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Procyclical endogenous taxation and aggregate instability

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Procyclical endogenous taxation and aggregate instability

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Abstract

The existing contributions on endogenous taxation, and balanced budget rules, suggest that countercyclical taxes should be avoided, because they may lead to aggregate instability (i.e. sunspot equilibria); on the other hand, procyclical taxes have always been praised for their stabilizing role. In this paper, we re-examine this issue in an endogenous growth model with productive government investment, and we prove that an economy with procyclical taxes, and a sufficiently large income effect, can still be characterized by i) global indeterminacy because two balanced growth paths may exist; ii) aggregate instability around the balanced growth path with the lowest growth rate. Finally, we show that this dynamics may emerge for reasonable choices of the parameters.

JEL Classification C62, E32, H20, O41

Keywords Endogenous growth, time-varying consumption tax, local and global indeterminacy.

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1 Introduction

Several contributions in the literature have shown that a balanced-budget rule together with endogenous distortionary taxes may lead to aggregate instability, once embedded in a neoclassical growth model.[1] Endogenous labor income taxes and capital income taxes are responsible of aggregate instability if sufficiently countercyclical with respect to output growth (Schmitt-Grohe and Uribe [13]). Although endogenous consumption taxes are preferred in such a setting, because they reduce the range of parameters leading to an indeterminate steady state and, therefore, to sunspot equilibria (Giannitsarou [5]), this sharp result cannot be extended to more general utility functions, as those proposed by Jaimovich [6] and Jaimovich and Rebelo [7]. In fact, local determinacy is guaranteed only if the consumption taxes are assumed to be countercyclical and the elasticity of intertemporal substitution in consumption is sufficiently large (Nourry et al. [10]). A common characteristic of these models is that government spending is never productive. Investigating the same issue in an endogenous growth model *à la* Barro [2], where government spending is productive, leads to a global form of indeterminacy when the consumption taxes are endogenous and countercyclical (Bambi and Venditti [1]).[2] Sunspot equilibria and, then aggregate instability, emerge also in this context, once extrinsic uncertainty is introduced.

Therefore, the existing literature points to procyclical endogenous taxes as the right policy to rule out aggregate instability in models where the government balances its budget in each period. Such a policy advice is relevant for at least two reasons.

First, balanced budget rules have been more and more advocated and even implemented as constitutional requirements in several European countries after the 2008 crisis (e.g. Art.81 of the Italian Constitutional Law 1/2012). In the period 2009-2013, for example, 16 European countries had a deficit as percentage of GDP below 3%.

Second, the same European countries and, in fact, several OECD countries have adopted and still adopt countercyclical taxes with respect to output growth (see Lane [8] among others).[3] Figure 1 shows, for example, how consumption taxes have

[1] Aggregate instability emerges because of the existence of (stationary) sunspot equilibria.

[2] The global indeterminacy found in this contribution is characterized by a unique stationary equilibrium in the variables consumption over capital and tax rate and a continuum of no-stationary equilibria in the same two variables.

[3] Consistently with the previously mentioned literature, we say that taxes are conutercyclical if

been adjusted countercyclically in several EU countries in the period 2009-2013, specifically those with a red and grey circle. Interestingly, eight of them kept a deficit as percentage of GDP below 3%.[4]

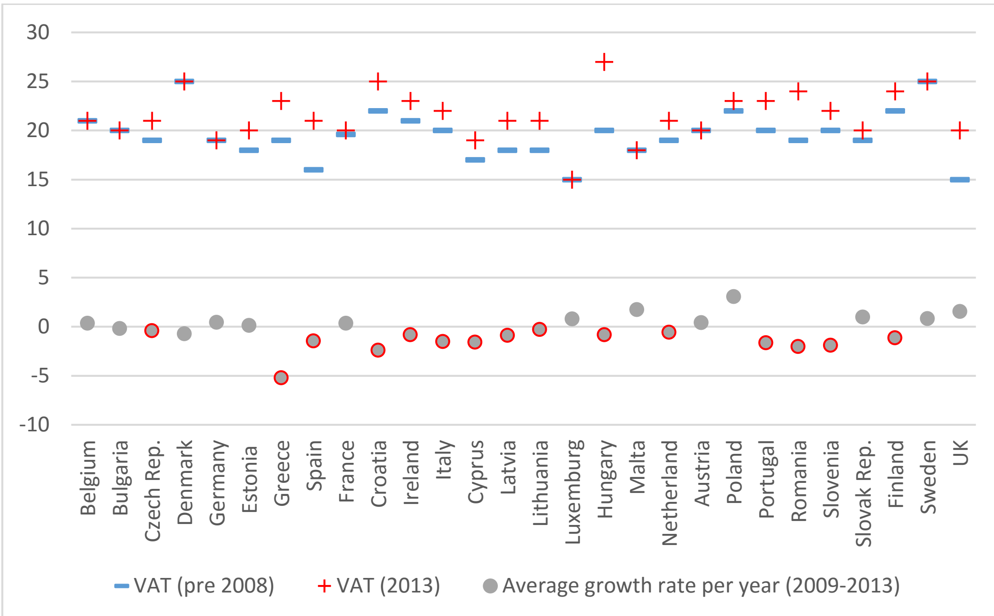

Figure 1: Evidence on Countercyclical VAT

For these reasons, these countries could benefit by switching to procyclical fiscal policies. In fact, this change would lead to a reduction of output volatility according to the prediction of existing contributions, that, as previously explained, have agreed on the stabilizing role of procyclical taxation.

The aim of this paper is to investigate how robust this prediction is in an endogenous growth model. More specifically, our setting differs from previous contributions on two dimensions.

First, the government finances productive public investments by levying endogenous consumption taxes; government spending is then a stock variable, not a flow as in Barro [2] and in Bambi and Venditti [1].[5] Consumption taxes are considered, instead of other types of taxation, because it is more difficult to generate local in-

the tax rate expands when output shrinks and viceversa. See also footnote 6.

[4]According to Eurostat, these countries are Czech Republic, Italy, Latvia, Lithuania, Hungary, Netherland, Romaina and Finland. Moreover, the countries adjusted often gradually the consumption tax rate in the period 2009-2013.

[5]Endogenous growth as a result of productive public investment financed by flat income taxes was originally investigated by Futagami et al. [4].

determinacy, and because consumption taxes have been adjusted countercyclically, as shown in the previous empirical evidence.

Second, the government sector is characterized by two equations: a balanced budget rule *and* a fiscal policy. The reason for the latter is that in our context both the government spending and the taxes are endogenous; therefore, we need to specify a fiscal policy, in addition to the balanced budget rule, to avoid a trivial form of global indeterminacy. Observe that this is a departure from what usually done in exogenous or no-growth models, where the government spending is exogenously given and the tax rate is endogenous, because it has to adjust, period by period, to balance the government budget.[6]

To design the fiscal policy, we follow Persson and Tabellini [12] and we consider a state-contingent and time-invariant fiscal policy rule and, specifically, a functional form which guarantees the existence of a balanced growth path.

The main finding of this article is that our economy may admit, for reasonable values of the parameters, two balanced growth paths and that the one characterized by the lowest growth rate can be locally indeterminate, even though the consumption tax is procyclical with output growth. Consequently, procyclical taxation may lead an economy to a poverty trap characterized by all the aggregate variables to fluctuate around the balanced growth path with the lowest growth. This result is surprising and it suggests to be cautious because the existing results in favor of procyclical taxation does not necessarily extend to an endogenous growth setting.

It also clarifies that in the presence of multiple balanced growth path a trade-off between output growth and output volatility may not exist because aggregate instability may emerge around the balanced growth path with the lowest growth rate.

Interestingly enough, the existence of multiple BGPs depends on the existence of a Laffer curve-type relationship between the tax rate and the (detrended) tax revenue. More precisely two balanced growth paths exist because detrended public investments may intersect twice the Laffer curve. The intersection is shown to

[6]For example, Schmitt-Grohe and Uribe [13] consider, at page 980, the balanced budget rule $G = \tau_t w_t \ell_t$ with G an exogenously given constant; the tax rate is endogenous because it has to adjust, in every period, to compensate the changes in labor income and to balance the government budget. Similarly, Giannitsarou [5] considers, in a model without growth, the balanced budget rule $G = \tau_t c_t$; the tax rate is, again, endogenous because G is an exogenously given constant and the budget needs to be balanced by varying the consumption tax. Clearly, in both cases, the taxation is, by construction, countercyclical respectively with output and consumption. In fact, if output or consumption increases the tax rate has to be reduced to keep the same government spending.

emerge on the left side of the Laffer curve and therefore both balanced growth paths can be empirically plausible. Although the existence of a Laffer curve was recently found by Nourry et al. [10], but in this contribution there is only one empirically plausible steady state. In addition, the reason behind its existence of a Laffer curve is very different since, in their case, it depends on the preferences' specification, while, in our case, the utility function is a standard CES and the existence of the Laffer curve depends on the combination of productive public investment together with the designed fiscal policy rule. In this framework, multiple BGPs emerge for a sufficiently large income effect together with a sufficiently large procyclical taxation.[7] In addition,

Similarly, our result on local indeterminacy is proved for a sufficiently large income effect together with a not too large procyclical taxation.

