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# Efficient Ascending Menu Auctions with Budget Constrained Bidders* 

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26 July 2018


#### Abstract

An auctioneer wishes to sell multiple heterogeneous indivisible items among many bidders. Each bidder has valuations over his interested bundles of items but may not be able to pay up to his valuations because of his budget constraint. We propose two ascending menu auctions in which bidders determine their own bids on their interested bundles. We prove that the first auction finds a core allocation when bidders are budget constrained, and it finds a strong core allocation when bidders face no budget constraints. The second auction improves the first one and finds a strongly Pareto efficient core allocation when bidders are budget constrained. The core allocation consists of an assignment of items and its associated supporting price for every assigned bundle and cannot be improved by any coalition of market participants.


Keywords: Dynamic menu auction, core, budget constraint, efficiency.
JEL classification: D44.

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## 1 Introduction

This paper addresses the problem of how to efficiently allocate multiple heterogeneous inherently indivisible commodities among a group of bidders who can be financially constrained. To be more precise, an auctioneer (or seller) wants to sell $n$ indivisible goods to $m$ potential bidders. Each bidder wishes to acquire several goods and has private valuations over his interested bundles of goods and may view some goods as substitutes and some other goods as complements. Valuation patterns of goods can be very general and arbitrary and may vary from one bidder to another. Every bidder is initially endowed with a budget but his budget is very limited and may not match his valuation over his interested bundles. In this setting, it is not possible to follow the traditional approach of using market-clearing prices as an effective means to allocate goods, as market-clearing prices are not guaranteed to exist due to budget constraints. We aim to develop a dynamic menu auction mechanism as an alternative way to overcome the nonexistence problem of market-clearing prices and still achieve an efficient market outcome.

Auctions have been long used for the sale of a variety of items since 2,500 years ago when they were applied by the Babylonians. Nowadays auctions can be conducted online and off-line. They are powerful market mechanisms and have been widely explored by both private and public sectors to carry out a broad range of and a huge volume of economic activities. For instance, at the heart of every stock market lie the double auctions. Auctions are used by governments to sell treasury bills, timber rights, off-shore oil leases, mineral rights and pollution permits, and to procure public projects including goods and services, and to privatize state companies (in the former Soviet Unions and other eastern European socialist states), and by firms and individuals to sell all kinds of commodities and services ranging from antiques, art works, flowers and fish, to airline routes, takeoff and landing slots, and keywords. The staggering sale of radio spectrum licenses in the United States and the United Kingdom and elsewhere in the 1990s and 2000s (see e.g., Klemperer (2004) and Milgrom (2004)) has seized public attention and made dynamic auction design for multiple items very popular.

A key assumption in auction theory has been that all potential bidders are not subject to any budget constraints so that they can pay up to their valuations on their interested item or bundles of items. Under this assumption a number of efficient dy-
namic auctions have been proposed, such as Crawford and Knoer (1981), Demange et al. (1986), and Andersson and Erlanson (2013) for the cases in which there are multiple items but each bidder demands at most one item, Kelso and Crawford (1982), Gul and Stacchetti (2000), Milgrom (2000), and Sun and Yang (2009) for the cases in which there are multiple items and each may demand several items. Ausubel $(2004,2006)$, Perry and Reny (2005), and Sun and Yang (2014) have introduced both efficient and strategyproof dynamic auctions for multiple items. In the models just mentioned prices are the same for every bidder and market-clearing prices exist. Ausubel and Milgrom (2002) have introduced package auctions for the environment where each bidder's valuations are not restricted to any particular pattern. In this case, prices have to be personalized and market-clearing personalized prices still exist. In all these models, prices are used in the auction process and are adjusted according to well-designed rules until marketclearing prices are reached to clear the markets.

Unfortunately, in reality, buyers may not always have enough cash or credit to buy those goods or bundles of goods that they want to buy. It is well-known that financial or budget constraints pose a serious obstacle to the efficient allocation of resources; see Che and Gale (1998), Laffont and Robert (1996), Maskin (2000) and Krishna (2010) among others. A longstanding guiding economic principle is that efficient allocation of goods can be achieved through market-clearing or Walrasian equilibrium prices. In the presence of budget constraints, this principle can no longer be applied, because market-clearing prices are not guaranteed to exist.

To overcome the absence of market-clearing prices caused by financial constraints, we have to adopt a more general approach-the notion of core-to the current challenging allocation problem. The concept of core is a generalization of Edgeworth's contract curve and is one of the most fundamental solution concepts in game theory and general equilibrium theory; see Gillies (1953), Debreu and Scarf (1963), Scarf (1967), Shapley and Shubik (1971), Shapley (1973), Shapley and Scarf (1974), Quinzii (1984), and Predtetchinski and Herings (2004) among others. A core allocation consists of an assignment of items and its supporting price system and is Pareto efficient. It specifies a feasible distribution of items and incomes among all market participants that is stable against every possible deviation from any coalition. Because of budget constraints, agents will not be able to transfer part of their utilities to others. In spite of budget constraints and non-transferable
utilities we prove that there exists at least one core allocation in the market and thus a Pareto-efficient allocation can be still achieved. Our major contribution goes further by designing auctions for actually locating such an efficient allocation.

We propose two ascending menu auctions in which bidders determine their own bids on their interested bundles of items. We prove that the first auction finds a core allocation when bidders are budget constrained, and it finds a strong core allocation when bidders face no budget constraints. The second auction improves the first one and finds a strongly Pareto efficient core allocation when bidders are budget constrained. Briefly speaking, in both auctions, initially every bidder sets a high enough target utility he aims to achieve and determines his bids on all his interested bundles according to this target utility and his budget. All bidders report their initial bids to the auctioneer. Then the auctioneer selects a provisional assignment based on reported bids from all bidders and her own valuations over all bundles. If a bidder gets no item from the provisional assignment and can reduce his current target utility to make new bids (such a bidder is called a provisional loser), then at least one of such bidders reduces his target utility and all other bidders keep their target utilities unchanged. Every bidder then updates his bids on his interested bundles according to his current target utility and his budget and reports his renewed bids. The auctioneer chooses again a provisional assignment based on currently reported bids and her own valuations. This process stops at the time when there is no provisional loser anymore.

In both auctions, every bidder's bidding price on each his interested bundle is weakly increasing over the time and is reported to the auctioneer and therefore the auction is called an ascending menu auction, whereas his target utility is weakly decreasing and is used by himself privately. It will be also shown that the core allocation found by our proposed auctions gives every bidder at least his target utility at the time when the auctions terminate. By definition, a core allocation need not be strongly Pareto efficient. To improve market efficiency, we modify our first ascending menu auction so that a strongly Pareto efficient core allocation can be always found. This modification requires each bidder to report his untransferable value, i.e., his valuation on his interested bundle minus his budget, when his bidding price on the bundle exceeds his budget. It should be stressed that a strongly Pareto efficient core allocation is not necessarily a strong core allocation, as the strong core may be empty, although the core is always nonempty.

A long-established pricing rule is that a seller will sell her good only if the price is no less than her valuation or reserve price. Because our current model allows the seller to have any kind of valuation on every bundle of items, we need to generalize this conventional rule in a way that a bundle can be sold only if its price is no less than its marginal value to the seller. We prove that at every core allocation the price of every sold bundle is at least equal to its marginal value to the seller when no condition is imposed on the seller's valuations, and that at every core allocation the price of every sold bundle is at least equal to the seller's valuation when her valuations are super-additive. Besides, our ascending auctions share several common features with other ascending auctions. In practice, business people are extremely reluctant to reveal their costs or values and may not always have complete information on the situation. Compared with the famous sealed-bid VCG auction mechanism, our ascending auctions have the advantage of demanding less information from bidders, allowing them to learn and adjust, being detailfree, and being independent of any probability distribution. According to the literature, see Wilson (1987), Rothkopf et al. (1990), Ausubel (2004, 2006), Perry and Reny (2005), Bergemann and Morris (2007), Milgrom (2007), and Rothkopf (2007) among others, this feature is very important and attractive for auction design.

Our auction model is a natural generalization of Ausubel and Milgrom (2002)'s seminal model without budget constraints to the one with budget constraints. In their model without budget constraints, they assume that the seller values every bundle of items at zero. We show that this assumption can be also dropped by using a generalization of the conventional pricing rule. Our first ascending menu auction can be viewed as an appropriate generalization and improvement of their package auction from the setting without budget constraints to the setting with budget constraints. Although most part of their paper focuses on the analysis of their auction without budget constraints, Ausubel and Milgrom also briefly discuss a model with budget constraints in their Section 8. In their model, bidders' budgets and utility functions are not explicitly given instead they require that every bidder and the seller each have a strict preference relation over a finite set of choices. They suggest to use the concept of core as solution and propose a procedure for finding a core element. Their model is different from ours and their auction cannot be applied to our current model, because our model permits every bidder to have a continuum of choices and be indifferent between many choices and examples of our
model will be given to show that their auction cannot guarantee to find a core allocation.
Our article also closely connects with Day and Milgrom (2008), Erdil and Klemperer (2010), Mishra and Parkes (2007), Talman and Yang (2015) and van der Laan and Yang (2016). Day and Milgrom (2008), Erdil and Klemperer (2010) have refined and improved the ascending package auctions of Ausubel and Milgrom (2002) without budget constraints. Mishra and Parkes (2007) introduce an ascending package auction for finding a Vickrey outcome which need not be a core allocation but has the strategy proof property. See also Bernheim and Whinston (1986) for an sealed-bid menu auction. Talman and Yang (2015) and van der Laan and Yang (2016) have examined the assignment market with budget constraints in which there are multiple items for sale but each bidder consumes only one item. The former paper proposes a dynamic auction that finds a core allocation. The latter one introduces an ascending auction that locates a constrained equilibrium. The constrained equilibrium possesses several interesting properties but is not necessarily a core allocation.

This article further relates to the literature on auctions of selling one or two items with budget constrained bidders. Che and Gale (1998), Laffont and Robert (1996), Maskin (2000), Krishna (2010), and Zheng (2001) have examined the cases of selling a single item when bidders face budget constraints. Hafalir et al. (2012) have studied a sealed-bid Vickrey auction for selling one divisible good to budget constrained bidders. Benoit and Krishna (2001), Brusco and Lopomo (2008), and Pitchik (2009) have analyzed auctions for selling two items under budget constraints. Pai and Vohra (2014) have considered the sale of a single item to ex-ante homogeneous bidders who have private information about their own valuations and budgets. They derive expected revenue maximizing and constrained efficient symmetric auctions and show an implementation through a modified all-pay auction. Beker and Hernando-Veciana (2015) have investigated an infinite horizon bidding market model with two financially constrained firms competing for a procurement project. They show that the effect of budget constraints does not disappear even if bidders can accumulate profits along time to alleviate them.

The rest of the paper is organized as follows. Section 2 presents the model and basic concepts. Section 3 introduces and analyzes the auctions. Section 4 concludes.

