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Efficiency, Stability, and Commitment in Senior Level Job Matching Markets[1]

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Abstract: We study a senior level job matching model with multiple heterogeneous incumbents and entrants. Incumbents can be committed or uncommitted (i.e., free). A committed agent is an incumbent (firm or worker) who has initially had a partner (worker or firm) and is bound by her commitment. A free agent is an entrant or an incumbent whose relation with by her initial partner is not binding. Every agent tries to find her best possible partner with contract. A committed agent cannot unilaterally dissolve her partnership unless her partner agrees to do so. We examine the problem of how to match workers and firms as well as possible and at the same time to set committed agents free as many as possible without violating their commitments to their partners. We show the existence of (strongly) stable and (strict) core matchings through a constructive algorithm and derive several properties of such outcomes.

Keywords: Matching state, core, stability, commitment, procedure.

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1 Introduction

This paper develops a new model of senior level job matching market with commitment. Following the classic work of Gale and Shapley (1962) substantial progress has been made in understanding entry level labor markets in which all market participants are new entries and firms try to hire workers to fill their positions and workers search for jobs; see e.g., Roth and Sotomayor (1990). However, there is not much study of senior level job matching markets. We attempt to fill a gap by investigating such a market. The market under consideration consists of multiple heterogenous incumbents and entrants. An incumbent can be a firm or a worker who has an initial partner (worker or firm) and an entrant can be also a firm or a worker. Every entrant has no initial partner and is free to hire or search for a position. An incumbent who has no commitment to her initial partner can act freely without concern about her partner. However, an incumbent who is committed to her initial partner cannot dissolve her partnership unless her partner agrees to do so.

Commitments exist in various forms and contexts and can influence people's behavior and affect the performance of the system involved. They can be imposed by law, by custom, by contract, by convention, or by morality. For instance, universities with a tenure track system are committed to their tenured faculty members in the sense that they generally cannot fire a tenured professor unless she/he is willing to leave voluntarily. On the other hand, a tenured professor can move rather freely without facing this kind of commitment constraint. In this case, commitment is imposed on one side. The civil service system is another typical example of one-sided commitment where government employees are free but their employers are committed. The mutual consent divorce law in many countries (see e.g., Voena 2015) permits divorce only when both husband and wife agree to it. Under this law, both husband and wife are committed to each other and can divorce if both consent to do so. This is a typical case of two-sided commitment. In some professions involving highly sensitive matters, a job contract may explicitly require an employee to commit to the position for at least a certain period of time. In this environment, commitment is two-sided with one side being explicit and the other being implicit. Schelling (1956, 1960) was the first to study how players may use commitments as tactics to advance their interest in bargaining or negotiation. Commitment has been further studied in the literature on repeated games and contracts and also been used in fiscal and monetary policies; see e.g., Laffont and Martimort (2002), Mailath and Samuelson (2006), and Lucas and Stokey (1983). The meaning of commitment may, however, vary from one situation to another.

In our model every firm has preferences over the workers with multiple contracts and the prospect of not hiring any one and every worker has preferences over the firms with multiple contracts and the prospect of being unemployed. Following the two-sided matching models with wages and relevant job characteristics given by Crawford and Knoer (1981), Kelso and Crawford (1982), Hatfield and Milgrom (2005), and Ostrovsky (2008), we represent the relation governing every worker and every firm by contracts. Every contract signifies a relationship between a firm and a worker and specifies the amount of remuneration to be paid by the firm, and the service to be supplied by the worker in return for the payment. There can be multiple contracts available between every firm and every worker. When a firm and a worker haggle over contracts and reach an agreement, only one contract will be taken by both sides. Every agent tries to find her best possible partner with contract from her opposite side of the market. However, a committed agent cannot unilaterally dissolve her partnership to find a new partner unless her initial partner agrees to it. The central issue is how to match workers and firms as well as possible and to set committed agents free as many as possible without violating their commitment constraints.

Our analysis focuses on what can be a reasonable solution to this new matching problem and how to design a procedure for finding the solution. In the two-sided matching literature, the notion of (pairwise) stability is the most widely used solution stemming from Gale and Shapley (1962); see also Roth and Sotomayor (1990). We need to adapt this concept to our setting which contains not only chains but also cycles induced by the presence of commitments. The existing literature typically deals with settings without cycles. In our model we call a firm and a worker a pair if they are matched by a common contract. A matching is a collection of contracts with their corresponding pairs and singles. A matching state consists of a matching and an associated set of committed agents. We say that a matching state is strongly stable if it is individually rational and not blocked by any its chain or cycle. Our second solution concerns the fundamental concept of core from game theory and equilibrium theory; see e.g., Gilles (1953), Scarf (1967), Shapley and Scarf (1974), and Predtetchinski and Herings (2004). The core of an economic model is the set of outcomes that can be achieved collectively by the entire group of market participants but cannot be blocked by the collusive action of any coalition of the participants, acting by themselves. Its analogue in the current model is introduced to accommodate the initial matching and its associated set of committed agents. The core does not coincide with the set of strongly stable matchings. This is in contrast with the Gale-Shapley model for which both sets are identical. Our main result establishes the existence of at least one strongly stable matching state whose matching is also in the strict core. This outcome is called a strongly stable core matching state. At this outcome every committed agent must be committed at the initial state and there can be less committed agents than at the initial state.

A key step in our analysis is to develop an algorithm for finding a strongly stable core matching state and thus to give a constructive proof of our main theorem. This algorithm to be called *the Hybrid Procedure* is a novel blend of two generalizations of the deferred acceptance (DA) procedure of Gale and Shapley (1962) and the top trading cycle (TTC) method of Shapley and Scarf (1974). It should be noticed that neither of the two generalizations is sufficient to reach a strongly stable core matching state but the Hybrid Procedure always finds it. In our generalized DA procedure, free workers make proposals to their favorite firms while any firm which receives any proposal rejects all proposals but her favorite one subject to the constraint that if she is a committed firm and receives a proposal from her initial partner worker, she should give him the priority over any other proposal. A committed worker can become free during the process if his initial partner has been provisionally matched to another worker. In our modified TTC procedure, each worker in an almost committed set will propose his favorite contract among all his mutually relatively acceptable contracts to its associated firm and each firm in the almost committed set points to her initially matched worker. In our Hybrid Procedure chains and cycles in which workers and firms appear alternatively and reveal their favourite choice and also their willingness to accept will be utilized for producing a strongly stable strict core matching. First, our generalized DA procedure will find a stable matching state. Then our modified TTC method will apply to an almost committed set of agents resulting from the generated stable matching state. Cycles will be found by our modified TTC method and will form part of a strict core matching. Agents involved in the core will leave. Then the modified TTC method applies again to a smaller almost committed set of agents who still remain in the market. Cycles will be generated and form a new part of the strict core, and agents involved leave the market. The same process will be repeatedly applied to the remaining agents until none is left. In this manner, a strongly stable core matching state will be found. In the process some committed agents will be set free.

Blum et al. (1997) are the first to examine a senior level labor market. In their model there are incumbents and new entries. An incumbent firm has hired a worker and an incumbent worker has worked at a firm. Each firm has preferences over all workers and herself and every worker has preferences over all firms and himself. All agents are free and have no commitment. They show that when the market is destabilized, it can regain stability in the sense of Gale and Shapley by a decentralized process of offers and acceptances. Our model contains theirs as a special case. Our model, motivation, solution and procedure differ from theirs. First, our model incorporates contracts which contain explicitly salaries and other characteristics concerning each job and each worker and which can be negotiated between firms and workers. This feature allows us to examine real life competitive markets with monetary transfers; see Crawford and Knoer (1981), Kelso and Crawford (1982), Crawford (1991), and Hatfield and Milgrom (2005). Second, we introduce commitments which enable us to handle a variety of practical situations and make our model markedly different from theirs. Third, while their stability is the traditional one concerning only pairs, our stability involves chains and circles. Our solution is the intersection of (strong) stability and (strict) core and our process is totally different from theirs.

Abdulkadiroğlu and Sönmez (1999) study a house allocation model such as college dormitories or subsidized public houses where there are both existing tenants and new applicants. They propose a modification of the TTC method that is individually rational, Pareto-efficient, and strategy-proof. Sun and Yang (2016) consider a marriage matching model in which there are many single men and single women and married couples. Married couples are two-sided committed and can divorce under the mutual consent divorce law. We show the existence of a stable and core matching between men and women via a constructive procedure. Diamantoudi et al. (2015) investigate the role of commitment in a dynamic matching model. Their commitment differs crucially from ours in that if both parties are committed to each other, they stay together permanently. Combe et al. (2017) examine a problem of teacher assignment in their independent study. Their model can be seen as a senior level market with only incumbents. Their teacher optimal block exchange algorithm modifies the TTC method, is strategy proof for teachers, and yields a two-sided maximal outcome.[4] Their model and solutions are different from ours.

This paper is organized as follows. Section 2 introduces the model and basic concepts. Section 3 establishes the main results. Section 4 concludes.

2 Model and Basic Concepts

Consider a senior level job matching labor market with many heterogenous workers and firms. In the model, let $W = \{w_1, \cdots, w_s\}$ denote the set of all workers and $F = \{f_1, \cdots, f_t\}$ the set of all firms. Some agents are incumbents and others are entrants. We will take several steps to give a full description of the model. Throughout the paper we treat any worker as male and any firm as female. When we talk about a generic agent which can be a firm or worker, we treat the agent as female. For easy exposition we assume that every firm hires at most one worker and every worker can work for at most one firm.[5] Relationships

[4]Their model implicitly implies that teachers and schools are two-sided committed in our sense. In their model each school can hire multiple teachers. A two-sided maximal matching that means individual rationality and Pareto-efficiency for both teachers and schools is not necessarily in the (strict) core nor (strongly) stable in our current model.