The paper is organized as it follows. In Section 2, we describe the economy and we identify the two key equations which describe the intertemporal equilibrium. Section 3 focuses on the existence of a balanced growth path and sufficient conditions, for global indeterminacy to emerge, are found in Proposition 2. The existence of a Laffer curve is discussed with the help of some figures, and numerical examples. The transitional dynamics around the balanced growth path with the lowest growth rate, in the case of global indeterminacy, are investigated in Section 4. In particular, sufficient conditions for local indeterminacy to emerge are found in Proposition 3. Again, numerical examples are proposed to show that this dynamic behavior is not only analytically possible, but also reasonable from a quantitative perspective. Section 5 emphasizes the role of the procyclical taxation in our setting and a comparison with existing results is proposed. Finally, Section 6 concludes the paper. The logical steps of the proofs appear in the main text, while a more detailed and rigorous version of the proofs can be found in the Appendix.

2 Model Setup

In this section, we present the decision problem faced by the households, by the firms and we also describe the role played by the government in the economy through its budget constraint and fiscal policy rule. The model setup is similar to the one proposed by Futagami et al. [4] with the exception that the fiscal policy rule consists of an endogenous, and (possibly) time-varying, consumption tax.

Households – There is a continuum of atomistic households. There is no pop-

[7] It is worth remembering that there is no Laffer curve with flat-rate consumption taxes.

ulation growth and its size is normalized to one, $L_t = L = 1$.[8] The representative household solves the following inter-temporal maximization problem

$$\max_c \int_0^\infty \mathrm{e}^{-\rho t} \cdot \frac{c^{1-\sigma} - 1}{1-\sigma} \,\mathrm{d}t$$

subject to

$$\dot{k} = w + Rk - \delta k - (1+\tau)c \tag{1}$$

$$c \geq 0, \qquad k \geq 0 \tag{2}$$

with the initial condition of capital, k_0, exogenously given. Gross income is the sum of the return on capital and of labor income, $y \equiv (R-\delta)k + w$, while net income is $y - \tau c$, with $\tau \in (0,1)$ indicating the consumption tax rate. Net income is allocated between consumption, c, and gross investment $i \equiv \dot{k} + \delta k$. The intertemporal preference discount factor, ρ, and the depreciation rate of capital, δ, are assumed, as usual, between zero and one while the inverse of the elasticity of intertemporal substitution in consumption, σ, is strictly greater than one.[9]

The Hamiltonian of this problem is

$$\mathcal{H} \equiv \frac{c^{1-\sigma} - 1}{1-\sigma} \cdot e^{-\rho t} + \lambda[(w + (R-\delta)k - (1+\tau)c]$$

whose first order conditions are:

$$\frac{\partial \mathcal{H}}{\partial c} = 0 \qquad \Leftrightarrow \qquad c^{-\sigma} e^{-\rho t} = \lambda(\tau + 1) \tag{3}$$

$$\frac{\partial \mathcal{H}}{\partial k} = -\dot{\lambda} \qquad \Leftrightarrow \qquad \lambda(R-\delta) = -\dot{\lambda} \tag{4}$$

Differentiating (3) and substituting into (4) leads to the Euler equation:

$$\frac{\dot{c}}{c} = \frac{1}{\sigma}\left(R - \delta - \rho - \frac{\dot{\tau}}{1+\tau}\right) \tag{5}$$

A standard transversality condition must also hold.

Firms – There is a continuum of atomistic firms. They demand capital and labor to the households and also received a public good from the government. The production function is assumed to be Cobb-Douglas, $y = Ak^\alpha G^{1-\alpha}$, with G the

[8] For this reason, a variable, X, and its per capita value, x, coincides.

[9] The cases $\sigma = 1$ and $\sigma < 1$ can be studied but have been excluded by the current analysis to make the presentation of the results less cumbersome.

public good. From the profit maximization problem of the firms, we get the following two conditions:

$$\alpha A k^{\alpha-1} G^{1-\alpha} = R \qquad and \qquad (1-\alpha) A k^{\alpha} G^{1-\alpha} = w \tag{6}$$

Government – Government spending is assumed to be a stock variable consistently with Futagami et al. [4]. Since public investment, $\mathcal{I}_G$, is financed by the consumption tax revenue, $\mathcal{T}$, and the government balances its budget in each period, we have that

$$\mathcal{I}_G \equiv \dot{G} + \delta G = \tau c \equiv \mathcal{T} \tag{7}$$

with the initial government spending, G_0, an exogenously given and positive constant. The depreciation of capital and of government spending are also assumed to be the same for simplicity.[10]

Since we want to allow both the government spending and the consumption tax to be endogenous and (possibly) time-varying, we need to specify, not only the government balance budget constraint (7), but also a fiscal policy rule to avoid a trivial form of global indeterminacy. Following Persson and Tabellini [12] (e.g. Chapter 11, page 279), we assume a state-contingent, and time-invariant policy rule:

$$\tau = \Psi(k, G) \equiv \tau_c \left(\frac{G}{k}\right)^{\eta} \tag{8}$$

where $\tau_c > 0$ and the elasticity of the consumption tax with respect to the government-capital ratio, i.e. η, can be a positive or negative constant.[11] The chosen functional form for the fiscal policy rule implies two important characteristics of the tax rate: i) it will be constant along any BGP, since $\frac{G}{k}$ will be constant; ii) it is predetermined, since a function of two state variables, k and G. This is, indeed, consistent with the fact that taxes are typically set in advance, as discussed in Schmitt-Grohe and Uribe [13]. The functional form differs from the one studied by Bambi and Venditti [1] since the tax does not depend on a control variable but rather on a state variable, namely the government-capital ratio. As it will result clear in the next section, this departure will lead to a dynamics substantially different and to remarkably different policy advices.

It is also worth noting that combining the production function with the fiscal policy rule leads to

$$\tau = \tau_c A^{\frac{\eta}{\alpha-1}} \left(\frac{y}{k}\right)^{\frac{\eta}{1-\alpha}}$$

[10] In this assumption, we depart from the setting of Futagami et al. [4] where government spending does not depreciate over time.

[11] A time-invariant policy rule means that the functional form, $\Psi(.)$, does not change over time.

which implies

$$g_y = g_k + \frac{1-\alpha}{\eta} g_\tau$$

where $g_x = \frac{\dot{x}}{x}$. Therefore, the tax growth rate, g_τ, is procyclical (countercyclical) with respect to output growth, g_y, if $\eta > 0$ ($\eta < 0$).

We are now ready to give a definition of an intertemporal equilibrium for the economy just described.

Definition 1 (Intertemporal Equilibrium) *Given an initial condition of capital* $k_0 > 0$ *and government spending* $G_0 > 0$*, an intertemporal equilibrium is any path* $(c(t); k(t); \tau(t); G(t))$ *which satisfies the system of equations (1), (5), (6), (7) and (2), respects the inequality constraints* $k > 0$ *and* $c > 0$*, and the transversality condition.*

As usual in the endogenous growth literature, the dynamics associated to such an equilibrium can be described by combining these equations to obtain a system of two ODEs:

$$\frac{\dot{x}}{x} = (\tau_c x^{\eta-1} + 1 + \tau_c x^\eta) y - A x^{1-\alpha} \quad (9)$$

$$\frac{\dot{y}}{y} = \frac{1}{\sigma}\left[\alpha A x^{1-\alpha} - \delta - \rho - \frac{\tau_c \eta x^\eta}{1+\tau_c x^\eta}\frac{\dot{x}}{x}\right] - \left[A x^{1-\alpha} - \delta - (1+\tau_c x^\eta) y\right] \quad (10)$$

in the state-like variable $x \equiv \frac{G}{k}$ and the control-like variable $y \equiv \frac{c}{k}$. The interested reader may look at Appendix B.1 for further details on the derivations leading to system (9)-(10).

3 Balanced Growth Paths

In this section, we investigate the existence and uniqueness of a balanced growth path for this economy. Two main results will be proved: first, that a unique balanced growth path always exists within reasonable parameter choices; secondly, that multiple balanced growth paths can also emerge for alternative but still plausible choices of the parameters.

A balanced growth path (from now on BGP) is a particular intertemporal equilibrium where consumption, government spending and capital grow exponentially at the same positive rate, γ:

$$c = c_0 e^{\gamma t}, \qquad G = G_0 e^{\gamma t} \qquad \text{and} \qquad k = k_0 e^{\gamma t}$$

Along a BGP, the government spending over capital ratio, and the consumption-capital ratio are constant, and their value, x^*, y^*, is a steady state for the system (9)-(10). In particular, along a BGP, equations (9)-(10) rewrite

$$\begin{aligned} y^* &= \frac{Ax^{*1-\alpha}}{\tau_c x^{*\eta-1} + 1 + \tau_c x^{*\eta}} && (11) \\ x^* &= \left(\frac{\sigma\gamma + \delta + \rho}{\alpha A}\right)^{\frac{1}{1-\alpha}} \quad \textit{with} && (12) \\ \gamma &= Ax^{*1-\alpha} - \delta - (1 + \tau_c x^{*\eta})y^*. && (13) \end{aligned}$$

Existence and uniqueness of a BGP can be investigated looking at the roots of the following equation in the variable $\gamma \in (0, +\infty)$:

$$\tilde{\mathcal{T}}(\gamma) \equiv \underbrace{\frac{A\tau_c}{\tau_c x^{*\alpha-1} + x^{*\alpha-\eta} + \tau_c x^{*\alpha}}}_{\textit{detrended tax revenue}} = \underbrace{\gamma + \delta}_{\textit{detrended public investment}} \equiv \tilde{\mathcal{I}}_G(\gamma) \qquad (14)$$

This equation can be obtained by solving (13) for y^*, substituting it into (11); it is worth noting that x^* is a one-to-one function of γ from (13). Alternatively, (14) can be obtained combining equation (11) with the government budget constraint (7) evaluated along a BGP. This last venue is more informative because, assuming without loss of generality $G_0 = 1$, the left hand side of (14) is detrended tax revenues, $\tilde{\mathcal{T}} \equiv \mathcal{T}e^{-\gamma t}$, while the right hand side is detrended public investment, $\tilde{\mathcal{I}}_G \equiv \mathcal{I}_G e^{-\gamma t}$. Therefore, equation (14) is the government balanced budget constraint, evaluated along a BGP.

