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Equilibrium in the Assignment Market under Budget Constraints[1]

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Abstract: We reexamine the well-known assignment market model in a more general and more practical environment where agents may be financially constrained. These constraints will be shown to have an important impact on the set of Walrasian equilibria. We prove that a price adjustment process will either find a unique minimal Walrasian equilibrium price vector, or exclusively validate the nonexistence of equilibrium.

Keywords: Assignment, Auction, Budget Constraint, Walrasian Equilibrium.

JEL classification: C62, C68, C71, D44.

1 Introduction

We reexamine the assignment market in a more general and more practical setting in which buyers may face financial constraints. In this market every buyer has a valuation on every indivisible item and also a limited amount of budget. His budget may be less than his valuation over some item. We are interested in the existence of a Walrasian equilibrium in this market, as such an equilibrium generates an efficient allocation of scarce resources. In particular, we attempt to address the following two basic questions: How budgets will affect the set of Walrasian equilibria? How to determine whether there is no Walrasian equilibrium or not. When no bidder faces any budget constraint, this model reduces to the celebrated assignment market model as studied by Koopmans and Beckmann (1957), Shubik and Shapley (1972), Crawford and Knoer (1981), Leonard (1983), and Demange, Gale, and Sotomayor (1986).

Koopmans and Beckmann (1957) introduced the assignment model and proved the existence of a Walrasian equilibrium. Shubik and Shapley (1972) formulated the problem as

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a cooperative game and showed that the core of the game coincides with the set of Walrasian equilibrium price vectors and is a closed lattice. The lattice contains a unique minimal price vector in the best interests of buyers and a unique maximum price vector in the best interests of sellers. Crawford and Knoer (1981) developed a price adjustment process which converges to a Walarasian equilibrium in the limit. Leonard (1983) discovered that the minimum Walrasian equilibrium price vector of the market is the Vickrey-Clarke-Grove payment vector. Demange, Gale, and Sotomayor (1986) developed an ascending auction process which finds this minimum price vector and thus can induce every bidder to bid truthfully. More recently, Andersson and Erlanson (2013) proposed a mixture of ascending and descending strategy-proof auctions. Andersson and Svensson (2014) studied non-manipulable allocation rules for an extended assignment model in which prices are restricted.

In the paper we will demonstrate that financial constraints can have an important impact on the set of Walrasian equilibria. We then establish that the set is a lower semi-lattice, which might be empty or not a closed set anymore, in contrast to the nonempty closed lattice of the classic case without financial constraints. To address the two basic problems mentioned above, we need to work on the set denoted by NOD of feasible integer price vectors at which there is no overdemanded set of items. Given a feasible price vector, a set of items is overdemanded if the number of buyers who demand only items in the set exceeds the number of items in the set. We prove that the set NOD is a nonempty semi-lattice and contains a unique minimal integer price vector p^{min}. We further show that if the set of Walrasian equilibrium price vectors is not empty, then the vector p^{min} must be the minimal integer vector in this set. Furthermore, we apply the Demange-Gale-Sotomayor auction to this problem and demonstrate that this auction always terminates with the price vector p^{min} at which there is no overdemand. If there is no underdemand at p^{min} either, then we find a Walrasian equilibrium. Otherwise, we validate the nonexistence of Walrasian equilibrium.

In the early literature, Che and Gale (1998) and Maskin (2000) pointed out that financial constraints can arise in a variety of situations. They examined how to sell a single item under budget constraints; see also Zheng (2001) and Krishna (2010). Illing and Klüh (2003), Klemperer (2004), and Milgrom (2004) discussed how budget constraints may have affected the sale of spectrum licenses in some situations. Hafalir, Ravi, and Sayedi (2012) studied a sealed-bid Vickrey auction for a divisible good that achieves a near Pareto efficiency. Benoît and Krishna (2001) and Pitchik (2009) studied auctions for selling two items to budget constrained bidders. Talman and Yang (2015) introduced a dynamic auction for the assignment model under budget constraints that finds a core allocation. Van der Laan and Yang (2016) proposed another auction for finding a constrained equilibrium which may not be in the core. None of these studies have investigated the two basic questions mentioned above which will be addressed in the current paper.