[5]This assumption has been used in Koopmans and Beckmann (1957), Gale and Shapley (1962), Shapley and Shubik (1972), Shapley and Scarf (1974), Crawford and Knoer (1981), Demange et al. (1986), Blum et al. (1997), Abdulkadiroğlu and Sönmez (1999), Chung (2000), Andersson and Svensson (2014), etc. Basic arguments and results here can be extended to the general case where every firm may hire many

between firms and workers are governed by bilateral contracts. Every contract α usually has several components including a firm and a worker and what service the worker should provide to the firm and what the worker can get from the firm in return for the service. The firm involved in contract α is denoted by f_α and the worker is denoted by w_α. This way of describing labor contracts is inspired by Hatfield and Milgrom (2005) and Ostrovsky (2008). In particular, if a worker w does not work for any firm or a firm f does not hire any worker, this state of standing alone will be represented by a simple contract $\alpha = \{w\}$ or $\alpha = \{f\}$. For simplicity, we also will use w and f to denote these two special contracts. If a firm has multiple contracts with a worker, then every contract has its own terms and conditions and differs from another contract. There will be no contracts between any two firms or between any two workers. Let Σ be the set of all possible but finite contracts given exogenously. For each firm f, let $\Sigma(f)$ denote the set of contracts in Σ in which the firm is involved. Observe that $\Sigma(f)$ contains the obvious contract $\alpha = \{f\}$. For each worker w, let $\Sigma(w)$ denote the set of contracts in Σ in which the worker is involved. $\Sigma(w)$ contains the obvious contract $\alpha = \{w\}$. For any subset S of $W \cup F$, define $\Sigma(S) = \cup_{x \in S}\Sigma(x)$.

Every worker w has strict preferences over all his possible contracts in $\Sigma(w)$. In other words, he ranks all contracts in $\Sigma(w)$ in certain linear order. This relationship can be represented by $\succeq_w$ and $\succ_w$. If $\alpha_1 \succeq_w \alpha_2$ for $\alpha_1, \alpha_2 \in \Sigma(w)$, we say that he likes contract α_1 at least as well as contract α_2. If $\alpha_1 \succ_w \alpha_2$ for $\alpha_1, \alpha_2 \in \Sigma(w)$, then he prefers contract α_1 to contract α_2. Similarly, every firm f has strict preferences over all her possible contracts in $\Sigma(f)$. This relationship will be represented by $\succeq_f$ and $\succ_f$.

Given any subset Ψ of Σ, let $\Psi(f)$ represent the set of contracts in Ψ in which the firm f is involved, and let $\Psi(w)$ be the set of contracts in Ψ in which the worker w is involved. Let $F(\Psi)$ be the set of those firms which are associated with some contracts in Ψ, and $W(\Psi)$ be the set of those workers which are associated with some contracts in Ψ.

A set Ψ of contracts in Σ is said to be *a matching* if every worker gets exactly one contract from Ψ so does every firm. For a matching Ψ and for any individual x, the set $\Psi(x)$ contains exactly one contract. For simplicity, we will also use $\Psi(x)$ to denote its unique element. If a firm f and a worker w share a common contract α in a matching

workers and every worker accepts at most one job.

Ψ, i.e., $\alpha \in \Psi(w) \cap \Psi(f)$, they will be called a partner of each other and described by a mapping μ_Ψ with $\mu_\Psi(f) = w$ and $\mu_\Psi(w) = f$; if the contract involves only one agent x, then the agent is said to be *self-matched* or *a single* and described by a mapping μ_Ψ with $\mu_\Psi(x) = x$. This mapping μ_Ψ is a one-to-one mapping from the set $W \cup F$ onto itself of order two and is uniquely determined by the matching Ψ. We call μ_Ψ a g-matching, or simply a matching. A matching μ_Ψ can be written as a collection of partners and singles at μ_Ψ. At matching Ψ, if firm f and work w form a partnership through a common contract α, i.e., $\alpha \in \Psi(w) \cap \Psi(f)$, we also say that *they are matched with the contract.*

2.1 Matching State and Commitment

In the classic marriage matching model of Gale and Shapley (1962), marriages are not binding on either side. In reality, however, marriages, or partnerships between workers and firms, or contracts between two parties can be binding on both sides, or one side. In other words, one or two parties involved may have commitments to their partner. We will incorporate this important feature into our matching model. For a given matching Ψ, let $P(\Psi) = \{x \in W \cup F \mid \mu_\Psi(x) \neq x\}$ denote the set of agents who have a partner under Ψ. An agent x in $P(\Psi)$ is said to be *committed* if she cannot unilaterally dissolve the relation with her partner $\mu_\Psi(x)$ unless her partner agrees to do so. In our context, $\mu_\Psi(x)$ would agree if doing so does not make $\mu_\Psi(x)$ worse off. Let $C(\Psi) = \{x \in P(\Psi) \mid x \text{ is committed}\}$ be the set of committed individuals under Ψ. Any agent in $W \cup F$ but not in $C(\Psi)$ is said to be *free*. A free agent is either a single or has a partner but has no commitment to her partner. Let $V(\Psi) = (W \cup F) \setminus C(\Psi)$ represent the set of all free agents. A free individual $x \in P(\Psi)$ can unilaterally rescind her relationship with her partner $\mu_\Psi(x)$. In our model, a matching state is described by a pair $(\Psi, C(\Psi))$ of matching Ψ and its set of committed agents $C(\Psi)$. In the Gale-Shapley model, for any matching Ψ we have $C(\Psi) = \emptyset$ and $V(\Psi) = W \cup F$.

Observe that in principle commitment can be embedded in a contract. Because committed agents and free agents play different roles in our model, we need to differentiate them clearly. If an agent $x \in P(\psi)$ and her partner $\mu_\Psi(x)$ are committed to each other, this will be called two-sided commitment. If either x or exclusively her partner $\mu_\Psi(x)$ is committed, this will be one-sided commitment. In our model, we allow the presence of free agents and committed agents (two-sided, one-sided, both, or mixed). This will enable us to consider a variety of situations including the most general case where some pairs of worker and firm are two-sided committed, some workers are committed but their partners are not, some firms are committed but their partners are not, and some agents are free. There are two typical special cases: the tenure-track system in universities is one-sided commitment under which tenured professors are free but their employers are committed, and the mutual consent divorce law is two-sided commitment. In our model, a committed agent cannot unilaterally change her partner unless her partner agrees to do so. This means that even if two parties are committed to each other, they can still dissolve their partnership as long as both consent to do so. This definition of commitment is very natural and consistent with the spirit of the mutual consent divorce law and can accommodate practical situations.

(Pairwise) stability was introduced by Gale and Shapley (1962) and is the most widely used solution concept in the two-sided matching literature. Their stability concept, however, cannot be directly applied to our current model due to the presence of commitments. To see this, let us consider the following example.

Example 1 *There are three workers w_0, w_1, w_2 and two firms f_1, f_2. We consider the simplest case in which there is at most one contract between every worker and every firm. We can therefore represent the preferences of each individual in the standard way as follows:*

$$\begin{array}{llll} \succ_{w_0}: & f_1, \quad w_0 & & \\ \succ_{w_1}: & f_2, \quad f_1, \quad w_1 & \qquad \succ_{f_1}: & w_0, \quad w_2, \quad w_1, \quad f_1 \\ \succ_{w_2}: & f_1, \quad f_2, \quad w_2 & \qquad \succ_{f_2}: & w_1, \quad w_2, \quad f_2 \end{array}$$

Here we read that w_0 prefers f_1 to standing alone or himself. All other firms which are not listed on his preferences are worse than standing alone.

Assume that we have the following matching and two different sets of committed agents

$$\mu = \begin{pmatrix} w_0, & w_1, & w_2 \\ w_0, & f_1, & f_2 \end{pmatrix}, \quad C(\mu) = \{w_1, w_2, f_1, f_2\} \quad \text{and} \quad C'(\mu) = \{w_1, w_2, f_1\}.$$

Consider the following two scenarios: Without commitment and with commitment.

Without any commitment, the pairs of worker and firm $\{w_0, f_1\}$, $\{w_1, f_2\}$ and $\{w_2, f_1\}$ would obviously form blocking pairs to the matching μ in the sense of Gale and Shapley, as doing so would make every individual of every blocking pair better off than they are at matching μ.

However, under commitment $C'(\mu)$, the pair $\{w_0, f_1\}$ cannot form a blocking pair to the matching μ because f_1 is committed to w_1 and cannot leave w_1 unilaterally. Nevertheless, it will be perfectly possible for w_0, f_1, w_1, f_2 to form a chain so that w_0 matches f_1, and w_1 matches f_2. As w_0 and f_2 are free, they can take the initiative to form this blocking chain so that everyone on the chain gets better off. Observe that f_1 is committed to w_1 and can dissolve his partnership with w_1 as w_1 finds a better replacement f_2 of f_1. Similarly, it is easy to see that $\{w_1, f_2\}$ and $\{w_2, f_1\}$ cannot form blocking pairs to μ as w_1 is committed to f_1 and w_2 is committed to f_2.

Under commitment $C(\mu)$, $\{w_0, f_1\}$, $\{w_1, f_2\}$ and $\{w_2, f_1\}$ cannot form blocking pairs to the matching μ because f_1 is committed to w_1 and cannot leave w_1 unilaterally, w_1 is committed to f_1 and cannot leave f_1 unilaterally and f_2 is committed to w_2 and cannot leave w_2 unilaterally. The same reason applies to $\{w_2, f_1\}$. Nevertheless, it will be perfectly plausible for w_1, f_2, w_2 and f_1 to form a cycle so that w_1 matches f_2, and w_2 matches f_1, as doing so will make everyone on the cycle better off without violating any commitment.