We have now all the preliminaries to prove, under which conditions, a unique balanced growth path exists.

Proposition 1 (Existence and Uniqueness) *A unique balanced growth path exists if*

$$A > \underline{A} \qquad \textit{and} \qquad \tau_c > \underline{\tau}_c \qquad (15)$$

where $\Gamma \equiv \left(\frac{\delta+\rho}{\alpha A}\right)^{\frac{1}{1-\alpha}}$, $\underline{A} \equiv \frac{\delta^{1-\alpha}(\delta+\rho)}{[(1-\alpha)\delta+\rho]^{1-\alpha}\alpha^\alpha} > 0$ *and* $\underline{\tau}_c \equiv \frac{\delta\Gamma^{\alpha-\eta}}{A-(\Gamma^{-1}+1)\Gamma^\alpha\delta} > 0$.

Proof. A unique BGP exists as long as $\tilde{\mathcal{T}}(\gamma)$ intersects *only* once the straight line $\gamma + \delta$. If $\tilde{\mathcal{T}}(0) \geq \delta$, there is always *at least* one intersection, since $\lim_{\gamma\to\infty} \tilde{\mathcal{T}}(\gamma) = \lim_{x\to\infty} \tilde{\mathcal{T}}(\gamma) = 0^+$ and $\tilde{\mathcal{T}}(\gamma)$ is continuous and differentiable in its domain. As shown in Appendix B.2.1:

$$\tilde{\mathcal{T}}(0) \geq \delta \qquad \Leftrightarrow \qquad A > \underline{A} \qquad \textit{and} \qquad \tau_c \geq \underline{\tau}_c \qquad (16)$$

Finally, there is only one intersection when $\tilde{\mathcal{T}}(0) > \delta$, because the function $\tilde{\mathcal{T}}(\gamma)$ has at most a unique critical point, $\hat{\gamma} > 0$, as shown in Appendix B.2.2. Therefore, (15) implies the existence of a unique BGP. ■

Discussion of these conditions is in order. The requirements of a sufficiently high level of technology, and of the tax rate, are crucial to guarantee a positive growth rate of the economy. In particular, the condition on A is similar to the one required in an AK model, while the condition on τ_c tells us that economic growth can be sustained only if a large enough government spending, in the form of a public good, is provided to the firms. These conditions are similar to those found in Bambi and Venditti [1].

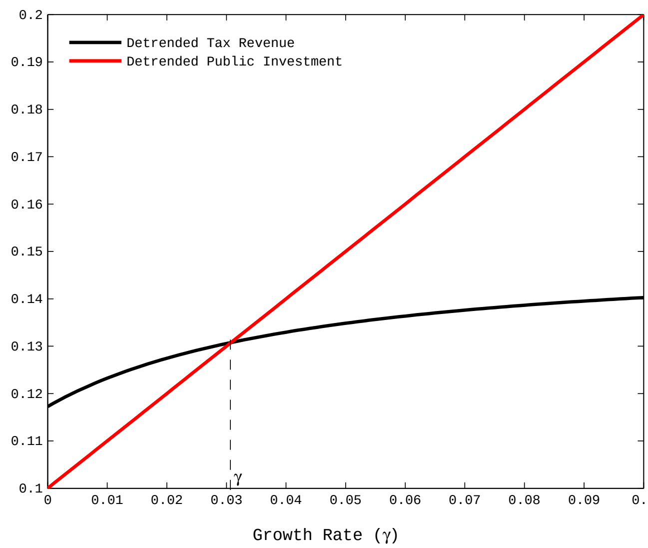

Figure 2: Existence and uniqueness of the BGP when conditions (15) hold.

It is also worth noting that, although the conditions on the parameters identified in this Proposition are sufficient but not necessary for the existence and uniqueness of a BGP, the set of parameters that are excluded is of zero measure. In fact, it is the set of parameters for which the function $\tilde{\mathcal{T}}(\gamma)$ is tangent to the straight line $\gamma + \delta$. In addition, for any positive value of γ both x^* and y^* are positive and, therefore, all the inequality constraints are respected. The transversality condition is also respected as long as $(1-\sigma)\gamma - \rho < 0$. This is indeed always the case since we have assumed $\sigma > 1$.

A numerical example is now proposed to show that the parameter values, to have a unique BGP, are plausible. Consider a yearly frequency of time, and suppose a depreciation rate $\delta = 0.1$, an intertemporal discount factor $\rho = 0.0101$ and a capital share $\alpha = 0.33$; then, it exists a unique BGP when $(A, \tau_c) = (0.94, 0.2)$. In fact, the two conditions in (15) are both respected given that $(\underline{A}, \underline{\tau}_c) = (0.19, 0.1477)$. Assuming $\sigma = 3$ and $\eta = 0.25$, the resulting growth rate is 3.28%, i.e. $\gamma = 0.0328$, the government-capital ratio is $x^* = 0.5527$ while the tax rate is 17.24%, i.e. $\tau^* = \tau_c x^{*\eta} = 0.1724$. Figure 2 shows how the growth rate of the economy has been found looking at the intersection of the detrended tax revenue with the detrended public investment curve.

We now proceed to find, if any, sufficient conditions for global indeterminacy to emerge.

Proposition 2 (Global Indeterminacy) *Two balanced growth paths exist if the following parameter's conditions hold:*

$$A > \underline{A}, \qquad \underline{\tau}_c - \epsilon < \tau_c \leq \underline{\tau}_c, \qquad \eta > \underline{\eta} \qquad \textit{and} \quad \sigma > \underline{\sigma} \tag{17}$$

with $\epsilon > 0$ and sufficiently small real number,

$$\underline{\eta} \equiv \frac{\rho\alpha A}{A - (\Gamma^{-1}+1)\Gamma^{\alpha}\delta} > 0 \qquad \textit{and} \qquad \underline{\sigma} \equiv \frac{\alpha(1-\alpha)\left[(\Gamma^{-1}+1)\tau_c + \Gamma^{-\eta}\right]^2 \Gamma}{\tau_c\{\tau_c[(1-\alpha)\Gamma^{-1} - \alpha] + (\eta-\alpha)\Gamma^{-\eta}\}}.$$

Proof. Given the properties of the function $\tilde{\mathcal{T}}(\gamma)$ found in Proposition 1, two BGPs exist as long as the following two conditions hold:

a) $\delta - \varepsilon \leq \tilde{\mathcal{T}}(0) < \delta$, for any $\varepsilon > 0$ sufficiently small real number;

b) $\left.\frac{d\tilde{\mathcal{T}}(\gamma)}{d\gamma}\right|_{\gamma=0} > 1$

In fact, condition *a)* means that the curve $\tilde{\mathcal{T}}(\gamma)$ is slightly below the straight line $\gamma + \delta$ at $\gamma = 0$ but it is steeper for condition *b)*. Therefore, the curve must intersect the straight line twice since it is continuous, it has a unique critical point, and $\lim_{\gamma\to\infty} \tilde{\mathcal{T}}(\gamma) = 0^+$. The steps to prove under which subset of parameters these two conditions hold can be found in Appendix A.

■

Interestingly, Proposition 2 suggests that global indeterminacy can emerge when the taxation is sufficiently procyclical, $\eta > \underline{\eta}$.

In the case $\tau_c = \underline{\tau}_c$, the BGP with the lowest growth rate, let us call it BGP_ℓ, has a zero growth rate, $\gamma_\ell = 0$, while the other, BGP_h, has a strictly positive rate, $\gamma_h > 0$. Also, in the case of two BGPs, the constraints of positive consumption and capital are respected. Regarding the transversality condition, the restriction $\sigma > \underline{\sigma}$ should be written as $\sigma > \max\{\underline{\sigma}, 1\}$ to have this condition always respected.

Remark 1 *Proposition 2 uses a continuity argument to prove that two BGPs may exist for an open set of parameters. The set of parameters found in Proposition 2 is, clearly, not the largest set for global indeterminacy to emerge. In particular, the lower bound for $\underline{\tau}_c$, namely $\underline{\tau}_c - \epsilon$, can be computationally enlarged.*

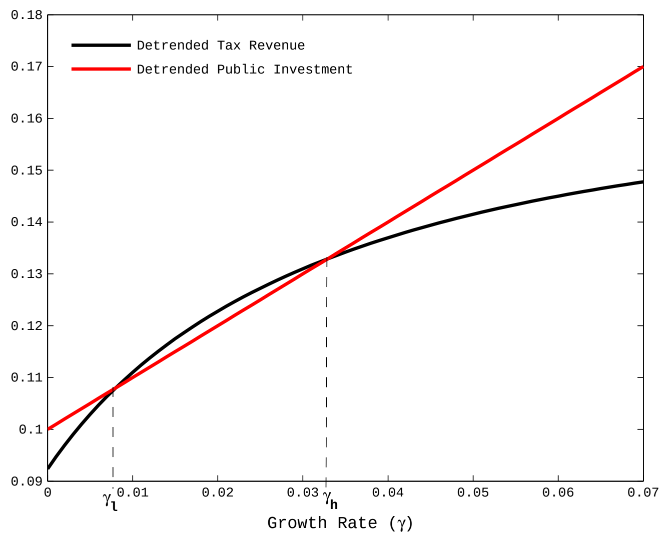

Figure 3: Existence of two BGPs when conditions (17) hold.

