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Abstract

We analyse optimal housing tenure choice (rent, buy or buy to let), consumption and a four asset portfolio in a life cycle model with uncertain labour income and asset returns (with a safe asset). Each period borrowing is possible only via a mortgage which is backed by housing collateral and which is itself subject to a loan to value and loan to income constraint. There is a minimum scale for house ownership. We derive some general theoretical properties of the solutions and closed-form solutions for specific preferences. To quantify the impact of uncertainty we simulate life cycle paths across random return and income realisations.

Keywords: housing tenure, mortgage, life cycle

1 Introduction

The paper examines individual housing tenure, housing finance and financial portfolio decisions in a life cycle framework in which utility each period depends on both consumption c_t and housing services h_t (the pleasure of living in a house). There are imperfect financial markets. Allowing for the tenure choice is theoretically and empirically important. First renting and buying with or without a mortgage have different risks (Deng et al., 2000); Sinai and Souleles (2005) find that empirically the volatility of rent can exceed the risk in house prices. Vigdor (2006) points out that housing finance constraints can also distort the relation between house prices and house rents, depressing the former. Moreover in recent times in the UK buy to let housing has become increasingly important as the percentage of renters in the population has increased, from 1999-2015 the number of new mortgages for buy to let (BTL) more than doubled (UK Finance, 2017). This is seen as the joint result of inequality in the income and wealth distribution and a rising real house price that makes it even more difficult for lower income individuals to afford the downpayment. In the UK there has also been a move towards more prudent lending policies with tighter control (Financial Services Authority, 2009) in contrast to the big increase in the supply of mortgage finance in the US prior to the subprime mortgage crisis (Mian and Sufi, 2009). In the UK the percentage of houseowners fell from around 71% in 2003 to around 64% in 2016. On the other hand the share of renters rose from about 18% in 2003 to around 27% in 2016. In 2000 only about 27% of

UK households had any direct participation in the stock market (Guiso et al., 2003), although there is indirect participation via pension schemes. Taking all these facts together, the problem has high policy importance in the UK. Indeed since 2010 there have increasingly been fiscal changes to reduce the return on buy to let by raising property taxes and conversely financial subsidies of various kinds to slacken the initial loan to income ratio constraint on first entry to owner occupied housing (Tucker, 2013).

In European and American households, a typical life cycle pattern of asset ownership between housing and financial assets (safe or risky), which arises partly because borrowing is allowed only against real assets and not against future income, is that in the young adult epoch households are renters. After saving from labour income or informal loans to finance the transaction cost, in middle age households become houseowners but with a finite term mortgage (typically 20-25 years). In this epoch households are typically also financing pre-employment children through education, financing mortgage debt and possibly elderly relatives. In the later epoch the financial demands on a households income have fallen: children have established their own households; elderly relatives are no more and the mortgage M_t has matured. However increasingly there is equity withdrawal in the older epoch either to finance consumption or to make gifts to others partly to avoid inheritance taxation. In the UK the instruments for this can be a life time mortgage, taking a new mortage or downsizing the house size owned. In the limit the old can choose to rent not own. If all this happens after retirement or a permanent income fall, the disposable income for savings may fall too. Thus typically we expect different stages of the life cycle to choose very different asset portfolios. But since idiosyncratic shocks are heterogeneous, there will be a variance within each life cycle group. Hirsch (2017) finds that the inequality of wealth by age particularly has risen.

We assume financial markets are imperfect: there is a safe return asset[1], a risky purely financial asset, housing mortgage debt. The only way of borrowing is through the mortgage which must be associated with house purchase but the amount that can be borrowed on the mortgage is the lower of a loan to house value ratio constraint and a loan to labour income constraint. Returns on these assets are uncertain over time, so are house prices and labour income. All random variables can be correlated.

Of course with any constrained optimisation under uncertainty, the realisations of random income variables like labour income or asset returns may be so bad that future feasibility is impossible and the individual must default on his obligations. With only mortgage debt, the only possible default is on this. There is a large literature on mortgage default on housing partly inspired by the subprime mortgage crisis in the US. Major advances have been made to understand voluntary and involuntary US mortgage default. A key study here is Campbell and Cocco (2015). But in the UK even with the impact of the global financial crisis, mortgage default has been found less important. The data shows that the arrears rate more or less hovers around 1% of mortgage loan debt (Building Societies Association) and Aron and Muellbauer (2016) reinforce both this and why it is so. Consequently we mainly abstract from these problems[2]. The reason is that we primarily want to see how the life cycle environment (basically the past of a household and its expected future) and the imperfect asset market restrictions affect the housing and financial asset portfolio choices of the household as it ages.

Housing is measured in quality adjusted physical size. The household can rent and/or buy units of housing. It can also buy housing to rent out. There is a minimum size of house that can be purchased, e.g.

[1] With fundamentally uncertain inflation and no pure indexed assets, it is arguable if there is a safe asset.

[2] But see subsequent discussions of preferences and distributions.

one cannot purchase one square meter of housing but it is possible to rent it. Housing and the financial asset portfolio are readjusted each period of time in light of changes in expectations and the realisations of past labour income and asset returns (including housing). The framework is rich in including most of the important empirical features of housing: it includes the housing tenure choice (rent or buy), the possibility of buy to let, part renting and part buying (e.g. shared ownership systems) and financial borrowing constraints especially the loan to value and loan to income constraints on the mortgage. We separate the consumption and investment sides of housing by specifying two variables, one for housing investment H_t and one for housing consumption h_t.

We measure all returns and the mortgage interest rate in real terms. This is consistent with the standard UK mortgage contract having an adjustable rate of interest[3](FCA, 2016). Each period of life, the consumer can costlessly readjust their portfolio and select nonhousing consumption. Thus mortgage refinancing is allowed within the loan to value (LTV) and loan to income (LTI) constraints. This allows for financing either nonhousing consumption or other asset investment via equity withdrawal at each date. It also allows the mortgage to be used within these limits to hedge income shocks. Empirical evidence shows that both of these are important in the US (Chen et al., 2013 ; Mian and Sufi, 2011). There is also some theoretical backing for using an adjustable rate mortgage with flexible refinancing and housing retraded every period (Piskorski and Tchistyi, 2010).

With no borrowing except in the mortgage, upper limits on the mortrgage size and a minimum house size purchase, the constraints on the portfolio decision each period are relatively complex. Even with simpler but less empirically relevant constraints, analytical solutions for general time additive preferences and distribution of uncertainty are not possible. A common approach is to fix the preferences and the main parameters involved (utility parameters, the discount rate, the joint distribution of uncertain asset returns and labour income, the rental income and the parameters in the mortgage constraints). Then numerically solve for the time path of optimal decisions (Cocco (2005), Attanasio et al. (2012)). A limit of this approach is that the solution inevitably depends on precise parameter assumptions and that the analytical causality may be hidden in the simulation. We compromise by using a general preference framework but making general qualitative assumptions on the randomness of returns. From this we can establish the solution regimes possible for the optimal portfolio, tenure, savings and housing services for arbitrary time additive concave preferences. We then specialise this to preferences from which closed form analytical solutions can be calculated but in which the dynamic time path of solutions still depends on realisations of random variables and so has to be simulated.

Net worth a_t is defined as the sum of the total value of assets (safe, risky and housing) net of mortgage debt. The optimal portfolio holdings for the safe and risky assets and for mortgage finance only has an effect on the future value function conditional on net worth a_t and housing investment H_t. Similarly the split of spending between c_t and h_t given values of a_t and H_t only affects the current utility. So we can solve the problem in two stages, first solving independently for the financial asset portfolio and then current spending decisions conditional on a_t and H_t. This reduces the value function to a function of a_t, H_t, m_t and other state variables where m_t is cash on hand at the start of period t.In the second step, we solve for a_t, H_t and from these for the optimal values of all decision variables. In terms of a_t, H_t we show

[3] In the UK, although around 50% of mortgage loans are termed fixed rate, in fact the rate is only fixed for a limited time period, usually two years.

the feasible set is either a polygon or a triangle; both have boumdary kinks.

The main distributional assumption we make is that the expected marginal value of returns on the risky asset dominates that on the safe asset and the cost of the mortgage. Given this, optimally so long as there is participation in housing it will be financed with a maximum possible mortgage so that one of the mortgage constraints will bind and there will be no investment in the safe asset. This allows us to reduce the decision problem from one with six choice variables to one with only two variables: net worth in a period and investment in housing.

Specialising the within period preferences to a Stone-Geary form, we derive closed form solutions of a_t and H_t in terms of m_t and other state variables in turn. The result is the value function and the optimal decisions of all the choice variables at any date. This allows explicit characterisation of the solution path in a framework with constraints preventing borrowing in all assets except the mortgage, mortgage constraints (based on loan to value and loan to income ratios) and housing tenure choices including renting, buy to let, part renting/part buying, or owning (with or without mortgage finance).

The results are:

(a) with general preferences and the return assumptions we use:

(i) so long as there is investment in housing, there is always a maximum constrained mortgage.

(ii) whether the mortgage is income or value constrained depends on planned housing investment at t reflected in H_t. The higher the planned housing investment is the more likely that the mortgage is loan to income constrained.

(iii) investing in the safe asset is not worthwhile.

(iv) there is a tradeoff between investing in the risky asset or housing which is partly conditional on the assumed marginal value of asset returns.

(v) the minimum house purchase size H^* makes the reduced feasible set in a_t and H_t non-convex so there may be multiple solutions.

(vi) in general there is a locus of a_t, H_t along which housing consumption is equal to housing purchase, which will usually be downward sloping. Increases in either a_t or H_t increase the propensity to BTL.

(b) specialising the preferences to a within period Stone Geary form:

(i) the indifference curves in a_t, H_t become linear and so any solution must pass through a boundary kink.

(ii) if the relevant (stochastically) discounted marginal utility of the future (taking the optimal portfolio of assets into account) is low enough, then it is optimal to consume most cash on hand today and save only enough to provide next period subsistence. Conversely if the current marginal utility of spending (after optimal allocation between housing and non-housing consumption) is low enough, despite the Inada conditions on current utility, it is optimal to limit the current period utility to the subsistence level.

(iii) even if the stochastically discounted risky asset return is higher than that on housing, it may still be optimal to invest in housing and even to invest all savings in housing if the return to buy to let/ savings on rental income are high enough.

(iv) in general the higher is planned net worth the more likely that there is buy to let holding H_t fixed. But with low planned net worth, the optimum may involve both purchase of some housing and in addition renting of some housing (shared ownership). We derive analytical conditions for buy to let to occur.

Depending on the position of the BTL locus (which depends on the relative preference between housing consumption and non-housing consumption, the shape of the feasible set and H^*, we can determine which kinks show positive BTL as compared with shared ownership. With an increase in the preference for housing consumption shared ownership will be more likely.

(v) in a life cycle context, the realisation of cash on hand each period depends on realisations of random outcomes each period. These then condition the nature of the available future time paths. So we provide calibrated stochastic life cycle simulations to show how heterogeneous realisations between individuals lead to variance in life cycle behaviour. The results indicate a lack of upward wealth mobility through life for the low educated who have worse lifetime labour income chances. In addition, less patient households invest less on housing compared to their more patient counterparts. And the indivisibility of housing purchase H^* has a strong effect on reducing house ownership.

Our paper is related to a set of literature that includes housing consumption and housing investment in the life cycle model. Attanasio et al. (2012) numerically solve a life-cycle model for households choosing consumption, saving and housing when they face uncertainties on both income and house prices with mortgage borrowing constraints. But there are only three discrete types of housing: renting, owning a flat and owning a house. And there is only one asset in their model. This implies the mortgage interest rate and safe saving interest rate are exactly the same, which is a restriction on modeling the interaction between different asset classes. Cocco (2005) studies the portfolio choice of homeowners by numerically solving a life cycle model with continuous housing, one riskless asset, one risky asset and mortgage debt. But he does not study the tenure choice (buy and/or rent). Brueckner (1997) also focuses on the behaviour of homeowners only. While including multiple assets and allowing for buy to let behaviour, he does not separate the mortgage and risky asset and does not solve the model explicitly. If the mortgage rate is identified with the safe rate, then there is no mortgage debt interest risk. Our paper differs from the existing literature in the following aspects. We derive some analytical properties for general preferences. Moreover, instead of solving the model numerically, we derive closed form solutions for the special preferences. This avoids the impreciseness of solution caused by interpolation and extrapolation. Second, we distinguish among a safe asset, mortgage debt, and a risky asset, which allows us to simulate the impact of uncertain asset returns for different assets on the individual choices. Third, we do not impose any restriction on the relative magnitude of housing consumption and housing investment, which allows for many different choices including renting only, owner occupation, buy to let, partly renting and partly owning (e.g. shared ownership scheme in the UK).