The above discussion motivates us to introduce concepts of blocking chain and blocking cycle as a necessary and plausible generalization of the Gale-Shapley concept of blocking pair. Given a matching state $(\Psi, C(\Psi))$, a *chain* of the state is an ordered sequence of an even number of distinct agents $\tau = (x_1, y_1, x_2, \ y_2, \cdots, x_K, y_K)$ with $K \geq 1$ such that $x_1, y_K \in V(\Psi)$ and $\mu_\Psi(y_k) = x_{k+1}$ for every $k = 1, 2, \cdots, K-1$. x_1 and y_K are called *end agents.* A *cycle* of the state is an ordered sequence of an even number of distinct agents $\tau = (x_1, y_1, x_2, \ y_2, \cdots, x_K, y_K)$ with $K \geq 1$ such that $\mu_\Psi(y_k) = x_{k+1}$ for every $k = 1, 2, \cdots, K$, where x_{K+1} becomes x_1 by convention. In particular, any pair (x_1, y_1) with $\mu_\Psi(y_1) = x_1$ can be looked as a cycle. For a chain or cycle $\tau = (x_1, y_1, x_2, y_2, \cdots, x_K, y_K)$ we will use $A(\tau) = \{x_1, y_1, x_2, y_2, \cdots, x_K, y_K\}$ to denote the set of all agents in τ. A chain $\tau = (x_1, y_1, x_2, y_2, \cdots, x_K, y_K)$ of $(\Psi, C(\Psi))$ is said to be *minimal* if $A(\tau)\backslash\{x_1, y_K\} \subset C(\Psi)$, i.e., all but the end agents x_1 and y_K have commitment to their partners at μ_Ψ. It is clear that every chain contains at least one minimal chain.[6] We also say a cycle of a matching

[6]For any chain $\tau = (x_1, y_1, x_2, y_2, \cdots, x_K, y_K)$ of a matching state $(\Psi, C(\Psi))$, we can find one of its

state $(\Psi, C(\Psi))$ is a *pure cycle* if it does not contain any chain.

We say a contract $\alpha \in \Sigma(x)$ is *acceptable* to an agent $x \in W \cup F$ if $\alpha \succeq_x x$. A matching state $(\Psi, C(\Psi))$ is *individually rational* if we have $\Psi(x) \succeq_x x$ for every $x \in V(\Psi)$, i.e., every free agent at $(\Psi, C(\Psi))$ is no worse than standing alone. A matching state $(\Psi, C(\Psi))$ is *blocked by a chain or cycle* $\tau = (x_1, y_1, x_2, y_2, \cdots, x_K, y_K)$, if for every $k = 1, \cdots, K$, there is a mutually acceptable contract α_k to x_k and y_k such that $\alpha_k \succ_{x_k} \Psi(x_k)$ and $\alpha_k \succ_{y_k} \Psi(y_k)$, i.e., agents x_k and y_k prefer contract α_k to their respective contract at Ψ. It is clear that if a matching state $(\Psi, C(\Psi))$ is blocked by a chain $\tau = (x_1, y_1, x_2, y_2, \cdots, x_K, y_K)$, then it is also blocked by every minimal chain contained in τ.

Definition 1 *A matching state* $(\Psi, C(\Psi))$ *is stable if it is individually rational and not blocked by any its chain. It is strongly stable if it is individually rational and not blocked by any its chain or cycle.*

Our chain stability is similar to the chain stability introduced by Ostrovsky (2008) for his general vertical supply chain model but our motivation and context are different from his. In particular, unlike his model which contains chains and no cycles, chains and cycles in our model are induced due to the presence of commitments. Our notion of strong stability is defined with respect to not only chains but also cycles and thus strengthens the notion of (chain) stability defined with respect to chains only. We have the following observation.

Lemma 1 *A matching state* $(\Psi, C(\Psi))$ *is stable if and only if it is individually rational and not blocked by any its minimal chain. A matching state is strongly stable if and only if it is individually rational and not blocked by any its minimal chain or pure cycle.*

Note that for any two matching states $(\Psi, C(\Psi))$ and $(\Psi, C'(\Psi))$ satisfying $C(\Psi) \subseteq C'(\Psi)$, each chain of $(\Psi, C'(\Psi))$ is also a chain of $(\Psi, C(\Psi))$, a pure cycle of $(\Psi, C'(\Psi))$ may contain a chain of $(\Psi, C(\Psi))$. Therefore, if $(\Psi, C(\Psi))$ is a (strongly) stable matching

minimal sub-chain as follows. We first find some $j \in \{1, 2, \cdots, K\}$ such that $y_j \in V(\Psi)$ and $y_k \notin V(\Psi)$ for all $k = 1, \cdots, j-1$, and next find some $i \in \{1, 2, \cdots, j\}$ such that $x_i \in V(\Psi)$ and $x_k \notin V(\Psi)$ for all $k = i+1, \cdots, j$. Then, we can show $(x_i, y_i, \cdots, x_j, y_j)$ is a minimal chain of $(\Psi, C(\Psi))$.

state, then every matching state $(\Psi, C'(\Psi))$ with $C(\Psi) \subseteq C'(\Psi) \subseteq P(\Psi)$ is also (strongly) stable. The converse may not be true.

Let us go back to Example 1. It is easy to verify that $(\Psi, C(\Psi))$ is a stable matching state, but not strongly stable because (w_1, f_2, w_2, f_1) is a blocking cycle. Moreover, the matching state $(\mu, C'(\mu))$ is not stable because (w_0, f_1, w_1, f_2) is a blocking chain. Nevertheless, the matching state $(\mu^1, C(\mu^1))$ is strongly stable with

$$\mu^1 = \begin{pmatrix} w_0, & w_1, & w_2 \\ w_0, & f_2, & f_1 \end{pmatrix}, \quad C(\mu^1) = \{f_1, f_2\}.$$

2.2 The Formal Model

In contrast to entry level matching models where all agents are entrants and there is no initial matching between any firm and any worker, our senior level matching model comprises many incumbents and also many entrants. Incumbents may be committed to their initial partners and every agent may be associated with multiple contracts. To reflect this situation, let $(\Psi^0, C(\Psi^0))$ denote the initial matching state, which is exogenously given. An agent $x \in W \cup F$ is said to be *contracted* if $\mu_{\Psi^0}(x) \neq x$. In other words, a contracted agent must initially match someone from the opposite group. For a contracted agent x, we say that x is a *contracted or initial partner* of $\mu_{\Psi^0}(x)$ and vice versa. So in our model there are initially many contracted agents and single agents. One can view each contracted agent as an incumbent. Single agents can be new entries or experienced agents. For instance, a single worker can be a person who just attains a professional qualification and starts to find a job, or an experienced worker who just lost his job. For convenience, let $\mu^0 = \mu_{\Psi^0}$ denote the initial g-matching, $P^0 = P(\Psi^0)$ denote the set of all contracted agents, $C^0 = C(\Psi^0) \subseteq P(\Psi^0)$ denote the set of all contracted agents who have committed to their initial partners, and $V^0 = V(\Psi^0) = (W \cup F) \setminus C(\Psi^0)$ be the set of all free agents at $(\Psi^0, C(\Psi^0))$. Observe that if a contracted agent x prefers remaining single to her initial partnership $\Psi^0(x)$, i.e., $x \succ_x \Psi^0(x)$, this agent wants to dissolve this partnership unconditionally and her partner $\mu^0(x)$ can do so freely. Therefore, for any contracted agent $x \in P^0$ with $x \succ_x \Psi^0(x)$, we can assume that her parter $\mu^0(x)$ is free, i.e., $\mu^0(x) \in V^0$. We use $\mathcal{M} = (W, F, \Psi^0, C(\Psi^0), \succ)$ to represent our current matching model.

In our model $\mathcal{M} = (W, F, \Psi^0, C^0, \succ)$, a contract $\alpha \in \Sigma(x)$ is *relatively acceptable* to an agent $x \in W \cup F$ if $\alpha \succeq_x \Psi^0(x)$. To an agent x with $\Psi^0(x) \succeq_x x$, each relatively acceptable contract is acceptable, but an acceptable contract need not be relatively acceptable. Conversely, to an agent x with $x \succ_x \Psi^0(x)$, any acceptable contract is relatively acceptable, but a relatively acceptable contract need not be acceptable. Given a matching Ψ, an agent x is said to be *rematched* if her associated contract $\Psi(x)$ is different from her initial contract $\Psi^0(x)$. Observe that rematching allows x to have the same partner, i.e., $\mu_\Psi(x) = \mu^0(x)$.

In our model $\mathcal{M} = (W, F, \Psi^0, C^0, \succ)$, there are incumbents and entrants and there are committed agents and free agents. New entries and free agents will most likely disturb and reshape the market. Every individual tries to find the best possible partner for herself and committed individuals strive to find better partners but have to comply with their commitment constraints. Naturally, commitment constraints whether imposed upon or chosen by individuals will certainly restrict the choices of the individuals who face the constraints. In a society, a state which is stable when every individual has less constraints to make her choice is intrinsically more desirable than a state which is stable when every individual has more constraints to make her choice. It will be in the best interest of every concerned individual to free individuals from the shackles of their commitment constraints as many as possible. Under this framework we study the problem of how to match workers and firms as well as possible and at the same time to give individuals as much freedom as possible. The latter goal means to reduce the number of committed individuals without violating their commitment constraints. In particular, we attempt to address two fundamental questions: Given a matching model $\mathcal{M} = (W, F, \Psi^0, C^0, \succ)$, what can be a desirable and natural outcome in terms of efficiency and stability and less commitments? and how can such an outcome be reached? One can well imagine that starting from an initial matching state, a dynamic process of matching and rematching between workers and firms will gradually improve efficiency and stability and reduce the number of commitment constraints until no further improvement can be made. The final outcome will be (strongly) stable, accepted by all agents and cannot be improved upon by any coalition of agents and will have a less number of committed agents.