To illustrate this last point further, and to show that global indeterminacy may raise for plausible values of the parameters, we propose now a numerical example. Similarly to the previous numerical exercise, we assume a depreciation rate $\delta = 0.1$, an intertemporal discount factor $\rho = 0.0101$, and a capital share $\alpha = 0.33$; then, global indeterminacy emerges for $(A, \tau_c, \eta, \sigma) = (0.9, 0.2, 0.5, 7)$, since all the conditions in (17) are respected. In particular, we have that $(\underline{A}, \underline{\tau}_c, \underline{\eta}, \underline{\sigma}) = (0.20, 0.22, 0.004, 1.72)$. As shown in Figure 3, the economy has two BGPs; the lowest is characterized by a growth rate $\gamma_\ell = 0.008$, implying a government spending over capital ratio $x_\ell^* = 0.4201$, and a tax rate $\tau_\ell^* = \tau_c x_\ell^{*\eta} = 0.13$; the highest is characterized by a growth rate $\gamma_h = 0.033$, implying a government spending over capital ratio $x_h^* = 1.22$, and a tax rate $\tau_h^* = \tau_c x_h^{*\eta} = 0.22$. Therefore, the growth rates of 0.8% and 3.3% are associated with plausible values, respectively 13% and 22%, of the consumption tax rates.

Remark 2 (Futagami et al. [4] case) *Consider the case with acyclical taxation, $\eta = 0$, which corresponds to the economy described in Futagami et al. [4] with the only difference that public investment is financed by levying a constant consumption tax, $\tau = \tau_c$, instead of an income tax. Assume also the same parameters' values* *of the last exercise; then it exists a unique BGP with a growth rate of the economy between γ_ℓ and γ_h and, precisely, equal to 2.67%, i.e.* $\gamma = 0.0267$.

We conclude this section by showing that a Laffer curve always exists in this economy. Observe that the existence of a Laffer curve can be investigated from equation (14), after rewriting it as a function of τ. This can be done easily by using equations (11)-(13) to write x, y and γ as functions of τ. Then equation (14) can be rewritten as

$$\tilde{\mathcal{I}}_G(\tau) \equiv \gamma(\tau) + \delta = \tau \cdot y(\tau) \equiv \tilde{\mathcal{T}}(\tau)$$

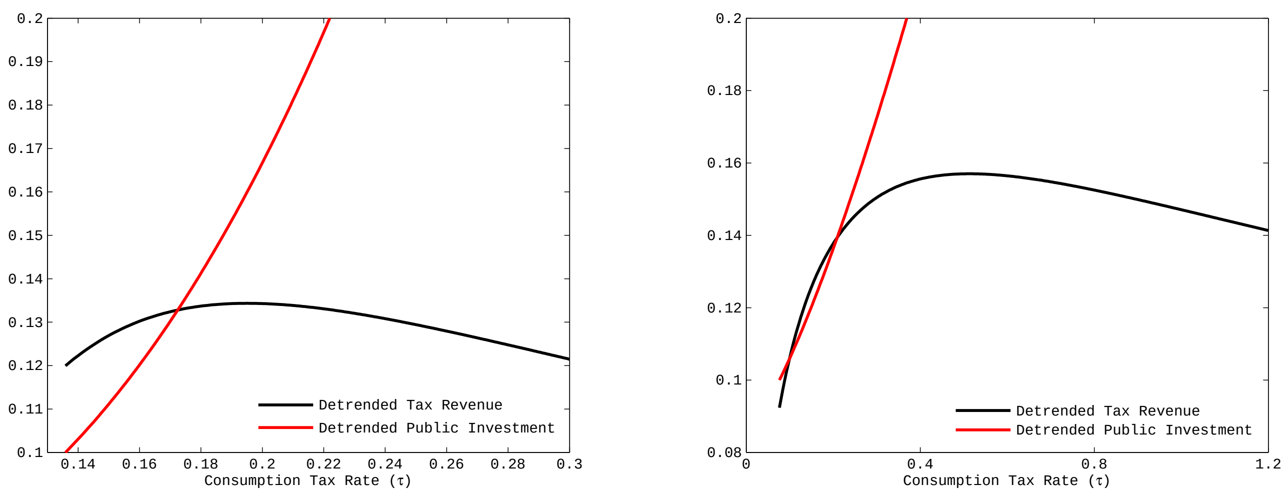

Figure 4: Laffer curve and a unique (left) or multiple (right) BGPs.

As it emerges clearly from Figure 4, the detrended tax revenue is always concave, meaning that a Laffer curve exists, both under the assumption of a unique BGP and under the assumption of two BGPs. In particular, Figure 4 has been obtained under the numerical choices of the parameters suggested previously but allowing the tax rate (and, therefore, the growth rate of the economy) to change. Consistently with the result of Proposition 1, the left side of Figure 4 shows that a unique BGP emerges when detrended public investment, i.e. the red curve, has only one positive intersection with detrended tax revenue, i.e. the black curve. Consistently with the numerical exercise proposed after Proposition 1, the consumption tax rate, at the intersection point, is slightly higher than 17%, implying a 3.28% growth rate of the economy.

On the other hand, the right side of Figure 4 shows, in line with the result of Proposition 2, that two BGPs emerge when detrended public investment, i.e. the red curve, has two positive intersections with detrended tax revenue, i.e. the black curve. As computed in the numerical exercise after Proposition 2, the consumption tax rates at these intersection are 13% and 22%, implying a growth rate of the economy equal to 0.8% and 3.3% respectively.

Looking at Figure 2, 3 and 4, and taking into account the proof of Proposition 2, it emerges that the consumption tax rate, which depends on the parameter τ_c, the elasticity of the consumption tax with respect to the government-capital ratio, η, and the inverse of the elasticity of intertemporal substitution in consumption, σ, plays a fundamental role to have global indeterminacy. In particular, global indeterminacy emerges when detrended tax revenues are low for low level of the tax rate while η and σ are sufficiently high to have a stronger variation in detrended tax revenue for small variation in the tax rate than in detrended public investment. Under these circumstances, the detrended tax revenue curve intersects the detrended public investment curve when the tax rate is very low (see right side of Figure 4) and, then, it intersects again at a higher level of the tax rate, because of the existence of a Laffer curve.

4 Transitional Dynamics

In this section, we study the transitional dynamics around the steady state(s). We begin by linearizing the system of ODEs (9) (10) around a generic steady state (x^*, y^*):[12]

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} \approx \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} \tag{18}$$

where

$$a = \left[\tau_c(\eta-1)x^{*\eta-1} + \tau_c\eta x^{*\eta}\right] y^* - (1-\alpha)Ax^{*1-\alpha} \tag{19}$$

$$b = \tau_c x^{*\eta} + x^* + \tau_c x^{*\eta+1} > 0 \tag{20}$$

$$c = y^* \left[-\frac{1}{\sigma}\left((1-\alpha)(\sigma-\alpha)Ax^{*-\alpha} + \frac{\tau_c\eta}{x^{*1-\eta}+\tau_c x^*}\cdot a\right) + \tau_c\eta x^{*\eta-1}y^*\right] \tag{21}$$

$$d = y^*\left(1 + \tau_c x^{*\eta} - \frac{1}{\sigma}\cdot\frac{\tau_c\eta}{x^{*1-\eta}+\tau_c x^*}\cdot b\right) \tag{22}$$

As usual, the stability of a steady state (x^*, y^*) can be unveiled looking at the sign of the determinant and trace of the Jacobian matrix, $J \equiv \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Long and

[12]Observe that both the equation have the form $\dot{z} = f(z, w)z$, whose first order Taylor approximation, around a steady state (z^*, w^*), is $\dot{z} \approx z^* \left(\frac{\partial f}{\partial z}(z^*, w^*)\cdot\tilde{z} + \frac{\partial f}{\partial w}(z^*, w^*)\cdot\tilde{w}\right)$ with tilde indicating the deviation from the steady state.

tedious computations, reported in the Supplementary Material, lead to the following determinant and trace of the Jacobian:

$$\det(J) \equiv y^{*2}x^{*2\eta}\left\{\tau_c x^{*-1}\left[(1-\alpha)\tau_c x^{*-1}-\tau_c\alpha+(\eta-\alpha)x^{*-\eta}\right]-\frac{(1-\alpha)\alpha}{\sigma}\left(\tau_c x^{*-1}+x^{*-\eta}+\tau_c\right)^2\right\}$$

$$\mathrm{tr}(J) \equiv y^{*}\left\{x^{*\eta}\tau_c\left[(\eta+\alpha-2)x^{*-1}+\eta+\alpha\right]+\alpha-\frac{1}{\sigma}\cdot\frac{\tau_c\eta\left(\tau_c x^{*\eta-1}+1+\tau_c x^{*\eta}\right)}{x^{*-\eta}+\tau_c}\right\}$$

The focus of this section is to study the local stability properties of the BGP with the lowest growth rate, in the case of global indeterminacy. The reason is that we want to see if the presence of an endogenous consumption tax, not only may induce aggregate instability (i.e. one of the steady state is locally indeterminate) but this dynamic behavior emerges around the BGP_ℓ, whose corresponding steady state is (x_ℓ^*, y_ℓ^*). This means that a poverty trap, characterized by a low growth rate and a high volatility, may exist in this model. We believe that this is a quite important result to unveil, because, otherwise, it could be argued that the aggregate instability around the highest BGP is the price to pay for a high growth rate; in other words, aggregate instability could be seen as the "necessary price" to pay for a high economic growth. To simplify the analysis, we will present the results when $\gamma_\ell = 0$ with the understanding that, by a continuity argument, such results still hold for any growth rate sufficiently close to zero. We begin presenting an intermediary result, that is crucial to unveil the transitional dynamics around (x_ℓ^*, y_ℓ^*). The next Lemma finds some sufficient conditions on the parameters such that the Jacobian matrix evaluated around (x_ℓ^*, y_ℓ^*) has a positive determinant and a negative trace.