## 2 The Model

An auctioneer (seller) wishes to sell a set of heterogeneous indivisible goods (items) $N=$ $\{1, \ldots, n\}$ to a group of potential bidders $M=\{1, \ldots, m\}$. Let 0 represent the seller (she) and let $M_{0}=M \cup\{0\}$ stand for all agents in the market. Let $2^{N}$ denote the collection of all subsets of $N$, and an element $A \in 2^{N}$ is called a bundle of goods. For every bidder $i \in M$, let $\mathcal{F}^{i} \subset 2^{N}$ represent the family of bundles in which he is interested. We assume that consuming nothing can be one of his options, i.e., $\varnothing \in \mathcal{F}^{i}$. Bidder $i$ attaches a monetary value (units of money) to each interested bundle of items, namely, each bidder $i$ has a valuation function $v^{i}: \mathcal{F}^{i} \rightarrow \mathbb{Z}_{+}$with $v^{i}(\varnothing)=0$. Every bidder $i$ is endowed with an amount $m^{i} \in \mathbb{Z}_{+}$of money. We say that bidder $i$ is budget or financially constrained if $m^{i}<\max _{A \in \mathcal{F}^{i}} v^{i}(A)$, that is, the valuation of bidder $i$ for some bundles exceeds what he can afford. Otherwise, bidder $i$ is not budget constrained. The seller has a valuation $v^{0}(S) \in \mathbb{Z}_{+}$for every bundle $S \in 2^{N}$ and an initial income $m^{0}=0$. Observe that for the seller we have $\mathcal{F}^{0}=\{S \mid S \subseteq N\}$. As in the literature, ${ }^{1}$ we also assume that money is a perfectly divisible commodity. Because all $v^{i}(S)$ and $m^{i}$ are integers, we shall prove that there will be an integral solution by using our proposed auctions and the size of price adjustment in the auctions can be 1 . It should be, however, pointed out that $v^{i}(S)$ and $m^{i}$ are allowed to be any real numbers.

A bundle $S$ can be sold only if its price $p(S)$ is no less than its marginal value to the seller, i.e., $p(S) \geq v^{0}(S \cup T)-v^{0}(T)$, where $T$ is the bundle that will not be sold but kept by the seller. This pricing rule is a natural and practical generalization of the traditional one that the seller is willing to sell her good only if its price is not below her reserve price. This new rule is needed because we allow the seller to have a very general pattern of valuations on her bundles of items. In the literature it is typically assumed that the seller has a valuation of zero on every item or bundle of items.

The following mild assumptions are imposed upon the model:
(A1) Private Values: Every bidder $i \in M$ knows privately his own feasible bundles $\mathcal{F}^{i}$, valuation function $v^{i}$, and budget $m^{i}$.
(A2) Quasilinear Utility: For any bidder $i \in M$, if he pays $p(A)$ in exchange for bundle

[^1]$A \in \mathcal{F}^{i}$, he gets utility of $v^{i}(A)+m^{i}-p(A)$ for $p(A) \leq m^{i}$ and utility of $-\infty$ for $p(A)>m^{i}$.
(A3) Monotonicity of the Seller Values: The seller's valuation function $v^{0}$ is weakly increasing with $v^{0}(\varnothing)=0$, i.e., $v^{0}(S) \leq v^{0}(T)$ for $S \subseteq T \subseteq N$.

Note that the valuation function of each agent (bidder or seller) can be arbitrary and very general. This can accommodate a variety of cases including the one in which some agents may view the items as substitutes but the other may see them as complements.

We use $\left(v^{i}, \mathcal{F}^{i}, m^{i}, i \in M_{0}\right)$ to represent this auction model. Observe that when no bidder is budget constrained, this model reduces to a more general version of the wellknown model of Ausubel and Milgrom (2002) without budget constraints; see also Day and Milgrom (2008), and Erdil and Klemperer (2010). They assume that no bidder is budget constrained and the seller valuates every bundle of items at zero and free disposal holds for every bidder. These three basic assumptions are dropped in the current model.

An assignment of items in $N$ is a partition $\pi=(\pi(i))_{i \in M_{0}}$ of items among all agents in $M_{0}$ such that $\pi(i) \cap \pi(j)=\varnothing$ for all $i \neq j$ and $\bigcup_{i \in M_{0}} \pi(i)=N$. Note that $\pi(i)=\varnothing$ is allowed. An assignment $\pi$ assigns the bundle $\pi(i)$ to agent $i$. If $\pi(0) \neq \varnothing$, then the bundle $\pi(0)$ is not sold and thus stays with the seller. An assignment $\pi$ is feasible if $\pi(i) \in \mathcal{F}^{i}$ for every bidder $i \in M$. Let $\mathcal{A}$ denote the family of all feasible assignments. A feasible assignment $\pi$ is fully efficient if for every feasible assignment $\rho$, we have

$$
\begin{equation*}
\sum_{i \in M_{0}} v^{i}(\pi(i)) \geq \sum_{i \in M_{0}} v^{i}(\rho(i)) . \tag{1}
\end{equation*}
$$

Given a fully efficient assignment $\pi^{*}$, let $w(N)=\sum_{i \in M_{0}} v^{i}\left(\pi^{*}(i)\right)$. We call $w(N)$ the potential market value. Clearly, this value is the same for any fully efficient assignment.

A vector $r=\left(r^{0}, r^{1}, \cdots, r^{m}\right) \in \mathbb{R}_{+}^{m+1}$ is a feasible income distribution if $\sum_{i=0}^{m} r^{i}=$ $\sum_{i=1}^{m} m^{i}$. A pair $(\pi, r)$ of a feasible assignment $\pi$ and a feasible income distribution $r$ is called an allocation. At $(\pi, r)$, agent $i \in M$ receives bundle $\pi(i)$ and holds $r^{i}$ a total amount of income. Then the utilities that the bidders and the seller achieve are given by

$$
u^{i}(\pi, r)=v^{i}(\pi(i))+r^{i}, \quad \forall i \in M,
$$

and

$$
u^{0}(\pi, r)=v^{0}(\pi(0))+r^{0}=v^{0}(\pi(0))+\sum_{i \in M}\left(m^{i}-r^{i}\right)
$$

respectively.
When there is no budget constraint, the Walrasian equilibrium has been the most widely used solution for auction and equilibrium models and Walrasian equilibrium or market-clearing prices are used in auction design.

Given a price vector $p \in \mathbb{R}^{n}$ which specifies a price for each item, the demand set of bidder $i$ is defined by

$$
D^{i}(p)=\left\{S \in \mathcal{F}^{i} \mid v^{i}(S)-\sum_{j \in S} p(\{j\})=\max _{T \in \mathcal{F}^{i}}\left(v^{i}(T)-\sum_{j \in T} p(\{j\})\right\} .\right.
$$

So $D^{i}(p)$ is the collection of his most preferred bundles at prices $p$.
Definition 1. A Walrasian equilibrium is a pair $(p, \pi)$ of prices $p$ and assignment $\pi$ such that $\pi(i) \in D^{i}(p)$ with $\sum_{j \in \pi(i)} p(\{j\}) \leq m^{i}$ for every bidder $i \in M$ and $v^{0}(\pi(0))=\sum_{j \in \pi(0)} p(\{j\})$ for the seller.

At equilibrium, every bidder gets his best bundle at the prices within his budget and the price of the unsold bundle is equal to the seller's valuation of the bundle.

If $(p, \pi)$ is a Walrasian equilibrium, then $p$ is called an equilibrium or market-clearing price vector and $\pi$ a Walrasian equilibrium allocation. It is well-known from Koopmans and Beckmann (1957) and Kelso and Crawford (1982) that there will be at least one Walrasian equilibrium when all indivisible goods are substitutes and traders face no budget constraints.

The following example shows, however, that when buyers are budget constrained and even if their budgets are different, the Walrasian equilibrium still cannot be guaranteed to exist.

Example 1. A seller has two items for sale and a valuation of zero on each item. There are three bidders 1, 2 and 3. Each bidder demands no more than one item and has valuation and budget as given in Table 1. We have $\mathcal{F}^{i}=\{\varnothing,\{1\},\{2\}\}$ for every bidder $i$. Observe that each bidder has a different budget and both bidders 2 and 3 are financially constrained.

We will prove there exists no Walrasian equilibrium due to budget constraints. Suppose there would be a Walrasian equilibrium price vector $p=\left(p_{1}, p_{2}\right)$. It is easy to see

Table 1: Valuation and budget

| Bidder | $\varnothing$ | Item 1 | Item 2 | Budget $m^{i}$ |
| :---: | :---: | :---: | :---: | :---: |
| Bidder 1 | 0 | 8 | 6 | 9 |
| Bidder 2 | 0 | 7 | 0 | 5 |
| Bidder 3 | 0 | 0 | 6 | 3 |

that both items must be sold. This means that it is necessary to have $p_{1} \leq 8$ and $p_{2} \leq 6$. We need to consider the following cases in which the two inequalities hold.

Case 1. When $p_{1}=p_{2}+2$, we have $D^{1}(p)=\{\{1\},\{2\}\}$. If $p_{1} \leq m^{2}=5$, then we have $D^{2}(p)=\{\{1\}\}$ and $D^{3}(p)=\{\{2\}\}$ and both items are over-demanded. If $p_{1}>m^{2}=5$, then $D^{2}(p)=D^{3}(p)=\{\varnothing\}$, then one item is over-supplied. In either case, there is no equilibrium.

Case 2. When $p_{1}<p_{2}+2$, we have $D^{1}(p)=\{\{1\}\}$. In order to have an equilibrium we must have $p_{2} \leq m^{3}=3$, which implies $p_{1}<5=m^{2}$. Then we have $D^{2}(p)=\{\{1\}\}$. So item 1 is over-demanded and we cannot have an equilibrium.

Case 3. When $p_{1}>p_{2}+2$, we have $D^{1}(p)=\{\{2\}\}$. In order to have an equilibrium we must have $p_{1} \leq m^{2}=5$, which implies $p_{2}<3=m^{2}$. Then we have $D^{3}(p)=\{\{2\}\}$. So item 2 is over-demanded and we cannot have an equilibrium.

This example motivates us to make use of a more general solution: the core. The notion of core has been widely used in general equilibrium theory and cooperative game theory. We now introduce this concept.

An allocation $(\pi, r)$ is individually rational if every agent $i \in M_{0}$ achieves no less utility than they stand alone, i.e., for every $i \in M, u^{i}(\pi, r) \geq m^{i}$ and for the sell $u^{0}(\pi, r) \geq$ $v^{0}(N)$. An allocation $(\pi, r)$ is Pareto efficient if there does not exist another allocation $(\rho, t)$ such that $u^{i}(\pi, r)>u^{i}(\rho, t)$ for all $i \in M_{0}$; otherwise, we say that $(\pi, r)$ is strongly Pareto dominated by $(\rho, t)$. An allocation $(\pi, r)$ is strongly Pareto efficient if there does not exist another allocation $(\rho, t)$ such that $u^{i}(\pi, r) \geq u^{i}(\rho, t)$ for all $i \in M_{0}$ with at least one strict inequality; otherwise, we say that $(\pi, r)$ is Pareto dominated by $(\rho, t)$. A nonempty subset $S \subseteq M_{0}$ is called a viable coalition if $S$ consists of either the seller with any number of bidders or a single bidder. Given a viable coalition $S$, an allocation $\left(\rho^{S}, t^{S}\right)$ with $t^{S} \in \mathbb{R}_{+}^{m+1}$ and $\sum_{i=0}^{m} t_{i}^{S}=\sum_{i=0}^{m} m^{i}$ is feasible for $S$ if $\rho^{S}(i)=\varnothing$ and $t_{i}^{S}=m^{i}$ for every bidder $i \in M_{0} \backslash S$. An allocation $(\pi, r)$ is blocked by a viable coalition $S$ if there exists a feasible allocation
( $\rho^{S}, t^{S}$ ) such that $u^{i}\left(\rho^{S}, t^{S}\right)>u^{i}(\pi, r)$ for all $i \in S$; the allocation $(\pi, r)$ is weakly blocked by a viable coalition $S$ if there exists a feasible allocation $\left(\rho^{S}, t^{S}\right)$ such that $u^{i}\left(\rho^{S}, t^{S}\right) \geq$ $u^{i}(\pi, r)$ for all $i \in S$ and with at least one strict inequality.
Definition 2. An allocation $(\pi, r)$ is in the core and is called a core allocation if it is not blocked by any coalition. It is in the strong core and is called a strong core allocation if it cannot be weakly blocked by any coalition.