As the market starts with the initial state and will be reshaped, one may wonder what new matchings will possibly emerge. The following definition introduces feasible matchings.

Definition 2 *A matching Ψ is feasible if $\Psi(x)$ is acceptable or relatively acceptable for every agent $x \in W \cup F$, and $\Psi(x)$ is further acceptable for every $x \in V^0$ and both acceptable and relatively acceptable for every x with $\mu^0(x) \in C^0$.*

Observe that a feasible matching Ψ is defined by comparing with the initial matching state (Ψ^0, C^0), and does not violate the initial commitments. So at a feasible matching Ψ, $\Psi(x)$ must be acceptable to x when x is initially free, and must be relatively acceptable to x when her partner $\mu^0(x)$ is initially committed. By the assumption that $\mu^0(x) \in V^0$ for every x with $x \succ_x \Psi^0(x)$, $\mu^0(x) \in C^0$ implies $\Psi^0(x) \succeq_x x$. Therefore, $\Psi(x)$ must be both acceptable and relatively acceptable to x when $\mu^0(x)$ is committed. For a committed agent $x \in C^0$ with $\Psi^0(x) \succ_x x$ and $\mu^0(x) \in V^0$, she may become worse off, i.e., $\Psi^0(x) \succ_x \Psi(x)$, but will be at least as well as being unmatched and thus $\Psi(x)$ will be acceptable to x when $\mu^0(x)$ finds a better contract than $\Psi^0(x)$. For a committed agent $x \in C^0$ with $x \succ_x \Psi^0(x)$ (and so $\mu^0(x) \in V^0$), x may accept a relatively acceptable contract $\Psi(x)$.

2.3 Core

The notion of core is one of the most fundamental solution concepts in game theory and equilibrium theory and can be seen as a generalization of the venerable Edgeworth's contract curve. We will adopt this solution to our current model.

A nonempty subset S of the set $W \cup F$ of workers and firms is called a coalition. $W \cup F$ itself is called the grand coalition. For the given initial matching state (Ψ^0, C^0) a coalition S is said to be *implementable*, if $x \in S \cap C^0$ implies her partner $\mu^0(x) \in S$. This means that when a committed individual in the coalition S contemplates dissolving her partnership, she cannot do so unilaterally but need to do collectively with her partner. We say that a coalition S *improves upon a matching* Ψ of the grand coalition $W \cup F$ if there exists a matching $\Phi \subseteq \Sigma(S)$ among workers and firms from the coalition alone such that everyone x in S weakly prefers $\Phi(x)$ to $\Psi(x)$ and at least one agent $y \in S$ prefers $\Phi(y)$ to $\Psi(y)$. A coalition S *strongly improves upon a matching* Ψ if there exists a matching Φ among workers and firms from the coalition alone such that every agent x in S prefers $\Phi(x)$ to $\Psi(x)$.

Definition 3 *In a matching model $\mathcal{M} = (W, F, \Psi^0, C^0, \succ)$, a feasible matching Ψ is in the strict core, called a strict core matching if it cannot be improved upon by any implementable coalition for the initial state (Ψ^0, C^0). It is in the core if it cannot be strongly improved upon by any implementable coalition.*

We denote the core and the strict core of the matching model by $Core(\Psi^0, C^0)$ and $SCore(\Psi^0, C^0)$ respectively.

Observe that the set of all agents in a chain or a cycle of the initial matching state (Ψ^0, C^0) is an implementable coalition for the initial matching state. A feasible matching Ψ is *improved upon by a chain or cycle* $\tau = (x_1, y_1, x_2, y_2, \cdots, x_K, y_K)$ of the initial state (Ψ^0, C^0) if for each $k = 1, \cdots, K$, there is a contract $\alpha_k \in \Sigma(x_k) \cap \Sigma(y_k)$ such that $\alpha_k \succeq_{x_k} \Psi(x_k)$ and $\alpha_k \succeq_{y_k} \Psi(y_k)$, and for some $k = 1, \cdots, K$, it holds $\alpha_k \succ_{x_k} \Psi(x_k)$ or $\alpha_k \succ_{y_k} \Psi(y_k)$. τ will be called *an improvement chain or cycle* of Ψ.

Let us go back to Example 1 and check if there is any core matching. The initial matching state (μ^0, C^0) is given by

$$\mu^0 = \begin{pmatrix} w_0, & w_1, & w_2 \\ w_0, & f_1, & f_2 \end{pmatrix} \quad \text{and} \quad C^0 = \{w_1, w_2, f_1, f_2\}.$$

In other words, w_1, w_2, f_1, and f_2 are incumbents and committed. Only w_0 is a new entry. In this example, there is a unique strict core matching μ given by

$$\mu = \begin{pmatrix} w_0, & w_1, & w_2 \\ w_0, & f_2, & f_1 \end{pmatrix}$$

At μ, no agent gets worse off and in fact w_1, w_2, f_1, and f_2 all get better off. Recall that w_1, w_2, f_1, and f_2 are committed at μ^0. The initial two partners (w_1, f_1) and (w_2, f_2) get dissolved and rematched and all become strictly better off!

Consider now a different scenario. If incumbents had no commitment to their partners at all and therefore could act as if they are new entries, then the unique stable matching in the sense of Gale and Shapley (1962) would make w_0 work at f_1, w_1 work at f_2, and w_2 stay single. In this case w_2 would be fired and get worse off.

In fact one can show that the matching state $(\mu, C(\mu))$ with the set $C(\mu) = \{f_1, f_2\}$ of committed agents is strongly stable.

All the above discussions indicate that commitment can affect the behavior of agents as well as the outcome considerably.

Lemma 2 *If a feasible matching Ψ is improved upon by an implementable coalition S, then it must be improved upon by a chain or cycle of the initial matching state (Ψ^0, C^0).*

Proof: By definition, there exists a matching Φ among agents from the coalition S alone such that every agent x in S weakly prefers $\Phi(x)$ to $\Psi(x)$ and at least one agent $y \in S$ prefers $\Phi(y)$ to $\Psi(y)$. Using these two matchings Ψ^0 and Φ, we define a directed bipartite graph $G = (S, E)$ on S by setting

$$\begin{aligned} E &= \Big\{(w, f) \,|\, w \in W \cap S, f \in F \cap S, \Phi(w) = \Phi(f)\Big\} \\ &\quad \cup \Big\{(f, w) \,|\, w \in W \cap S, f \in F \cap S, \Psi^0(w) = \Psi^0(f)\Big\}. \end{aligned}$$

That is, there is a directed arc from a worker $w \in S$ to a firm $f \in S$ if and only if they are partners under Φ, and there is a directed arc from a firm $f \in S$ to a worker $w \in S$ if and only if they are partners under Ψ^0.

Choose any individual $y \in S$ such that $\Phi(y) \succ_y \Psi(y)$. Note that in this graph G every vertex's degree is less than or equal to 2. Let G' denote the component (the maximal connected subgraph) of G which contains y. Then, G' is a directed chain or cycle. If G' is a directed cycle, then it is an improvement cycle of Ψ. If G' is a directed chain, then every its end vertex must be one of the following three cases: (i) a single agent under Ψ^0; (ii) a free contracted agent in the initial state (Ψ^0, C^0) whose contracted partner is not in S; (iii) a committed agent in the initial state (Ψ^0, C^0) who becomes a single under Φ. Consider Case (iii). In this case for an end agent $x \in C^0 \cap S$ with $x = \Phi(x) \succeq_x \Psi(x)$, it satisfies $\mu^0(x) \in V^0 \cap S$. This is because $\Psi^0(x) \succ_x x$ implies $\Psi^0(x) \succ_x x = \Phi(x) \succeq_x \Psi(x)$. By definition of feasible matching we see $\mu^0(x) \in V^0$. Otherwise, from $x \succ_x \Psi^0(x)$ and by assumption we see $\mu^0(x) \in V^0$. In all these three cases we can find an improvement chain of Ψ contained in the directed chain G'.

As a result, we can always find an improvement chain or cycle of Ψ contained in G'.

□

The next lemma follows immediately from Lemma 2.

Lemma 3 *A feasible matching Ψ is in the strict core if it cannot be improved upon by any chain or cycle of the initial state (Ψ^0, C^0).*

3 Main Results

In this section we are going to present two major results of this paper. With regard to our matching model, our first result (Theorem 1) establishes the existence of a stable matching state with a feasible matching Ψ in which every committed agent at Ψ must be committed at the initial matching Ψ^0. This implies that the number of committed agents at this stable state can be reduced and thus some committed agents at Ψ^0 will be set free. Our second result (Theorem 2) is a stronger theorem stating that there exists at least one strongly stable matching state $(\Omega, C(\Omega))$ in which the matching Ω is in the strict core and the family of committed agents at Ω is a subset of the family of committed agents at the initial matching Ψ^0.

We will prove these two results through two constructive algorithms which will actually generate a precise solution for each case within a finite number of steps. The first algorithm is a generalization of the deferred acceptance (DA) procedure of Gale and Shapley (1962) and the second one is a generalization of the top trading cycle (TTC) method from Shapley and Scarf (1974). We will use the first algorithm to prove Theorem 1. However, to prove the second theorem we will have to use a combination of the two algorithms-our Hybrid Procedure. It will be shown that neither of the two algorithms alone suffices to serve the purpose.

We first introduce the following modification of the deferred acceptance (DA) procedure of Gale and Shapley (1962). Dubins and Freedman (1981) have shown that the DA procedure is strategy proof for men in the face of honest women. Kojima and Manea (2010) have axiomatized the DA procedure.