Lemma 1 *Consider the case (x_ℓ^*, y_ℓ^*) with $\gamma_\ell = 0$. Then the following results hold:*

i) if $A > \bar{A}$, $\eta > \alpha$ and $\sigma > \underline{\sigma}$ then $\det(J) > 0$;

ii) if $-\alpha < \eta < 2-\alpha$, $A > \hat{A}$ and $\tau_c > \hat{\tau}_c$ then $tr(J) < 0$;

where $\hat{A} \equiv \frac{\delta+\rho}{\alpha}\left(\frac{\eta+\alpha}{2-\eta-\alpha}\right)^{1-\alpha}$ *and* $\hat{\tau}_c \equiv \frac{\alpha}{\Gamma^{\eta}[\Gamma^{-1}(2-\eta-\alpha)-\eta-\alpha]}$.

Proof. See Appendix A. ■

We are now ready to combine the two conditions found in this Lemma with the conditions on parameters found in Proposition 2 to have global indeterminacy. The next proposition shows that the intersection of these different sets of parameters is non-empty and, therefore, we may have global indeterminacy with the lowest steady state being locally indeterminate.

Proposition 3 (Local Indeterminacy) *The steady state* (x_ℓ^*, y_ℓ^*)*, with* $\gamma_\ell = 0$*, is locally indeterminate if*

$$A > \hat{A}, \qquad 0 < \rho < \epsilon, \qquad \alpha < \eta < \eta^\circ, \qquad \tau_c = \underline{\tau}_c, \qquad \textit{and} \;\; \sigma > \underline{\sigma} \tag{23}$$

with ϵ *a sufficiently small and positive real number, and* $\eta^\circ \equiv \frac{\delta-\rho}{\delta(1+\Gamma)}$.

Proof. The proof consists in finding the open set of parameters under which there is global indeterminacy and, at the same time, the Jacobian, evaluated at the lowest steady state is characterized by a $\det(J) \geq 0$ and a $\mathrm{tr}(J) < 0$. The complete proof can be found in Appendix A. ■

Therefore, local indeterminacy around the lowest BGP can emerge for a mild procyclical taxation. This means that procyclical taxation could lead to a poverty trap (see next section, for further considerations on this point). As already explained at the beginning of this section, we observe that:

Remark 3 *By continuity, the result stated in Proposition 3 holds for any* τ_c *lower than, but still sufficiently close to,* $\underline{\tau}_c$*. Therefore, the result holds for any growth rate sufficiently close to zero. This is shown computationally in the following numerical exercise.*

We now propose a numerical exercises to show that the lowest BGP can be indeterminate within reasonable parameter choices. Suppose that the parameters are set exactly as in the numerical exercise proposed earlier, to show the possibility of global indeterminacy. The only difference is that, now, $\tau_c = 0.208$. Then, the growth rate on the lowest BGP is 0.43% and the corresponding steady state is locally indeterminate. In fact the conditions on Proposition 3 are respected since $(\hat{A}, \eta^\circ, \hat{\sigma}) = (0.265, 0.678, 3.55)$. In particular $\det(J) = 0.0055 > 0$ while $\mathrm{tr}(J) = -0.0027$ and, therefore, two negative eigenvalues emerge.

5 Procyclical versus Countercyclical Taxation

In the introduction, we observed that the existing literature, on time varying endogenous taxation, has often argued in favour of procyclical taxation (or government spending). It was indeed shown, in different settings, that, while countercyclical taxes may induce aggregate instability, procyclical taxes should be preferred, because they guarantee local determinacy of the steady state. Examples of this result include Nourri et al. [10] where the authors observe at page 1989 bullet v) that the consumption tax has to be countercyclical with output growth to have a locally indeterminate steady state; Schmitt-Grohe and Uribe [13] who observe at page 977 that "the rational expectations equilibrium is more likely to be indeterminate (...) the less procyclical government expenditure." More recently Bambi and Venditti [1] confirm that procyclical taxation should be implemented to stabilize an economy characterized by productive government spending and endogenous time-varying consumption taxes.

Then, it is compelling, in our framework, to re-address this issue and see if our results suggest a similar policy advice.

According to Propositions 2, 3 and the numerical exercises presented throughout the paper, it is evident that multiple BGPs, as well as aggregate instability around the lowest BGP, may arise even if the taxation is procyclical.

In particular, one of the sufficient conditions to have two BGPs is that the consumption tax growth rate has to be sufficiently procyclical, namely higher than $\underline{\eta} = 0.004$. Althoug it is true that this is not a necessary but only a sufficient condition, our numerical exercises, built upon conditions (17), show that multiple BGPs emerge for plausible choices of the parameters when the consumption tax growth rate is procyclical, i.e. $\eta = 0.5$. Interestingly, making the tax rate less procyclical by slightly reducing η from 0.5 to 0.4, while keeping unchanged the other parameters, leads to a unique BGP because one of the conditions (17) is violated, namely $\tau_c > \underline{\tau}_c = -0.97$.

Regarding the possibility of local indeterminacy of the lowest steady state, i.e. the one associated to BGP_ℓ, a numerical exercise reveals that both the steady states are locally determinate when all the parameters are chosen as in the numerical exercise proposed to show the existence of two BGPs. Under this choice of the parameters, the sufficient condition (23), to have a locally indeterminate (low) steady state, is violated. On the other hand, a slight increase of τ_c from 0.2 to 0.208 changes the stability properties of the lowest BGP and now local indeterminacy emerges. Therefore, aggregate instability around the lowest BGP emerges once extrinsic uncertainty is introduced in the model. Most interestingly, this dynamics can be obtained assuming a procyclical taxation, namely $\eta = 0.5$. Then, assuming an initial condition for the government spending over capital ratio, i.e. $x(0) \equiv \frac{g_0}{k_0}$, sufficiently close to x^*_ℓ, we have that the economy converges toward the lowest BGP with the lowest growth rate. In addition, such a convergence is characterized by volatility in the main aggregate variables once extrinsic uncertainty is added to the model. Summing up, procyclical taxation may lead to a poverty trap characterized by an asymptotic low growth rate and a high volatility.

6 Conclusion

In this paper, we have proposed an endogenous growth model with productive public investments and endogenous consumption taxes and we have shown that procyclical taxation is compatible with global and local indeterminacy. Most interestingly, the local indeterminacy has been proved to emerge around the balanced growth path with the lowest growth rate, implying the existence of a poverty trap. Adding extrinsic uncertainty to the model, an economy, in this poverty trap, would be characterized by high volatility and low growth rate of the aggregate variables. Therefore, our results suggest a careful re-consideration of the procyclical taxation policies suggested by the existing literature.

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A Appendix: Proofs

Proof of Proposition 2. Given the properties of the function $\tilde{\mathcal{T}}(\gamma)$ found in Proposition 1, two BGPs exist as long as the following two conditions hold:

a) $\delta - \varepsilon \leq \tilde{\mathcal{T}}(0) < \delta$, for any $\varepsilon > 0$ sufficiently small real number;

b) $\left.\frac{d\tilde{\mathcal{T}}(\gamma)}{d\gamma}\right|_{\gamma=0} > 1$

In fact, condition *a)* means that the curve $\tilde{\mathcal{T}}(\gamma)$ is slightly below the straight line $\gamma+\delta$ at $\gamma = 0$ but it is steeper for condition *b)*. Therefore the curve must intersect the straight line twice since is continuous, has a unique critical point and $\lim_{\gamma\to\infty} \tilde{\mathcal{T}}(\gamma) = 0^+$.