The rest of the paper is organised as follows. Section 2 introduces the general framework. Section 3 specialises the model to Stone Geary within period utility and proves the linearity of the value function under some assumptions. Closed form solutions are derived for this model. Section 4 simulates the life cycle paths of consumption and asset allocation with stochastic income and asset return processes. Finally, Section 5 concludes.

2 The general framework

In each period t utility depends on consumption c_t and the use of housing h_t. It is strictly concave and increasing in these variables.

$$u(c_t - \overline{c}, h_t - \overline{h})$$

$\overline{c}$ and $\overline{h}$ are time invariant subsistence levels[4]. This includes preferences satisfying Inada conditions on each of c_t and h_t at $c_t = 0$ or $h_t = 0$ or generating a strictly concave value function. Subsequent individual choices may optimally avoid the risk of default. Cash on hand at period start is m_t measured in units of consumption used for investing (in the safe asset A_t, the risky asset F_t) or the purchase of housing units $p_t H_t$ (p_t is the price of a unit of housing measured in units of consumption, H_t is the units purchased) or c_t and h_t. For each unit of housing rented, rental income/cost is y_{rt} in units of consumption. If housing is purchased the consumer can take out a mortgage in amount M_t in units of consumption. This gives a budget constraint on the use of cash on hand at t.

$$m_t = c_t + F_t + A_t + p_t H_t - M_t + y_{rt}(h_t - H_t) \tag{1}$$

We have assumed there is an asset portfolio providing zero mortgage default; in addition, we assume a lower bound for labour income ensuring that the subsistence level can always be attained so $m_t \geq \overline{c} + y_{rt}\overline{h}$.

$H_t - h_t$ is rented out housing, $p_t(H_t - h_t)$ is rented housing in units of consumption which generates a rental income $y_{rt}(H_t - h_t)$. Note we don't have any constraint on the relative values of H_t and h_t. When the housing owned is bigger than the housing consumed, i.e. $H_t > h_t$, the household rents out $(H_t - h_t)$ (buy-to-let) and gains the rental income $y_{rt}(H_t - h_t)$. This can be interpreted as either renting space in the principal residence or a household having some buy-to-let housing and at the same time living in rented space which is smaller than the buy-to-let house. When the housing owned is smaller than the housing consumed, i.e. $H_t < h_t$, the household would need to pay rent $y_{rt}(H_t - h_t)$. This can be interpreted as either an equity sharing scheme in the UK or a household having some buy-to-let housing and at the same time living in a different rented house which is bigger than the buy-to-let house. Finally, if the housing owned is the same as the housing consumed, i.e. $H_t = h_t$, then the household has neither rental income nor rental expenditure. The model is more general than Brueckner (1997) and Henderson and Ioannides (1983) that require housing owned to be bigger than housing consumed.

Define the rate of return on housing owned by $r_{Ht} = (p_{t+1} - p_t)/p_t$ so we can write $p_{t+1}H_t = (1 + r_{Ht})p_t H_t$. Then next period's cash on hand evolves from the decisions at t according to

$$\begin{aligned} m_{t+1} &= y_{t+1} + (1 + r_{At+1})A_t + (1 + r_{Ft+1})F_t - (1 + r_{Mt+1})M_t + p_{t+1}H_t \\ &= y_{t+1} + R_{At+1}A_t + R_{Ft+1}F_t - R_{Mt+1}M_t + R_{Ht+1}p_t H_t \end{aligned} \tag{2}$$

where $r_{At+1}, r_{Ft+1}, r_{Mt+1}, r_{Ht+1}$ are the realised real interest rates on the various assets and R_{At+1}, R_{Ft+1}, R_{Mt+1}, R_{Ht+1} are the gross returns on safe asset, risky asset, mortgage and housing wealth ($R_{it} = 1 + r_{it}$); y_t is labour income of period t. Equation (2) is the intertemporal budget constraint describing how wealth accumulates through time. At date t the returns on assets $F_{t+1}, H_{t+1}, M_{t+1}$, the unit cost of rental y_{rt+1} and labour income in the future are uncertain. However the future interest rate on the safe asset is certain. And the current house price and rental cost are known at the start of the period.

[4] $\overline{h}$ reflects minimum housing preferences but H^* reflects a supply condition of the minimum house size available partly set by physical engineering conditions, partly by the need for a supplier to cover fixed market costs of selling.

It is convenient to define the net worth in period t, $a_{t,}$ as the sum of the values of safe asset A_t, risky asset F_t, mortgage M_t, and housing wealth $p_t H_t$ owned by households.

$$a_t = A_t + F_t - M_t + p_t H_t$$

Then we can write (1) as

$$m_t = c_t + a_t + y_{rt}(h_t - H_t) \tag{3}$$

Equation (3) is the within-period budget constraint describing how resources are allocated among housing and non-housing consumption, and net worth.

The time line of the model is shown in Figure 1.

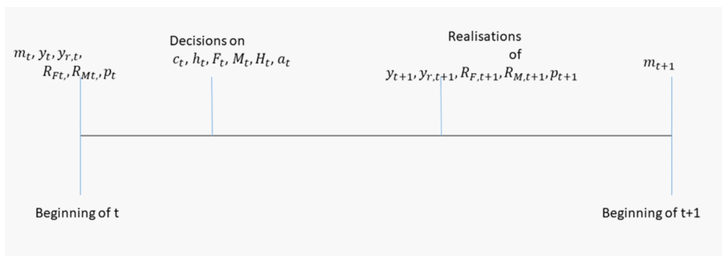

Figure 1: Timeline of the model

Here we distinguish between two types of state variables, some are purely exogenous but cash on hand m_t is not. We define the purely exogenous state variables at t in our model as $S_t = (y_t, y_{rt}, R_{Ft}, R_{Mt}, p_t)$.

The constraints on asset variables (H_t, A_t, F_t, M_t) reflect the borrowing constraints. All the assets must be nonnegative. The only borrowing possible is in the mortgage which can only be accessed if housing is purchased. The constraints on the mortgage reflect the facts that a mortgage can only be taken against the value of housing purchased but the upper limit on the amount borrowed is the smaller of some percentage τ_1 $(\tau_1 < 1)$ of the house value purchased and some multiple τ_2 of current labour income. Thus $M_t \leq \min(\tau_1 p_t H_t, \tau_2 y_t)$. In addition $M_t \geq 0$ and since $\tau_1 < 1$, $M_t < p_t H_t$.

We assume that there is a minimum size of house H^* that can be purchased, which creates a threshold for owner occupation. That is if $H_t > 0$ then $H_t \geq H^*$. But houses of any divisible size above H^* can be purchased.

The asset constraints are

$$M_t \leq \min(\tau_1 p_t H_t, \tau_2 y_t) \tag{4a}$$
$$H_t, A_t, F_t, M_t \geq 0 \tag{4b}$$
$$H_t \geq H^* \text{ if } H_t > 0 \tag{4c}$$

The general Bellman equation is

$$v_t(m_t, S_t) = \max_{c_t, h_t, M_t, H_t, A_t, a_t} u(c_t - \bar{c}, h_t - \overline{h}) + \beta E_t v_{t+1}(m_{t+1}, S_{t+1}), t < T$$

s.t.

$$m_t = c_t + y_{rt}(h_t - H_t) + a_t \tag{5}$$

$$\begin{aligned} m_{t+1} &= y_{t+1} + R_{At+1}A_t + R_{Ft+1}F_t - R_{Mt+1}M_t + R_{Ht+1}p_tH_t \\ &\quad \text{the asset constraints (4)} \end{aligned} \tag{6}$$

Here $v_t(m_t, S_t)$ is the value function, of course implicitly it also depends on the probability distribution of the future uncertain variables $y_{rt+1}, y_{t+1}, R_{it+1}$[5].

2.1 The Reduced Feasible Set

We start each period with decision variables $c_t, h_t, A_t, F_t.M_t, H_t, a_t$. We can write the optimsation problem at t so that given a_t, H_t the remaining decisions only have within period effects. Conditional on a_t, H_t we can find c_t, h_t from optimisation within the current period. Similarly we can characterise the optimal portfolio A_t, F_t, M_t conditional on a_t, H_t by maximising the future value at $t + 1$. Having done this (still for general time additive preferences) we reduce the problem to only two variables a_t, H_t. The underlying financial budget and asset constraints generate a feasible set in a_t, H_t. In this subsection we characterise this feasible set.

Since there is no bequest motive, optimally $m_{T+1} = 0$ and so in the final period it is always better to rent than buy. Also there is no value in saving so $A_T = F_T = H_T = M_T = 0$. At T, the only choices are of c_T, h_T which are chosen within the current budget constraint $c_T + y_{rT}h_T = m_T$ to maximise final period utility.

$$\begin{aligned} &\max_{c_T, h_T} u(c_T - \bar{c}, h_T - \bar{h}) \\ \text{s.t. } c_T + y_{rT}h_T &= m_T \end{aligned}$$

So in the final period the feasible set for a_T, H_T is just the origin. There will be an indirect utility for the final period $u^*(m_T - \bar{c} - y_{rT}\bar{h}, y_{rT})$ measuring the maximum final period utility given $y_{rT}, m_{T.}$

2.2 Generic period t

For a generic period $t<T$ the optimisation problem is

$$v_t(m_t, S_t) = \max_{c_t, h_t, F_t, M_t, H_t, A_t, a_t} u(c_t - \bar{c}, h_t - \bar{h}) + \beta E_t v_{t+1}(m_{t+1}, S_{t+1})$$

s.t.

$$m_t = c_t + y_{rt}(h_t - H_t) + a_t \tag{7}$$

$$m_{t+1} = y_{t+1} + R_{At+1}A_t + R_{Ft+1}F_t - R_{Mt+1}M_t + R_{Ht+1}p_tH_t \tag{8}$$

the asset constraints (4)

[5] Given the properties of $u()$, it is strictly increasing and concave in m_t (Bobenrieth et al., 2012).

Again conditional on a_t and H_t we can determine c_t and h_t to maximise current period utility.

$$\begin{aligned} & \max_{c_T.h_T} u(c_t - \overline{c}, h_t - \overline{h}) \\ \text{s.t. } c_t + y_{rt}h_t &= m_t - a_t + y_{rt}H_t \end{aligned}$$

2.3 Optimal conditional portfolio allocation

At $t < T$ conditional on a_t and H_t, the optimal A_t, F_t, M_t must solve

$$\begin{aligned} & \max_{A_t,F_t,M_t} E_t v_{t+1}(m_{t+1}, S_{t+1}) \\ \Leftrightarrow \quad & \max_{A_t,F_t,M_t} E_t v_{t+1}(y_{t+1} + R_{At+1}A_t + R_{Ft+1}F_t - R_{Mt+1}M_t + R_{Ht+1}p_t H_t, S_{t+1}) \end{aligned}$$

$s.t.$

$$\begin{aligned} M_t &\leq \min(\tau_1 p_t H_t, \tau_2 y_t) \\ A_t, F_t, M_t &\geq 0 \end{aligned}$$

We assume that the covariation of the marginal value of m_{t+1} with the risky asset return exceeds its covariation with either the mortgage rate or the safe asset rate.

$$E_t \frac{\partial v_{t+1}}{\partial m_{t+1}}(R_{Ft+1} - R_{Mt+1}) > 0 \tag{9a}$$

$$E_t \frac{\partial v_{t+1}}{\partial m_{t+1}}(R_{Ft+1}) - R_{At+1} E \frac{\partial v_{t+1}}{\partial m_{t+1}} > 0 \tag{9b}$$

Since $E_t R_{Ft+1} > E_t R_{Mt+1}, E_t R_{Ft+1} > R_{At+1}$ are weak assumptions, the overall assumption (9) holds if $cov(\frac{\partial v_{t+1}}{\partial m_{t+1}}, R_{Ft+1}) > \max(0, cov(\frac{\partial v_{t+1}}{\partial m_{t+1}}, R_{Mt+1}))$. That is variations in the risky rate of return have a bigger impact on the marginal future value than variations in the mortgage rate.

A sufficient condition for this is $R_{Ft+1} > \max(R_{Mt+1}, R_{At+1})$ with probability 1, i.e. always the realised risky return is above the mortgage and the safe rate. Then just so long as $\frac{\partial v_{t+1}}{\partial m_{t+1}} > 0$ with probability 1 (which is a weak assumption) it follows that $E_t \frac{\partial v_{t+1}}{\partial m_{t+1}} R_{Ft+1} > \max(E_t \frac{\partial v_{t+1}}{\partial m_{t+1}} R_{Mt+1}, R_{At+1} E_t \frac{\partial v_{t+1}}{\partial m_{t+1}})$.

Given (9):

(i) optimally $A_t = 0$. If $A_t > 0$ it would raise utility to reduce A_t a little and use this reduction to further invest in the risky asset[6];

(ii) when possible, increasing the mortgage and using the extra funds to invest in the risky asset must raise expected value[7]. This process can continue until M_t reaches its upper bound.

