\Gamma|)w$. Since $(n - |N_\Gamma|)w$ is an upper bound on the sum of weights assigned to arcs ending in patients belonging to $N \setminus N_\Gamma$, MAXCARD-FAIR's primary goal is to match as many patients in N_Γ as possible and its secondary goal is to match as many patients in $N \setminus N_\Gamma$ as possible. In other words, the set of MAXCARD-FAIR solutions is given by $\mathcal{M}_\Gamma$. If the planner disallows desensitization, then $c_{i,j} \neq 1$ for all $i, j \in N$ and $B(\mu) = |N^*(\mu)|$ for all $\mu \in \mathcal{M}$. Since $|N^*(\mu)| = |N^*(\mu')|$ for all $\mu, \mu' \in \mathcal{M}_\Gamma$, $\mathcal{M}_E = \mathcal{M}_\Gamma$ in this case. Consequently, the set of egalitarian threshold matchings is the set of MAXCARD-FAIR solutions. □

Proposition 7. Suppose that the planner disallows desensitization. Consider some priority order π and partition N such that $N_t = \{\pi^{-1}(t)\}$ for all $t \in \{1, \ldots, n\}$. Then $\mathcal{M}^* = \mathcal{E}_n$.

Proof. Since $m = n$, it follows that $\mathcal{E}_t^m = \mathcal{E}_t$ for all $t \in \{1, \ldots, n\}$. The definition of $\mathcal{M}_T^m$

can therefore be rewritten in the following way.

$$\text{For } t \in \{1,\ldots,n\},\ \ \mathcal{E}_t = \begin{cases} \{\mu \in \mathcal{E}_{t-1} \mid \pi^{-1}(t) \in N^*(\mu)\} \text{ if } \pi^{-1}(t) \in N^*(\mu) \text{ for some } \mu \in \mathcal{E}_{t-1} \\ \mathcal{E}_{t-1} \text{ otherwise} \end{cases}$$
$$\mathcal{M}_T^m = \arg\max_{\mu \in \mathcal{E}_n} B(\mu)$$

Note that $|N^*(\mu)| = |N^*(\mu')|$ for all $\mu, \mu' \in \mathcal{E}_n$, since for each $t \in \{1,\ldots,n\}$, $\mathcal{E}_t$ is merely the subset of $\mathcal{E}_{t-1}$ containing matchings at which the unique patient with priority t is matched (if such a matching exists). Furthermore, as $c_{i,j} \neq 1$ for all $i,j \in N$, $B(\mu) = |N^*(\mu)|$ for all $\mu \in \mathcal{M}$. These two observations imply that $B(\mu) = B(\mu')$ for all $\mu, \mu' \in \mathcal{E}_n$. Hence, $\mathcal{M}_T^m = \mathcal{E}_n$. Then, ny Proposition 3, $\mathcal{M}^* = \mathcal{M}_T^m = \mathcal{E}_n$. □

Proposition 8. For any priority order π, $N^*(\mu) = N^*(\mu')$ for all $\mu, \mu' \in \mathcal{E}_n$.

Proof. Consider some $\mu, \mu' \in \mathcal{E}_n$ and suppose that $N^*(\mu) \neq N^*(\mu')$ to reach a contradiction. Then there exists either some $i \in N^*(\mu) \setminus N^*(\mu')$, some $i \in N^*(\mu') \setminus N^*(\mu)$ or both. Without loss of generality, let $i \in N^*(\mu) \setminus N^*(\mu')$ be the highest priority patient among all such patients. Since $\mu \in \mathcal{E}_n$, it follows that $\mu \in \mathcal{E}_{\pi(i)-1}$. Furthermore, since $\pi^{-1}(\pi(i)) = i \in N^*(\mu)$ by assumption, $\mathcal{E}_{\pi(i)} = \{\nu \in \mathcal{E}_{\pi(i)-1} \mid i \in N^*(\nu)\}$. However, since $i \notin N^*(\mu')$, $\mu' \notin \mathcal{E}_{\pi(i)}$. As $\mathcal{E}_n \subset \mathcal{E}_{\pi(i)}$, this contradicts the assumption that $\mu' \in \mathcal{E}_n$. □

Proposition 9. Consider some priority order π and partition N such that $N_t = \{\pi^{-1}(t)\}$ for all $t \in \{1,\ldots,n\}$. Then $\mathcal{M}^* = \mathcal{M}_B$.