Clearly, every core allocation or element is Pareto efficient and every strong core allocation is strongly Pareto efficient. It can be shown that if no bidder is budget constrained, then every strongly Pareto efficient allocation is fully efficient. However, when bidders face budget constraints, a strongly Pareto efficient need not be fully efficient.

Let us return to Example 1 which has no Walrasian equilibrium due to budget constraints. However, it is easy to verify that this example has the following core allocations $\left(\pi^{1}, r^{1}\right)=((\varnothing,\{1\}, \varnothing,\{2\}),(9,3,5,0))$, and $\left(\pi^{2}, r^{2}\right)=((\varnothing,\{2\},\{1\}, \varnothing),(9,5,0,3))$. These are not in the strong core as they can be weakly blocked by a coalition. ${ }^{2}$

In Example 1, every bidder consumes at most one item. Next we we consider a more general case in which bidders may consume more than one item.
Example 2. There are three items $\{a, b, c\}$ and three bidders $\{1,2,3\}$. Each bidder's interested bundles, corresponding valuations, and budget are given in Table 2. Observe that for ease of notation, we express a bundle $\{a, b\}$ as $a b$.

Table 2: Valuations and budgets.

| Bidders | $\mathcal{F}^{i}$ | $v^{i}$ |
| :---: | :---: | :---: |
| 1 | $\{a b, a c, \varnothing\}$ | $v^{1}(a b)=4, v^{1}(a c)=4$ |
| 2 | $\{b, a b, \varnothing\}$ | $v^{2}(b)=3, v^{2}(a b)=4$ |
| 3 | $\{c, a c, \varnothing\}$ | $v^{3}(c)=4, v^{3}(a c)=6$ |

We first consider the case with budgets $\left(m^{1}, m^{2}, m^{3}\right)=(4,4,6)$. In this case no bidder is budget constrained. The market value is $w(0123)=9$, and the corresponding efficient assignment is $\pi^{*}=(\varnothing, \varnothing, b, a c)$ which allocates $b$ to bidder 2 and $a c$ to bidder 3 . By giving a feasible income distribution $r^{*}=(9,4,1,0)$, we construct a strong core allocation $\left(\pi^{*}, r^{*}\right)$. To show this, consider the utility distribution $\left(u_{0}, u_{1}, u_{2}, u_{3}\right)=(9,4,4,6)$, which

2 The strong core of this problem is not empty. For example, allocations $\left(\pi^{1}, r^{3}\right)=$ $((\varnothing,\{1\}, \varnothing,\{2\}),(10,2,5,0))$ and $\left(\pi^{2}, r^{4}\right)=((\varnothing,\{2\},\{1\}, \varnothing),(10,4,0,3))$ are both strong cores.
satisfies $\sum_{i \in M_{0}} u_{i}=23=w(0123)+\sum_{i \in M_{0}} m^{i}$ and $\sum_{i \in S} u_{i} \geq w(S)+\sum_{i \in S} m^{i}$ for any proper subset $S \subset\{0,1,2,3\} .{ }^{3}$

We now show that the market has no strong core under budgets $\left(m^{1}, m^{2}, m^{3}\right)=$ $(3,2,2)$. Since any wasteful allocation, in which the seller holds a bundle unsold, can be Pareto improved by assigning the unsold object(s) to the right person without changing the income distribution, so we focus on the four non-wasteful assignments: $\pi^{*}=$ $(\varnothing, \varnothing, b, a c), \pi^{\prime}=(\varnothing, a c, b, \varnothing), \pi^{\prime \prime}=(\varnothing, a b, \varnothing, c)$, and $\pi^{\prime \prime \prime}=(\varnothing, \varnothing, a b, c)$.

We first consider the fully efficient assignment $\pi^{*}=(\varnothing, \varnothing, b, a c)$. For any feasible income distribution $r=\left(7-r^{1}-r^{2}-r^{3}, r^{1}, r^{2}, r^{3}\right)$, individual rationality implies that $r^{1} \geq m^{1}=3 .\left(\pi^{*}, r\right)$ is weakly blocked by coalition $\{0,1,2\}$, and the blocking allocation is $\left(\pi^{\prime}, t(r)\right)$ with $t(r)=\left(8-r^{1}-r^{2}, r^{1}-3, r^{2}, 2\right)$ and

$$
\begin{aligned}
& u^{0}\left(\pi^{\prime}, t(r)\right)=8-r^{1}-r^{2}>7-r^{1}-r^{2}-r^{3}=u^{0}\left(\pi^{*}, r\right), \\
& u^{1}\left(\pi^{\prime}, t(r)\right)=v^{1}(a c)+r^{1}-3>r^{1}=u^{1}\left(\pi^{*}, r\right), \\
& u^{2}\left(\pi^{\prime}, t(r)\right)=v^{2}(b)+r^{2}=v^{2}(b)+r^{2}=u^{2}\left(\pi^{*}, r\right) .
\end{aligned}
$$

In a similar way, we can show that any individually rational allocation $\left(\pi^{\prime}, r\right)$ is weakly blocked by coalition $\{0,1,3\}$ and the proposed allocation is $\left(\pi^{\prime \prime},\left(7-r^{1}-r^{3}, r^{1}, 2, r^{3}-2\right)\right.$ ). The individually rational allocation $\left(\pi^{\prime \prime}, r\right.$ ) will be weakly blocked by coalition $\{0,1,2\}$ with the proposed allocation $\left(\pi^{\prime},\left(7-r^{1}-r^{2}, r^{1}, r^{2}-2,2\right)\right)$. Finally, $\left(\pi^{\prime \prime \prime}, r\right)$ will be weakly blocked by coalition $\{0,1,3\}$ with allocation $\left(\pi^{\prime \prime},\left(8-r^{1}-r^{2}, r^{1}-3,2, r^{3}\right)\right)$.

By contrast, the core is not empty. In fact, $\left(\pi^{\prime},(5,0,0,2)\right)$ and $\left(\pi^{\prime \prime},(5,0,2,0)\right)$ are both in the core. Because there is always a bidder receiving nothing and thus holding his budget by individual rationality, the seller cannot get more than an income of 5 . Then no coalition can improve the seller's revenue based on the two given core allocations.

We will establish the following core existence theorem for our auction model.
Theorem 1. There exists at least one core allocation in the auction model ( $v^{i}, \mathcal{F}^{i}, m^{i}, i \in M_{0}$ ).
In the next section we will propose a dynamic auction mechanism which always finds a core allocation, thus giving a constructive proof of the theorem.

[^2]
## 3 The Design of Dynamic Auction

In this section we will present two dynamic auctions for our auction model. The first auction can always find a core allocation under budget constraints and find a strong core allocation when no bidder is financially constrained. The second auction modifies the first one but needs to use more information from bidders and can always find a strongly Pareto efficient core allocation. Roughly speaking, in both auctions, at each round every bidder determines his bids on his interested bundles according to his current target utility and his budget and reports his bids. Then the auctioneer selects a provisional assignment based on reported bids from all bidders and her own valuations. If a bidder gets no item from the provisional assignment and his current target utility is still above his budget (such a bidder is called a provisional loser), then at least one of such bidders reduces his target utility and other bidders keep their target utilities unchanged, and the auction continues until there is no provisional loser.

### 3.1 The First Dynamic Auction

At each round of the auction, every bidder $i \in M$ has a target utility $\hat{u}^{i}$ being at least as high as his budget $m^{i}$ and will bid according to this target utility as follows. For every feasible bundle $A \in \mathcal{F}^{i}$, he sets an intermediate price as $\hat{p}^{i}\left(A \mid \hat{u}^{i}\right)=v^{i}(A)+m^{i}-\hat{u}^{i}$, and proposes his bidding price on $A$ as

$$
p^{i}\left(A \mid \hat{u}^{i}\right)= \begin{cases}0, & \text { if } \hat{p}^{i}\left(A \mid \hat{u}^{i}\right)<0 \\ \min \left\{\hat{p}^{i}\left(A \mid \hat{u}^{i}\right), m^{i}\right\}, & \text { if } \hat{p}^{i}\left(A \mid \hat{u}^{i}\right) \geq 0\end{cases}
$$

For example, suppose that bidder $i$ is interested in bundles $A$ and $B$ with $v^{i}(A)=10$ and $v^{i}(B)=5$, and his budget is 2 . We list all the target utilities and bidding prices in Table 3 .

Bidder anchors his bidding prices to his target utility, so he increases his bids by decreasing his target utility. We define the decrement of target utility, $\Delta^{i}\left(\hat{u}^{i}\right)$, as the minimal integer that leads to a new bidding profile. Formally,

$$
\Delta^{i}\left(\hat{u}^{i}\right)=\min \left\{k \in \mathbb{Z}_{+} \mid p^{i}\left(A \mid \hat{u}^{i}-k\right) \neq p^{i}\left(A \mid \hat{u}^{i}\right) \text { for some } A \in \mathcal{F}^{i}\right\}
$$

Table 3: Illustration of the target utilities and bidding prices.

| $\hat{u}^{i}$ | $\hat{p}^{i}\left(A \mid \hat{u}^{i}\right)$ | $\hat{p}^{i}\left(B \mid \hat{u}^{i}\right)$ | $p^{i}\left(A \mid \hat{u}^{i}\right)$ | $p^{i}\left(B \mid \hat{u}^{i}\right)$ |
| :---: | :---: | :---: | :---: | :---: |
| 12 | 0 | -5 | 0 | 0 |
| 11 | 1 | -4 | 1 | 0 |
| 10 | 2 | -3 | 2 | 0 |
| 9 | 3 | -2 | 2 | 0 |
| 8 | 4 | -1 | 2 | 0 |
| 7 | 5 | 0 | 2 | 0 |
| 6 | 6 | 1 | 2 | 1 |
| 5 | 7 | 2 | 2 | 2 |
| 4 | 8 | 3 | 2 | 2 |
| 3 | 9 | 4 | 2 | 2 |
| 2 | 10 | 5 | 2 | 2 |

Commonly, the decrement is one. However, when the bidding price of some bundle reaches the budget, the decrement may be larger than one. And when all the bidding prices reach the budget or the target utility reaches the budget, his bids cannot be increased further and consequently there will be no decrement and we denote this by $\Delta^{i}\left(\hat{u}^{i}\right)=\varnothing$. For instance, the minimal decrement of the above example is $\Delta^{i}\left(\hat{u}^{i}=11\right)=1$ since $p^{i}(A \mid 10)=2 \neq p^{i}(A \mid 11)=1$. The bidding prices keep unchanged when the target utility decreases from 10 to 9,8 , and 7 , so $\Delta^{i}\left(\hat{u}^{i}=10\right)=4$ because $p^{i}(B \mid 6)=1 \neq$ $p^{i}(B \mid 10)=0$. When $\hat{u}^{i}=5$, all the bidding prices reach the budget and no further bid increase is possible, so $\Delta^{i}\left(\hat{u}^{i}=5\right)=\varnothing$.