[6] Variations satisfying $dF_t + dA_t - dM_t$ are feasible. So setting $dF_t = -dA_t > 0$ will move to another feasible point which generates higher value.

[7] Similarly if the mortgage is not yet constrained choosing variations $dF_t = -dM_t > 0$ will raise the value but retain feasibility.

Thus conditional on given values of a_t and H_t, the optimum for general preferences has

$$\begin{aligned} M_t &= \min(\tau_1 p_t H_t, \tau_2 y_t) \\ A_t &= 0 \end{aligned}$$

This also means that $F_t = a_t + \min(\tau_1 p_t H_t, \tau_2 y_t) - p_t H_t \geq 0$. Hence we require $p_t H_t - a_t \leq \min(\tau_1 p_t H_t, \tau_2 y_t)$.We must also ensure that subsistence can be attained at t which requires $m_t - \bar{c} - y_{rt}\bar{h} + y_{rt}H_t - a_t \geq 0$. Finally we need $H_t \geq H^*$ if $H_t > 0$. From this we can describe the feasible set in a_t, H_t. Summarising the constraints:

$$\begin{aligned} p_t H_t - a_t &\leq \tau_1 p_t H_t && (10) \\ p_t H_t - a_t &\leq \tau_2 y_t && (11) \\ y_{rt} H_t &\geq a_t - m_t + \bar{c} + y_{rt}\bar{h} && (12) \\ H_t &\geq H^* \text{ if } H_t > 0, a_t \geq 0 \end{aligned}$$

The constraints are all linear inequalities. The full range of patterns for the feasible set is in the Appendix (which also provides a labelling for feasible critical corners which we use in the sequel). In all cases when $H^* = 0$ it is either a polygon or triangle. When any of the constraints bind we are on a linear part of the boundary of the feasible set. H_t must be to the south east of the inequalities (10) and (11) and to the northwest of (12). It must also be either zero or above the H^* line. There are fundamentally two possible forms for the feasible set. When $H^* = 0$, under Condition 1 there are labour income levels (with other exogenous parameters fixed) for which the feasible set contains areas with a LTI and LTV constrained mortgage; in (i) for moderate labour income levels there are both LTI and LTV areas but above some labour income level the LTI constraint never binds, in (ii) for any labour income level there is a LTI region. With Condition 1 the feasible set is polygonal. With Condition 2 there is no LTI ratio constrained feasible region, and so with Condition 2 the feasible set is triangular.

Algebraically conditions 1,2 can be expressed as

Condition 1: either (i) $\frac{m_t - \bar{c} - y_{rt}\bar{h}}{1 - y_{rt}\frac{1}{(1-\tau_1)p_t}} > \frac{\tau_2 y_t (1-\tau_1)}{\tau_1}$ and $1 - y_{rt}\frac{1}{(1-\tau_1)p_t} > 0$

or (ii) $1 - y_{rt}\frac{1}{(1-\tau_1)p_t} < 0$

Condition 2: $\frac{m_t - \bar{c} - y_{rt}\bar{h}}{1 - y_{rt}\frac{1}{(1-\tau_1)p_t}} \leq \frac{\tau_2 y_t (1-\tau_1)}{\tau_1}$ and $1 - y_{rt}\frac{1}{(1-\tau_1)p_t} > 0$

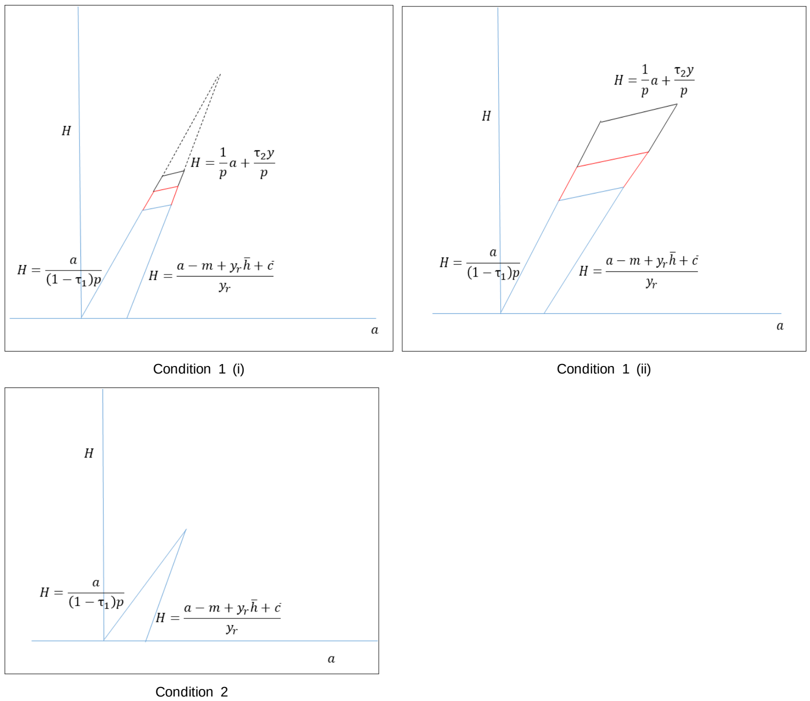

Figure 2: Examples of feasible sets with H*=0 and varying y

Figure 2 shows the examples of feasible sets under Condition 1 (i), Condition 1 (ii) and Condition 2 with $H^* = 0$ and varying y_t. Under Condition 1 (i), the feasible set is a polygon. It can be enlarged by raising y, when y is big enough so that $\frac{m_t-\bar{c}-y_{rt}\bar{h}}{1-y_{rt}\frac{1}{(1-\tau_1)p_t}} = \frac{\tau_2 y_t(1-\tau_1)}{\tau_1}$, the feasible set becomes a triangle. Under Condition 1 (ii), no matter how big y_t is, the feasible set is always a polygon. But the feasible set can also be enlarged by raising y_t. Under Condition 2, the feasible set is a triangle. Raising y_t alone cannot enlarge the feasible set any further and the mortgage will always be constrained by LTV ratio, making the LTI ratio constraint irrelevant. Table 1 gives the a_t, H_t values at each boundary kink.

There is also the effect of the lower bound set by $H^* > 0$ when some housing is purchased. The feasible set must have $H_t \geq H^*$ with housing purchase. But $H_t = 0$ is always a viable option. Thus the part of the axis $H_t = 0$ with $a_t \in [0, m_t - \bar{c} - y_{rt}\bar{h}]$ is also feasible thus making the overall feasible set non-convex. In all cases the feasible set has boundary kinks.

Table 1: Values of net worth and housing investment (at kinks) and corresponding H* for general preference with subsistence level

Kink	H^*	C1,C2	a_t	H_t
1	any	any	$a_{1t} = 0$	$H_{1t} = 0$
2	$[0, H_{2t}]$	2	$a_{2t} = \frac{m_t - \bar{c} - y_{rt}\bar{h}}{1 - y_{rt}\frac{1}{(1-\tau_1)p_t}}$	$H_{2t} = \frac{1}{p_t(1-\tau_1)} \frac{m_t - \bar{c} - y_{rt}\bar{h}}{(1 - y_{rt}\frac{1}{(1-\tau_1)p_t})}$
3	$[0, H_{3t}]$	1	$a_{3t} = \frac{\tau_2 y_t(1-\tau_1)}{\tau_1}$	$H_{3t} = \frac{1}{p_t}\frac{\tau_2 y_t}{\tau_1}$
4	$[0, H_{4t}]$	1	$a_{4t} = \frac{m_t - \bar{c} - y_{rt}\bar{h} + \frac{y_{rt} y_t \tau_2}{p_t}}{1 - \frac{y_{rt}}{p_t}}$	$H_{4t} = \frac{m_t - \bar{c} - y_{rt}\bar{h} + \tau_2 y_t}{p_t - y_{rt}}$
5	any	any	$a_{5t} = m_t - \bar{c} - y_{rt}\bar{h}$	$H_{5t} = 0$
6	$[0, H_{3t}]$	1	$a_{6t} = m_t - \bar{c} - y_{rt}\bar{h} + \frac{\tau_2 y_{rt} y_t}{\tau_1 p_t}$	$H_{6t} = \frac{\tau_2 y_t}{\tau_1 p_t}$
7	(H_{3t}, H_{4t})	1	$a_{7t} = p_t H^* - \tau_2 y_t$	$H_{7t} = H^*$
8	(H_{3t}, H_{4t})	1	$a_{8t} = H^* y_{rt} + m_t - \bar{c} - y_{rt}\bar{h}$	$H_{8t} = H^*$
9	$(0, H_{2t})$	2	$a_{9t} = H^* p_t (1 - \tau_1)$	$H_{9t} = H^*$
10	$(0, H_{2t})$	2	$a_{10t} = H^* y_{rt} + m_t - \bar{c} - y_{rt}\bar{h}$	$H_{10t} = H^*$

For any point in the feasible set there will be corresponding optimal c_t, h_t, F_t, M_t and of course $A_t = 0$. Then knowing H_t and h_t we can determine if there is buy to let or if a house owner rents additional housing. In general there is a downward sloping buy to let locus along which optimally $h_t = H_t$ if consumption and housing are normal goods [8]. The intercept of this locus along the the east side of the boudary of the feasible set defined by (12) with maximal a_t is at $a_t = m_t - \bar{c}, H_t = h_t = \bar{h}$. This gives us some general properties of possible optimal behaviour in housing. First if $\bar{h} \geq H_{4t}$ (Condition 1) or $\bar{h} \geq H_{2t}$ (Condition 2) the household does not buy to let, and an increase in $\bar{h}$ reduces the feasible area involving BTL (allowing for the dependence of the right hand boundary of the feasible set on $\bar{h}$). Second if $H_{4t} > \bar{h} \geq H_{3t}$ then the whole BTL locus falls in the LTI mortgage part of the feasible set and hence any BTL individual finances his house purchase with a LTI mortgage. If $\bar{h} < H_{3t}$ an individual who has BTL positive may finance housing with an LTI or LTV mortgage.

In general there are various factors affecting the decision to buy to let:

(i) The amount of cash on hand which is one determinant of the amount of net worth at t.

(ii) The allocation of savings between housing and the risky asset which depends on the relative value of their marginal returns but also on the cost/income of rental per unit of space.

(iii) The rate of time preference which is an important determinant of the amount of savings.

(iv) The relative preference for housing for general consumption and housing service consumption

[8] For the current period and conditional on a_t and H_t, write the solution for h_t as $h_t = f(w_t, y_{rt})$ where $w_t = m_t + y_{rt}H_t - a_t - \bar{c} - y_{rt}\bar{h}$. The buy to let locus satisfies $H_t = f(w_t, y_{rt})$. This gives us a locus in a_t, H_t space whose slope is defined by

$$\frac{dH_t}{da_t} = -\frac{\partial f/dw_t}{1 - y_{rt}\partial f/dw_t}$$

If both c_t, h_t are normal goods then $\partial f/dw_t > 0$ and also $\partial c_t/dw_t > 0$. Since $dc_t/dw_t + y_{rt}dH_t/dw_t = 1$ this means that $1 - y_{rt}\partial f/dw_t > 0$ and hence the zero BTL locus is downward sloping; it also requires $H_t = 0$ when $a_t = m_t - \bar{c} - y_{rT}\bar{h}$.

at t.

The exact tradeoff of these factors depends on the form of preferences. Without specialising preferences we cannot go further in either generating closed form life cycle paths or simulating actual realised optimal life cycle paths. We also cannot clearly distinguish the solutions in which buy to let occurs. So next we specialise preferences.

3 Closed form solutions for special preferences

Cocco's within period preferences (2005) are $\frac{(c_t^{1-\theta} h_t^{\theta})^{1-\gamma}}{1-\gamma}, 0 < \theta < 1, \gamma < 1$ and are homothetic. Generally these do not yield closed form solutions for the value function. But if we specialise the form to Stone-Geary $u(c_t - \overline{c}, h_t - \overline{h}) = (c_t - \overline{c})^{1-\rho}(h_t - \overline{h})^{\rho}$ we can derive closed form solutions, essentially because the utility function is homogeneous of degree one in translated c_t, h_t[9].

For these preferences we derive the value function by backward induction: assume a form for the value function $v_{t+1}(m_{t+1})$, use Bellmans equation to solve the problem at t and then verify that the value function does indeed have the assumed form. We conjecture that the value function at t is linear in m_t for $t \leq T$

$$v_t(m_t, S_t) = B_{1t} m_t + B_{2t}$$

where both B_{1t}, B_{2t} are realisations at t of random functions. First we consider the final period and then recursively move backward in time.