This paper is organized as follows. Section 2 introduces the model and Section 3 presents the main results.

2 The Model

Consider the assignment market where a seller or auctioneer wants to sell n indivisible items to m bidders. Let $N = \{1, \ldots, n\}$ denote the set of real items for sale and let $M = \{1, 2, \cdots, m\}$ denote the set of bidders. It is assumed that every bidder consumes at most one real item. In addition to the set of real items there is a *dummy item*, denoted by 0. The dummy item can be assigned to any number of bidders simultaneously, while any real item can be assigned only to at most one bidder. The set N_0 denotes the set $N \cup \{0\}$ of all items. The seller has for each item $j \in N_0$ a nonnegative reservation price c_j below which the item will not be sold. By convention, the reservation price of the dummy item is known to be $c_0 = 0$. A price vector $p = (p_0, p_1, \ldots, p_n) \in \mathbb{R}_+^{n+1}$ gives a price $p_j \geq 0$ for each item $j \in N_0$. A price vector $p \in \mathbb{R}_+^{n+1}$ is *feasible* if $p_0 = 0$ and $p_j \geq c_j$ for every $j \in N$.

Every bidder $i \in M$ attaches a monetary value to each item given by the valuation function $V^i \colon N_0 \to \mathbb{R}$. Also by convention, the value of the dummy item for every buyer $i \in M$ is known to be $V^i(0) = 0$. Every bidder $i \in M$ has a nonnegative budget m^i, being the maximum amount of money he can spend. Without loss of generality we assume that all seller's reservation prices and all bidders' valuations and budgets are integers, as they all are naturally measured in units of some currency. Because all V^i, m^i, and c_j are integers, we can prove that an integer solution will be found by our proposed auction process in which the size of price adjustment is fixed at 1. This assumption can be relaxed by allowing these primitives to be real numbers and by adjusting the size of price increment in the auction process.

Buying item $j \in N_0$ against price p_j by bidder $i \in M$ yields this bidder a utility U^i equal to

$$U^i = \begin{cases} V^i(j) + m^i - p_j & \text{if } p_j \leq m^i, \\ -\infty & \text{if } p_j > m^i. \end{cases}$$

By the assumption that bidders are not allowed to have a deficit in money, no bidder is willing to pay a price for any item above his budget m^i. We say that bidder $i \in M$ is *budget or financially constrained* if $m^i < \max_{j \in N_0} V^i(j)$, i.e., the valuation of bidder i for some items exceeds what he can afford. Otherwise, bidder i faces *no financial constraint*. When no bidder is financially constrained, the model reduces to the classical assignment market model as studied by Koopmans and Beckmann (1957), Shapley and Shubik (1972), Crawford and Knoer (1981), Demange, Gale and Sotomayor (1986), and Andersson and Erlanson (2013).

A *feasible assignment* $\pi : M \to N_0$ assigns to every bidder $i \in M$ precisely one item $\pi(i) \in N_0$ such that no real item $j \in N$ is assigned to more than one bidder. A feasible assignment may assign the dummy item to several bidders. A real item $j \in N$ is *unassigned* at feasible assignment π if there is no $i \in M$ such that $\pi(i) = j$. $N_\pi = \{j \in N \mid j \neq \pi(i) \text{ for all } i \in M\}$ denotes the set of unassigned real items at feasible assignment π.