Step 1 – Parameters Conditions for a) to hold. Using the same argument of the proof of Proposition 1, it can be proved that for any given ε sufficiently small positive constant we have that $\tilde{\mathcal{T}}(0) \geq \delta - \varepsilon$ if $A > \underline{A}(\varepsilon)$ and $\tau_c \geq \underline{\tau}_c(\varepsilon)$ where $\underline{\tau}_c(\varepsilon) \equiv \frac{(\delta-\varepsilon)\Gamma^{\alpha-\eta}}{A-(\Gamma^{-1}+1)\Gamma^{\alpha}(\delta-\varepsilon)}$, $\underline{A}(\varepsilon) \equiv \frac{(\delta-\varepsilon)^{1-\alpha}(\delta+\rho)}{[(1-\alpha)\delta+\rho+\alpha\varepsilon]^{1-\alpha}\alpha^{\alpha}}$, $\underline{\tau}_c(\varepsilon) \leq \underline{\tau}_c$ and $\underline{A}(\varepsilon) \leq \underline{A}$ with equality when $\varepsilon = 0$ as shown in Appendix B.3.1. In the same Appendix, we also show that $\epsilon \equiv \underline{\tau}_c - \underline{\tau}_c(\varepsilon)$ and ε are infinitesimals of the same order.[13]

Based on previous results, it is also the case that $\tilde{\mathcal{T}}(0) < \delta$ if $A > \underline{A}$ and $\tau_c < \underline{\tau}_c$. Summing up, condition *a)* always holds if

$$A > \underline{A} \qquad \textit{and} \qquad \underline{\tau}_c(\varepsilon) = \underline{\tau}_c - \epsilon < \tau_c < \underline{\tau}_c.$$

Step 2 – Parameters Conditions for b) to hold. Taking into account Appendix B.2.2, we have that

$$\left.\frac{d\tilde{\mathcal{T}}(\gamma)}{d\gamma}\right|_{\gamma=0} > 1 \quad \Leftrightarrow \quad \frac{-A\tau_c \left.\frac{dx^*}{d\gamma}\right|_{\gamma=0} \Gamma^{\alpha-1}}{[\tau_c\Gamma^{\alpha-1} + \Gamma^{\alpha-\eta} + \tau_c\Gamma^{\alpha}]^2}[\tau_c(\alpha-1)\Gamma^{-1} + (\alpha-\eta)\Gamma^{-\eta} + \tau_c\alpha] > 1$$

Given that $\left.\frac{dx^*}{d\gamma}\right|_{\gamma=0} = \frac{\sigma\Gamma^{\alpha}}{\alpha(1-\alpha)A}$, the last inequality can be rewritten as follows:

$$\frac{\tau_c\frac{\sigma}{\alpha(1-\alpha)}\Gamma^{-1}}{[\tau_c\Gamma^{-1} + \Gamma^{-\eta} + \tau_c]^2}\{\tau_c[(1-\alpha)\Gamma^{-1} - \alpha] + (\eta-\alpha)\Gamma^{-\eta}\} > 1 \tag{24}$$

Then condition *b)* hold as long as this inequality is satisfied. Clearly the inequality is never satisfied if the term inside the curly brackets is negative. To avoid that, we look for condition on τ_c such that

$$\eta - \alpha + \tau_c[(1-\alpha)\Gamma^{-1} - \alpha]\Gamma^{\eta} > 0 \tag{25}$$

[13] The infinitesimals have the same order if their speed of convergence toward zero is the same. This is indeed shown in Appendix B.3.1.

We need to distinguish two cases.

Case 1: $(1-\alpha)\Gamma^{-1}-\alpha>0$ which is indeed the case when $A>\bar{A}\equiv\frac{\delta+\rho}{(1-\alpha)^{1-\alpha}\alpha^{\alpha}}$.[14] In this case (25) implies

$$\tau_c > \frac{\alpha-\eta}{[(1-\alpha)\Gamma^{-1}-\alpha]\Gamma^{\eta}} \equiv \bar{\tau}_c$$

Once this inequality is imposed, it follows that (24) holds as long as $\sigma>\underline{\sigma}$. Summing up, Condition *b)* - case 1 is satisfied when

$$A>\bar{A}, \qquad \tau_c>\bar{\tau}_c \qquad \textit{and} \qquad \sigma>\underline{\sigma}.$$

Case 2: $(1-\alpha)\Gamma^{-1}-\alpha<0$ which is indeed the case when $A<\bar{A}$. In this case (25) implies

$$\tau_c < \frac{\alpha-\eta}{[(1-\alpha)\Gamma^{-1}-\alpha]\Gamma^{\eta}} \equiv \bar{\tau}_c$$

Once this inequality is imposed, it follows that (24) holds as long as $\sigma>\underline{\sigma}$. Summing up, Condition *b)* - case 2 is satisfied when

$$A<\bar{A}, \qquad \tau_c<\bar{\tau}_c \qquad \textit{and} \qquad \sigma>\underline{\sigma}.$$

Step 3 – Combining Steps 1 and 2. The following inequalities are proved in Appendix B.3.2:

- $\underline{A}<\bar{A}$ always;
- if $A>\bar{A}$ and $\eta>\underline{\eta}$ then $\bar{\tau}_c<\underline{\tau}_c$;
- if $A<\bar{A}$ and $\eta>\underline{\eta}$ then $\bar{\tau}_c>\underline{\tau}_c$;

Taking into account these results, both conditions *a)* and *b)* - case 1 hold if

$$A>\bar{A}, \qquad \eta>\underline{\eta}, \qquad \underline{\tau}_c-\epsilon<\tau_c<\underline{\tau}_c, \qquad \textit{and} \;\; \sigma>\underline{\sigma}. \tag{26}$$

On the other hand, both conditions *a)* and *b)* - case 2 hold if

$$\underline{A}<A<\bar{A}, \qquad \eta>\underline{\eta}, \qquad \underline{\tau}_c-\epsilon<\tau_c<\underline{\tau}_c, \qquad \textit{and} \;\; \sigma>\underline{\sigma}. \tag{27}$$

But then, it follows immediately that conditions *a)* and *b)* hold when (17) is satisfied. Finally observe that if the condition $\eta>\underline{\eta}$ is replaced by $\eta>\alpha$ then the result of Case 1 is unchanged since equation (25) is respected for any choice of τ_c. ■

[14] This last inequality and the value of $\bar{A}$ can be easily found by combining $(1-\alpha)\Gamma^{-1}-\alpha>0$ with the definition of Γ given in Proposition 1.

Proof of Lemma 1. From Step 2 of Proposition 2 we know that $(1-\alpha)\Gamma^{-1}-\alpha > 0 \quad \Leftrightarrow \quad A > \bar{A}$. Based on that, condition i) and ii) follow immediately.

On the other hand, $\mathrm{tr}(J) < 0$ if the sum of the first two terms within the curly parenthesis is negative:

$$x^{*\eta}\tau_c\left[(\eta+\alpha-2)x^{*-1}+\eta+\alpha\right]+\alpha<0 \tag{28}$$

Clearly, this never happens if the term in square brackets is positive. Therefore, let us consider the case when it is negative. This is indeed possible when, for example, $-\alpha < \eta < 2-\alpha$ and $\Gamma < \frac{2-\eta-\alpha}{\eta+\alpha}$. The last condition can be rewritten in term of A by taking into account the definition of Γ and leads to $A > \hat{A}$. Under these assumptions on the parameters, the inequality (28) is respected when $\tau_c > \hat{\tau}_c$ and therefore $\mathrm{tr}(J) < 0$. ■

Proof of Proposition 3. To have local indeterminacy we need that $\det(J) \geq 0$. We begin combining the conditions on the parameters $(A, \eta, \tau_c, \sigma)$ which guarantee this sign of the determinant with those for multiple BGPs. From step 3 of the proof of Proposition 2 we know that $\underline{A} < \bar{A}$ and therefore the resulting condition on A for having $\det(J) \geq 0$ and multiple BGPs is $A > \bar{A}$. Regarding τ_c and η the conditions which must hold are those for having multiple BGPs since $\det(J) \geq 0$ independently on their values. Finally we have $\sigma > \underline{\sigma}$. Summing up, we need

$$A > \bar{A}, \qquad \eta > \alpha, \qquad \tau_c = \underline{\tau}_c, \quad \textit{and} \quad \sigma > \hat{\sigma} \tag{29}$$

We now need to combine these inequalities with those to have $\mathrm{tr}(J) < 0$. Let us begin with the condition on A. It is immediate to see that it must be that $A > \max\{\bar{A}, \hat{A}\}$ where $\hat{A} > \bar{A}$ if $\eta > \alpha$, see Appendix B.4.1. Regarding the condition on η, let us consider the case $\rho \to 0^+$; in this case $\underline{\eta} \to 0^+$ and then, by continuity, we have that $\exists \epsilon > 0 : \forall \rho \in (0, \epsilon)$ we have that the resulting condition on η is $\alpha < \eta < 2-\alpha$. Finally we need to find conditions under which $\hat{\tau}_c < \underline{\tau}_c$ otherwise a negative trace is incompatible with multiple BGPs. After tedious computations, reported in Appendix B.4.2, it emerges that $\hat{\tau}_c < \underline{\tau}_c$ if $\eta < \eta^\circ$. We need then to verify that $\eta^\circ \in (\alpha, 2-\alpha)$. As shown in Appendix B.4.3, $\eta^\circ > \alpha$ if $\rho < \delta$ and $A > \frac{(\delta+\rho)\delta^{1-\alpha}}{[(1-\alpha)\delta-\rho]^{1-\alpha}\alpha^\alpha} \equiv A^\bullet$; we need then to check under which conditions $A^\bullet < \hat{A}$; as shown in Appendix B.4.4, this inequality is always respected when $\eta > \alpha$ as $\rho \to 0^+$.[15] On the other hand, $\eta^\circ < 2-\alpha$ always since it implies $(\alpha-1)\delta-\rho < (2-\alpha)\Gamma\delta$ which is clearly always respected. Combining the resulting inequalities lead to (23).

■

[15] As a consequence, we have that $\lim_{\rho\to 0^+}\eta^\circ = \frac{1}{1+\Gamma} > \alpha$.