3.1 The Final Period

From Section 2 we have seen that renting is best in the final period to allow all resources to be consumed. Hence in the final period T, the only choices are of c_T, h_T which are chosen within the budget constraint $c_T + y_{rT} h_T = m_T$ to maximise final period utility.

$$\begin{aligned} & \max_{c_T . h_T} (c_t - \overline{c})^{1-\rho}(h_t - \overline{h})^{\rho} \\ \text{s.t. } c_T + y_{rT} h_T &= m_T \end{aligned}$$

This yields

$$\begin{aligned} c_T &= \overline{c} + (1-\rho)(m_T - \overline{c} - y_{rT}\overline{h}) \\ h_T &= \overline{h} + \rho \frac{(m_T - \overline{c} - y_{rT}\overline{h})}{y_{rT}} \end{aligned}$$

and there is a final period value function

$$\begin{aligned} v_T(m_T, S_T) &= (1-\rho)^{(1-\rho)}(\rho / y_{rT})^{\rho}(m_T - \overline{c} - y_{rT}\overline{h}) \\ &= B_{1T} m_T + B_{2T} \end{aligned}$$

which is certainly linear in m_T with $B_{1T} = (1-\rho)^{(1-\rho)}(\rho/y_{rT})^{\rho}, B_{2T} = -(1-\rho)^{(1-\rho)}(\rho/y_{rT})^{\rho}(\overline{c} + y_{rT}\overline{h})$

[9] In this case the within period Inada conditions hold at the origin but there is only a finite loss at t in having zero cash on hand.

3.2 Generic Period t

From Section 2.3 optimally $A_t = 0, M_t = \min(\tau_1 p_t H_t . \tau_2 y_t), F_t = a_t + M_t - p_t H_t$. Also conditionally on a_t and H_t optimally

$$
\begin{aligned}
c_t &= \bar{c} + (1-\rho)(m_t - a_t - \bar{c} - y_{rt}\bar{h} + y_{rt}H_t) \\
h_t &= \bar{h} + \frac{\rho}{y_{rt}}(m_t - a_t - \bar{c} - y_{rt}\bar{h} + y_{rt}H_t)
\end{aligned}
$$

and a current period indirect utility of $(1-\rho)^{(1-\rho)}(\frac{\rho}{y_{rt}})^{\rho}(m_t + y_{rt}H_t - y_{rt}\bar{h} - a_t - \bar{c})$.

Assume that the value function is linear at $t+1 : v_{t+1}(m_{t+1}, S_{t+1}) = B_{1t+1}m_{t+1} + B_{2t+1}$ with $B_{1t+1} > 0$.

The problem at t is

$$
\begin{aligned}
v_t &= \max_{a_t, H_t} V_t(a_t, H_t) \\
&= \max_{a_t, H_t}[(1-\rho)^{(1-\rho)}(\frac{\rho}{y_{rt}})^{\rho}(m_t + y_{rt}H_t - y_{rt}\bar{h} - a_t - \bar{c}) \\
&\quad +\beta E B_{1t+1}(y_{t+1} + R_{Ft+1}(\min\{\tau_1 p_t H_t, \tau_2 y_t\} - p_t H_t + a_t) - R_{Mt+1}\tau_1 p_t H_t + p_{t+1}H_t) + \beta E B_{2t+1}]
\end{aligned}
$$

$$
\begin{aligned}
\text{s.t. } a_t &\geq 0, H_t \geq H^* \text{ or } H_t = 0 \\
y_{rt}H_t &\geq a_t - m_t + \bar{c} + y_{rt}\bar{h}
\end{aligned}
$$

Let V_{1t} be the objective when $H_t \geq H^*$ and $\tau_1 p_t H_t < \tau_2 y_t$ (at any point in the LTV mortgage constrained feasible region), and V_{2t} the objective when $H_t \geq H^*$ and $\tau_1 p_t H_t > \tau_2 y_t$ (at any point in the LTI mortgage constrained feasible region). Along the common boundary between the LTI and LTV areas, we have $\tau_1 p_t H_t = \tau_2 y_t$ and so at any point (a_t, H_t) on this boundary $V_{1t} = V_{2t}$.

The derivatives of the objective function at t are

$$
\begin{aligned}
\frac{\partial V_{1t}(a_t, H_t)}{\partial a_t} &= \frac{\partial V_{2t}(a_t, H_t)}{\partial a_t} \\
&= \beta E B_{1t+1} R_{Ft+1} - (1-\rho)^{1-\rho}\rho^{\rho} y_{rt}^{-\rho} \\
\frac{\partial V_{1t}(a_t, H_t)}{\partial H_t} &= (1-\rho)^{1-\rho}\rho^{\rho} y_{rt}^{1-\rho} \\
&\quad + p_t \beta E_t B_{1t+1}(R_{Ht+1} - R_{Ft+1} + \tau_1(R_{F,t+1} - R_{M,t+1})) \\
\frac{\partial V_{2t}(a_t, H_t)}{\partial H_t} &= (1-\rho)^{1-\rho}\rho^{\rho} y_{rt}^{1-\rho} + p_t \beta E_t B_{1t+1}(R_{Ht+1} - R_{Ft+1}) < \frac{\partial V_{1t}(a_t, H_t)}{\partial H_t}
\end{aligned}
$$

All these are constants independent of a_t, H_t so the indifference curves within each of the LTI and LTV areas are linear. But the signs of all the derivatives are ambiguous. The indifference curves have slopes $MRS_{it} = -\partial V_{it}(a_t, H_t)/\partial a_t / \partial V_{it}(a_t, H_t)/\partial H_t$ $(i = 1, 2)$.so in the LTI mortgage constrained part of the feasible set the indifference curves have smaller slope than in the LTV mortgage constrained area of the feasible set.

There are only three possible rankings of $\frac{\partial V_{1t}(a_t,H_t)}{\partial H_t}$, $\frac{\partial V_{2t}(a_t,H_t)}{\partial H_t}$, 0. i.e.

(1) $\frac{\partial V_{1t}(a_t,H_t)}{\partial H_t} > \frac{\partial V_{2t}(a_t,H_t)}{\partial H_t} > 0$

(2) $0 > \frac{\partial V_{1t}(a_t,H_t)}{\partial H_t} > \frac{\partial V_{2t}(a_t,H_t)}{\partial H_t}$

(3) $\frac{\partial V_{1t}(a_t,H_t)}{\partial H_t} > 0 > \frac{\partial V_{2t}(a_t,H_t)}{\partial H_t}$

Combiniing these with the sign of $\frac{\partial v_t(a_t,H_t)}{\partial a_t}$, there are six different shapes of indifference curve. The arrows in the Figure 3 show the direction of increase of utility.

(Panel A) $\frac{\partial v_t(a_t,H_t)}{\partial a_t} > 0, \frac{\partial V_{1t}(a_t,H_t)}{\partial H_t} > \frac{\partial V_{2t}(a_t,H_t)}{\partial H_t} > 0 \Longrightarrow 0 > MRS_{1t} > MRS_{2t}$

(Panel B) $\frac{\partial v_t(a_t,H_t)}{\partial a_t} < 0, 0 > \frac{\partial V_{1t}(a_t,H_t)}{\partial H_t} > \frac{\partial V_{2t}(a_t,H_t)}{\partial H_t} \Longrightarrow 0 > MRS_{2t} > MRS_{1t}$

(Panel C) $\frac{\partial v_t(a_t,H_t)}{\partial a_t} > 0, 0 > \frac{\partial V_{1t}(a_t,H_t)}{\partial H_t} > \frac{\partial V_{2t}(a_t,H_t)}{\partial H_t} \Longrightarrow MRS_{1t} > MRS_{2t} > 0$

(Panel D) $\frac{\partial v_t(a_t,H_t)}{\partial a_t} < 0, \frac{\partial V_{1t}(a_t,H_t)}{\partial H_t} > \frac{\partial V_{2t}(a_t,H_t)}{\partial H_t} > 0 \Longrightarrow 0 < MRS_{1t} < MRS_{2t}$

(Panel E) $\frac{\partial v_t(a_t,H_t)}{\partial a_t} > 0, \frac{\partial V_{1t}(a_t,H_t)}{\partial H_t} > 0 > \frac{\partial V_{2t}(a_t,H_t)}{\partial H_t} \Longrightarrow MRS_{1t} < 0 < MRS_{2t}$

(Panel F) $\frac{\partial v_t(a_t,H_t)}{\partial a_t} < 0, \frac{\partial V_{1t}(a_t,H_t)}{\partial H_t} > 0 > \frac{\partial V_{2t}(a_t,H_t)}{\partial H_t} \Longrightarrow MRS_{1t} > 0 > MRS_{2t}$

Since the indifference curves are linear then if the slope of the indifference curves are never equal to the slope of any part of the boundary of the feasible set, all solutions must be at one of the corners of the boundary of the feasible set. Which corner depends on the derivatives of V_{1t}, V_{2t} and the value of H_t^*. Figure 3 shows the possible indifference curve configurations.

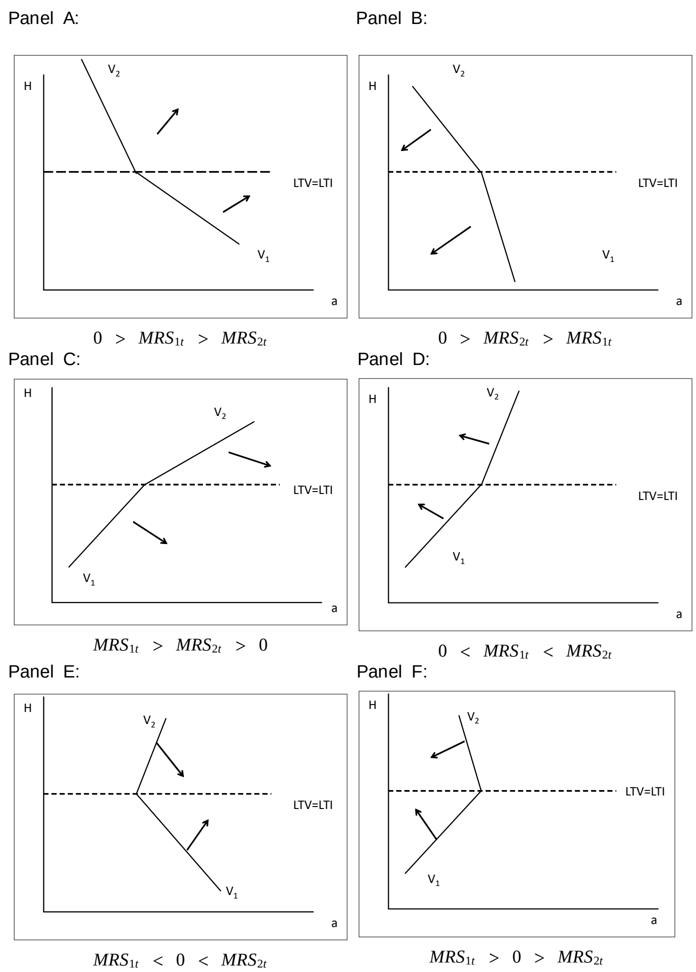

Figure 3: Examples of indifference curves at switchpoints

The pattern of preferences determines which kink will be optimal.

From Table 1 we see that at each kink the values of a_t, H_t are linear functions of m_t. Hence for any slope of the indifference curves, the solutions for a_t, H_t (ranging through the optimal kinks for each indifference curve slope configuration) are linear in m_t. We can characterise the conditions on the feasible set and parameters of preferences under which each feasible corner kink is optimal[10]. This involves comparing the

[10] With $H_t^* > 0$, the feasible set of a_t, H_t is not convex and consists of the union of a convex set with nonempty interior and part of the line segment ($H_t = 0$). But we rule out the cases when the indifference curves coincide with lines that join kink points on the upper half of the feasible and the end points of the feasible part of the line segment where $H_t = 0$. We could allow this but then for particular parameters there could be an infinite number of possible optima, one at each point of the relevant part of the feasible set boundary.

constant MRS with the slopes of the boundary segments of the different feasible set configurations and the direction of increase of utility. The conditions define if the feasible set is a polygon or triangle, (i.e. conditions C1 and C2) the relative preference and constraint slopes and finally the value of H^*. For each indifference curve slope configuration, replacing a_t, H_t by their optimal values yields a value function at t which is linear in m_t

$$v_t(m_t, S_t) = B_{1t}m_t + B_{2t.}$$

In each regime, irrespective of H^* the value function at t<T is linear in m_t. Hence by backward recursion, this verifies our conjecture that the value function at $t+1$ is linear in m_{t+1}

$$E_t v_{t+1}(m_{t+1}, S_{t+1}) = E_t[B_{1t+1}m_{t+1} + B_{2t+1}]$$

Theorem 1 *For the preference* $u(c_t - \overline{c}, h_t - \overline{h}) = (c_t - \overline{c})^{1-\rho}(h_t - \overline{h})^{\rho}$ *,if Assumption (9) holds, then the value function is linear in cash on hand, i.e.* $v_t(m_t, S_t) = B_{1t}(S_t)m_t + B_{2t}(S_t)$.