Given a feasible price vector p, the budget set of bidder $i \in M$ is defined by

$$B^i(p) = \{j \in N_0 \ \mid p_j \leq m^i\},$$

and the demand set of bidder i is defined by

$$D^i(p) = \{j \in B^i(p) \ \mid \ V^i(j) - p_j = \max_{k \in B^i(p)} \ (V^i(k) - p_k)\},$$

i.e., $D^i(p)$ is the collection of most preferred items at p by bidder i within his budget set. An item $j \in N_0$ is in the demand set $D^i(p)$ of bidder $i \in M$ at feasible price vector p if and only if at p item j can be afforded and maximizes the surplus $V^i(k) - p_k$ over all affordable items k. When the demand set contains multiple items, then at the given prices of the items the bidder is indifferent between any two items in his demand set.

A pair (p, π) of a feasible price vector p and a feasible assignment π is said to be *implementable* if $p_{\pi(i)} \leq m^i$ for all $i \in M$, i.e., every bidder $i \in M$ can afford to buy the item $\pi(i)$ assigned to him. Note that an implementable pair (p, π) yields allocation (π, x) with $x_i = m^i - p_{\pi(i)} \geq 0$ the money amount of bidder $i \in M$.

Definition 2.1 An implementable pair (p^*, π^*) is a *Walrasian equilibrium* (WE) if
(a) $\pi^*(i) \in D^i(p^*)$ for all $i \in M$,
(b) $p^*_j = c_j$ for every $j \in N_{\pi^*}$.

If (p^*, π^*) is a Walrasian equilibrium, then p^* is called a (Walrasian) equilibrium or WE price vector and π^* a (Walrasian) equilibrium or WE assignment.

The following example shows that financial constraints can have a significant impact on the set of Walrasian equilibria.

Example 1. Consider a market with two bidders and two real items. The seller's reservation prices are given by $(c_0, c_1, c_2) = (0, 0, 0)$ and the bidders' values are given in Table 1.

For this example we consider three cases of budget constraints.

Table 1: Bidders' values in Example 1.

Item	0	1	2
Bidder 1	0	8	2
Bidder 2	0	6	3

Case 1. When both bidders are not financially constrained, the set of Walrasian equilibrium price vectors is a nonempty closed lattice given by the set

$$\{(p_1, p_2) \mid p_2 + 3 \le p_1 \le p_2 + 6,\ 3 \le p_1 \le 8,\ 0 \le p_2 \le 3\}.$$

The equilibrium assignment is $(\pi(1), \pi(2)) = (1, 2)$ by which bidder 1 gets item 1 and bidder 2 gets item 2, resulting in a total social value of 11.

Case 2. When the two bidders are financially constrained by $m^1 = 1$ and $m^2 = 2$, the set of Walrasian equilibrium price vectors is still a nonempty lattice given by the set

$$\{(p_1, p_2) \mid 1 < p_1 \le 2,\ 0 \le p_2 \le 1\},$$

which is open from below for the price of item 1. The equilibrium assignment is $(\pi(1), \pi(2)) = (2, 1)$ by which bidder 1 gets item 2 and bidder 2 gets item 1, resulting in a total social value of 8, less than value 11 of Case 1.

Case 3. When the two bidders have budget constraints $m^1 = 1$ and $m^2 = 1$, the set of Walrasian equilibrium price vectors is empty. The market has no equilibrium.

Case 3 in Example 1 may suggest that the emptiness of the set of Walrasian equilibria is caused by the fact that the bidders' budgets are equal to each other. The next example shows that this need not to be the case. Budget constraints can affect considerably the set of Walrasian equilibria regardless of whether they are equal or unequal.

Example 2. Consider a market with three bidders and two real items. The seller's reservation prices are given by $(c_0, c_1, c_2) = (0, 0, 0)$, bidders' budgets are given by $m^1 = 3$, $m^2 = 1$, $m^3 = 6$, and their values are given in Table 2.

Suppose p is a Walrasian equilibrium price vector. Then we must have $p_1 \le 3$ and $p_2 \le 5$. Otherwise no bidder would demand any real item. When $p_2 = p_1 + 2$, we have $D^3(p) = \{1, 2\}$. If $p_2 = p_1 + 2 > 3$, then $D^1(p) = D^2(p) = \{0\}$, and if $p_2 = p_1 + 2 \le 3$, then $D^1(p) = \{2\}$ and $D^2(p) = \{1\}$. In both cases there is no equilibrium with $p_2 = p_1 + 2$.