Now we give a complete description of the rules of the first dynamic auction mechanism.

Initialization: Every bidder $i \in M$ sets a target utility $\hat{u}_{1}^{i} \in \mathbb{Z}_{+}$that he wishes to achieve

$$
\hat{u}_{1}^{i} \geq \max _{A \in \mathcal{F}^{i}} v^{i}(A)+m^{i}
$$

Set $t=1$ and go to the Bidding Step.
Bidding Step: For any bidder $i \in M$, if it is the first round $t=1$ or he revises his target utility $\hat{u}_{t}^{i}<\hat{u}_{t-1}^{i}$, he makes new bids as follows. For every feasible bundle $A \in \mathcal{F}^{i}$, bidder $i$ sets an intermediate bidding price $\hat{p}_{t}^{i}\left(A \mid \hat{u}_{t}^{i}\right)=v^{i}(A)+m^{i}-\hat{u}_{t}^{i}$ and adjusts
his bidding prices as

$$
p_{t}^{i}\left(A \mid \hat{u}_{t}^{i}\right)\left(=p_{t}^{i}(A)\right)= \begin{cases}0, & \text { if } \hat{p}_{t}^{i}\left(A \mid \hat{u}_{t}^{i}\right)<0 \\ \min \left\{\hat{p}_{t}^{i}\left(A \mid \hat{u}_{t}^{i}\right), m^{i}\right\}, & \text { if } \hat{p}_{t}^{i}\left(A \mid \hat{u}_{t}^{i}\right) \geq 0\end{cases}
$$

Let $p_{t}^{j}=p_{t-1}^{j}$ for any $t>1$ and any other bidder $j \in M$ who does not revise his target utility $\hat{u}_{t}^{j}=\hat{u}_{t-1}^{j}$. Then all prices $p_{t}^{h}$ for $h \in M$ are reported to the auctioneer. Go to the Assigning Step.

Assigning Step: Based on the current bidding prices $P_{t}=\left(p_{t}^{0}, p_{t}^{1}, \ldots, p_{t}^{m}\right)$ with $p_{t}^{0}(S)=$ $v^{0}(S)$ for every $S \in 2^{N}$, the auctioneer announces a provisional assignment $\pi_{t}$ yielding the highest revenue, i.e., an element of the following optimal set:

$$
C\left(P_{t}\right)=\left\{\pi \in \mathcal{A} \mid \pi=\underset{\rho \in \mathcal{A}}{\arg \max }\left(\sum_{i \in M_{0}} p_{t}^{i}(\rho(i))\right) \text { s.t. } p_{t}^{i}(\rho(i))>0 \forall \rho(i) \neq \varnothing \text { and } i>0\right\} .
$$

We require $p_{t}^{i}(\rho(i))>0$ because zero-price offers lead less utilities to bidders ${ }^{4}$ and no revenue to seller. Then go to the following step.

Continue or Stop: At the provisional assignment $\pi_{t}$, if there is a bidder $i \in M$ with $\pi_{t}(i)=\varnothing$ and $\Delta^{i}\left(\hat{u}_{t}^{i}\right) \neq \varnothing$ (which means that this bidder still has incentive to make new bids), at least one such bidder $i$ updates his target utility by setting $\hat{u}_{t+1}^{i}=\hat{u}_{t}^{i}-\Delta^{i}\left(\hat{u}_{t}^{i}\right)$, and every other bidder $j \in M$ keeps his target utility unchanged as $\hat{u}_{t+1}^{j}=\hat{u}_{t}^{j}$. Then set $t=t+1$ and go back to the Bidding Step. Otherwise, the auction stops with the provisional assignment $\pi_{t}$. Then every bidder $i \in M$ receives bundle $\pi_{t}(i)$ and pays his bidding price $p_{t}^{i}\left(\pi_{t}(i)\right)$. If a bidder $i$ gets no item, he updates his target utility by setting $\hat{u}_{t}^{i}=m^{i}$. The seller keeps bundle $\pi_{t}(0)$ and receives all payments from bidders for sold items.

In the auction, bidder $i \in M$ is called a provisional loser (at round $t$ ) if he gets no item from the provisional assignment $\pi_{t}$ and his target utility $\hat{u}_{t}^{i}$ can be further reduced to offer new bids, i.e., $\Delta^{i}\left(\hat{u}_{t}^{i}\right) \neq \varnothing$. Observe that in the auction, every bidder reports only

[^3]his bidding prices to the auctioneer and does not need to reveal any other information, and that no bidder will be assigned a nonempty bundle with a zero bidding price.

To have a better understanding of our proposed auction mechanism, we illustrate it through Example 2 with the constrained budgets $\left(m^{1}, m^{2}, m^{3}\right)=(3,2,2)$. Table 4 collects the data generated by the proposed auction mechanism and shows that the auction finds the core allocation $\left(\pi^{\prime \prime},(5,0,2,0)\right) .{ }^{5}$ At the end of the auction, bidder 2 gets no item and adjusts his target utility as $\hat{u}_{10}^{2}=m^{2}=2$.

Table 4: Illustration of the proposed auction mechanism via Example 2.

| $t$ | $\hat{u}_{t}^{1}$ | $p_{t}^{1}(a b, a c, \varnothing)$ | $\hat{u}_{t}^{2}$ | $p_{t}^{2}(b, a b, \varnothing)$ | $\hat{u}_{t}^{3}$ | $p_{t}^{3}(c, a c, \varnothing)$ | $\pi_{t}(0,1,2,3)$ | $r_{t}(0,1,2,3)$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 1 | 7 | $(0,0,0)$ | 6 | $(0,0,0)$ | 9 | $(0,0,0)$ | $(a b c, \varnothing, \varnothing, \varnothing)$ | $(0,3,2,2)$ |
| 2 | 6 | $(1,1,0)$ | 5 | $(0,1,0)$ | 9 | $(0,0,0)$ | $(c, a b, \varnothing, \varnothing)$ | $(1,2,2,2)$ |
| 3 | 6 | $(1,1,0)$ | 4 | $(1,2,0)$ | 9 | $(0,0,0)$ | $(\varnothing, a c, b, \varnothing)$ | $(2,2,1,2)$ |
| 4 | 6 | $(1,1,0)$ | 4 | $(1,2,0)$ | 7 | $(0,1,0)$ | $(\varnothing, a c, b, \varnothing)$ | $(2,2,1,2)$ |
| 5 | 6 | $(1,1,0)$ | 4 | $(1,2,0)$ | 6 | $(0,2,0)$ | $(\varnothing, \varnothing, b, a c)$ | $(3,3,1,0)$ |
| 6 | 5 | $(2,2,0)$ | 4 | $(1,2,0)$ | 6 | $(0,2,0)$ | $(\varnothing, \varnothing, b, a c)$ | $(3,3,1,0)$ |
| 7 | 4 | $(3,3,0)$ | 4 | $(1,2,0)$ | 6 | $(0,2,0)$ | $(\varnothing, a c, b, \varnothing)$ | $(4,0,1,2)$ |
| 8 | 4 | $(3,3,0)$ | 4 | $(1,2,0)$ | 5 | $(1,2,0)$ | $(\varnothing, a b, \varnothing, c)$ | $(4,0,2,1)$ |
| 9 | 4 | $(3,3,0)$ | 3 | $(2,2,0)$ | 5 | $(1,2,0)$ | $(\varnothing, a c, b, \varnothing)$ | $(5,0,0,2)$ |
| 10 | 4 | $(3,3,0)$ | 3 | $(2,2,0)$ | 4 | $(2,2,0)$ | $(\varnothing, a b, \varnothing, c)$ | $(5,0,2,0)$ |

Having the above discussion we will prove that our proposed auction mechanism is well-designed and will always find a core allocation in finite rounds.
Lemma 1. The proposed auction terminates in finitely many rounds with each bidder's target utility no less than his budget.

Proof. Observe that for every bidder $i \in M$, his initial target value $\hat{u}_{1}^{i}$ is an integer no less than $\max _{A \in \mathcal{F}^{i}} v^{i}(A)+m^{i} \geq m^{i}$. His target utility decreases at least by 1 if he is a provisional loser and his target utility can be further reduced so that he can make new bids. Because $m^{i}$ is an integer and the target utility is also an integer, the target utility can at most reduce to $m^{i}$. Therefore, the auction must terminate in finitely many rounds.

[^4]In the auction process, every bidder $i^{\prime}$ s bidding price $p_{t}^{i}$ is an increasing function of time $t$. Because $\hat{u}_{t}^{i} \geq m^{i}$ for all $t$, by the pricing formula his bidding price for every feasible bundle will never be higher than his valuation of the bundle nor above his budget, i.e., $p_{t}^{i}(A) \leq \min \left\{m^{i}, v^{i}(A)\right\}$ for all $t$ and $A \in \mathcal{F}^{i}$.

Suppose the auction stops in round $T$. The final assignment is $\pi_{T}$, and the corresponding income distribution is $r_{T}^{0}=\sum_{i \in M} p_{T}^{i}\left(\pi_{T}(i)\right)$ for the seller, and $r_{T}^{i}=m^{i}-$ $p_{T}^{i}\left(\pi_{T}(i)\right)$ for every bidder $i \in M$. Bidder $i \in M$ is said to be a loser if his assigned bundle $\pi_{T}(i)$ is empty; otherwise, he is a winner. Let $\left(\pi_{T}, r_{T}\right)$ denote the final outcome generated by the auction. Clearly, a loser gets no item and pays nothing.

The next result shows that the price of every sold bundle by the auction is at least as high as its marginal value to the seller.
Lemma 2. At the outcome $\left(\pi_{T}, r_{T}\right)$ generated by the proposed auction, every bidder $i \in M$ pays a price for his assigned bundle no less than its marginal value to the seller, i.e.,

$$
p_{T}^{i}\left(\pi_{T}(i)\right) \geq v^{0}\left(\pi_{T}(0) \cup \pi_{T}(i)\right)-v^{0}\left(\pi_{T}(0)\right)
$$

Proof. It follows immediately from the Assigning Step of the auction that the seller is maximizing her revenue.