Note that B_{1t+1}, B_{2t+1} have forms varying with the optimal future kink.

Since the indifference curves are piecewise linear with slopes independent of labour income, the optimal kink does not vary with household labour income. i.e. if a low labour income household finds kink 3 best, then so does a higher labour income household so long as the shape of the feasible set does not change. Under Condition 1 (i) a large enough rise in labour income can switch the feasible set from a polygon to a triangle; in this case the high labour income family prefers kink 2 but the lower labour income household can prefer kink 3, kink 4 or kink 6. This implies the optimal a_t, H_t are monotonic in y_t.

For each kink Table 2 shows the conditions on H^* and the patterns of preferences, making each kink optimal. In some cases, the pattern of preferences has to have restricted slopes in relation to the slopes of the boundaries of the feasible set.

Table 2: Conditions for corner solutions (at kinks) 1-8 with Stone Geary preferences

Corner	condition	preference pattern	H_t^*	MRS
1	any	B	any	
1	C1	B,D,F	$H_t^* > H_{4t}$	
1	C2	B,D,F	$H_t^* > H_{2t}$	
1	C2	D,F	$0 \leq H^* \leq H_{2t}$	$MRS_{1t}(0,0) > \frac{1}{(1-\tau_1)p_t}$
1	C1	D,F	$0 \leq H^* \leq H_{3t}$	$MRS_{1t}(0,0) > \frac{1}{(1-\tau_1)p_t}$
2	C2	A,E	$0 \leq H^* \leq H_{2t}$	
2	C2	D,F	$0 \leq H^* \leq H_{2t}$	$MRS_{1t}(a_{2t}, H_{2t}) < \frac{1}{(1-\tau_1)p_t}$
2	C2	C	$0 \leq H^* \leq H_{2t}$	$MRS_{1t}(a_{2t}, H_{2t}) > \frac{1}{y_{rt}}$
3	C1	F	$0 \leq H^* \leq H_{3t}$	
3	C1	D	$0 \leq H^* \leq H_{3t}$	$MRS_{1t}(a_{3t}, H_{3t}) < \frac{1}{p_t} < \frac{1}{(1-\tau_1)p_t}$ and $MRS_{2t}(a_{3t}, H_{3t}) > \frac{1}{(1-\tau_1)p_t} > \frac{1}{p_t}$
4	C1	A	$0 \leq H^* \leq H_{4t}$	
4	C1	D	$0 \leq H^* \leq H_{4t}$	$MRS_{2t}(a_{4t}, H_{4t}) < \frac{1}{p_t}$
4	C1	C,E	$0 \leq H^* \leq H_{4t}$	$MRS_{2t}(a_{4t}, H_{4t}) < \frac{1}{y_{rt}}$
5	any	C	any	$MRS_{1t}(a_{5t}, H_{5t}) < \frac{1}{y_{rt}}$
5	C1	A,E	$H^* > H_{4t}$	
5	C2	A,E	$H^* > H_{2t}$	
6	C1	C	$0 \leq H^* \leq H_{3t}$	$MRS_{1t}(a_{6t}, H_{6t}) > \frac{1}{y_{rt}}$ and $MRS_{2t}(a_{6t}, H_{6t}) < \frac{1}{y_{rt}}$
6	C1	E	$0 \leq H^* \leq H_{3t}$	$MRS_{2t}(a_{6t}, H_{6t}) < \frac{1}{y_{rt}}$
7	C1	D	$H_{3t} < H^* < H_{4t}$	$MRS_{2t}(a_{7t}, H_{7t}) < \frac{H^*}{p_t H_7 - \tau_2 y_t}$
8	C1	C	$H_{3t} < H^* < H_{4t}$	$MRS_{2t}(a_{8t}, H_{8t}) < \frac{1}{y_{rt}}$

The value functions $v_t(m_t, S_t)$ are in Table 3 and the optimal values of the choice variables are in Table 4.

Table 3: Value functions for different regimes for Stone Geary preferences

Regime	B_{1t}	B_{2t}	v_t
1	$(1-\rho)^{1-\rho}(\frac{\rho}{y_{rt}})^{\rho}$	$(1-\rho)^{1-\rho}(\frac{\rho}{y_{rt}})^{\rho}(-\bar{c}-y_{rt}\bar{h})$ $+\beta E_t(B_{1t+1}y_{t+1}+B_{2t+1})$	$(1-\rho)^{1-\rho}(\frac{\rho}{y_{rt}})^{\rho}(m_t-\bar{c}-y_{rt}\bar{h})$ $+\beta E_t(B_{1t+1}y_{t+1}+B_{2t+1})$
2	$\beta E_t B_{1t+1}([\frac{(R_{Ht+1}-\tau_1 R_{Mt+1})}{1-\tau_1}]$ $\frac{1}{(1-y_{rt}\frac{1}{(1-\tau_1)p_t})})$	$\beta E_t[B_{1t+1}(y_{t+1}+[\frac{(R_{Ht+1}-\tau_1 R_{Mt+1})}{1-\tau_1}]$ $\frac{-\bar{c}-y_{rt}\bar{h}}{(1-y_{rt}\frac{1}{(1-\tau_1)p_t})})+B_{2t+1}]$	$\beta E_t[B_{1t+1}(y_{t+1}+[\frac{(R_{Ht+1}-\tau_1 R_{Mt+1})}{1-\tau_1}]$ $\frac{m_t-\bar{c}-y_{rt}\bar{h}}{(1-y_{rt}\frac{1}{(1-\tau_1)p_t})})+B_{2t+1}]$
3	$(1-\rho)^{1-\rho}(\frac{\rho}{y_{rt}})^{\rho}$	$(1-\rho)^{1-\rho}(\frac{\rho}{y_{rt}})^{\rho}$ $[-\bar{c}-y_{rt}\bar{h}--\frac{\tau_2 y_t(1-\tau_1)}{\tau_1}(1-\frac{y_{rt}}{p_t(1-\tau_1)})]$ $+\beta E_t[B_{1t+1}(y_{t+1}+\frac{R_{Ht+1}-\tau_1 R_{Mt+1}}{1-\tau_1}$ $\frac{\tau_2 y_t(1-\tau_1)}{\tau_1})+B_{2t+1}]$	$(1-\rho)^{1-\rho}(\frac{\rho}{y_{rt}})^{\rho}[m_t-\bar{c}-y_{rt}\bar{h}-$ $\frac{\tau_2 y_t(1-\tau_1)}{\tau_1}(1-\frac{y_{rt}}{p_t(1-\tau_1)})]$ $+\beta E_t[B_{1t+1}(y_{t+1}+\frac{R_{Ht+1}-\tau_1 R_{Mt+1}}{1-\tau_1}$ $\frac{\tau_2 y_t(1-\tau_1)}{\tau_1})+B_{2t+1}]$
4	$\beta E_t[B_{1t+1}(R_{Ht+1}\frac{1}{(1-\frac{y_{rt}}{p_t})}]$	$\beta E_t[B_{1t+1}(y_{t+1}+R_{Ht+1}\frac{-\bar{c}-y_{rt}\bar{h}+\frac{\tau_2 y_t y_{rt}}{p_t}}{(1-\frac{y_{rt}}{p_t})}$ $+(R_{Ht+1}-R_{Mt+1})\tau_2 y_t)+B_{2t+1}]$	$\beta E_t[B_{1t+1}(y_{t+1}+R_{Ht+1}\frac{m_t-\bar{c}-y_{rt}\bar{h}+\frac{\tau_2 y_t y_{rt}}{p_t}}{(1-\frac{y_{rt}}{p_t})}$ $+(R_{Ht+1}-R_{Mt+1})\tau_2 y_t)+B_{2t+1}]$
5	$\beta E_t(B_{1t+1}R_{Ft+1})$	$\beta E_t[B_{1t+1}(y_{t+1}+R_{Ft+1}(-\bar{c}-y_{rt}\bar{h}))+B_{2t+1}]$	$\beta E_t[B_{1t+1}(y_{t+1}+R_{Ft+1}(m_t$ $-\bar{c}-y_{rt}\bar{h}))+B_{2t+1}]$
6	$\beta E_t(B_{1t+1}R_{Ft+1})$	$\beta E_t[B_{1t+1}(y_{t+1}+R_{Ft+1}(-\bar{c}-y_{rt}\bar{h}$ $+\frac{\tau_2 y_{rt} y_t}{\tau_1 p_t}-\frac{\tau_2 y_t(1-\tau_1)}{\tau_1})+$ $R_{Mt+1}(-\tau_2 y_t)+$ $R_{Ht+1}\frac{\tau_2 y_t}{\tau_1})+B_{2t+1}]$	$\beta E_t[B_{1t+1}(y_{t+1}+R_{Ft+1}(m_t-\bar{c}-y_{rt}\bar{h}$ $+\frac{\tau_2 y_{rt} y_t}{\tau_1 p_t}-\frac{\tau_2 y_t(1-\tau_1)}{\tau_1})+$ $R_{Mt+1}(-\tau_2 y_t)+$ $R_{Ht+1}\frac{\tau_2 y_t}{\tau_1})+B_{2t+1}]$
7	$(1-\rho)^{1-\rho}(\frac{\rho}{y_{rt}})^{\rho}$	$(1-\rho)^{1-\rho}(\frac{\rho}{y_{rt}})^{\rho}(-\bar{c}-y_{rt}\bar{h}+y_{rt}H_t^*-$ $p_t H_t^*+\tau_2 y_t)+\beta E_t[B_{1t+1}(y_{t+1}$ $+R_{Ht+1}H_t^*-R_{Mt+1}\tau_2 y_t)+B_{2t+1}]$	$(1-\rho)^{1-\rho}(\frac{\rho}{y_{rt}})^{\rho}(m_t-\bar{c}-y_{rt}\bar{h}+$ $y_{rt}H_t^*-p_t H_t^*+\tau_2 y_t)$ $+\beta E_t[B_{1t+1}(y_{t+1}+R_{Ht+1}p_t H_t^*-$ $R_{Mt+1}\tau_2 y_t)+B_{2t+1}]$
8	$\beta E_t(B_{1t+1}R_{Ft+1})$	$\beta E_t[B_{1t+1}(y_{t+1}+R_{Ht+1}p_t H_t^*+$ $R_{Ft+1}(H_t^*(y_{rt}-p_t)-\bar{c}-y_{rt}\bar{h}$ $+\tau_2 y_t)-R_{Mt+1}\tau_2 y_t)+B_{2t+1}]$	$\beta E_t[B_{1t+1}(y_{t+1}+R_{Ht+1}p_t H_t^*+$ $R_{Ft+1}(H_t^*(y_{rt}-p_t)+m_t$ $-\bar{c}-y_{rt}\bar{h}$ $+\tau_2 y_t)-R_{Mt+1}\tau_2 y_t)+B_{2t+1}]$

Table 4: The solutions for different regimes for Stone Geary Preferences

Regime	a_t	H_t	F_t	M_t	c_t	h_t
1	a_{1t}	H_{1t}	0	0	$\overline{c}+(1-\rho)(m_t-\overline{c}-y_{rt}\overline{h})$	$\overline{h}+\frac{\rho}{y_{rt}}(m_t-\overline{c}-y_{rt}\overline{h})$
2	a_{2t}	H_{2t}	0	$\frac{\tau_1}{1-\tau_1}\frac{m_t-\overline{c}-y_{rt}\overline{h}}{(1-y_{rt}\frac{1}{(1-\tau_1)p_t})}$	$\overline{c}$	$\overline{h}$
3	a_{3t}	H_{3t}	0	$\tau_2 y_t$	$\overline{c}+(1-\rho)(m_t- \frac{\tau_2 y_t(1-\tau_1)}{\tau_1}+\frac{y_{rt}y_t\tau_2}{p_t\tau_1} -\overline{c}-y_{rt}\overline{h})$	$\overline{h}+\frac{\rho}{y_{rt}}(m_t- \frac{\tau_2 y_t(1-\tau_1)}{\tau_1}+\frac{y_{rt}y_t\tau_2}{p_t\tau_1} -\overline{c}-y_{rt}\overline{h})$
4	a_{4t}	H_{4t}	0	$\tau_2 y_t$	$\overline{c}$	$\overline{h}$
5	a_{5t}	H_{5t}	$m_t-\overline{c}-y_{rt}\overline{h}$	0	$\overline{c}$	$\overline{h}$
6	a_{6t}	H_{6t}	$m_t-\overline{c}-y_{rt}\overline{h}-(1-\tau_1-\frac{y_{rt}}{p_t})\frac{\tau_2 y_t}{\tau_1}$	$\tau_2 y_t$	$\overline{c}$	$\overline{h}$
7	a_{7t}	H_{7t}	0	$\tau_2 y_t$	$\overline{c}+ (m_t+y_{rt}H_t^*-\overline{c}-y_{rt}\overline{h} -p_tH_t^*+\tau_2 y_t)(1-\rho)$	$\overline{h}+ (m_t+y_{rt}H_t^*-\overline{c}-y_{rt}\overline{h} -p_tH_t^*+\tau_2 y_t))\frac{\rho}{y_{rt}}$
8	a_{8t}	H_{8t}	$H_t^*(y_{rt}-p_t)+m_t-\overline{c}-y_{rt}\overline{h}+\tau_2 y_t$	$\tau_2 y_t$	$\overline{c}$	$\overline{h}$

3.3 Buy to let or rent?

From this we can deduce the explicit form of the BTL=0 locus. It is linear:

$$H_t = h_t = \overline{h} + \frac{\rho}{(1-\rho)y_{rt}}(m_t - a_t - \overline{c})$$

This only applies when $H_t > 0$ and $H^* \leq H_4$ (since otherwise the household cannot afford to purchase any available housing).