Workers Proposing Deferred Acceptance (WP-DA) Procedure

- At the beginning, every committed worker in C^0 is provisionally matched to his original partner under their initial contract.

- At the first step, every free worker in V^0 proposes his most-liked contract among all his acceptable contracts to its associated firm. Every firm rejects any proposed unacceptable contract, and also rejects all but her most preferred among those proposed acceptable contracts she has received, as well as the contract she is provisionally matched with, subject to the constraint that if she is a committed firm in C^0 and has received her initial contract at Ψ^0 proposed from her initial partner, she should treat it as her unique favourite contract and be provisionally matched with this contract. Any worker whose proposed contract has not been rejected should be provisionally matched with his proposed contract.

- At any step, any worker who either has just become a *free agent*[7] or was rejected at the previous step proposes his favourite contract among those which are acceptable to him and which he has not yet proposed, to its associated firm. Every firm rejects all proposed unacceptable contracts and also rejects all but her most preferred contract among those proposed acceptable contracts she has received, as well as the contract she is provisionally matched with, subject to the constraint that if she is a committed firm in C^0 and has received her initial contract at Ψ^0 proposed from her initial partner, she should treat it as her unique favourite contract and be provisionally matched with this contract. Any worker whose proposed contract has not been rejected should be provisionally matched with his proposed contract.

- When there is no new proposal from any worker, we then arrive at a matching state as follows: The matching Ψ consists of all those contracts that remain in force and are currently provisionally matched with some firm and some worker, and of those trivial contracts involving only single agents who are currently not provisionally matched with any other agent. Let W_V denote the set of those workers who are free at

[7] In this procedure, we say that a worker w is *free at a step*, if he is a free worker or his partner $\mu^0(w)$ has being provisionally matched with a new contract rather than her initial contract at Ψ^0 at the beginning of the current step.

Ψ^0 or have become free in the procedure, $W_C = W \setminus W_V$ denote the set of those workers who are committed at Ψ^0 and have never become free in the procedure, and $F_C = \{f \in C^0 \cap F \mid \Psi(f) = \Psi^0(f)\}$ denote the set of those committed firms which keep the same partner with the same contract as the initial state $(\Psi^0, C(\Psi^0))$. We assign the same commitment constraint for every worker in W_C and for every firm in F_C as in the initial state $(\Psi^0, C(\Psi^0))$ and obtain the new set of committed agents $C(\Psi) = W_C \cup F_C \subset C^0$ for the matching Ψ. This yields the matching state $(\Psi, C(\Psi))$.

Several remarks are in order. First, in this procedure, any committed worker in C^0 can make a proposal only if his contracted partner has received some proposal and been provisionally matched with a new contract rather than her initial contract at Ψ^0. Any committed firm in C^0 cannot reject her contracted partner's proposal if the proposed contract is her initial contract at Ψ^0. Second, free workers have no restriction of proposing their most preferred contracts to their associated firms nor have free firms any restriction of tentatively accepting their received favorite proposals. Third, when a committed firm receives a proposal of a better contract than her initial contract, she will reject her initial contract and let her initial partner free if her initial partner is committed. However, when her initial partner regardless of being committed or free later comes to propose to her, she will accept her initial partner's proposal, i.e., initial contract and reject all other proposals including the one she provisionally holds. Fourth, a committed worker becomes free when he is set free in the procedure, and a committed firm becomes also free when she is finally rematched with a new contract rather than her initial contract.

Analogously one can introduce the firms proposing deferred acceptance (FP-DA) procedure. To facilitate a better understanding of the above procedure, we illustrate it by the following example. Again we consider the simplest case in which there is at most one contract between every worker and one firm.

Example 2 *There are six workers w_1, w_2, $\cdots$, w_6, and six firms f_1, f_2, $\cdots$, f_6. w_6 and f_6 are new entries but the rest are incumbents, and the initially matched pairs are $\{f_1, w_1\}$, $\{f_2, w_2\}$, $\{f_3, w_3\}$, $\{f_4, w_4\}$, and $\{f_5, w_5\}$. We have the initial matching $\mu^0(w_1) = f_1$,*

$\mu^0(w_2) = f_2$, $\mu^0(w_3) = f_3$, $\mu^0(w_4) = f_4$, $\mu^0(w_5) = f_5$, $\mu^0(w_6) = w_6$, and $\mu^0(f_6) = f_6$ and the set of committed agents $C^0 = \{f_1, f_2, f_3, f_5, w_1, w_2, w_3, w_4, w_5\}$. Observe that the incumbent firm f_4 is free. Each agent's preferences are given by

$$
\begin{array}{llll}
\succ_{w_1}: & f_2, \quad f_1, \quad w_1 & \qquad \succ_{f_1}: & w_6, \quad w_2, \quad w_1, \quad f_1 \\
\succ_{w_2}: & f_1, \quad f_2, \quad w_2 & \qquad \succ_{f_2}: & w_5, \quad w_1, \quad w_2, \quad f_2 \\
\succ_{w_3}: & f_6, \quad f_3, \quad w_3 & \qquad \succ_{f_3}: & w_6, \quad w_3, \quad f_3 \\
\succ_{w_4}: & f_5, \quad f_4, \quad w_4 & \qquad \succ_{f_4}: & w_1, \quad w_5, \quad w_4, \quad f_4 \\
\succ_{w_5}: & f_2, \quad f_4, \quad f_5, \quad w_5 & \qquad \succ_{f_5}: & w_4, \quad w_5, \quad f_5 \\
\succ_{w_6}: & f_1, \quad f_3, \quad w_6 & \qquad \succ_{f_6}: & w_3, \quad f_6
\end{array}
$$

The steps of the procedure is stated below in detail.

Step 1: Free worker w_6 proposes to f_1 who will set w_1 free and be provisionally matched to w_6. Now w_1 is free.

Step 2: w_1 proposes to f_2 who will let w_2 free and be provisionally matched to w_1. Now w_2 is free.

Step 3: w_2 proposes to f_1. f_1 declines w_2 and is provisionally matched to w_6.

Step 4: w_2 proposes to his initial partner f_2. Observe that by the rule of the procedure f_2 has to reject w_1 and is provisionally matched to w_2 because f_2 is committed to w_2.

Step 5: w_1 proposes to his initial partner f_1. f_1 has to reject w_6 and is provisionally matched to w_1 because f_1 is committed to her initial partner w_1.

Step 6: w_6 proposes to f_3. f_3 sets w_3 free and is provisionally matched to w_6. Now w_3 is free.

Step 7: w_3 proposes to f_6 who will be provisionally matched to w_3.

Step 8: Now there is no new proposal. The procedure terminates with a stable matching state $(\Psi, C(\Psi))$ where the matching is given by

$$\Psi = \begin{pmatrix} f_1 & f_2 & f_3 & f_4 & f_5 & f_6 \\ w_1 & w_2 & w_6 & w_4 & w_5 & w_3 \end{pmatrix},$$

and the set of committed agents is $C(\Psi) = \{f_1, f_2, f_5, w_4, w_5\}$. In the process, w_1, w_2, and w_3 have been set free, and also f_3 has been set free as f_3 is rematched to w_6 instead of w_3. So in total we have now seven free agents f_3, f_4, f_6, w_1, w_2, w_3, and w_6. Recall that initially we have only three free agents f_4, f_6, and w_6.

Theorem 1 *In the model $\mathcal{M} = (W, F, \Psi^0, C^0, \succ)$, there exists at least one stable matching state $(\Psi, C(\Psi))$ with a feasible matching Ψ and a set of committed agents $C(\Psi) \subseteq C^0$.*

Proof: The WP-DA procedure will generate a matching state $(\Psi, C(\Psi))$ in a finite number of rounds because there is only a finite number of contracts, and no worker proposes one contract to any firm more than once. In the procedure every worker who has become free proposes only acceptable contracts, and every firm rejects all unacceptable contracts that she has received except that she is a committed firm and has received her initial contract proposed by her initial partner. So no agent is provisionally matched with a new but unacceptable contract. By assumption that $\mu^0(x) \in V^0$ for each contracted agent x with $x \succ_x \Psi^0(x)$, the matching Ψ is feasible and the matching state $(\Psi, C(\Psi))$ is individually rational.

We will show that $(\Psi, C(\Psi))$ is stable.

Suppose to the contrary that the matching state $(\Psi, C(\Psi))$ is blocked by one of its minimal chain $(\bar{w}_1, \bar{f}_1, \bar{w}_2, \bar{f}_2, \cdots, \bar{w}_K, \bar{f}_K)$. Then, for each $k = 1, \cdots, K$, there exists a mutually acceptable contract α_k to $\bar{w}_k$ and $\bar{f}_k$ such that $\alpha_k \succ_{\bar{w}_k} \Psi(\bar{w}_k)$ and $\alpha_k \succ_{\bar{f}_k} \Psi(\bar{f}_k)$, i.e., both $\bar{w}_k$ and $\bar{f}_k$ prefer contract α_k to their contracts under Ψ. Since $\bar{w}_1 \in W_V$ and $\alpha_1 \succ_{\bar{w}_1} \Psi(\bar{w}_1)$, worker $\bar{w}_1$ must have previously proposed the contract α_1 to firm $\bar{f}_1$. If $K = 1$, then $\bar{f}_1$ is a free firm or a rematched firm. In both cases, $\bar{f}_1$ should not have rejected the proposal α_1 from worker $\bar{w}_1$, yielding a contradiction. In the case of $K \geq 2$, a minimal chain implies that $\bar{f}_1$ and $\bar{w}_2$ are in $C(\Psi)$. Thus, $\bar{f}_1 \in C^0$ and $\Psi(\bar{f}_1) = \Psi^0(\bar{f}_1)$, and so $\bar{w}_2 = \mu_\Psi(\bar{f}_1) = \mu^0(\bar{f}_1)$. Note that $\alpha_1 \succ_{\bar{f}_1} \Psi(\bar{f}_1) = \Psi^0(\bar{f}_1)$. Thus α_1 is both acceptable and relatively acceptable to firm $\bar{f}_1$. This implies that firm $\bar{f}_1$ should have freed her initial partner $\bar{w}_2 = \mu^0(\bar{f}_1)$ at some step in the procedure, namely, $\bar{w}_2$ must be in $V(\Psi)$, leading to a contradiction as well.