Recall that we do not impose any condition upon the valuation functions of bidders and the seller. The following result as a corollary of Lemma 2 proves that when the seller's valuation function is super-additive, the price of every sold bundle by the auction is at least as high as the seller's valuation or reserve price of the bundle.
Corollary 1. When the seller's valuation function $v^{0}$ is super-additive, i.e., $v^{0}(K \cup L) \geq$ $v^{0}(K)+v^{0}(L)$ for any two disjoint sets $K$ and $L$ of items, then at the outcome $\left(\pi_{T}, r_{T}\right)$ generated by the proposed auction, every bidder $i \in M$ pays a price for his assigned bundle no less than the seller's valuation or reserve price, i.e., $p_{T}^{i}\left(\pi_{T}(i)\right) \geq v^{0}\left(\pi_{T}(i)\right)$.

Proof. It follows that

$$
\begin{aligned}
p_{T}^{i}\left(\pi_{T}(i)\right) & \geq v^{0}\left(\pi_{T}(0) \cup \pi_{T}(i)\right)-v^{0}\left(\pi_{T}(0)\right) \\
& \geq v^{0}\left(\pi_{T}(0)\right)+v^{0}\left(\pi_{T}(i)\right)-v^{0}\left(\pi_{t}(0)\right) \\
& =v^{0}\left(\pi_{T}(i)\right) .
\end{aligned}
$$

Lemma 3. The outcome $\left(\pi_{T}, r_{T}\right)$ generated by the proposed auction is individually rational and gives every bidder $i$ at least his target utility $\hat{u}_{T}^{i}$.

Proof. Observe that each loser $i$ achieves utility $u^{i}\left(\pi_{T}, r_{T}\right)=v^{i}(\varnothing)+m^{i}=m^{i}=\hat{u}_{T}^{i}$. Each winner $i^{\prime}$ s final target utility $\hat{u}_{T}^{i} \geq m^{i}$. We prove that each winner achieves his final utility $u^{i}\left(\pi_{T}, r_{T}\right) \geq \hat{u}_{T}^{i}$. To see this, let $A=\pi_{T}(i)$ denote $i^{\prime}$ s assigned bundle. Since the winner's price is not zero, we have $p_{T}^{i}(A)=\min \left\{v^{i}(A)-\hat{u}_{T}^{i}+m^{i}, m^{i}\right\}$. We need to consider the following two cases. Firstly, if it holds $0<\hat{p}_{T}^{i}(A) \leq m^{i}$, then bidder $i^{\prime}$ s utility is $u^{i}\left(\pi_{T}, r_{T}\right)=v^{i}(A)+m^{i}-p_{T}^{i}(A)=v^{i}(A)+m^{i}-\hat{p}_{T}^{i}(A)=\hat{u}_{T}^{i}$. Secondly, if it holds $\hat{p}_{T}^{i}(A)>m^{i}$ implying $v^{i}(A)>\hat{u}_{T}^{i}$, then $p_{T}^{i}(A)=m^{i}$ and bidder $i^{\prime}$ s utility equals $u^{i}\left(\pi_{T}, r_{T}\right)=v^{i}(A)+m^{i}-m^{i}=v^{i}(A)>\hat{u}_{T}^{i}$.

For the seller, consider the no-sale assignment $\bar{\rho}$ such that $\bar{\rho}(0)=N$ and $\bar{\rho}(i)=\varnothing$ for all $i \in M$. It is feasible, i.e., $\bar{\rho} \in \mathcal{A}$, and gives her the utility of $\sum_{i \in M_{0}} p_{T}^{i}(\bar{\rho}(i))=$ $p_{T}^{0}(N)=v^{0}(N)$. The seller's optimal choice guarantees her rationality, $u^{0}\left(\pi_{T}, r_{T}\right)=$ $\max _{\rho \in \mathcal{A}} \sum_{i \in M_{0}} p_{T}^{i}(\rho(i)) \geq v^{0}(N)$.

We are ready to establish the first major result concerning the auction. It states that the outcome generated by the auction is in the core and thus is a Pareto efficient allocation.
Theorem 2. The outcome $\left(\pi_{T}, r_{T}\right)$ yielded by the proposed auction is in the core and thus Pareto efficient.

Proof. By Lemma 3, the outcome $\left(\pi_{T}, r_{T}\right)$ is individually rational. Suppose to the contrary that $\left(\pi_{T}, r_{T}\right)$ is not in the core. Then there exists a coalition $S$ consisting of the seller and at least one bidder with an allocation $\left(\rho^{S}, t\right)$ such that $u^{i}\left(\rho^{S}, t\right)>u^{i}\left(\pi_{T}, r_{T}\right)$ for all $i \in S$. Without loss of generality, we can assume that $t^{i} \leq m^{i}$ for all $i \in S \backslash\{0\}$. To see
this, suppose that there is a nonempty subset of bidders $J \subseteq S \backslash\{0\}$ such that $t^{j}>m^{j}$ for all $j \in J$. Then some bidder in $S$ must transfer his money to bidders in $J$. Let $K$ be the set of all such bidders. We can construct a new blocking coalition $S^{\prime}=S \backslash J$, which obviously contains the set $K$. Let every agent in $S^{\prime}$ keep the same bundle and the same income as at $\left(\rho^{S}, t\right)$, while for every bidder $i \in K \cap S^{\prime}$, beside his income $t_{i}$ let him also take back his transferred income from the set $J$, and let the seller get back her items from the bidders in $J$. In this way we obtain a new allocation $\left(\rho^{S^{\prime}}, t^{\prime}\right)$ at which $u^{i}\left(\rho^{S^{\prime}}, t^{\prime}\right) \geq u^{i}\left(\rho^{S}, t\right)>u^{i}\left(\pi_{T}, r_{T}\right)$ for every $i \in S^{\prime}$ and $t^{i} \leq m^{i}$ for all $i \in S^{\prime} \backslash\{0\}$.

For each bidder in the coalition $i \in S \backslash\{0\}$, if he is a winner of the auction, then his final target utility in the auction is lower than the improved utility, i.e.,

$$
\hat{u}_{T}^{i} \leq u^{i}\left(\pi_{T}, r_{T}\right)<u^{i}\left(\rho^{S}, t\right)=v^{i}\left(\rho^{S}(i)\right)+t^{i} .
$$

This implies that he sets a higher intermediate price on $\rho^{S}(i)$, i.e.,

$$
\hat{p}_{T}^{i}\left(\rho^{S}(i)\right)=v^{i}\left(\rho^{S}(i)\right)+m^{i}-\hat{u}_{T}^{i}>m^{i}-t^{i} \geq 0
$$

and proposes a higher bidding price on $\rho^{S}(i)$, i.e.,

$$
p_{T}^{i}\left(\rho^{S}(i)\right)=\min \left\{m^{i}, \hat{p}_{T}^{i}\left(\rho^{S}(i)\right)\right\} \geq m^{i}-t^{i}
$$

If $i \in S \backslash\{0\}$ is a loser of the auction, he is also improved in the coalition, i.e.,

$$
m^{i}=u^{i}\left(\pi_{T}, r_{T}\right)<u^{i}\left(\rho^{S}, t\right)=v^{i}\left(\rho^{S}(i)\right)+t^{i} .
$$

The loser cannot increase his bids anymore, i.e., $\Delta^{i}\left(\hat{u}_{T}^{i}\right)=\varnothing$, so either his bidding prices on all the feasible bundle reach his budget, which implies that

$$
p_{T}^{i}\left(\rho^{S}(i)\right)=m^{i} \geq m^{i}-t^{i}
$$

or his target utility reaches his budget, which implies that $\hat{p}_{T}^{i}\left(\rho^{S}(i)\right)=v^{i}\left(\rho^{S}(i)\right)$ and

$$
p_{T}^{i}\left(\rho^{S}(i)\right)=\min \left\{m^{i}, v^{i}\left(\rho^{S}(i)\right)\right\} \geq m^{i}-t^{i}
$$

Then for the seller we have

$$
\begin{aligned}
\sum_{i \in S} p_{T}^{i}\left(\rho^{S}(i)\right) & \geq p_{T}^{0}\left(\rho^{S}(0)\right)+\sum_{i \in S \backslash\{0\}}\left(m^{i}-t^{i}\right) \\
& =u^{0}\left(\rho^{S}, t\right) \\
& >u^{0}\left(\pi_{T}, r\right) \\
& =\sum_{i \in M_{0}} p_{T}^{i}\left(\pi_{T}(i)\right)
\end{aligned}
$$

It contradicts that $\pi_{T}$ maximizes the seller's revenue based on bidding prices $P_{T}$.
The previous theorem shows that the auction always generates a core allocation no matter whether bidders are budget constrained or not, yielding a constructive proof of Theorem 1. We know that when bidders face budget constraints, we can guarantee to find a core and Pareto efficient allocation but we cannot expect to have a strong core allocation and therefore we have to accept some loss of market efficiency. This raises an important question whether the auction can find a strong core allocation when bidders are not budget constrained. Our next result establishes that the auction will always generate a strong core allocation if bidders can afford to pay up to their valuations. Every strong core allocation must be strongly Pareto efficient. When no bidder is budget constrained, every strongly Pareto efficient allocation must be fully efficient and thus every strong core allocation must be fully efficient.
Theorem 3. When no bidder faces a budget constraint, the outcome $\left(\pi_{T}, r_{T}\right)$ yielded by the proposed auction is in a strong core and thus strongly Pareto efficient and fully efficient.

Proof. We first prove that at the final outcome $\left(\pi_{T}, r_{T}\right)$ every bidder $i \in M$ achieve his target utility $u^{i}\left(\pi_{T}, r_{T}\right)=\hat{u}_{T}^{i}$. If a bidder $i \in M$ is a loser, he receives nothing such that $u^{i}\left(\pi_{T}, r_{T}\right)=m^{i}$. According to the definition of the auction, when the loser is not budget constrained, he stops to decrease his target utility until $\hat{u}_{T}^{i}=m^{i}$. We then prove
that each winner's final utility $u^{i}\left(\pi_{T}, r_{T}\right)$ also equals his target utility $\hat{u}_{T}^{i}$. To see this, let $A=\pi_{T}(i)$ denote $i$ 's assigned bundle. Since the winner's price is not zero, we have $p_{T}^{i}(A)=\min \left\{v^{i}(A)-\hat{u}_{T}^{i}+m^{i}, m^{i}\right\}$. There are two cases: We first consider the case of $0<\hat{p}_{T}^{i}(A) \leq m^{i}$. By the bidding rule we have $p_{T}^{i}(A)=\hat{p}_{T}^{i}(A)$ and then bidder $i^{\prime}$ s final utility equals $u^{i}\left(\pi_{T}, r_{T}\right)=v^{i}(A)+m^{i}-p_{T}^{i}(A)=v^{i}(A)+m^{i}-\hat{p}_{T}^{i}(A)=\hat{u}_{T}^{i}$. The second case is $\hat{p}_{T}^{i}(A)>m^{i}$, which implies $v^{i}(A)>\hat{u}_{T}^{i}$. By Lemma 1 we have $\hat{u}_{T}^{i} \geq m^{i}$. By this we would have $v^{i}(A)>m^{i}$. On the other hand, because bidder $i$ is not budget constrained, we must have $v^{i}(A) \leq m^{i}$ contradicting $v^{i}(A)>m^{i}$. Therefore this case is not possible. We can conclude that for every bidder $i \in M$, no matter whether he is a winner or loser, his final utility $u^{i}\left(\pi_{T}, r_{T}\right)$ equals his final target utility $\hat{u}_{T}^{i}$.