When $p_2 > p_1 + 2$, then $D^3(p) = \{1\}$. For an equilibrium to exist, we must have $p_1 > 1$, which implies $p_2 > 3$ and $D^1(p) = D^2(p) = \{0\}$. But then there is no equilibrium. On the

Table 2: Bidders' values in Example 2.

Item	0	1	2
Bidder 1	0	0	4
Bidder 2	0	3	0
Bidder 3	0	3	5

other hand, when $p_2 < p_1 + 2$, then $D^3(p) = \{2\}$. In order to have an equilibrium we must have $p_1 \leq 1$, which implies $p_2 < 3$ and $D^1(p) = \{2\}$. Again there is no equilibrium.

3 Main Results

For a nonempty set of real items $S \subseteq N$ and a feasible price vector p, define the *lower inverse demand set* of S at p by

$$D_S^-(p) = \{i \in M \mid D^i(p) \subseteq S\},$$

i.e., $D_S^-(p)$ is the set of bidders who demand at p only items in S. Note that S is a subset of real items, so any bidder i in the lower inverse demand set of S at p does not demand the dummy item and thus has a positive surplus $V^i(j) - p_j$ for any item j in his demand set $D^i(p)$. Define the *upper inverse demand set* of S at p by

$$D_S^+(p) = \{i \in M \mid D^i(p) \cap S \neq \emptyset\},$$

i.e., $D_S^+(p)$ is the set of bidders that demand at p at least one of the items in S. Note that the latter set contains the former set. Let $|A|$ denote the cardinality of a finite set A.

Definition 3.1
Let p be a feasible price vector.
1. A nonempty set of real items $S \subseteq N$ is *overdemanded* at p if $|D_S^-(p)| > |S|$. An overdemanded set S at p is *minimal* if no proper subset of S is overdemanded at p.
2. A nonempty set of real items $S \subseteq N$ is *underdemanded* at p if both $S \subseteq \{j \in N \mid p_j > c_j\}$ and $|D_S^+(p)| < |S|$.

The notion of minimal overdemanded set is used in Demange, Gale, and Sotomayor (1986) and the notion of underdemanded set can be found in Sotomayor (2002) and Mishra and Talman (2010). Note that a feasible price vector p is a Walrasian equilibrium price vector if and only if at p there is neither underdemand nor overdemand.

The first lemma is due to van der Laan and Yang (2016, p.124, Lemma 3.4) and will be invoked.

Lemma 3.2 *Let T be a minimal overdemanded set at feasible price vector p and let T' be a nonempty proper subset of T. Then*

$$|\{i \in D_T^-(p) \mid D^i(p) \cap T' \neq \emptyset\}| \geq |T'| + |D_T^-(p)| - |T|.$$

For any $p, q \in \mathbb{R}^{n+1}$, let $\min(p, q)$ denote the componentwise minimum of p and q, and $\max(p, q)$ the componentwise maximum of p and q. A set $X \subseteq \mathbb{R}^{n+1}$ is a lower semi-lattice if X contains $\min(p, q)$ for any $p, q \in X$. The set X is a lattice if X contains both $\min(p, q)$ and $\max(p, q)$ for any $p, q \in X$. Let NOD denote the set of feasible integer price vectors at which there is no overdemand. The next lemma shows that NOD is a lower semi-lattice.