This shows that $(\Psi, C(\Psi))$ cannot be blocked by any minimal chain. By Lemma 1, $(\Psi, C(\Psi))$ is a stable matching state. □

We now turn to present a modification of the top trading cycle (TTC) method of Shapley and Scarf (1974) which will be an integral part of our Hybrid Procedure for finding a strongly stable matching state with a strict core matching. Roth (1982) has shown that the TTC procedure is strategy proof. Ma (1994) has given an axiomatic characterization of the TTC procedure. Abdulkadiroğlu and Sönmez (1999, 2003) have adapted the procedure to a house allocation model with both existing tenants and new applicants, and to the school choice. Combe et al. (2017) have transformed the TTC procedure to the block exchange and teacher optimal block exchange algorithms in order for teachers and schools to improve their welfare.

A key feature of the TTC procedure and its variants is to generate (top trading) cycles in which agents can get better offers or positions and thus improve their welfare. In our modification this feature will be maintained but we need to apply this modified procedure to *an almost committed set* which is defined next. This modified procedure will be implemented only after our modified DA procedure is executed. A subset of partners $S \subseteq P^0$ is called an *almost committed set* if for every $x \in S$ it holds $\mu^0(x) \in S$ and $\{x, \mu^0(x)\} \cap C^0 \neq \emptyset$. It follows that if a contracted agent x is in an almost committed set, then her partner $\mu^0(x)$ must be also in the almost committed set and at least one of the two agents must be committed. Clearly, every almost committed subset S and its complementary set $(W \cup F) \setminus S$ are both implementable coalitions for the initial matching state (Ψ^0, C^0). The following procedure will be applied to any given almost committed set S. In the procedure, every worker in the set S points to the firm involved in his favourite contract among all his mutually relatively acceptable contracts in $\Sigma(S)$, and each firm f in the set S points to her partner $\mu^0(f)$ in the set S.

Workers Proposing TTC (WP-TTC) Procedure
on an Almost Committed Set S

- Every worker in the almost committed set S points to the firm involved in his favourite contract among all his mutually relatively acceptable contracts in $\Sigma(S)$[8], and each firm in S points to her worker partner under μ^0.

[8]In a decentralized market where each individual knows only their own preferences but not others, a worker can find such a contract as follows: He first proposes his favourite contract among all his relatively acceptable contract in $\Sigma(S)$. If his proposal is not relatively acceptable to his proposed firm, then his proposal will be rejected. Repeat this process to the remaining contracts that he has not yet proposed until he will not be rejected.

- There exists at least one directed cycle[9]. In each directed cycle, match every worker to his pointed firm under the worker's favorite mutually relatively acceptable contracts between them. All such matched work-firm pairs leave the market.

- Repeat this process to all the remaining agents which form an almost committed set as well, until no agent is left. Let Π be the matching consisting of all matched contracts.

Similarly we can have the firms proposing top trading cycle (FP-TTC) procedure.

Next we introduce a Hybrid Procedure of the WP-DA and WP-TTC procedures for finding a strongly stable matching state with a strict core matching.

The Hybrid Procedure for Finding a Strongly Stable Matching State with a Strict Core Matching

Step 1: Apply the WP-DA procedure to the model $\mathcal{M} = (W, F, \Psi^0, C^0, \succ)$ and give a stable matching state $(\Psi, C(\Psi))$.

Step 2: Define an almost committed set by $S = \{x \in P^0 \mid \Psi(x) = \Psi^0(x) \text{ and } \{x, \mu^0(x)\} \cap C^0 \neq \emptyset\}$. Apply the WP-TTC procedure to the set S and generate a matching Π.[10]

Step 3: Based on matchings Ψ and Π, construct a matching Ω by

$$\Omega(x) = \begin{cases} \Psi(x) & \text{if } \ x = (W \cup F) \setminus S, \\ \Pi(x) & \text{if } \ x \in S. \end{cases}$$

Set $C(\Omega) = W_C \cup \{f \in S \cap F \mid \mu^0(f) \in W_V\}$, where W_V is the set of those workers who are free at the state $(\Psi, C(\Psi))$ given at Step 1, and W_C is the set of those workers who are committed at the state $(\Psi, C(\Psi))$. The procedure stops with the matching state $(\Omega, C(\Omega))$.

[9]We say a cycle $(\bar{w}_1, \bar{f}_1, \bar{w}_2, \bar{f}_2, \cdots, \bar{w}_K, \bar{f}_K)$ of the initial state $(\mu^0, C(\mu^0))$ with mutually relatively acceptable pairs $(\bar{w}_k, \bar{f}_k)$ $(k = 1, \cdots, K)$ is a *directed cycle* if each $\bar{w}_k$ points to $\bar{f}_k$ and each $\bar{f}_k$ points to $\bar{w}_{k+1} = \mu^0(\bar{f}_k)$ for $k = 1, \cdots, K$, where $\bar{w}_{K+1}$ denotes $\bar{w}_1$.

[10]If the set S is empty, let the matching Π be empty.

In order to have a better understanding of the Hybrid Procedure, we explain how some committed incumbents will maintain their commitment and how other committed incumbents can be freed from their commitment. First, observe that the almost committed set $S = \{x \in P^0 \mid \Psi(x) = \Psi^0(x) \text{ and } \{x, \mu^0(x)\} \cap C^0 \neq \emptyset\}$ is the family of agents who have gone through the WP-DA procedure but still retained the same contracts as they had at the initial state $(\Psi^0, C(\Psi^0))$, and who had partners at $(\Psi^0, C(\Psi^0))$ and are committed to their partners or whose partners are committed. Recall that $C^0 = C(\Psi^0)$. Second, we can decompose the set S into two disjoint subsets S_1 and S_2 as follows. Let $S_1 = \{x \in S \mid \{x, \mu^0(x)\} \cap W_V \neq \emptyset\}$ denote the set of agents in S who did not rematch in the WP-DA procedure but had opportunities to participate in the rematching process. Let $S_2 = \{x \in S \mid \{x, \mu^0(x)\} \cap W_C \neq \emptyset\}$ denote the set of agents in S who did not have any opportunity to rematch in the WP-DA procedure. Then, we have $S_1 \cap S_2 = \emptyset$ and $S_1 \cup S_2 = S$. Let $S_3 = (W \cup F) \setminus S$. Clearly S_1, S_2 and S_3 form a partition of $W \cup F$. Note that $C(\Omega) = (S_2 \cap W) \cup (S_1 \cap F)$. This means that at the matching state $(\Omega, C(\Omega))$, only workers in S_2 and firms in S_1 will maintain their commitment but all other agents are free.

Recall that $S_2 \cap W \subseteq C(\Psi)$, i.e., no worker $w \in S_2$ has become free in the WP-DA procedure at Step 1 of the Hybrid Procedure. For every worker $w \in S_1 \cup S_3$ and every firm $f \in S_2$ there is no mutually acceptable contract α to them such that $\alpha \succ_w \Psi(w)$ and $\alpha \succ_f \Psi^0(f) \succ_f f$. Otherwise firm f would have received the proposal α from worker w and set her partner $\mu^0(f) \in S_2$ free. Moreover, for any worker $w \in S_1 \cup S_3$ and any firm $f \in S_2$, since $\alpha \neq \Psi(w)$ and $\alpha \neq \Psi(f) = \Psi^0(f)$ for any $\alpha \in \Sigma(w) \cap \Sigma(f)$, there is no mutually acceptable contract α to f and w such that $\alpha \succeq_w \Psi(w)$ and $\alpha \succeq_f \Psi^0(f) \succ_f f$. As a result, in the WP-TTC procedure at Step 2 of the Hybrid Procedure, no worker in S_1 points to a firm in S_2. This implies that every directed cycle is contained either in S_1 or in S_2 and cannot be across both S_1 and S_2. So every matched pair from every directed cycle must be either in S_1 or in S_2.

We use again Example 2 to illustrate the Hybrid Procedure. The detailed steps are given as follows:

Step 1: The WP-DA procedure runs and generates the matching state $(\Psi, C(\Psi))$ where

$$\Psi = \begin{pmatrix} f_1 & f_2 & f_3 & f_4 & f_5 & f_6 \\ w_1 & w_2 & w_6 & w_4 & w_5 & w_3 \end{pmatrix},$$

and $C(\Psi) = \{f_1, f_2, f_5, w_4, w_5\}$.

Step 2: We identify the almost committed set $S = \{f_1, f_2, f_4, f_5, w_1, w_2, w_4, w_5\}$. We apply the WP-TTC procedure to the set S.

Step 3: w_1 points to f_2, w_2 to f_1, w_4 to f_5, and w_5 to f_2, while f_1 points to w_1, f_2 to w_2, f_4 to w_4, and f_5 to w_5. We have a directed cycle (w_1, f_2, w_2, f_1) from which two matched pairs $\{w_1, f_2\}$ and $\{w_2, f_1\}$ are generated. They leave the market.

Step 4: w_4 points to f_5, and w_5 to f_4, while f_4 points to w_4, and f_5 to w_5. We have a directed cycle (w_4, f_5, w_5, f_4) from which two matched pairs $\{w_4, f_5\}$ and $\{w_5, f_4\}$ are made. They leave the market.