Now suppose to the contrary that $\left(\pi_{T}, r_{T}\right)$ is not in the strong core. By Lemma 3, as the outcome $\left(\pi_{T}, r_{T}\right)$ is individually rational, it cannot be blocked by any single agent. Then there would exist a coalition $S$ with the seller and at least one bidder and an implementable pair $\left(\rho^{S}, t\right)$ such that $u^{i}\left(\rho^{S}, t\right) \geq u^{i}\left(\pi_{T}, r_{T}\right)$ for all $i \in S$ with at least one strict inequality. Without loss of any generality, we can assume that $t^{i} \leq m^{i}$ for all $i \in S \backslash\{0\}$.

Consider the case in which the seller is strictly improved, i.e., $u^{0}\left(\rho^{S}, t\right)>u^{0}\left(\pi_{T}, r_{T}\right)$. For every bidder $i \in S \backslash\{0\}$, the blocking condition implies that $\hat{u}_{T}^{i}=u^{i}\left(\pi_{T}, r_{T}\right) \leq$ $u^{i}\left(\rho^{S}, t\right)=v^{i}\left(\rho^{S}(i)\right)+t^{i}$, and hence $p_{T}^{i}\left(\rho^{S}(i)\right) \geq \hat{p}_{T}^{i}\left(\rho^{S}(i)\right)=v^{i}\left(\rho^{S}(i)\right)+m^{i}-\hat{u}_{T}^{i} \geq$ $m^{i}-t^{i} \geq 0$. For the seller, we have

$$
\begin{aligned}
\sum_{i \in S} p_{T}^{i}\left(\rho^{S}(i)\right) & \geq p_{T}^{0}\left(\rho^{S}(0)\right)+\sum_{i \in S \backslash\{0\}}\left(m^{i}-t^{i}\right) \\
& =u^{0}\left(\rho^{S}, t\right) \\
& >u^{0}\left(\pi_{T}, r\right) \\
& =\sum_{i \in M_{0}} p_{T}^{i}\left(\pi_{T}(i)\right)
\end{aligned}
$$

It contradicts the fact that $\pi_{T}$ maximizes the seller's revenue on bidding prices $P_{T}$.
Consider the other case in which at least one bidder is strictly improved. We have the inequalities $p_{T}^{i}\left(\rho^{S}(i)\right) \geq m^{i}-t^{i}$ for all $i \in S \backslash\{0\}$, and $p_{T}^{i}\left(\rho^{S}(i)\right)>m^{i}-t^{i}$ for the
bidder being strictly improved. Then for the seller, we have the strict inequality

$$
\begin{aligned}
\sum_{i \in S} p_{T}^{i}\left(\rho^{S}(i)\right) & >p_{T}^{0}\left(\rho^{S}(0)\right)+\sum_{i \in S \backslash\{0\}}\left(m^{i}-t^{i}\right) \\
& =u^{0}\left(\rho^{S}, t\right) \\
& \geq u^{0}\left(\pi_{T}, r\right) \\
& =\sum_{i \in M_{0}} p_{T}^{i}\left(\pi_{T}(i)\right)
\end{aligned}
$$

It yields also a contradiction.

### 3.2 A Comparison with Ausubel and Milgrom's Auction

As mentioned previously, although Ausubel and Milgrom (2002) focus on their package auction without budget constraints, they also briefly discuss in their Section 8 a model with budget constrained bidders; see also Milgrom (2004). Their model is different from ours. The key assumption in their model is that every bidder has only a finite number of choices and has strict preferences over those choices and the seller also has strict preferences over her choices. Each choice may include a bundle of items and an amount of money. The basic idea of their auction bears some similarity with the Gale-Shapley deferred acceptance procedure. At each round of their auction, each bidder makes his most preferred offer to the seller. The seller compares the offers she has in hand, including the offers she has received from the previous rounds, and rejects all but her most preferred offers. At the next round, each bidder whose offer was rejected previously offers his next most preferred choice on his preference list to the seller. Offers not rejected remain in force. The process continues until no new offers are made. And when the process stops, those offers in force become finally accepted.

We use Example 1 (a simple example) and Example 2 (a bit more general example) to show that Ausubel-Milgrom auction is too restrictive to work properly, as in the current model and also in any practical model, bidders and the seller can be often indifferent between many choices.

We use our Example 1 to demonstrate that the outcome of Ausubel-Milgrom auction
lies outside the core. This is not because of integer prices. In fact, using the same tiebreaking rule below, the outcome of their auction still lies outside the core even if the increment of offer prices is 0.5 or other smaller number. Consider the following tiebreaking rules: (i) every bidder $i$ always bids for item 1 first when he is indifferent between item 1 and another item, and (ii) when several assignments yield same revenue, the seller chooses the one in which bidder 3 is a winner. If there are still ties, the seller chooses the assignment in which bidder 1 is a winner. ${ }^{6}$ Table 5 shows the process of the Ausubel-Milgrom auction under the tie-breaking rule.

Table 5: Illustration of the Ausubel-Milgrom auction via Example 1.

| Round | Bidding Prices |  |  | Assignment | Income | Winners |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $t$ | $p^{1}(1,2, \varnothing)$ | $p^{2}(1,2, \varnothing)$ | $p^{3}(1,2, \varnothing)$ | $\pi_{t}(0,1,2,3)$ | $r_{t}(0,1,2,3)$ |  |
| 1 | $(0,0,0)$ | $(0,0,0)$ | $(0,0,0)$ | $(123, \varnothing, \varnothing, \varnothing)$ | $(0,9,5,3)$ |  |
| 2 | $(1,0,0)$ | $(1,0,0)$ | $(0,1,0)$ | $(\varnothing, 1, \varnothing, 2)$ | $(2,8,5,2)$ | 1,3 |
| 3 | $(1,0,0)$ | $(2,0,0)$ | $(0,1,0)$ | $(\varnothing, \varnothing, 1,2)$ | $(3,9,4,2)$ | 2,3 |
| 4 | $(2,0,0)$ | $(2,0,0)$ | $(0,1,0)$ | $(\varnothing, 1, \varnothing, 2)$ | $(3,7,5,2)$ | 1,3 |
| 5 | $(2,0,0)$ | $(3,0,0)$ | $(0,1,0)$ | $(\varnothing, \varnothing, 1,2)$ | $(4,9,2,2)$ | 2,3 |
| 6 | $(3,0,0)$ | $(3,0,0)$ | $(0,1,0)$ | $(\varnothing, 1, \varnothing, 2)$ | $(4,6,5,2)$ | 1,3 |
| 7 | $(3,0,0)$ | $(4,0,0)$ | $(0,1,0)$ | $(\varnothing, \varnothing, 1,2)$ | $(5,9,1,2)$ | 2,3 |
| 8 | $(3,1,0)$ | $(4,0,0)$ | $(0,1,0)$ | $(\varnothing, \varnothing, 1,2)$ | $(5,9,1,2)$ | 2,3 |
| 9 | $(4,1,0)$ | $(4,0,0)$ | $(0,1,0)$ | $(\varnothing, 1, \varnothing, 2)$ | $(5,5,5,2)$ | 1,3 |
| 10 | $(4,1,0)$ | $(5,0,0)$ | $(0,1,0)$ | $(\varnothing, \varnothing, 1,2)$ | $(6,9,0,2)$ | 2,3 |
| 11 | $(4,2,0)$ | $(5,0,0)$ | $(0,1,0)$ | $(\varnothing, 2,1, \varnothing)$ | $(7,7,0,3)$ | 1,2 |
| 12 | $(4,2,0)$ | $(5,0,0)$ | $(0,2,0)$ | $(\varnothing, \varnothing, 1,2)$ | $(7,9,0,1)$ | 2,3 |
| 13 | $(5,2,0)$ | $(5,0,0)$ | $(0,2,0)$ | $(\varnothing, 1, \varnothing, 2)$ | $(7,4,5,1)$ | 1,3 |

Let $\left(\pi^{A M A}, r^{A M A}\right)=((\varnothing, 1, \varnothing, 2),(7,4,5,1))$ denote the allocation yielded by AusubelMilgrom auction. To show that $\left(\pi^{A M A}, r^{A M A}\right)$ is not in the core, consider the coalition

[^5]$\{0,1,2\}$ and the blocking allocation $\left(\pi^{2}, t\right)=((\varnothing, 2,1, \varnothing),(7.5,6.5,0,3))$ in which
\[

$$
\begin{gathered}
u^{0}\left(\pi^{2}, t\right)=7.5>7=u^{0}\left(\pi^{A M A}, r^{A M A}\right) \\
u^{1}\left(\pi^{2}, t\right)=12.5>12=u^{1}\left(\pi^{A M A}, r^{A M A}\right) \\
u^{2}\left(\pi^{2}, t\right)=7>5=u^{2}\left(\pi^{A M A}, r^{A M A}\right)
\end{gathered}
$$
\]

Our dynamic auction runs as follows, and finds a core allocation. The auction of Talman and Yang (2015) finds also the same core allocation.

Table 6: Illustration of the proposed auction mechanism via Example 1.

| $t$ | $\hat{u}_{t}^{1}$ | $p_{t}^{1}(1,2, \varnothing)$ | $\hat{u}_{t}^{2}$ | $p_{t}^{2}(1,2, \varnothing)$ | $\hat{u}_{t}^{3}$ | $p_{t}^{3}(1,2, \varnothing)$ | $\pi_{t}$ | $r_{t}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 1 | 17 | $(0,0,0)$ | 12 | $(0,0,0)$ | 9 | $(0,0,0)$ | $(12, \varnothing, \varnothing, \varnothing)$ | $(0,9,5,3)$ |
| 2 | 16 | $(1,0,0)$ | 11 | $(1,0,0)$ | 8 | $(0,1,0)$ | $(\varnothing, 1, \varnothing, 2)$ | $(2,8,5,2)$ |
| 3 | 16 | $(1,0,0)$ | 10 | $(2,0,0)$ | 8 | $(0,1,0)$ | $(\varnothing, \varnothing, 1,2)$ | $(3,9,3,2)$ |
| 4 | 15 | $(2,0,0)$ | 10 | $(2,0,0)$ | 8 | $(0,1,0)$ | $(\varnothing, \varnothing, 1,2)$ | $(3,9,3,2)$ |
| 5 | 14 | $(3,1,0)$ | 10 | $(2,0,0)$ | 8 | $(0,1,0)$ | $(\varnothing, 1, \varnothing, 2)$ | $(4,6,5,2)$ |
| 6 | 14 | $(3,1,0)$ | 9 | $(3,0,0)$ | 8 | $(0,1,0)$ | $(\varnothing, 1, \varnothing, 2)$ | $(4,6,5,2)$ |
| 7 | 14 | $(3,1,0)$ | 10 | $(4,0,0)$ | 8 | $(0,1,0)$ | $(\varnothing, \varnothing, 1,2)$ | $(5,9,1,2)$ |
| 8 | 13 | $(4,2,0)$ | 10 | $(4,0,0)$ | 8 | $(0,1,0)$ | $(\varnothing, 2,1, \varnothing)$ | $(6,7,1,3)$ |
| 9 | 13 | $(4,2,0)$ | 10 | $(4,0,0)$ | 7 | $(0,2,0)$ | $(\varnothing, 2,1, \varnothing)$ | $(6,7,1,3)$ |
| 10 | 13 | $(4,2,0)$ | 10 | $(4,0,0)$ | 6 | $(0,3,0)$ | $(\varnothing, 1, \varnothing, 2)$ | $(7,5,5,0)$ |
| 11 | 13 | $(4,2,0)$ | 9 | $(5,0,0)$ | 6 | $(0,3,0)$ | $(\varnothing, \varnothing, 1,2)$ | $(8,9,0,0)$ |
| 12 | 12 | $(5,3,0)$ | 9 | $(5,0,0)$ | 6 | $(0,3,0)$ | $(\varnothing, \varnothing, 1,2)$ | $(8,9,0,0)$ |
| 13 | 11 | $(6,4,0)$ | 9 | $(5,0,0)$ | 6 | $(0,3,0)$ | $(\varnothing, 1, \varnothing, 2)$ | $(9,3,5,0)$ |

Let us turn to Example 2 with the constrained budgets $\left(m^{1}, m^{2}, m^{3}\right)=(3,2,2)$. The following tie-breaking rules will be used.