Proof. Since $m = n$, the definition of $\mathcal{M}_T^n$ can be rewritten in the following way.

$$\text{For } t \in \{1,\ldots,n\},\ \ \mathcal{E}_t = \begin{cases} \{\mu \in \mathcal{E}_{t-1} \mid \pi^{-1}(t) \in N^*(\mu)\} \text{ if } \pi^{-1}(t) \in N^*(\mu) \text{ for some } \mu \in \mathcal{E}_{t-1} \\ \mathcal{E}_{t-1} \text{ otherwise} \end{cases}$$
$$\mathcal{M}_T^n = \arg\max_{\mu \in \mathcal{E}_n} B(\mu)$$

Thus, $\mathcal{M}_B = \mathcal{M}_T^n$. By Proposition 3, $\mathcal{M}^* = \mathcal{M}_B$. □

Proposition 10. Let $N_1 = N$. Then $\mathcal{M}^* = \mathcal{M}_{\max}^{\varepsilon}$.

Proof. When $N_1 = N$, the definition of $\succsim$ reduces to the following statement.

$$\mu \succ \mu' \iff \begin{cases} |N^*(\mu)| > |N^*(\mu')|, \\ |N^*(\mu)| = |N^*(\mu')| \text{ and } B(\mu) > B(\mu'), \end{cases}$$
$$\mu \sim \mu' \iff |N^*(\mu)| = |N^*(\mu')| \text{ and } B(\mu) = B(\mu').$$

Consider some $\mu \in \mathcal{M}^*$. By definition, $\mu \succsim \mu'$ for all $\mu' \in \mathcal{M}$. This implies that $|N^*(\mu)| \geq |N^*(\mu')|$ for all $\mu' \in \mathcal{M}$. That is, $\mu \in \mathcal{M}_{\max}$. Furthermore, for all $\mu' \in \mathcal{M}$ such that $|N^*(\mu)| = |N^*(\mu')|$, $\mu \succsim \mu'$ implies that $B(\mu) \geq B(\mu')$. Together, these observations imply that $\mu \in \mathcal{M}_{\max}^{\varepsilon}$. Thus, $\mathcal{M}^* \subseteq \mathcal{M}_{\max}^{\varepsilon}$.

Suppose that $\mathcal{M}^* \subset \mathcal{M}^\varepsilon_{\max}$ to reach a contradiction. Then, there exists some $\mu \in \mathcal{M}^\varepsilon_{\max} \setminus \mathcal{M}^*$. Since $\mathcal{M}$ is assumed to be non-empty, $\mathcal{M}^*$ is guaranteed to be non-empty. Consider some $\mu' \in \mathcal{M}^*$. Since $\mu \notin \mathcal{M}^*$, it must either be the case that $|N^*(\mu')| > |N^*(\mu)|$, or $|N^*(\mu')| = |N^*(\mu)|$ and $B(\mu') > B(\mu)$. Suppose that $|N^*(\mu')| > |N^*(\mu)|$. Then $\mu \notin \mathcal{M}_{\max}$. This contradicts $\mu \in \mathcal{M}^\varepsilon_{\max}$ since $\mathcal{M}^\varepsilon_{\max} \subseteq \mathcal{M}_{\max}$. Thus, $|N^*(\mu')| = |N^*(\mu)|$ and $B(\mu') > B(\mu)$. Since $\mathcal{M}^\varepsilon_{\max} \subseteq \mathcal{M}_{\max}$, $\mu \in \mathcal{M}_{\max}$. It must therefore be the case that $\mu' \in \mathcal{M}_{\max}$ as well. Then $B(\mu') > B(\mu)$ contradicts $\mu \in \arg\max_{\nu \in \mathcal{M}_{\max}} B(\nu)$. Hence, $\mathcal{M}^* \not\subset \mathcal{M}^\varepsilon_{\max}$. $\mathcal{M}^* \subseteq \mathcal{M}^\varepsilon_{\max}$ and $\mathcal{M}^* \not\subset \mathcal{M}^\varepsilon_{\max}$ imply that $\mathcal{M}^* = \mathcal{M}^\varepsilon_{\max}$. □

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