Step 5: We obtain the matching $\Pi(w_1) = f_2$, $\Pi(w_2) = f_1$, $\Pi(w_4) = f_5$, and $\Pi(w_5) = f_4$.

Step 6: We construct the final matching

$$\Omega = \begin{pmatrix} f_1 & f_2 & f_3 & f_4 & f_5 & f_6 \\ w_2 & w_1 & w_6 & w_5 & w_4 & w_3 \end{pmatrix}.$$

and a set of committed agents $C(\Omega) = \{f_1, f_2, w_4, w_5\}$. Thus, we obtain a strongly stable matching state $(\Omega, C(\Omega))$ with a strict core matching Ω.

In this example, we have $W_C = \{w_4, w_5\}$ and $W_V = \{w_1, w_2, w_3, w_6\}$ from $(\Psi, C(\Psi))$. Moreover, we have $S_1 = \{f_1, f_2, w_1, w_2\}$, $S_2 = \{f_4, f_5, w_4, w_5\}$, and $S_3 = \{f_3, f_6, w_3, w_6\}$ forming a partition of $F \cup W$.

We are now ready to establish the following major existence theorem.

Theorem 2 *The matching Ω generated by the Hybrid Procedure is a strict core matching of the model $\mathcal{M} = (W \cup F, \Psi^0, C^0, \succ)$. Furthermore, the generated matching state $(\Omega, C(\Omega))$ is strongly stable with $C(\Omega)$ being a subset of C^0.*

Proof: We first show that Ω is a strict core matching.

In Step 1 of the Hybrid Procedure, by Theorem 1 the WP-DA procedure produces a stable matching state $(\Psi, C(\Psi))$. Ψ is a feasible matching. In Step 2 of the Hybrid

Procedure, the WP-TTC procedure generates a matching Π. At Π every agent in the almost committed set S is matched to a partner through a mutually relatively acceptable contract. Observe that $\Psi^0(x) \succ_x x$ for every free agent $x \in S \cap V^0$ (or else $\mu^0(x) \in V^0$ as well) and also for every agent x with $\mu^0(x) \in C^0$. $\Pi(x)$ is acceptable to every free agent $x \in S \cap V^0$ and also to every agent x with $\mu^0(x) \in C^0$. So the matching Ω given at Step 3 of the Hybrid Procedure is feasible. If $S = \emptyset$, then $\Omega = \Psi$ and $C(\Omega) = C(\Psi) = \emptyset$. And $\Omega = \Psi$ is stable in the sense of Gale and Shapley and is in the strict core.

Now consider the general case of $S \neq \emptyset$. Assume by way of contradiction that Ω is not in the strict core of the market. By Lemma 2, Ω must be improved upon by a chain or cycle of the initial state (Ψ^0, C^0). Suppose that Ω could be improved upon by a chain $\tau = (\bar{w}_1, \bar{f}_1, \bar{w}_2, \bar{f}_2, \cdots, \bar{w}_K, \bar{f}_K)$ of (Ψ^0, C^0). This means that (i) for each $k = 1, \cdots, K$, there is a contract $\alpha_k \in \Sigma(\bar{w}_k) \cap \Sigma(\bar{f}_k)$ such that $\alpha_k \succeq_{\bar{w}_k} \Omega(\bar{w}_k)$ and $\alpha_k \succeq_{\bar{f}_k} \Omega(\bar{f}_k)$; and (ii) there is some $\bar{k} = 1, \cdots, K$ such that $\alpha_{\bar{k}} \succ_{\bar{w}_{\bar{k}}} \Omega(\bar{w}_{\bar{k}})$ or $\alpha_{\bar{k}} \succ_{\bar{f}_{\bar{k}}} \Omega(\bar{f}_{\bar{k}})$. It follows from strict preferences that $\alpha_{\bar{k}} \succ_{\bar{w}_{\bar{k}}} \Omega(\bar{w}_{\bar{k}})$ and $\alpha_{\bar{k}} \succ_{\bar{f}_{\bar{k}}} \Omega(\bar{f}_{\bar{k}})$. We have the following possibilities:

(i) $A(\tau) \not\subset S_3$, or else $(\bar{w}_{\bar{k}}, \bar{f}_{\bar{k}})$ is a blocking pair of the stable matching state $(\Psi, C(\Psi))$.

(ii) $A(\tau) \not\subset S_1$, because there is no firm in S_1 who is free at the initial state.

(iii) $A(\tau) \not\subset S_2$, because there is no worker in S_2 who is free at the initial state. Furthermore, $A(\tau) \cap S_2 = \emptyset$, because for any worker $w \in S_1 \cup S_3$ and any firm $f \in S_2$ there is no contract $\alpha \in \Sigma(w) \cap \Sigma(f)$ such that $\alpha \succeq_w \Psi(w)$ and $\alpha \succeq_f \Psi(f) \succeq_f \Psi^0(f)$.

We will further show that $A(\tau) \not\subset S_1 \cup S_3$. Suppose to the contrary that $A(\tau) \subset S_1 \cup S_3$. Then, $A(\tau) \cap S_1 \neq \emptyset$ and $A(\tau) \cap S_3 \neq \emptyset$. From $S_1 \cap F \subset C(\Omega) \subset C^0$, we see $\bar{f}_K \in S_3$. Thus, there is some $k = 1, \cdots, K$ such that $\bar{w}_k \in S_1 \cap W \subset V(\Psi)$ and $\bar{f}_k \in S_3 \subset V(\Psi)$. From $\alpha_k \neq \Omega(\bar{w}_k)$ and $\alpha_k \neq \Omega(\bar{f}_k)$, we further have $\alpha_k \succ_{\bar{w}_k} \Omega(\bar{w}_k) \succeq_{\bar{w}_k} \Psi^0(\bar{w}_k) = \Psi(\bar{w}_k)$ and $\alpha_k \succ_{\bar{f}_k} \Omega(\bar{f}_k) = \Psi(\bar{f}_k) \succeq_{\bar{f}_k} \bar{f}_k$. Note that $\Psi^0(\bar{w}_k) \succeq_{\bar{w}_k} \bar{w}_k$, or else by assumption his initial parter $\mu^0(\bar{w}_k) \in S_1 \cap F \subset C_0$ is in V_0, yielding a contradiction. This implies that $(\bar{w}_k, \bar{f}_k)$ is a blocking pair of the stable matching state $(\Psi, C(\Psi))$. We have proved that Ω cannot be improved upon by any chain of the initial state (Ψ^0, C^0).

Suppose that Ω could be improved upon by a cycle $\tau = (\bar{w}_1, \bar{f}_1, \bar{w}_2, \bar{f}_2, \cdots, \bar{w}_K, \bar{f}_K)$ of (Ψ^0, C^0). This means that (i) for each $k = 1, \cdots, K$, there is a contract $\alpha_k \in \Sigma(\bar{w}_k) \cap \Sigma(\bar{f}_k)$

such that $\alpha_k \succeq_{\bar{w}_k} \Omega(\bar{w}_k)$ and $\alpha_k \succeq_{\bar{f}_k} \Omega(\bar{f}_k)$; and (ii) there is some $\bar{k} = 1, \cdots, K$ such that $\alpha_{\bar{k}} \succ_{\bar{w}_{\bar{k}}} \Omega(\bar{w}_{\bar{k}})$ or $\alpha_{\bar{k}} \succ_{\bar{f}_{\bar{k}}} \Omega(\bar{f}_{\bar{k}})$. It follows from strict preferences that $\alpha_{\bar{k}} \succ_{\bar{w}_{\bar{k}}} \Omega(\bar{w}_{\bar{k}})$ and $\alpha_{\bar{k}} \succ_{\bar{f}_{\bar{k}}} \Omega(\bar{f}_{\bar{k}})$.

Obviously, $A(\tau) \not\subset S_3$, or else $(\bar{w}_{\bar{k}}, \bar{f}_{\bar{k}})$ is a blocking pair of the stable matching state $(\Psi, C(\Psi))$.

We claim that $A(\tau) \not\subset S_1$. Suppose to the contrary that $A(\tau) \subset S_1$. Observe that for every agent $x \in S_1$, agents x, $\mu^0(x)$, $\Pi(x)$ and $\mu^0(\Pi(x))$ must be matched and removed at the same round in the WP-TTC procedure. Let A_t denote the set of all agents in S_1 matched and removed at round t of the WP-TTC procedure. Assume that each worker $\bar{w}_k$ is removed at round t_k, i.e., $\bar{w}_k \in A_{t_k}$. Then $\bar{f}_k = \mu^0(\bar{w}_{k+1})$ is removed at round t_{k+1}. Recall that α_k is mutually relatively acceptable to $\bar{w}_k$ and $\bar{f}_k$, and $\alpha_k \succeq_{\bar{w}_k} \Omega(\bar{w}_k)$, the WP-TTC procedure matches $\bar{w}_k \in A_{t_k}$ to his pointed firm under the worker's favorite mutually relatively acceptable contracts in $\Sigma(\cup_{s\geq t_k} A_s)$. We see that $\alpha_k \in \Sigma(\cup_{s\leq t_k} A_s)$ and $\bar{f}_k \in \cup_{s\leq t_k} A_s$. Therefore, $\bar{f}_k$ and $\bar{w}_{k+1} = \mu^0(\bar{f}_k)$ must be removed no later than $\bar{w}_k$. Thus, it must hold $t_k \geq t_{k+1}$ for all $k = 1, 2, \cdots, K$, where $K+1$ denotes 1. This implies $t_1 = t_2 = \cdots = t_K$. However, $\alpha_{\bar{k}} \succ_{\bar{w}_{\bar{k}}} \Omega(\bar{w}_{\bar{k}}) = \Pi(w_{\bar{k}})$ implies that $\alpha_{\bar{k}} \in \Sigma(\cup_{s<t_{\bar{k}}} A_s)$ and $t_{\bar{k}} > t_{\bar{k}+1}$. This contradiction shows that $A(\tau) \not\subset S_1$.