- If bidder $i$ is indifferent between many offers, he prefers the offer(s) containing item $a$ to those without $a$. If there are still ties, he prefers the offers containing item $b$ to those without $b$, and so on.
- If multiple outcomes yield the same revenue, the seller always prefers an outcome in which she can keep $c$ unsold to any choice without $c$. If still multiple outcomes
yield the same revenue and allow the seller to hold the same items, the seller prefers bidder 1 to bidder 3, and to bidder 2 .

Table 7: Illustration of the Ausubel-Milgrom auction via Example 2.

| $t$ | $p^{1}(a b, a c, \varnothing)$ | $p^{2}(b, a b, \varnothing)$ | $p^{3}(c, a c, \varnothing)$ | $\pi_{t}(0,1,2,3)$ | $r_{t}(0,1,2,3)$ | Winner |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 1 | $(0,0,0)$ | $(0,0,0)$ | $(0,0,0)$ | $(a b c, \varnothing, \varnothing, \varnothing)$ | $(0,3,2,2)$ |  |
| 2 | $(1,0,0)$ | $(0,1,0)$ | $(0,1,0)$ | $(c, a b, \varnothing, \varnothing)$ | $(1,2,2,2)$ | 1 |
| 3 | $(1,0,0)$ | $(0,2,0)$ | $(0,2,0)$ | $(c, \varnothing, a b, \varnothing)$ | $(2,3,0,2)$ | 2 |
| 4 | $(1,1,0)$ | $(0,2,0)$ | $(1,2,0)$ | $(\varnothing, \varnothing, a b, c)$ | $(3,3,0,1)$ | 2,3 |
| 5 | $(2,1,0)$ | $(0,2,0)$ | $(1,2,0)$ | $(\varnothing, a b, \varnothing, c)$ | $(3,1,2,1)$ | 1,3 |
| 6 | $(2,1,0)$ | $(1,2,0)$ | $(1,2,0)$ | $(\varnothing, a b, \varnothing, c)$ | $(3,1,2,1)$ | 1,3 |
| 7 | $(2,1,0)$ | $(2,2,0)$ | $(1,2,0)$ | $(\varnothing, \varnothing, b, a c)$ | $(4,3,0,1)$ | 2,3 |
| 8 | $(2,2,0)$ | $(2,2,0)$ | $(1,2,0)$ | $(\varnothing, \varnothing, b, a c)$ | $(4,3,0,1)$ | 2,3 |
| 9 | $(3,2,0)$ | $(2,2,0)$ | $(1,2,0)$ | $(\varnothing, a b, \varnothing, c)$ | $(4,0,2,1)$ | 1,3 |

Table 7 shows the operation of the Ausubel-Milgrom auction. Their auction ends with accepting bidder 1 and 3's offers. The outcome is given by the assignment $\pi^{A M A}=$ $\pi^{\prime \prime}$ and the corresponding income distribution $r^{A M A}=(4,0,2,1)$. This outcome is, however, not a core allocation as it is blocked by coalition $\{0,1,2\}$ and allocation $\left(\pi^{\prime}, t\right)$ with $t=(4.5,0.5,0,2)$, and

$$
\begin{aligned}
& u^{0}\left(\pi^{\prime}, t\right)=4.5>u^{0}\left(\pi^{A M A}, r^{A M A}\right)=4 \\
& u^{1}\left(\pi^{\prime}, t\right)=4.5>u^{1}\left(\pi^{A M A}, r^{A M A}\right)=4 \\
& u^{2}\left(\pi^{\prime}, t\right)=3>u^{2}\left(\pi^{A M A}, r^{A M A}\right)=2
\end{aligned}
$$

### 3.3 A Modified Dynamic Auction

The proposed auction mechanism in the previous can always find a core allocation. This allocation need not be strongly Pareto efficient, as it is a core allocation. In this section we improve the auction so that a strongly Pareto efficient core allocation can be always found. As a result, more efficiency will be gained. This modification requires each bidder to report his untransferable value when he is constrained by his budget. Observe that in this modified auction the bidding process is simpler for bidders than the first auction, but
the bidding process will take more rounds as in each round a provisional loser reduces his target utility just by one.

We are going to describe the rules of this modified dynamic auction mechanism.
Initialization: Every bidder $i \in M$ sets a target utility $\hat{u}_{1}^{i} \in \mathbb{Z}_{+}$that he wishes to achieve

$$
\hat{u}_{1}^{i} \geq \max _{A \in \mathcal{F}^{i}} v^{i}(A)+m^{i}
$$

Set $t=1$ and go to the Bidding Step.
Bidding Step: For any bidder $i \in M$, if it is the first round $t=1$ or he revises his target utility $\hat{u}_{t}^{i}<\hat{u}_{t-1}^{i}$, he makes new bids as follows. For every feasible bundle $A \in \mathcal{F}^{i}$, bidder $i$ sets an intermediate bidding price $\hat{p}_{t}^{i}(A)=v^{i}(A)+m^{i}-\hat{u}_{t}^{i}$ and adjusts his bidding prices as

$$
p_{t}^{i}(A)= \begin{cases}0, & \text { if } \hat{p}_{t}^{i}(A)<0 \\ \min \left\{\hat{p}_{t}^{i}(A), m^{i}\right\}, & \text { if } \hat{p}_{t}^{i}(A) \geq 0\end{cases}
$$

Let $p_{t}^{j}=p_{t-1}^{j}$ for any $t>1$ and any bidder $j \neq i$. Beside this, every bidder $h \in M$ calculates his untransferable value $\hat{v}_{t}^{h}(A)$ on bundle $A \in \mathcal{F}^{h}$ at price $p_{t}^{h}(A)$. That is

$$
\hat{v}_{t}^{h}(A)= \begin{cases}0, & \text { if } \hat{p}_{t}^{h}(A) \leq m^{h} \\ \hat{p}_{t}^{h}(A)-m^{h}, & \text { if } \hat{p}_{t}^{h}(A)>m^{h}\end{cases}
$$

The prices $p_{t}^{h}$ and untransferable utilities $\hat{v}_{t}^{h}$ for all bidders $h \in M$ are reported to the auctioneer. Go to the Assigning Step.

Assigning Step: Based on the current bidding prices $P_{t}=\left(p_{t}^{0}, p_{t}^{1}, \ldots, p_{t}^{m}\right)$ with $p_{t}^{0}(S)=$ $v^{0}(S)$ for every $S \in 2^{N}$, the auctioneer announces a provisional assignment $\pi_{t}$ yielding the highest revenue, i.e., an element of the following optimal set:

$$
C\left(P_{t}\right)=\left\{\pi \in \mathcal{A} \mid \pi=\underset{\rho \in \mathcal{A}}{\arg \max }\left(\sum_{i \in M_{0}} p_{t}^{i}(\rho(i))\right) \text { s.t. } p_{t}^{i}(\rho(i))>0 \forall \rho(i) \neq \varnothing \text { and } i>0\right\} .
$$

If $C\left(P_{t}\right)$ contains only one assignment, the assignment will be a provisional assignment. If $C\left(P_{t}\right)$ contains more than one assignment, the auctioneer chooses an assignment with the highest total untransferable utility. That is

$$
\pi_{t}=\underset{\pi \in C\left(P_{t}\right)}{\arg \max } \sum_{i \in M} \hat{v}_{t}^{i}(\pi(i))
$$

Then go to the following step.
Continue or Stop: At the provisional assignment $\pi_{t}$, if there is a bidder $i \in M$ with $\pi_{t}(i)=\varnothing$ and $\hat{u}_{t}^{i}>m^{i}$, at least one such bidder $i$ updates his target utility by setting $\hat{u}_{t+1}^{i}=\hat{u}_{t}^{i}-1$, and every other bidder $j \in M$ keeps his target utility unchanged as $\hat{u}_{t+1}^{j}=\hat{u}_{t}^{j}$. Then set $t=t+1$ and go back to the Bidding Step. Otherwise, the auction stops with the provisional assignment $\pi_{t}$. Then every bidder $i \in M$ gets bundle $\pi_{t}(i)$ and pays his bidding price $p_{t}^{i}\left(\pi_{t}(i)\right)$. The seller keeps bundle $\pi_{t}(0)$ and receives all payments from bidders in exchange for her sold bundles of items.

In the auction, bidder $i \in M$ is called a provisional loser (at round $t$ ) if he gets no item from the provisional assignment $\pi_{t}$ and his target utility $\hat{u}_{t}^{i}$ is still above his budget $m^{i}$. Suppose the auction stops in round $T$. The final assignment is $\pi_{T}$, and the corresponding income distribution is $r_{T}^{0}=\sum_{i \in M} p_{T}^{i}\left(\pi_{T}(i)\right)$ for the seller, and $r_{T}^{i}=m^{i}-p_{T}^{i}\left(\pi_{T}(i)\right)$ for each bidder $i \in M$. Bidder $i \in M$ is a loser of the auction if he is assigned nothing when the auction ends; otherwise, he is a winner. Each loser $i$ sets his final target utility as $\hat{u}_{T}^{i}=m^{i}$ and achieves utility $u^{i}\left(\pi_{T}, r_{T}\right)=\hat{u}_{T}^{i}=m^{i}$; while each winner $i$ achieves utility $u^{i}\left(\pi_{T}, r_{T}\right)=\hat{u}_{T}^{i}+\hat{v}_{T}^{i}\left(\pi_{T}(i)\right)$, which may be larger than his final target utility. Clearly, each bidder achieves a utility no less than his budget, so the outcome is individually rational.
Lemma 4. The modified auction stops in finitely many rounds with an individually rational outcome that gives every bidder at least his target utility being no less than his budget.

We are now ready to prove the following major theorem concerning the modified auction.
Theorem 4. When bidders are financially constrained, the outcome $\left(\pi_{T}, r_{T}\right)$ yielded by the proposed modified auction is strongly Pareto efficient and in the core.