Lemma 3.3 *The set NOD is a nonempty lower semi-lattice.*

Proof. Clearly, the set NOD is not empty. Consider any $p, q \in NOD$. Then $r = \min(p, q)$ is also a feasible integer price vector. Denote $S^+ = \{j \in N \mid p_j < q_j\}$, $S^- = \{j \in N \mid p_j > q_j\}$, and $S = \{j \in N \mid p_j = q_j\}$. Suppose there is overdemand at r and let T be a minimal overdemanded set at r. Thus $|D_T^-(r)| > |T|$. Denote $T^+ = T \cap (S^+ \cup S)$ and $T^- = T \cap (S^- \cup S)$. By definition at least one of these two sets is not empty. Suppose T^+ is not empty. According to Lemma 3.2 we have that

$$|\{i \in D_T^-(r) \mid D^i(r) \cap T^+ \neq \emptyset\}| \geq |T^+| + |D_T^-(r)| - |T| \geq |T^+| + 1.$$

Since $r_j = p_j$ for every $j \in S^+ \cup S$ and $r_j = q_j < p_j$ for every $j \in S^-$, it follows that $D^i(p) \subseteq T^+$ if $D^i(r) \cap T^+ \neq \emptyset$. Hence,

$$|\{i \in M \mid D^i(p) \subseteq T^+\}| \geq |T^+| + 1,$$

implying that T^+ is overdemanded at p. This contradicts that $p \in NOD$. Similarly, when T^- is not empty, we obtain that T^- is overdemanded at q, contradicting that $q \in NOD$. $\square$

The proof of Lemma 3.3 and Example 1 show that the set of Walrasian equilibrium price vectors is a lower semi-lattice, which is not necessarily closed. Lemma 3.3 implies the following corollary, saying that there is a minimal integer price vector at which there is no overdemand. Note that this generalizes the well-known fact that for the model without budget constraints there is a unique minimal Walrasian equilibrium (integer) price vector. Let the price vector $p^{\min}$ be defined by $p_j^{\min} = \min_{p \in NOD}\ p_j$ for all $j \in N_0$.

Corollary 3.4 *It holds that $p^{\min} \in NOD$.*

The next lemma states that when p is an integer Walrasian equilibrium price vector, then any $q \in NOD$ satisfying $q \leq p$ is also a Walrasian equilibrium price vector.

Lemma 3.5 *If $q, p \in NOD$, $q \lneq p$, and p is a WE price vector, then q is a WE price vector.*

Proof. Let π be a Walrasian equilibrium assignment at p. We show that there exists a Walrasian equilibrium assignment ρ at q. Define $S = \{j \in N \mid q_j < p_j\}$. Note that S is nonempty. Since p is a WE price vector, we have that S is neither overdemanded nor underdemanded at p, i.e.,

$$|D_S^-(p)| \leq |S| \text{ and } |D_S^+(p)| \geq |S|.$$

Because $q_j < p_j$ for every $j \in S$, we have that $D^i(q) \subseteq S$ for every $i \in D_S^+(p)$. Hence, we must have that $|D_S^+(p)| \leq |S|$, otherwise S is overdemanded at q. It follows that $|D_S^+(p)| = |S|$. Since $q_j < p_j$ for every $j \in S$, we have that $p_j > c_j$ for every item $j \in S$. Therefore, at p every item $j \in S$ is assigned to a bidder. However, only the bidders in the set $D_S^+(p)$ demand items from S at p. With $|D_S^+(p)| = |S|$ it then follows that $\pi(i) \in S$ for every $i \in D_S^+(p)$ and that every $j \in S$ is assigned to a bidder in $D_S^+(p)$.

From $|D_S^+(p)| = |S|$ it also follows that there is no bidder $h \in M \setminus D_S^+(p)$ with $D^h(q) \subseteq S$, again because otherwise S is overdemanded at q. This implies that for every bidder $h \in M \setminus D_S^+(p)$ it must hold that $D^h(q)$ contains an item j not in S and thus an item j with $q_j = p_j$. From this it follows that for every bidder $h \in M \setminus D_S^+(p)$ it holds that