Below the locus the homeowner rents additional space; above the locus there is buy to let activity. With Stone Geary preferences we can also derive comparisons between households with varying exogenous variables. The absolute value of the slope of the locus falls with ρ and rises with y_{rt} The intercept rises with $m_t - \overline{c}, \overline{h}, \rho$ and falls with y_{rt}. Hence in addition to the results for general preferences, as ρ falls, the intercept of the locus falls and the slope becomes flatter so the chance of BTL rises. If $H^* \leq H_{3t}$ there will be a critical value of ρ under condition 1[11] which ensures that H_{3t} is on the BTL locus. This ρ is given by

$$\rho_3 = \frac{y_{rt}(\overline{h}p_t\tau_1 - \tau_2 y_t)}{y_{rt}(\overline{h}p_t\tau_1 - \tau_2 y_t) + (-p_t\tau_1\tau_2 + p_t\tau_2)y_t + \overline{c}p_t\tau_1 - m_t p_t\tau_1}$$

If $\rho < \rho_3$ then there is BTL at kinks 3, 4, 6 under condition 1. If preference for housing consumption falls, it is more likely that the optimium will involve BTL (Figure 4).

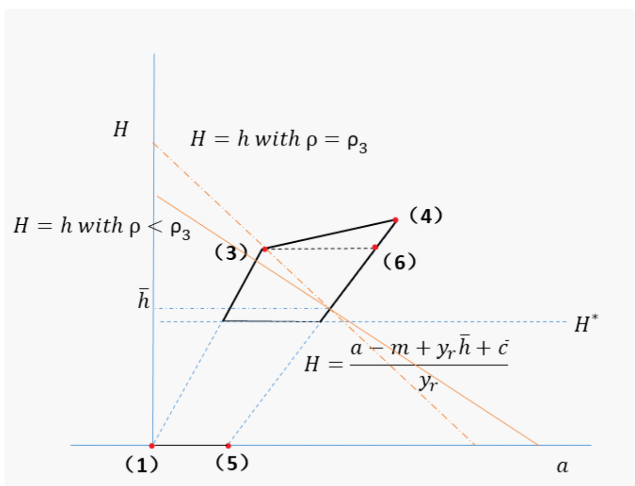

Figure 4: Examples of feasible sets and BTL locus with different rho's

Another example is variation in labour income[12]. This affects the polygonal feasible set and especially the kinks 3,4,6 but not the BTL locus; it can also cause the feasible set to switch from a polygon to a

[11] With other parameters constant.

[12] Although y_t enters m_t we keep m_t constant by an offsetting change in the asset income of the households.

triangle as labour income rises far enough. The Figure 5 assumes $H^* = 0$ and shows three feasible sets for varying y_t and a single BTL locus common to the three households. As y_t rises. the slopes of the boundary segments are unaffected, so are the positions of the BTL locus and the righthand boundary of the feasible set. But the LTI part of the lefthand boundary rises in a verical direction causing kinks (3),(4),(6) to move away from the origin. Starting with the feasible set defining kinks (3'),(4'),(6') the household just consumes all the housing owned at kink (4') with a LTI mortgage and also zero investment in the risky asset. At kinks (3') or (6') the household is a net renter. An increase in y_t leads to an enlargement of the feasible set and kinks at (3"),(4"),(6"). Note that kink (6") is on the BTL locus and also with exactly the same house purchase as the lower income household at (4'). The extra labour income in this case is all invested in the risky asset; the household can raise the funds for financing housing with a LTV or LTI mortgage. A further labour income increase gives the feasible set with kink (3"'),(4"'),(6"'). This household is on the BTL locus at kink (3"') with zero risky investment and again can raise the same funds from either a LTV or LTI mortgage. Both the two upper income households could choose to buy to let if they wished but this choice is unavailable to the lowest income household. As income continues increasing eventually the point is reached at which $\frac{m_t-\bar{c}-y_{rt}\bar{h}}{1-y_{rt}\frac{1}{(1-\tau_1)p_t}} = \frac{\tau_2 y_t(1-\tau_1)}{\tau_1}$ and the LTI area of the feasible set disappears leading to the triangle.

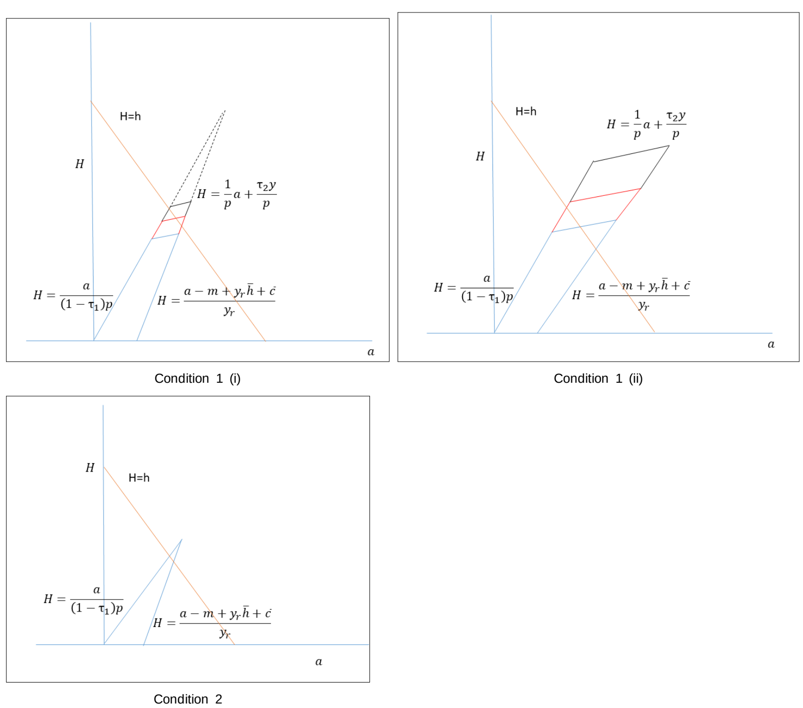

Figure 5: Examples of feasible sets and BTL locus with H*=0 and and varying y

The example shows various things:

(i) under condition 1, with a fixed BTL locus, as income increases, more of the kinks are likely to be in the BTL area.

(ii) kinks (3),(4),(6) each show higher H_t and a_t as labour income increases. If kink 3 has BTL positive then so do kink 6 and kink 4. Similarly, if kink 7 has BTL then so do kinks 4 and 8 (Figure 6).

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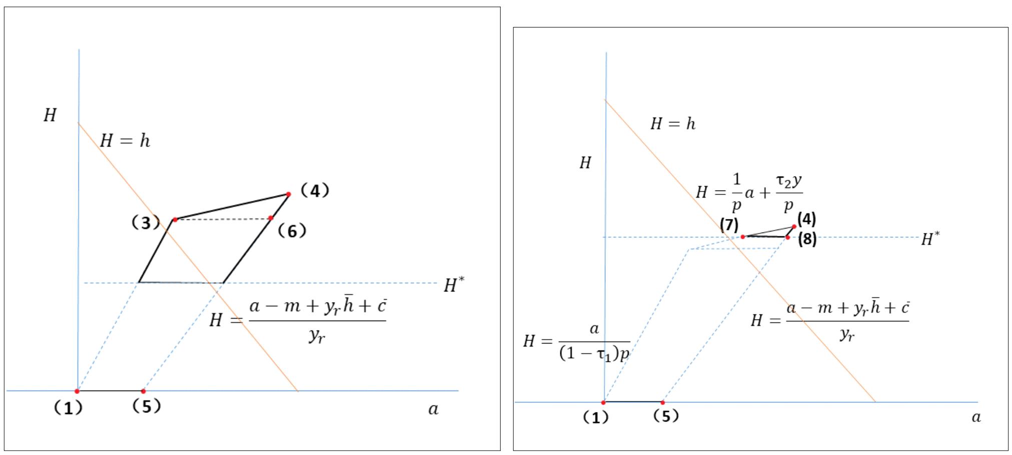

Figure 6: Examples of feasible sets and BTL locus

4 Simulation of the Stone Geary preference model

We simulate behaviour for heterogeneous households. Heterogeneity is reflected in two respects. First, there are different expectations and realisations of labour income among individuals. To be specific, we allow the labour income process to vary by individual education and age. In addition, even for two individuals with the same age and education their realisations will vary. Second, we let initial cash on hand at the start of life differ among people. Other than these two aspects, people are identical; they have the same preferences and share the same expectations and realisation of asset returns, house prices. The planning horizon is also the same for everyone in each period.

For one realisation of the aggregate shocks[13], we generate realisations for the shocks to the labour income process for 100 individuals who differ in education qualification and initial cash on hand. Then we compute the optimal consumption and investment decisions for these 100 individuals. We repeat this process for 80 paths for the aggregate variables, each path with 100 individuals. This gives 8000 different paths in total for each of the four sets of calibrated parameters (models (i)-(iv) as shown below).

[13] Here aggregate shocks include the shocks for house prices, rental, mortgage interest rate and risky asset return, which are assumed to be common for everyone.

4.1 Stochastic processes

We assume the five sources of uncertainty (the risky asset return, the mortgage rate, house prices, rental and labour income) are mutually independent. There is always a portfolio which has a zero probability of default when the distributions of gross asset returns and income variables have lognormal distributions.

We select stochastic processes for each of them based on UK data (see Table 5). The house price follows a random walk with a deterministic upward trend ($\gamma > 0$).

$$\ln(p_t) = \gamma + \ln(p_{t-1}) + \varepsilon_{pt}$$

where $\varepsilon_{pt} \tilde{} N(0, \sigma^2_{\varepsilon p})$.

The labour income process is assumed to be i.i.d. with a hump shape. The coefficients of age, the intercept and the distribution of the shocks are different for different education groups.

For people with higher education, the income process is

$$\ln(y_h) = \alpha_{0h} + \alpha_{1h} age + \alpha_{2h} age^2 + \varepsilon_{yh}$$

where $\varepsilon_{yh} \tilde{} N(0, \sigma^2_{\varepsilon yh})$.

For people with lower education, the income process is

$$\ln(y_l) = \alpha_{0l} + \alpha_{1l} age + \alpha_{2l} age^2 + \varepsilon_{yl}$$

where $\varepsilon_{yl} \tilde{} N(0, \sigma^2_{\varepsilon yl})$.

Rental income follows an AR(1) process.

$$\ln(y_{rt}) = \eta_0 + \eta_1 \ln(y_{rt-1}) + \varepsilon_{y_r t}$$

where $\varepsilon_{yrt} \tilde{} N(0, \sigma^2_{\varepsilon yr})$.

The risky asset return is assumed to be i.i.d. (in real terms) and log normally distributed over time

$$\ln(R_F) \tilde{} N(\mu_{RF}, \sigma^2_{RF})$$

The mortgage interest rate is assumed to be an AR(1) process.

$$\ln(R_{Mt}) = \delta_0 + \delta_1 \ln(R_{Mt-1}) + \varepsilon_{RMt}$$

where $\varepsilon_{RMt} \tilde{} N(0, \sigma^2_{\varepsilon RM})$.

4.2 Calibration

The calibrated parameters are shown in Table 5. Note that all the monetary values are in real terms. All the parameters in the calibration are from data.