Proof. Suppose to the contrary that $\left(\pi_{T}, r_{T}\right)$ is Pareto dominated by an allocation $(\rho, t)$
such that $u^{i}(\rho, t) \geq u^{i}\left(\pi_{T}, r_{T}\right)$ for all $i \in M_{0}$ with at least one strict inequality. Because every bidder is (weakly) better off at $(\rho, t)$, similar to the coalition members in proof of Theorem 2, in the auction each bidder $i \in M$ sets a comparatively lower target utility $\hat{u}_{T}^{i} \leq u^{i}(\rho, t)=v^{i}(\rho(i))+t^{i}$ such that a higher intermediate price $\hat{p}_{T}^{i}(\rho(i))=v^{i}(\rho(i))+$ $m^{i}-\hat{u}_{T}^{i} \geq m^{i}-t^{i}$. If $t^{i} \leq m^{i}$, we have that $\hat{p}_{T}^{i}(\rho(i)) \geq m^{i}-t^{i} \geq 0$ and $p_{T}^{i}(\rho(i))=$ $\min \left\{m^{i}, \hat{p}_{T}^{i}(\rho(i))\right\} \geq m^{i}-t^{i}$; if $t^{i}>m^{i}$, it is obvious that $p_{T}^{i}(\rho(i)) \geq 0>m^{i}-t^{i}$. Anyhow, we have $p_{T}^{i}(\rho(i)) \geq m^{i}-t^{i}$. The seller is also (weakly) better off, so we have

$$
\begin{aligned}
\sum_{i \in M_{0}} p_{T}^{i}(\rho(i)) & \geq p_{T}^{0}(\rho(0))+\sum_{i \in M}\left(m^{i}-t^{i}\right) \\
& =u^{0}(\rho, t) \\
& \geq u^{0}\left(\pi_{T}, r\right) \\
& =\sum_{i \in M_{0}} p_{T}^{i}\left(\pi_{T}(i)\right) \\
& \geq \sum_{i \in M_{0}} p_{T}^{i}(\rho(i)) .
\end{aligned}
$$

The last inequality comes from that $\pi_{T}$ maximizes seller's utility on prices $P_{T}$. So we have $\sum_{i \in M_{0}} p_{T}^{i}\left(\pi_{T}(i)\right)=\sum_{i \in M_{0}} p_{T}^{i}(\rho(i))$, and $p_{T}^{i}(\rho(i))=m^{i}-t^{i}$ for all $i \in M$. These imply that, in the final round of the auction, $\rho$ yields the same highest revenue as $\pi_{T}$ does (i.e., $\rho \in C\left(P_{T}\right)$ ), and if $\rho$ is chosen, the corresponding income distribution is just $t$.

Let's come back to Assigning step of the final round. The seller is indifferent between the two assignments, $\rho$ and $\pi_{T}$. For each bidder $i \in M$, his utility of assignment $\rho$ equals $u^{i}(\rho, t)=\hat{u}_{T}^{i}+\hat{v}_{T}^{i}(\rho)$, and his utility of $\pi_{T}$ is given by $u^{i}\left(\pi_{T}, r_{T}\right)=\hat{u}_{T}^{i}+\hat{v}_{T}^{i}\left(\pi_{T}(i)\right)$. Allocation $(\rho, t)$ Pareto dominates $\left(\pi_{T}, r_{T}\right)$, it implies that $\hat{v}_{T}^{i}(\rho) \geq \hat{v}_{T}^{i}\left(\pi_{T}(i)\right)$ for all $i \in M$ with at least one strict inequality. Then we have

$$
\sum_{i \in M} \hat{v}_{T}^{i}(\rho(i))>\sum_{i \in M} \hat{v}_{T}^{i}\left(\pi_{T}(i)\right)
$$

It contradicts that $\pi_{T}$ yields the highest total untransferable utility among the assignment in $C\left(P_{T}\right)$.

We will use the following simple example to show that the modified auction can find
a strongly Pareto efficient core allocation. Perhaps more importantly, we will demonstrate that this core allocation is not a strong core allocation, although the strong core is not empty.
Example 3. There are two items $a$ and $b$, and two bidders 1 and 2. Valuations and budgets of bidders are given in Table 8.

Table 8: Valuation and budget with two items.

| Bidder | $\varnothing$ | Item $a$ | Item $b$ | Package $a b$ | Budget |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 1 | 0 | 2 | 0 | 10 | 5 |
| 2 | 0 | 0 | 5 | 6 | 5 |

Table 9 shows the process of the modified auction in the example. The numbers in parentheses are untransferable utilities reported by the bidders.

Table 9: Illustration of the modified auction via Example 8.

| $t$ | $\hat{u}_{t}^{1}$ | $p_{t}^{1}(\varnothing, a, a b)$ | $\hat{u}_{t}^{2}$ | $p_{t}^{2}(\varnothing, b, a b)$ | $\pi_{t}(0,1,2)$ | $r_{t}(0,1,2)$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 1 | 15 | $(0,0,0)$ | 11 | $(0,0,0)$ | $(a b, \varnothing, \varnothing)$ | $(0,5,5)$ |
| 2 | 14 | $(0,0,1)$ | 10 | $(0,0,1)$ | $(\varnothing, a b, \varnothing)$ | $(1,4,5)$ |
| 3 | 14 | $(0,0,1)$ | 9 | $(0,1,2)$ | $(\varnothing, \varnothing, a b)$ | $(2,5,3)$ |
| $\vdots$ |  |  |  |  |  |  |
| 11 | 10 | $(0,0,5)$ | 6 | $(0,4,5)$ | $(\varnothing, \varnothing, a b)$ | $(5,5,0)$ |
| 12 | 9 | $(0,0,5(1))$ | 6 | $(0,4,5)$ | $(\varnothing, a b, \varnothing)$ | $(5,0,5)$ |
| 13 | 9 | $(0,0,5(1))$ | 5 | $(0,5,5(1))$ | $(\varnothing, \varnothing, a b)$ | $(5,5,0)$ |
| 14 | 8 | $(0,0,5(2))$ | 5 | $(0,5,5(1))$ | $(\varnothing, a b, \varnothing)$ | $(5,0,5)$ |

When the auction ends with the allocation $\left(\pi_{T}, r_{T}\right)=((\varnothing, a b, \varnothing),(5,0,5))$. This outcome assigns the two items to bidder 1, maximizes the seller's revenue and the total untransferable utility and yields utilities $\left(u^{0}, u^{1}, u^{2}\right)=(5,10,5)$. The assignment is fully efficient.

The allocation $\left(\pi_{T}, r_{T}\right)$ is in the core but not in the strong core. To see this, consider the coalition $\{0,2\}$ with a feasible allocation $\rho^{S}=(\varnothing, \varnothing, a b)$ and a feasible income distribution $t=(5,5,0)$. The new allocation yields utilities $(5,5,6)$ weakly blocking $\left(\pi_{T}, r_{T}\right)$. Actually, allocations $\left(\pi_{T}, r_{T}\right)$ and $(\rho, t)$ weakly block each other.

To show that the strong core is not empty, consider the allocation

$$
(\pi, r)=((\varnothing, a, b),(6,4,0))
$$

It offers utilities $\left(u^{0}, u^{1}, u^{2}\right)=(6,6,5)$. Clearly, it is individually rational. We show why it cannot be weakly blocked by any coalition. The first viable coalition is $S=\{0,1\}$. In this bidder 1 cannot pay more than 5 and thus the seller's revenue decreases. The second viable coalition $S=\{0,2\}$ faces a similar problem. In it bidder 2 cannot pay more than 5 and so the seller gets less. The final viable coalition is $S=\{0,1,2\}$. In this case, each bidder pays money to sustain the seller's revenue no less than 6; in return, each bidder gets one item. The only possible assignment is $\pi=(\varnothing, a, b)$. Using the same assignment, any redistribution of income favors one but also hurts someone else in the coalition.

It is interesting to notice that the strong core assignment $\pi$ is not fully efficient. In contrast, the assignment $\pi_{T}$ found by the modified auction is fully efficient.

The above example shows that under budget constraints, the modified auction can always find a strongly Pareto efficient core allocation but may not guarantee to locate a strong core allocation even if it exists. This does not contradicts Theorem 3, which says that the proposed auction can always find a strong core allocation when no bidder is financially constrained.

## 4 Concluding Remarks

In this paper we have studied a package auction model, in which bidders may demand several objects but have budget constraints. Besides the absence of substitution in demand, budget constraints can also fail the existence of a Walrasian equilibrium, so we adopt a more general solution-the core which is one of the most widely used and most fundamental solution concepts in the fields of game theory and general equilibrium theory. A core allocation consists of an assignment of items and a supporting price system and is a prime strategic equilibrium solution that is robust against any possible coalition deviation. We prove that the core of our auction model is always nonempty via a constructive proof through our proposed two ascending menu auctions which can guarantee to find a core allocation. Our first auction finds a core allocation when bidders
are budget constrained, and it finds a strong core allocation when bidders are not budget constrained. The second auction improves the first one and finds a strongly Pareto efficient core allocation when bidders are budget constrained. Our auctions can be seen as a natural generalization of the auctions proposed by Ausubel and Milgrom (2002) and further studied by Day and Milgrom (2008) and Erdil and Klemperer (2010) from the setting without budget constrained bidders to the setting with budget constrained bidders. The current auctions can be used to tackle more challenging and more practical resource allocation problems involving significant indivisibilities, heterogeneity in preferences and shortage of financial resources.

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[^0]:    *This paper has been presented at the 2018 Conference on Mechanism and Institution Design (Durham), the 3rd Workshop on Mechanism Design and Behavioural Economics (Glasgow), 2018 International Conference on Economic Theory and Application (Chengdu), the 9th Shanghai Microeconomics Workshop, 2018 York Annual Symposium on Game Theory, and City University of Hong Kong. Acknowledgement to be added.
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[^1]:    ${ }^{1}$ Koopmans and Beckmann (1957), Shapley and Shubik (1971), Crawford and Knoer (1981), Kelso and Crawford (1982), Quinzii (1984), Demange et al. $(1986)$, and Gul and Stacchetti $(1999,2000)$ among others.

[^2]:    ${ }^{3}$ The values of all the coalitions are listed here: $w(0)=w(1)=w(2)=w(3)=w(12)=w(13)=$ $w(23)=w(123)=0, w(01)=4, w(02)=4, w(03)=6, w(012)=7, w(013)=8, w(023)=9$, and $w(0123)=9$.

[^3]:    ${ }^{4}$ Commonly bidder $i$ sets $p_{t}^{i}(A)=0$ because of $0>\hat{p}_{t}^{i}=v^{i}(A)+m^{i}-\hat{u}_{t}^{i}$. So assigning $i$ the bundle $A$ with zero-price is harmful to him, i.e., $v^{i}(A)+m^{i}<\hat{u}_{t}^{i}$.

[^4]:    ${ }^{5}$ Another core allocation $\left(\pi^{\prime},(5,0,0,2)\right)$ is also possible to be selected in step $t=10$. If so, the auction also terminates at this step. Then bidder 3 , as a loser, updates his target utility by $\hat{u}_{10}^{3}=m^{3}=2$.

[^5]:    ${ }^{6} \mathrm{~A}$ different tie-breaking rule can also be used by adding small perturbations to bidding prices. For example, we can add -0.02 to bidder 1's bidding prices of offers related to item $1,-0.01$ to his bidding prices of offers related to item 2 , and -0.03 to bidder 2 's bidding prices.