B Appendix: Other material

B.1 Further details on how to obtain the system (9) (10)

Equation (9) can be obtained as it follows. Combining the firms' FOC with the intertemporal budget constraint of the households and rewriting it in the new variables $x \equiv \frac{g}{k}$ and $y \equiv \frac{c}{k}$ leads to:

$$\frac{\dot{k}}{k} = Ax^{1-\alpha} - \delta - (1+\tau)y = Ax^{1-\alpha} - \delta - (1+\tau_c x^\eta)y$$

where the last equality follows from the fact that $\tau = \tau_c x^\eta$. Similarly the government budget constraint can be rewritten as it follows:

$$\frac{\dot{g}}{g} = \tau y x^{-1} - \delta = \tau_c y x^{\eta-1} - \delta$$

Since $\frac{\dot{x}}{x} = \frac{\dot{g}}{g} - \frac{\dot{k}}{k}$ it must be that

$$\frac{\dot{x}}{x} = (\tau_c x^{\eta-1} + 1 + \tau_c x^\eta)y - Ax^{1-\alpha}$$

Regarding equation (10) the steps to obtain it are the following. Combining the Euler equation with the firms' FOCs and rewriting it in the new variables leads to

$$\frac{\dot{c}}{c} = \frac{1}{\sigma}\left(\alpha A x^{1-\alpha} - \delta - \rho - \frac{\dot{\tau}}{1+\tau}\right)$$

From the fiscal policy (2) we have that

$$\log(1+\tau) = \log(1+\tau_c x^\eta) \qquad \Rightarrow \qquad \frac{\dot{\tau}}{1+\tau} = \frac{\tau_c \eta x^\eta}{1+\tau_c x^\eta}\frac{\dot{x}}{x}$$

Substituting this last expression into the Euler equation and considering that $\frac{\dot{y}}{y} = \frac{\dot{c}}{c} - \frac{\dot{k}}{k}$ it must be that

$$\frac{\dot{y}}{y} = \frac{1}{\sigma}\left[\alpha A x^{1-\alpha} - \delta - \rho - \frac{\tau_c \eta x^\eta}{1+\tau_c x^\eta}\frac{\dot{x}}{x}\right] - \left[Ax^{1-\alpha} - \delta - (1+\tau_c x^\eta)y\right]$$

B.2 Further details on the proof of Proposition 1

B.2.1 Conditions for $\tilde{\mathcal{T}}(0) \geq \delta$

Regarding the conditions which emerge from imposing $\tilde{\mathcal{T}}(0) \geq \delta$. First

$$\tilde{\mathcal{T}}(0) = \frac{\tau_c A}{\tau_c \Gamma^{\alpha-1} + \Gamma^{\alpha-\eta} + \tau_c \Gamma^\alpha} \geq \delta$$

Observe that the denominator is always strictly positive so we may rewrite the inequality as

$$\tau_c[A - (\Gamma^{\alpha-1} + \Gamma^{\alpha})\delta] \geq \delta\Gamma^{\alpha-\eta}$$

Since $\Gamma = \left(\frac{\delta+\rho}{\alpha A}\right)^{\frac{1}{1-\alpha}}$, it follows immediately that

$$A - (\Gamma^{\alpha-1} + \Gamma^{\alpha})\delta > 0 \qquad \Leftrightarrow \qquad A > \frac{\delta^{1-\alpha}(\delta+\rho)}{[(1-\alpha)\delta+\rho]^{1-\alpha}\alpha^{\alpha}} \equiv \underline{A}$$

Assuming $A > \underline{A}$, then $\tilde{\mathcal{T}}(0) \geq \delta$ if and only if

$$\tau_c \geq \frac{\delta\Gamma^{\alpha-\eta}}{A - (\Gamma^{\alpha-1} + \Gamma^{\alpha})\delta} \equiv \underline{\tau}_c$$

where $\underline{\tau}_c > 0$. On the other hand, the case $0 < A \leq \underline{A}$ is not admissible since it implies that $\tilde{\mathcal{T}}(0) \geq \delta$ only if x_0 or δ are negative for any positive values of τ_c.

B.2.2 Existence of (at most) a critical point for $\tilde{\mathcal{T}}(\gamma)$

Second, the function $\tilde{\mathcal{T}}(\gamma)$ has a unique critical point. In fact

$$\frac{d\tilde{\mathcal{T}}}{d\gamma} = \frac{d\tilde{\mathcal{T}}}{dx^*} \cdot \frac{dx^*}{d\gamma} = \underbrace{\frac{-A\tau_c\frac{dx^*}{d\gamma}x^{*\alpha-1}}{[\tau_c x^{*\alpha-1} + x^{*\alpha-\eta} + \tau_c x^{*\alpha}]^2}}_{<0}[\tau_c(\alpha-1)x^{*-1} + (\alpha-\eta)x^{*-\eta} + \tau_c\alpha]$$

and therefore

$$\frac{d\tilde{\mathcal{T}}}{d\gamma} = 0 \qquad \Leftrightarrow \qquad \underbrace{\tau_c(\alpha-1)x^{*-1} + (\alpha-\eta)x^{*-\eta} + \tau_c\alpha}_{\equiv g(x^*)} = 0$$

Now it exists always a unique positive root $\hat{x}$ of $g(x^*) = 0$. This is a direct consequences of the following arguments. First, the function $g(x^*)$ is not continuous in $x^* = 0$ and has a critical point at $x^{\bullet} = \left(\frac{\eta(\alpha-\eta)}{\tau_c(1-\alpha)}\right)^{\frac{1}{\eta-1}}$. Second,

$$\lim_{x^*\to+\infty} g(x^*) = \begin{cases} (\tau_c\alpha)^- & if \quad \eta > \alpha \\ (\tau_c\alpha)^+ & if \quad 0 < \eta < \alpha \\ +\infty & if \quad \eta < 0 \end{cases} \qquad and \qquad \lim_{x^*\to 0^+} g(x^*) = -\infty \quad always$$

Combining this information it is always the case that $g(x^*)$ intersects the x^*-axis only once. Since x^* is a one-to-one function of γ then it always exits a unique critical point $\hat{\gamma}$ of $\tilde{\mathcal{T}}(\gamma)$ in the extended domain $\gamma > -\frac{\delta+\rho}{\alpha A\sigma}$.[16] Therefore, $\tilde{\mathcal{T}}(\gamma)$ has *at most* a critical point in the domain $\gamma \geq 0$.

[16] In fact, looking at equation (12) we have that $\gamma \to -\frac{\delta+\rho}{\alpha A\sigma}$ as $x^* \to 0^+$

B.3 Further details on the proof of Proposition 2

B.3.1 Details for Step 1

We prove that $\underline{A} \geq \underline{A}(\varepsilon)$ by contradiction. Suppose that $\underline{A} < \underline{A}(\varepsilon)$ then it follows that

$$\underline{A} \equiv \frac{(\delta+\rho)\delta^{1-\alpha}}{[\rho+(1-\alpha)\delta]^{1-\alpha}\alpha^{\alpha}} < \frac{(\delta+\rho)(\delta-\varepsilon)^{1-\alpha}}{[\rho+(1-\alpha)\delta+\alpha\varepsilon]^{1-\alpha}\alpha^{\alpha}} \equiv \underline{A}(\varepsilon)$$

Simplifying these expressions the inequality boils down to $-\varepsilon(\rho+\delta) > 0$ which is clearly impossible since $\varepsilon, \delta, \rho$ are positive.

We prove now by contradiction that $\underline{\tau}_c \geq \underline{\tau}_c(\varepsilon)$. Suppose that $\underline{\tau}_c < \underline{\tau}_c(\varepsilon)$ then it follows that

$$\underline{\tau}_c(\varepsilon) \equiv \frac{\delta-\varepsilon}{A-(\Gamma^{\alpha-1})+\Gamma^{\alpha}\delta} > \frac{\delta}{A-(\Gamma^{\alpha-1})+\Gamma^{\alpha}\delta} \equiv \underline{\tau}_c$$

Simplifying the inequality boils down to $-\varepsilon A > 0$ which is clearly impossible since ε, A are positive.

Finally we want to show that $\epsilon \equiv \underline{\tau}_c - \underline{\tau}_c(\varepsilon)$ and ε are infinitesimals of the same order. To do so, we need to show that $\lim_{\varepsilon\to 0} \frac{\underline{\tau}_c-\underline{\tau}_c(\varepsilon)}{\varepsilon}$ is a positive constant. Since the argument of the limit has the indeterminate form $\frac{0}{0}$ then we apply the Hopital's rule and we find that

$$\lim_{\varepsilon\to 0} \frac{\underline{\tau}_c-\underline{\tau}_c(\varepsilon)}{\varepsilon} = \lim_{\varepsilon\to 0} -\tau_c'(\varepsilon) = \frac{\Gamma^{\alpha-\eta}A}{[A-(\Gamma^{-1}+1)\delta\Gamma^{\alpha}]^2} > 0.$$

B.3.2 Details for Step 3

We want here to show that $\underline{A} < \bar{A}$. In fact

$$\underline{A} < \bar{A} \qquad \Leftrightarrow \qquad \frac{\delta^{1-\alpha}(\delta+\rho)}{[(1-\alpha)\delta+\rho]^{1-\alpha}\alpha^{\alpha}} < \frac{\delta+\rho}{(1-\alpha)^{1-\alpha}\alpha^{\alpha}} \qquad \Leftrightarrow \qquad \rho > 0.$$

Moreover, we want to find conditions such that $\underline{\tau}_c < \bar{\tau}_c$. Taking into account their definition, observe that,

$$\underline{\tau}_c \equiv \frac{\delta\Gamma^{\alpha-\eta}}{A-(\Gamma^{-1}+1)\Gamma^{\alpha}\delta} < \frac{\alpha-\eta}{[(1-\alpha)\Gamma^{-1}-\alpha]\,\Gamma^{\eta}} \equiv \bar{\tau}_c \tag{30}$$

From Appendix B.2.1, we know that $A > \underline{A}$ implies $A-(\Gamma^{-1}+1)\Gamma^{\alpha}\delta > 0$; let us first focus on case 1, i.e. $A > \bar{A}$ and then $(1-\alpha)\Gamma^{-1}-\alpha > 0$, then (30) holds if and only if

$$\eta < \underline{\eta} \equiv \frac{\rho\alpha A}{A-(\Gamma^{-1}+1)\Gamma^{\alpha}\delta} \equiv \underline{\eta}$$

On the other hand, on case 2, i.e. $A < \bar{A}$ and then $(1-\alpha)\Gamma^{-1}-\alpha < 0$, then (30) holds if and only if $\eta > \underline{\eta}$. Combining these results lead to the last two inequalities listed in Step 3 of Proposition 2.