Table 5: Calibrated parameters for simulation

Parameter	Value	Source
Utility parameter ρ	0.5	
LTV ratio τ_1	0.9	
LTI ratio τ_2	3	
Initial cash on hand range	$[15000, 100000]$	
House price process		
Initial house price p_1	281032.8	ONS
γ	0.024	ONS
σ_p	0.052	ONS
Income process		
α_{0h}	7.96	ONS
α_{1h}	0.11	ONS
α_{2h}	-0.001	ONS
$\sigma_{\varepsilon_{yh}}$	0.057	ONS
α_{0l}	8.65	ONS
α_{1l}	0.06	ONS
α_{2l}	-0.0007	ONS
$\sigma_{\varepsilon_{yl}}$	0.023	ONS
Rental process		
Initial rent y_{r1}	9930.41	
η_0	0.195	VOA
η_1	0.981	VOA
ε_{yr}	0.03	VOA
Risky asset return distribution		
μ_{RF}	0.086	FTSE
σ_{RF}	0.152	FTSE
Mortgage interest rate distribution		
Initial mortgage interest rate R_{M1}	0.99	BSA
δ_0	-0.005	BSA
δ_1	0.71	BSA
ε_{RM}	0.0192	BSA
Subsistence level		
$\overline{c}$	2700	
$\overline{h}$	0.5	

Note: ONS: Office for National Statistics; VOA: Valuation Office Agency; BSA: Building Societies Association; FTSE: Financial Times Stock Exchange 100 Index

Note that this leaves two parameters that we can vary: β, H^* to see how the optimal life cycle path varies with each of these. Obviously β affects the intertemporal MRS whilst H^* is measured in terms of the average house size (for example, $H^* = 0.5$ means the minimum size of house for purchase is half the average house size) and is an attempt to capture indivisibility in the house ownership market in a model in which housing is otherwise treated as continuous[14].

4.3 The method

We simulate solution of the optimal life cycle path calibrated to fit UK data for three epochs of 15 years each with different samples of realisations of shocks. There are two broad reasons for working with epochs of the young (age range 21-35), the middle aged (age range 36-50), and the old (age range 51-65).

First it brings the theoretical framework closer to the real world. In the theory the financial markets have no transaction costs in housing or financial assets (including housing debt) markets and in the rental market. In reality there are costs in most of these markets e.g. stamp duty and legal costs in house purchase; mortgage fees in mortgage markets and in equity markets spreads and broker fees. In the rental markets there are security deposit bonds and contract costs to pay. In theory the household adjusts its portfolio every period but in reality housing tenure, ownership of a particular house and taking a particular mortgage are adjusted less frequently than this, presumably partly because of these transaction costs. On average, houses changed hands once every 23 years now (Intermediary Mortgage Lenders Associations, 2015).

A second reason for concentrating on just 3 epochs rather than say a $40 - 70$ year life cycle path is the computational complexity involved. The theory model has 8 regimes each period because of the market and other constraints. The choice of optimal regime in a period depends on the cash on hand at the start of that period, in turn this depends on the outcome of exogenous random variables in that period and past savings/portfolio decisions in the preceding period. We have to find the optimal future regime for each realisation of future random variables and each level of saving today in order to decide the optimal savings portfolio today. With 8 regimes, T periods and N random realisations this means comparing payoffs between $8^{T-1}N^T$ paths of possible outcomes. Other studies (Carroll, 2012) overcome this problem by using interpolation of the solution path for choice variables between points of a finite grid. This obviously introduces some additional approximation errors.

We solve for B_{1t} and B_{2t} by backward induction using their recurrence relations from the closed form solutions in the previous section. This allows us to simulate data for different people. For any period t an individual starts with cash on hand and exogenous state variables which reflect their own past history of realisations. At t they have to make decisions which maximise their current utility and their future expected utility, taking into account that in the future they will replan depending on how the future realisations work out. But the future B_{1t+1} and B_{2t+1} are determined recursively backwards from T and depend on what turns out to be the optimal decision at each future date. So we first solve for the best decisions in the final period T for given cash on hand and other exogenous variables at T. Then come back to $T - 1$. At $T - 1$ the individual knows that his period T realisations will affect his best choices at T, so at $T - 1$ he takes expectations over these maximal utilities at T to determine his best choices at $T - 1$. Similarly at any prior period t, the future expected value function depends on the expected future course of optimal choices conditional on future realisations of random variables.

[14]Similarly $\overline{h} = 0.5$ means the subsistence level of housing enjoyed is half that of the average sized house.

To be specific, in our code solving the three epoch model ($T = 3$), random exogenous state variables denoted S_t are realised at the start of period t; the available cash on hand m_t is also known. The objective at t is the current indirect utility u_t^* plus the discounted expected future value. The latter depends on which of the possible 8 regimes (corners) it is optimal to take at $t+1$ and the optimal decisions within that regime; in turn these depend on the realisations of the state variables S_{t+1} and the optimal choices each period ahead. The available cash on hand depends on decisions of the preceding period about asset accumulation and on the current realisation of labour income and asset returns. The final period is special, the only choice is a static one of c_T, h_T as there is no bequest motive. To work out simulations of the optimal life cycle path we need to calculate the future expected values. The main idea here is to use Monte Carlo simulation to compute the expectation of the future utility over the the future state variables $E_t(B_{1t+1}(S_{t+1})m_{t+1} + B_{2t+1}(S_{t+1}))$ i.e. we use the mean of simulated function values to approximate the expectation of the function. We use Matlab to do this. To find the optimal corner for any given realisations at a period, we compare the objective (value) functions for each corner.

4.4 Simulated result

Assuming rational expectations, expectations of future realisations are computed by Monte Carlo integration. We simulate the model for four different combinations of parameters (H^* and β). We do it for 8000 different paths for each of the four sets of calibrated parameters. In our implementation, 50% of the households are high educated and 50% are low educated in each path. Within each of the education groups, the initial cash on hand is uniformly distributed over the same range.

4.4.1 Impact of minimum house to purchase and time preference on decision

One of the aims is to investigate the impact of these two parameters on household decision making, especially housing purchase behaviour. Table 6 shows the simulated life cycle decisions for different combinations of H^* and β (models (i)-(iv))[15]. When any house size can be purchased ($H^* = 0$), all the households purchase housing in the first two epochs of their life since it is the most profitable investment and can supplement income by saving rental cost and/or earning rental income from buy to let. This is true even if people discount the future heavily with $\beta = 0.7$. In comparison, a minimum house size to buy ($H^* = 0.5$) discourages all households from holding the housing asset in the first epoch while some households enter homeownership in the second epoch (models (iii) and (iv)). Whenever households buy housing, they borrow as much mortgage debt as they can. In our simulation the loan to income ratio constraint is binding for everyone who takes a mortgage[16]. This implies that whenever households decide to buy housing, the exogenous parameters always result in the polygonal feasible set of a_t and H_t. As we set the income in the first epoch to be the same for all the households, everyone with $H^* = 0$ borrows the same amount of mortgage in the first epoch. For this reason those who buy more housing in the first epoch will have more housing equity and get more capital gain from housing if the house price increases. Since our simulated house price does rise over time, we can expect to see the initially richer households

[15] Note that the average LTV and LTI are computed for mortgage borrowers only. And the average housing wealth are computed for homeowners only.

[16] With $H^* = 0, \beta = 0.7$, both loan to value and loan to income ratio constraints bind for mortgage borrowers.

(with higher initial cash on hand) who can afford a bigger house size to be richer in the second epoch in terms of capital gains compared with the poorer counterpart if we ignore the different income processes due to education. The transition of social status will be discussed in the next subsection.

Less patient households ($\beta = 0.7$) invest less on housing compared to their more patient counterparts ($\beta = 0.95$) because they derive relatively more utility from current consumption rather than future consumption. The intertemporal marginal rate of substitution (IMRS) is defined as

$$IMRS = \frac{\beta E v'_{t+1}(m_{t+1}, S_{t+1})}{u_t^{*\prime}(m_t - a_t + y_{rt} H_t)} = \frac{\beta E B_{1t+1}}{(1-\rho)^{1-\rho}(\frac{\rho}{y_{rt}})^{\rho}}$$

The IMRS is in general higher for more patient people with $\beta = 0.95$ as they care about the future more.

Occasionally in the simulations c_t and h_t are held down to their subsistence levels due to the form of the value functions shown in Theorem 1.

Table 6: Simulated life cycle decisions for different parameter combinations

minimum housing to buy	$H^* = 0$		$H^* = 0.5$	
Time discount factor	$\beta = 0.95$	$\beta = 0.7$	$\beta = 0.95$	$\beta = 0.7$
	model (i)	model (ii)	model (iii)	model (iv)
Homeownership in epoch 1	100%	100%	1%	0
Homeownership in epoch 2	100%	100%	7%	6%
Average housing wealth for home owners in epoch 1	97690.1	49215.9	141047.6	N/A
Average housing wealth for home owners in epoch 2	213128.3	99766.9	162806.7	148738.8
Average BTL in epoch 1	-43357.5	-787114.4	-779718.3	-888505.6
Average BTL in epoch 2	11217.1	-763470.2	-201929.5	-270981.1
BTL percentage in epoch 1	2%	0	0	0
BTL percentage in epoch 2	58%	0	2%	0.2%
Average LTV in epoch 1	0.49	0.9	0.3	N/A
Average LTV in epoch 2	0.43	0.9	0.7	0.8
Average LTI in epoch 1	3	3	3	N/A
Average LTI in epoch 2	3	3	3	3
Average risky asset holding in epoch 1	0	0	7862.2	1816.1
Average risky asset holding in epoch 2	0	0	26690	16982.9
Average non-housing consumption in epoch 1	2700	26152.1	23429.7	26907.9
Average non-housing consumption in epoch 2	2700	22985.2	3987.7	5159.9
Average IMRS in epoch 1	1.05	0.61	0.8	0.6
Average IMRS in epoch 2	0.86	0.63	0.9	0.6

4.4.2 Heterogeneous households

Figure 7 shows the distribution of loan to value ratio for mortgage borrowers for model (i) in the first two epochs. Although everyone in model (i) purchases houses in the first two epochs and are loan to income constrained in mortgage, their equity in housing is different.

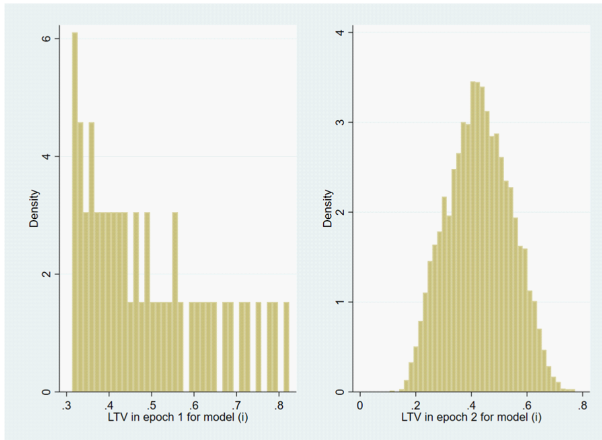

Figure 7: Simulated distribution of loan to value ratios for mortgage borrowers for model (i) in the first two epochs
Note: Left panel: LTV ratio in epoch 1; right panel: LTV ratio in epoch 2

Tables 7, 8, 9, 10 show for each education level the dynamics of cash on hand over time for each model. In the tables there are four notations: HH denotes high education and high cash on hand, HL denotes high education and low cash on hand, LH denotes low education and high cash on hand, LL denotes low education and low cash on hand. The number (1 and 2) following these notations means the epoch of these states. To define membership of the high or low cash on hand group, we use the median of cash on hand in the corresponding epoch as the threshold. i.e. if the cash on hand is less than or equal to the median, cash on hand is defined as low, otherwise it is defined as high. Since education attainment and initial cash on hand is calibrated the same way as stated above for each model, each of the categories HH1, HL1, LH1, LL1 must account for the same percentage (25% each in epoch 1). As education attainment is assumed to be constant through time for a particular household, some elements such as (HL1, LH2) must be zero. As there are 8000 households in each of the model simulations, the sum of all the elements for each transition matrix must be 8000. From the tables we can see that the most common pattern is transition from HH to HH, i.e. if one household is high educated with high cash on hand in the current epoch, then it is likely to remain in the high cash on hand group in the next epoch. Although some transition between high and low cash on hand happens, it is interesting to notice that for every model in the last epoch, most of the households are in either HH or LL group, which means in the end the high educated tend to get rich and the low educated tend to get poor. Life time movement in the disparity of cash on hand are reinforced by the high share of highly educated people amongst house owners in epoch 2 (it is on average over 90% between models (iii) and (iv)) who can access housing capital gains.