Using the same argument above, we can show that $A(\tau) \not\subset S_2$. Moreover, $A(\tau) \cap S_2 = \emptyset$, because for any worker $w \in S_1 \cup S_3$ and any firm $f \in S_2$ there is no contract $\alpha \in \Sigma(w) \cap \Sigma(f)$ such that $\alpha \succeq_w \Psi(w)$ and $\alpha \succeq_f \Psi(f) \succeq_f \Psi^0(f)$.

Finally, we will show that $A(\tau) \not\subset S_1 \cup S_3$. Suppose to the contrary that $A(\tau) \subset S_1 \cup S_3$. Then, $A(\tau) \cap S_1 \neq \emptyset$ and $A(\tau) \cap S_3 \neq \emptyset$. Thus, there is some $k = 1, \cdots, K$ such that $\bar{w}_k \in S_1 \cap W \subset V(\Psi)$ and $\bar{f}_k \in S_3 \cap F \subset V(\Psi)$. We can again show that $(\bar{w}_k, \bar{f}_k)$ is a blocking pair of the stable matching state $(\Psi, C(\Psi))$, leading to a contradiction.

In summary, we have proved that Ω cannot be improved upon by any chain or cycle of the initial state (Ψ^0, C^0). Consequently, Ω must be a strict core matching of the market.

We shall prove that $(\Omega, C(\Omega))$ is a strongly stable matching state.

By Lemma 1, it is sufficient to show that $(\Omega, C(\Omega))$ is individually rational and not blocked by any its minimal chain or pure cycle. First, note from the WP-DA procedure that $\Omega(x) = \Psi(x)$ is acceptable to every agent $x \in S_3$. For every $w \in S_1$, we have

$\Omega(w) \succeq_w \Psi^0(w) \succeq_w w$, i.e., $\Omega(w)$ is acceptable to him. Similarly, $\Omega(f)$ is acceptable to every firm $f \in S_2$. Thus, $\Omega(x)$ is acceptable to every free agent $x \in V(\Omega)$, and hence $(\Omega, C(\Omega))$ is individually rational.

Next, note that at matching state $(\Omega, C(\Omega))$ there are no mutually committed partners. Therefore, a minimal chain of $(\Omega, C(\Omega))$ must be a free pair of worker and firm (w, f). We can assume to the contrary that $(\Omega, C(\Omega))$ is blocked by a free pair of worker and firm (w, f). Then, $w \in S_3 \cup S_1$ and $f \in S_3 \cup S_2$, and there is a mutually acceptable contract α to them such that $\alpha \succ_w \Omega(w)$ and $\alpha \succ_f \Omega(f)$. Thus, if $\{w, f\} \subset S_3$, then (w, f) is a blocking pair of the stable matching state $(\Psi, C(\Psi))$. If $w \in S_3, f \in S_2$, then $\alpha \succ_w \Omega(w) = \Psi(w)$ and $\alpha \succ_f \Omega(f) \succeq_f \Psi^0(f) = \Psi(f)$, and (w, f) is a blocking pair of $(\Psi, C(\Psi))$. Similarly, we can show (w, f) is a blocking pair of $(\Psi, C(\Psi))$ for $w \in S_1$ and $f \in S_2 \cup S_3$. We have shown that $(\Omega, C(\Omega))$ is stable.

Suppose that $(\Omega, C(\Omega))$ could be blocked by a pure cycle $\tau = (\bar{w}_1, \bar{f}_1, \bar{w}_2, \bar{f}_2, \cdots, \bar{w}_K, \bar{f}_K)$ of $(\Omega, C(\Omega))$. We first see that $A(\tau) \cap S_3 = \emptyset$, or else τ becomes a blocking chain of the stable matching state $(\Omega, C(\Omega))$. Moreover, cycle τ cannot be across both S_1 and S_2, or else it becomes a blocking chain of $(\Omega, C(\Omega))$ as well. Thus, it is only possible $A(\tau) \subset S_1$ or $A(\tau) \subset S_2$.

Assume that $A(\tau) \subset S_1$. Let A_t denote the set of all agents in S_1 matched and removed at round t of the WP-TTC procedure. Suppose that every worker $\bar{w}_k$ is removed at round t_k, i.e., $\bar{w}_k \in A_{t_k}$. Then $\bar{f}_k = \Pi(\bar{w}_{k+1})$ is removed at round t_{k+1}. Note that for each $k = 1, \cdots, K$, there is a mutually acceptable contract α_k to $\bar{w}_k$ and $\bar{f}_k$ such that $\alpha_k \succ_{\bar{w}_k} \Omega(\bar{w}_k) \succeq_{\bar{w}_k} \Psi^0(\bar{w}_k)$ and $\alpha_k \succ_{\bar{f}_k} \Omega(\bar{f}_k) \succeq_{\bar{f}_k} \Psi^0(\bar{f}_k)$, the WP-TTC procedure matches $\bar{w}_k \in A_{t_k}$ to her pointed firm under the worker's favorite mutually relatively acceptable contracts in $\Sigma(\cup_{s \geq t_k} A_s)$. We see that $\alpha_k \in \Sigma(\cup_{s<t_k} A_s)$ and $\bar{f}_k \in \cup_{s<t_k} A_s$. Therefore, $\bar{f}_k$ and $\Pi(\bar{f}_k) = \bar{w}_{k+1}$ must have been removed earlier than $\bar{w}_k$ i.e., $t_k > t_{k+1}$. This yields a contradiction that $t_1 > t_2 > \cdots > t_{K-1} > t_K > t_1$. Similarly, we can show that $A(\tau) \not\subset S_2$. Thus, we have proved that $(\Omega, C(\Omega))$ cannot be blocked by any pure cycle.

We can now conclude that the matching state $(\Omega, C(\Omega))$ is individually rational and cannot be blocked by any minimal chain or pure cycle. By Lemma 1, $(\Omega, C(\Omega))$ is a strongly stable matching state. □

4 Concluding Remarks

Entry level job matching markets have been extensively studied in the literature along the line of Gale and Shapley (1962). In such a market all participants (firms and workers) are new entries and they each try to find a best possible partner to match. The celebrated deferred acceptance procedure or its variant has been used to find a stable matching in the market. In this paper we have developed a senior level job matching model with multiple heterogeneous incumbents and new entries. New entries are free to make their choices. However a committed incumbent cannot dissolve her partnership to find an alternative unless her partner agrees to do so. An uncommitted incumbent is free to seek a new partner. Every firm has preferences over workers with contracts and the prospect of not hiring any worker and every worker has preferences over firms with contracts and the prospect of not working at any firm. Each contract specifies a service and a payment between one firm and one worker. There can be multiple different contracts between every firm and every worker. Firms and workers haggle over contracts. When a firm reaches a deal with a worker, they both are agreed on a common contract to implement.

Our model is general enough to cover a variety of situations involving commitment. A typical example is one-sided commitment including the civil service system and the tenure-track system. Another typical example is two-sided commitment such as the mutual consent divorce law. When no incumbent is committed and there is at most one contract between every firm and every worker, the model reduces to the model of Blum et al. (1997). When there is no incumbent and there are sufficient many contracts between every firm and every worker, the model becomes identical to Crawford and Knoer (1981). When there is no incumbent and there is at most one contract between every firm and every worker, the model coincides with Gale and Shapley (1962). Our model can accommodate the most general environment where there are new entries and incumbents, some pair of incumbents are two-sided committed, some incumbent firms are committed, some incumbent workers are committed, and some incumbents are uncommitted. Commitment has been previously studied in several economic subjects since Schelling (1956). Our notion of commitment is a natural and practical one fitting well the mutual consent divorce law and other situations.

In our model every agent strives to find her best possible partner with contract from her opposite side of the market but a committed agent has to honour her commitment. The key question was how to match workers and firms with contracts and possibly set committed agents free without violating their commitment constraints in order to achieve efficiency and stability of the market.

To address the question we have adopted and adapted two fundamental solution concepts to the current model. We introduced the concept of matching state comprising a matching between firms and workers with the corresponding contracts and a set of committed agents. The presence of commitments induces chains and cycles generating a rich structure beneath the problem. Our notion of strong stability is defined with respect to chains and cycles going beyond (pairwise) stability of Gale and Shapley (1962) and (chain) stability of Ostrovsky (2008). Our concept of core has to take the initial matching state into account ruling out some infeasible coalitions. The (strict) core does not coincide with the set of (strongly) stable matchings. Our major result shows that there exists at least one strongly stable matching state whose matching is in the strict core, and moreover the family of committed agents in this matching state is contained by the family of committed agents at the initial matching state and therefore some committed agents at the initial state will be freed in this final state. We have developed an algorithm for finding this solution and thus given a constructive proof of our main result. This algorithm is a novel combination of two generalizations of the deferred acceptance (DA) procedure and the top trading cycle (TTC) method. We have proved that neither of the two generalizations suffices to find a desired solution but our Hybrid Procedure guarantees to discover one.

In this paper we did not discuss the issue of incentive compatibility. It is already known from Ostrosky (2008, p.914) that in general incentive compatibility cannot be achieved in a setting involving chains. The presence of both chains and cycles due to commitments in our model makes it even harder to obtain such a result. In practice, however, as long as the number of participants is relatively large, it would be extremely difficult for any participant to manipulate her preferences in order to make a profitable gain even in two-sided job matching markets (see Kojima and Pathak 2009). Nevertheless, it is possible to show that in the current model our Hybrid Procedure will be strategy-proof for certain participants.

We hope that the current study has shed some new insights into senior level job matching markets.

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