B.4 Further details on the proof of Proposition 3

B.4.1 Details on $\hat{\sigma} > \underline{\sigma}$ and $\hat{A} > \bar{A}$ when $\eta > \alpha$

The inequality $\hat{A} > \bar{A}$ holds too when $\eta > \alpha$; in fact,

$$\hat{A} \equiv \frac{\delta + \rho}{\alpha} \left(\frac{\eta + \alpha}{2 - \eta - \alpha} \right)^{1-\alpha} > \frac{\delta + \rho}{\alpha} \left(\frac{\alpha}{1 - \alpha} \right)^{1-\alpha} \equiv \bar{A} \tag{31}$$

which is satisfied as long as $\frac{\eta+\alpha}{2-\eta-\alpha} > \frac{\alpha}{1-\alpha}$ which is indeed always the case when $\eta > \alpha$.

B.4.2 Details on $\hat{\tau}_c < \underline{\tau}_c$ if $\eta < \eta^\circ$

Taking into account the definition of $\hat{\tau}_c$ and $\underline{\tau}_c$ we have that

$$\hat{\tau}_c \equiv \frac{\alpha}{\Gamma^\eta [\Gamma^{-1}(2 - \eta - \alpha) - \eta - \alpha]} < \frac{\delta \Gamma^{\alpha - \eta}}{A - (\Gamma^{-1} + 1)\Gamma^\alpha \delta} \equiv \underline{\tau}_c$$

After some simplifications, it emerges that such inequalities lead to

$$\alpha A < -\delta \Gamma^\alpha \left[\Gamma^{-1}(\eta - 2) + \eta \right]$$

and solving for η we obtain that

$$\eta < \frac{\alpha A(\delta - \rho)}{\delta(\alpha A + (\delta + \rho)\Gamma^\alpha)} = \frac{\delta - \rho}{\delta(1 + \Gamma)} \equiv \eta^\circ$$

here the last equality is obtained by dividing both sides of the right hand side of the inequality by αA and using the definition of Γ.

B.4.3 Details on $\eta^\circ > \alpha$ if $A > A^\bullet$

Using the definition of x_0^* and rearranging the terms we have that the inequality $\eta^\circ > \alpha$ is equivalent to

$$\frac{(1 - \alpha)\delta - \rho}{\delta \alpha} > \left(\frac{\delta + \rho}{\alpha A} \right)^{\frac{1}{1-\alpha}}$$

Assuming $\rho < \delta$ and solving for A leads to $A > A^\bullet$.

B.4.4 Details on $A^\bullet < \hat{A}$ as $\rho \to 0^+$

We want to show under which conditions we have that

$$A^\bullet \equiv \frac{(\delta + \rho)\delta^{1-\alpha}}{[(1 - \alpha)\delta - \rho]^{1-\alpha} \alpha^\alpha} < \frac{\delta + \rho}{\alpha} \left(\frac{\eta + \alpha}{2 - \eta - \alpha} \right)^{1-\alpha} \equiv \hat{A}$$

Assuming $\rho < \delta$ this inequality is equivalent to

$$\eta > \alpha \frac{\delta + \rho}{\delta - \rho}$$

which clearly implies $\eta > \alpha$ as $\rho \to 0^+$.

Supplementary Material

Determinant of the Jacobian

From the system of differential equations, the determinant of Jacobian matrix can be found from its components; a, b, c and d.

$$det(J) = ad - bc$$

Recall the value of a, b, c and d, the determinant of Jacobian matrix (det(J)) is,

$$\begin{aligned}
&\det(J) = a \cdot y^* \left(1 + \tau_c x^{*\eta} - \frac{b}{\sigma} \cdot \frac{\tau_c \eta}{x^* + \tau_c x^*}\right) \\
&-b \cdot y^* \left[-\left(\frac{1}{\sigma}(1-\alpha)(\sigma - \alpha) A x^{*-\alpha} + \frac{\tau_c \eta}{x^{*1-\eta} + \tau_c x^*} \cdot a \right) + \tau_c \eta x^{*\eta-1} y^* \right]
\end{aligned}$$

Rewriting component a as a function of b,

$$a = \left[\eta(b - x) - \tau_c x^{*\eta}\right] y^* x^{*-1} - (1-\alpha) A x^{*1-\alpha}$$

Substituting a into the equation, the determinant of Jacobian matrix can be rewritten in term of b.

$$\begin{aligned}
&\det(J) = y^* \left(\tau_c x^{*\eta-1} y^* (\eta - 1 - \tau_c x^{*\eta})\right) \\
&-(1-\alpha) A x^{*1-\alpha} y^* \left(1 + \tau_c x^{*\eta} - b x^{*-1} - \frac{b\alpha}{\sigma} x^{*-1}\right)
\end{aligned}$$

Extracting b out,

$$\begin{aligned}
&\det(J) = y^* \left(\tau_c x^{*\eta-1} y^* (\eta - 1 - \tau_c x^{*\eta})\right) \\
&-(1-\alpha) A x^{*1-\alpha} y^* \left(-\tau_c x^{*\eta-1} + \frac{\alpha}{\sigma}(\tau_c x^{*\eta-1} + 1 + \tau_c x^{*\eta})\right)
\end{aligned}$$

Using the fact that along a BGP it must be that

$$y^* = \frac{A x^{*1-\alpha}}{\tau_c x^{*\eta-1} + 1 + \tau_c x^{*\eta}}$$

some algebraic manipulation leads to the following result

$$\begin{aligned}
&\det(J) = y^{*2} x^{*2\eta} \left\{\tau_c x^{*-1}[(1-\alpha)\tau_c x^{*-1} - \alpha\tau_c + (\eta - \alpha) x^{*-\eta}]\right\} \\
&-y^{*2} x^{*2\eta} \left\{\frac{(1-\alpha)\alpha}{\sigma}(\tau_c x^{*-1} + x^{*-\eta} + \tau_c)^2\right\}
\end{aligned}$$

Trace of the Jacobian Matrix

The Trace of the Jacobian is the sums of the diagonal components.

$$tr(J) = a + d$$

That is,

$$\begin{aligned} tr(J) &= (\tau_c(\eta - 1)x^{*\eta-1} + \tau_c\eta x^{*\eta}) \cdot y^* - (1-\alpha)Ax^{*1-\alpha} \\ &+y^* \left[1 + \tau_c x^{*\eta} - \frac{1}{\sigma} \cdot \frac{\tau_c\eta}{x^{*1-\eta} + \tau_c x^*} \cdot b\right] \end{aligned}$$

Extracting b out,

$$\begin{aligned} tr(J) &= (\tau_c(\eta - 1)x^{*\eta-1} + \tau_c\eta x^{*\eta}) \cdot y^* - (1-\alpha)Ax^{*1-\alpha} \\ &+y^* \left[1 + \tau_c x^{*\eta} - \frac{1}{\sigma} \cdot \frac{\tau_c\eta}{x^{*1-\eta} + \tau_c x^*}(\tau_c x^{*\eta} + x^* + \tau_c x^{*\eta+1})\right] \end{aligned}$$

Adding and subtracting $(1-\alpha)[\tau_c x^{*\eta-1} + 1 + \tau_c x^{*\eta}]y^*$ on the right hand side, the value of tr(J) will be,

$$\begin{aligned} tr(J) &= y^* \left[\tau_c(\eta - 1)x^{*\eta-1} + \tau_c\eta x^{*\eta} - \frac{1}{\sigma}\frac{\tau_c\eta}{x^{*1-\eta} + \tau_c x^*}(\tau_c x^{*\eta} + x^* + \tau_c x^{*\eta+1})\right] \\ &+(1 + \tau_c x^{*\eta})y^* - (1-\alpha)Ax^{*1-\alpha} + (1-\alpha)(\tau_c x^{*\eta-1} + x^* + \tau_c x^{*\eta}) \\ &-(1-\alpha)(\tau_c x^{*\eta-1} + x^* + \tau_c x^{*\eta}) \end{aligned}$$

Using the fact that $\frac{\dot{x}}{x} = 0$ along a BGP, the trace of Jacobian Matrix can be rewritten as follows,

$$\begin{aligned} tr(J) &= y^*\{\tau_c x^{*\eta}[(\eta - 2 + \alpha)x^{*-1} + \eta + \alpha] + \alpha\} \\ &-y^* \left\{\frac{1}{\sigma} \cdot \frac{\tau_c\eta(\tau_c x^{*\eta-1} + 1 + \tau_c x^{*\eta})}{x^{*-\eta} + \tau_c}\right\} \end{aligned}$$