Table 7: Transition of education attainment and cash on hand combinations through time for model (i)

epoch 1 to epoch 2				
	HL1	HH1	LH1	LL1
HL2	47 (5.9%)	52 (0.6%)	0	0
HH2	1953 (24.4%)	1948 (24.4%)	0	0
LH2	0	0	9 (0.1%)	90 (1.1%)
LL2	0	0	1991 (24.9%)	1910 (23.9%)
epoch 2 to epoch 3				
	HL2	HH2	LH2	LL2
HL3	0	2 (0.03%)	0	0
HH3	99 (1.2%)	3899 (48.7%)	0	0
LH3	0	0	2 (0.03%)	0
LL3	0	0	97 (1.2%)	3901 (48.8%)

Table 8: Transition of education attainment and cash on hand combinations through time for model (ii)

epoch 1 to epoch 2				
	HL1	HH1	LH1	LL1
HL2	1526 (19.1%)	175 (2.2%)	0	0
HH2	474 (5.9%)	1825 (22.8%)	0	0
LH2	0	0	1522 (19%)	179 (2.2%)
LL2	0	0	478 (6%)	1821 (22.8%)
epoch 2 to epoch 3				
	HL2	HH2	LH2	LL2
HL3	1028 (12.9%)	226 (2.8%)	0	0
HH3	673 (8.4%)	2073 (25.9%)	0	0
LH3	0	0	1100 (13.8%)	154 (1.9%)
LL3	0	0	601 (7.5%)	2145 (26.8%)

Table 9: Transition of education attainment and cash on hand combinations through time for model (iii)

epoch 1 to epoch 2				
	HL1	HH1	LH1	LL1
HL2	501 (6.3%)	873 (10.9%)	0	0
HH2	1499 (18.7%)	1127 (14.1%)	0	0
LH2	0	0	84 (1.1%)	1290 (16.2%)
LL2	0	0	1916 (24%)	710 (8.9%)
epoch 2 to epoch 3				
	HL2	HH2	LH2	LL2
HL3	158 (2%)	22 (0.3%)	0	0
HH3	1216 (15.2%)	2604 (32.6%)	0	0
LH3	0	0	180 (2.3%)	0
LL3	0	0	1194 14.9%)	2626 (32.8%)

Table 10: Transition of education attainment and cash on hand combinations through time for model (iv)

epoch 1 to epoch 2				
	HL1	HH1	LH1	LL1
HL2	47 (5.9%)	52 (0.7%)	0	0
HH2	1953 (24.4%)	1948 (24.4%)	0	0
LH2	0	0	9 (0.1%)	90 (1.1%)
LL2	0	0	1991 (24.9%)	1910 (23.9%)
epoch 2 to epoch 3				
	HL2	HH2	LH2	LL2
HL3	0	2 (0.03%)	0	0
HH3	99 (1.2%)	3899 (48.7%)	0	0
LH3	0	0	2 (0.03%)	0
LL3	0	0	97 (1.2%)	3901 (48.8%)

5 Conclusion

There are some stylised facts for UK housing and financial asset decisions:

(i) Mortgage constraints combined with a minimum scale of house purchase can ration the cash poor (especially the young) out of house ownership. A related phenomenon is the growth of parental cash contributions to their offspring to facilitate initial house purchase but only when the parents can afford and choose to do this.

(ii) On average UK households own a relatively small proportion of their wealth in financial assets compared with the US (Banks et al., 2002).

(iii) In recent decades buy to let has become increasingly important.

To try to understand why these patterns arise and especially the underlying forces which lead to heterogeneity across the population in the life cycle paths followed in housing and financial asset decisions, we set up a life cycle model where individuals derive utility from housing and non-housing consumption, and make decisions about consumption and investment under uncertainty. The constraint set is relatively complicated, including mortgage borrowing constraint (LTV and LTI ratio constraints), and no-short-selling constraints for the safe and risky asset and also a fixed minimum level of house purchase. Applying life cycle models under uncertainty with investment constraints is difficult. With a general setting often just the first order Kuhn Tucker conditions can be characterised but explicit closed form solutions cannot be found analytically. Thus in general it is difficult to answer questions such as which market constraints hurt the most or which types of household (with heterogeneous life cycle income profiles, varying within period and intertemporal preferences, varying initial endowments) are going to be constrained in particular ways or specialise their portfolio in particular directions. An alternative in much of the literature is to abandon the search for general analytical characterisations and with specific parametrised but still quite general preferences and uncertainty, derive numerical solutions through simulation. Here we take a compromise between these. In the first half of the paper we take quite general concave preferences, additive over time and with within period utility depending on housing and non-housing consumption. We find a solution strategy which allows us to solve for the portfolio allocation and the current period allocations between consumption and housing services just within one period problems. These solutions are conditional on the variables with intertemporal effects: the housing stock H_t and net worth a_t. In fact we show that conditional on net worth in a period and investment in house purchase, the remaining decisions can be found in two independent blocks: current period decisions on consumption and housing consumption; current decisions on the investment portfolio of nonhousing assets. With general concave preferences we sketch some general properties of the optimal solution patterns and conditions under which different solution patterns can hold. However in this part we impose quite strong restrictions on the distributions of asset returns: one (overstrong) interpretation of these is that the distributions are such that with probability one the housing mortgage rate is always below the returns on housing and risky financial assets and then that again with probability one the return on the safe investment is always below that on the risky financial asset. Another interpretation is that the expected marginal value function return on the risky asset is always above that on either the safe asset or the mortgage. A consequence is that the optimal portfolio choice has some clear strong features: the safe asset is never held, if there is investment in home ownership then it is always with a maximum possible mortgage (which can be either LTV or LTI constrained). This leaves the choice of net worth and house purchase both of which have intertemporal utility effects. We can characterise the set of possible optimal choices of housing and net worth including which, if any, market constraints bind. The solutions may show zero marginal value in these two variables i.e. intertemporal smoothing of marginal values is achieved or inequalities in the FOC's due to the constraints. But in this approach we cannot explicitly derive optimal life cycle consumption and asset paths except in a highly conditional way so they would have limited interpretability.

So in the second half of the paper we add a functional form on preferences (concave but homogenous degree one within a period) which simplifies the problem enormously since it makes the intertemporal MRS (each of the current marginal utility of total current spending and the expected future marginal value of spending) independent of the levels of current spending or wealth carried forward.

It generates a value function linear in cash on hand. The main competing investments are housing and the risky asset. While housing investment will save the rent or bring rental income for the current period, the risky asset can only bring income for the future period. For this reason, even if the return of the risky asset is higher than housing, it can be optimal to hold some housing asset. Depending on the minimum available size of house to buy, some poor households can be rationed out of buying housing over life.

We can derive explicit conditions on preference parameters and intertemporal MRS under which different regimes with their constraints are optimal. We can also identify in which situations buy to let will occur or households will choose to rent rather than purchase any housing. One implication of the constant intertemporal MRS is that it may be optimal to transfer most resources to the future, staying at subsistence level today, or conversely consume all available cash on hand today, transferring no wealth into the future.

Combining the special preferences, the basic distributional assumptions on asset returns with calibrated preference parameters and empirically estimated stochastic processes and initial conditions for all the exogenous random variables (three asset returns, labour income, housing rent, house prices), we can numerically solve for optimal life cycle paths for different realisations of the random variables. The stochastic processes matter both in determining how agents form their expected future value functions and, for a given realisation of the random variables at time t, the cash on hand available at t. In this part we maintain the assumptions on the distribution of asset returns referred to previously so that it is still true that the safe asset is dominated by the risky financial asset and potentially housing and that if there is investment in home ownership, it is always financed with a maximum mortgage. We find that in general in any period there are eight possible configurations of optimal decisions with different sets of binding constraints. How the optimal life cycle switches between these depends on the distributions of random variables and their realisations, the preference parameters especially the rate of time preference and then key parameters in the constraints like the minimum house size available for purchase, the maximal mortgage loan to value and to labour income ratios. Within a given framework of key constraint parameter values and time preference rate (we take four alternative frameworks for these) we take 100 households with varying education (50 high and 50 low level individuals) with varying initial cash on hand. For each household we take 80 realisation paths of the future uncertain variables and determine the optimal life cycle profile regimes for each set of realisations and household. Comparing these our main simulation findings are:

- Social status switching (education and cash on hand effect): In our simulation, almost all the high educated people with high initial cash on hand remain in the high cash on hand group. Some of the high educated with low initial cash on hand switch to the richer group, but luck is important for them. On the other hand, almost all the low educated with low initial cash on hand remain in the low cash on hand group all through their life. The low educated with high initial cash on hand tend to switch to the poorer group either in the second or final epoch of their life. Most of those who climb up the housing ladder are all high educated.

- The minimum house purchase size effect: With no minimum, every household invests in housing in the first two epochs. On the other hand, when the minimum size is 0.5, very few or no household invests in housing in the first epoch.

- The time preference effect: The more patient house owners tend to buy to let while the less patient tend to enjoy shared ownership (they both buy some housing and also rent some housing). With a zero minimum house purchase size and low discounting of the future ($\beta = 0.95$), everyone is a pure rentier ($H > 0, h = \overline{h}$) in the first two epochs.

We also ignore the possibility of mortgage default (essentially by assuming the housing return always exceeds the mortgage rate). In fact in the data mortgage arrears and house repossessions by lenders are now below their 2007 level and have been falling since the financial crisis (BoE, 2017).

There are some possible extensions to this paper. One immediate possibility is to simulate for more than 3 epochs/periods. We have worked with reallocating the portfolio and housing tenure/ownership each period and effectively with one period adjustable rate mortgages. There are no transaction costs of changing tenure or portfolio in the approach here. An obvious extension is to allow for these. Similarly we could add a bequest motive at T or a random time horizon (date of death). Within our framework further simulation results could be calculated, e.g. we could compute the probability of each regime being chosen at any date t for given m_t and hence from this the Markov chain for cash on hand as regimes switch between adjacent periods. Furthermore, other parameters such as the maximal LTI or LTV could be varied between simulations to generate further comparative static results. As this is an individual decision model, the house price and interest rates are taken as given, but using the features here for the demand side (net demand if there is no new build or demolition) model and aggregating the net demand over individuals we could try to determine equilibrium house purchase and rental prices. The range of decision variables could be extended e.g. labour income is partly determined by an individual's choice of working hours. But using the Stone-Geary preferences, we could in fact avoid imposing dominance type restrictions on the distributions of returns. Then in some circumstances the safe asset could dominate. And finally in principle simulated and numerically solved paths could be calculated for a general preference case as in Cocco (2005).

A Feasible Sets

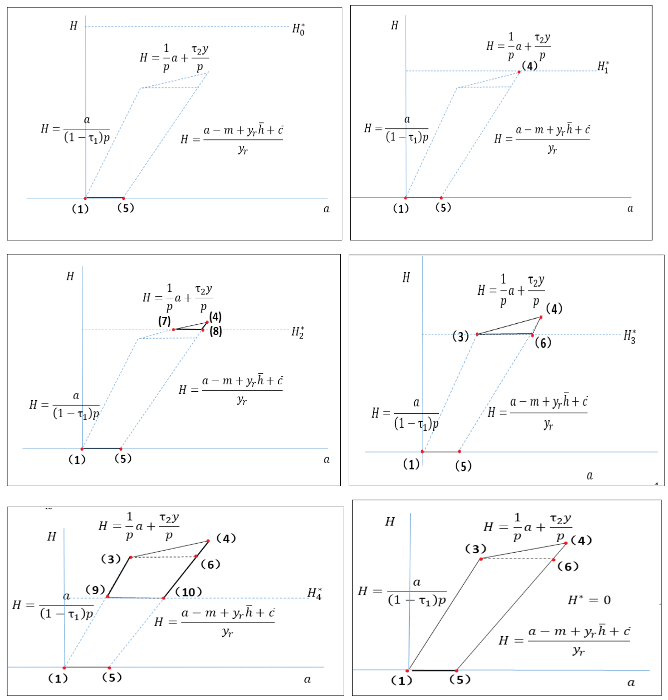

Figure 8: Possible shapes of feasible sets under Condition 1 for different values of H*

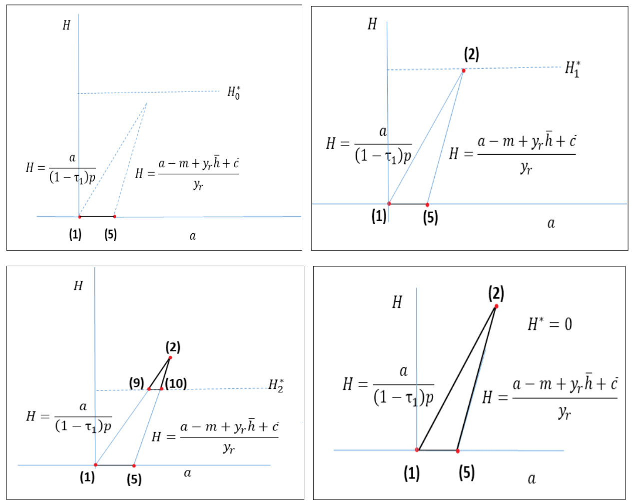

Figure 9: Possible shapes of feasible sets under Condition 2 for different values of H*

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