$$D^h(p) \subseteq D^h(q).$$

Therefore, for every $h \in M \setminus D_S^+(p)$ it holds that $\pi(h) \in D^h(q)$. Hence, for every $h \in M \setminus D_S^+(p)$ we can set $\rho(h) = \pi(h)$, i.e. every bidder not in $D_S^+(p)$ gets at q the same item as at p. Moreover, for every bidder $h \in M \setminus D_S^+(p)$ it holds that $\pi(h) \in N_0 \setminus S$. Further, for any item $j \in N_\pi$ we have that $q_j = p_j = c_j$ and $j \in N \setminus S$, because all items in S are assigned in π. Also at q these items can stay at the seller. It remains to show that all items in S can be assigned at q to the bidders in $D_S^+(p)$. Suppose there is a nonempty subset $T \subseteq S$ that is underdemanded at q, so the number of bidders that demand items from T at q is less than $|T|$. Because the bidders in $D_S^+(p)$ only demand items from S at q, this implies that at q at least $|D_S^+(p)| - (|T| - 1)$ bidders only demand items from $S \setminus T$. With $|D_S^+(p)| = |S|$ it then follows that $S \setminus T$ is overdemanded at q, which contradicts that $q \in NOD$. So, at q there is no underdemanded subset T of S and there is also no overdemand. Further only the bidders in $D_S^+(p)$ demand items from S. This implies that we can choose ρ such that it assigns any item in S to a bidder in $D_S^+(p)$ and reversely assigns to every bidder h in $D_S^+(p)$ an item from S in its demand set $D^h(q)$. Hence, ρ is a Walrasian equilibrium assignment at q. $\square$

The next corollary follows immediately from Corollary 3.4 and Lemma 3.5.

Corollary 3.6 *If there exists an integer WE price vector, then $p^{\min}$ is an integer WE price vector.*

The next lemma says that if there exists a Walrasian equilibrium, then there also exists an integer Walrasian equilibrium price vector.

Lemma 3.7 *If the set of WE price vectors is not empty, it contains an integer WE price vector.*

Proof. Let (q, π) be a Walrasian equilibrium and let p be the integer price vector with, for every $j \in N_0$, p_j the smallest integer larger than or equal to q_j. We show that (p, π) is also a Walrasian equilibrium. Since $p \geq q$, we have that p is a feasible price vector. Moreover, for every $j \in N_\pi$ it holds that $p_j = q_j = c_j$, so condition (b) of Definition 2.1 is satisfied. To show condition (a) of Definition 2.1, take any $i \in M$. Since m^i is an integer, it holds that $B^i(p) = B^i(q)$. Let $j \in B^i(p)$. Since $p_j \geq q_j$ and $\pi(i) \in D^i(q)$, we obtain

$$V^i(\pi(i)) - q_{\pi(i)} \geq V^i(j) - q_j \geq V^i(j) - p_j. \tag{3.1}$$

Because $p_{\pi(i)}$ is the smallest integer larger than or equal to $q_{\pi(i)}$ and p_j is an integer, this implies

$$V^i(\pi(i)) - p_{\pi(i)} \geq V^i(j) - p_j. \tag{3.2}$$

This shows that (p, π) also satisfies condition (a) of Definition 2.1 and is therefore a WE. $\square$

The next corollary follows immediately from Corollary 3.6 and Lemma 3.7.

Corollary 3.8 *If $p^{\min}$ is not a WE price vector, then a Walrasian equilibrium does not exist.*

In the remaining of this section we prove that the auction of Demange, Gale, and Sotomayor (1986) (DGS auction in short) terminates with the minimal integer price vector p^{min} at which there is no overdemand. If there is no underdemand at $p^{\min}$, then a Walrasian equilibrium exists and the auction terminates with the minimal integer Walrasian equilibrium price vector $p^{\min}$. If there is underdemand at $p^{\min}$, then $p^{\min}$ is not a Walrasian equilibrium price vector and therefore according to Corollary 3.8 there exists no Walrasian equilibrium in the market.

The DGS Auction

Step 1 (Initialization): Set $t := 0$, $p^t_j := c_j$, $j \in N_0$. Go to Step 2.

Step 2: Every bidder $i \in M$ reports his demand set $D^i(p^t)$. If there is no overdemanded set at p^t, stop. Otherwise, go to Step 3.

Step 3: The auctioneer chooses a minimal overdemanded set $O^t \subseteq N$ of items. Then set $p_j^{t+1} := p_j^t + 1$ for every $j \in O^t$, $p_j^{t+1} := p_j^t$ for every $j \in N_0 \setminus O^t$. Set $t := t + 1$ and return to Step 2.

The next theorem generalizes the fact that for the setting without budget constraints the DGS auction ends with the minimal Walrasian equilibrium price vector to the setting with budget constrained bidders.

Theorem 3.9 *The DGS auction above terminates with $p^{\min}$ in a finite number of steps.*

Proof. Because every bidder has only a bounded valuation on every item and the price of each item is monotonically increasing, the DGS auction will clearly stop at some round, say t^*. Let t^* be the last round of the DGS auction and let $p = p^{t^*}$. So, $p \in NOD$ and $p^t \notin NOD$ for every $0 \leq t < t^*$. If $t^* = 0$, then p^0 is the minimal integer Walrasian equilibrium price vector $p^{\min}$, because by definition there is no underdemand at p^0.

Otherwise, suppose that $p \neq p^{\min}$. By Lemma 3.3 we must have that $p \geq p^{\min}$ and $p_j > p_j^{\min}$ for some $j \in N$. By construction, we have that $p^0 \leq p^{\min}$. Let $t < t^*$ be the last round at which $p^t \leq p^{\min}$. Since $t < t^*$, there must be overdemand at p^t, otherwise the auction would have terminated at t. Let $O^t = O$ and $S = \{j \in N | p_j^{t+1} > p_j^{\min}\}$. Note that S is nonempty. Since at t the price of an item $j \in N$ is increased with one if and only if $j \in O$, it follows that $S \subseteq O$. Moreover, since all prices are integers, $p_j^t = p_j^{\min}$ for all $j \in S$. Because O is overdemanded at p^t, we have that $|D_O^-(p^t)| > |O|$.

Let L be the subset of bidders in $D_O^-(p^t)$ that demand at least one item from S. By Lemma 3.2 it holds that

$$|L| \geq |S| + |D_O^-(p^t)| - |O| > |S|. \tag{3.3}$$

So, L is not empty. Consider a bidder $i \in L$, an item $j \in S \cap D^i(p^t)$, and an item $k \neq j$. First, consider the case that $k \notin O$, then $k \notin D^i(p^t)$ and either bidder i prefers j to k at p^t, or k is not affordable for i at p^t. Because $p_j^t = p_j^{\min}$ and $p_k^t \leq p_k^{\min}$, we have that also at $p^{\min}$ item j is preferred above k or k is not affordable for i and so $k \notin D^i(p^{\min})$. Second, let $k \in O \setminus S$. Then bidder i likes item j at least as well as item k, because i demands j and maybe also k at price p^t, or item k is not affordable for i at p^t. Since $k \in O$ but not in S, we have that $p_k^{t+1} = p_k^t + 1 \leq p_k^{\min}$. On the other hand $p_j^t = p_j^{\min}$. Again, we have that at $p^{\min}$ item j is preferred above k or k is not affordable for i and so $k \notin D^i(p^{\min})$. Hence, $D^i(p^{\min})$ does not contain any item from $N \setminus S$. Finally, since $j \in D^i(p^t)$, the dummy item is not in $D^i(p^t)$, because $D^i(p^t) \subseteq O$, and $p_j^t = p_j^{\min}$, it follows that j is affordable at $p^{\min}$

and is preferred above the dummy item at $p^{\min}$. Hence, $D^i(p^{\min}) \subseteq S$. Because there is no overdemand at $p^{\min}$, we must have that $|L| \leq |S|$, which contradicts inequality (3.3). Hence, there can not be overdemand at p^t, contradicting that $t \leq t^*$. Therefore, $t = t^*$ and $p^{t^*} = p^{\min}$. $\square$

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