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Abstract: This paper considers a general and practical kidney exchange model with compatible or incompatible patient-donor pairs, single donors, and patients on the waiting list. Efficient exchange procedures are proposed with dichotomous preferences in which only one-way, two-way, three, or four-way chains or cycles of exchange are used. We derive a tight upper bound of the possible number of feasible kidney transplants in each case of exchange and provide substantial simulation results. We find that two-way cycles and chains of exchange can substantially increase the number of feasible transplants, three-way can have a visible effect, and at most four-way cycles and chains suffice to capture all potential gains of exchange. Our results are not only theoretically interesting but also have important and novel policy implications.

Keywords: Kidney Exchange, Efficiency, Matching, Simulation.
JEL classification: C78, D47.

1 Introduction

Every year in the world hundreds and thousands patients of severe kidney disease need a kidney transplant. The difficulty of having suitable kidney transplants arises in three major aspects. Firstly, there is a significant shortage of kidneys from deceased donors. For instance, in the United States in 2005 more than 60,000 patients were waiting for kidney transplants and only about 9,900 received transplants from deceased donors and 6,563 received transplants from living donors. While in waiting, over 4,000 patients passed away and about 1,000 were getting too sick to have a transplant and were therefore removed from the waiting list (see Roth, Sönmez and Ünver 2007, p.828). In the United Kingdom during the period of 2013-2014 which is the best year over the previous ten years, 5881 active patients were on the waiting list and 2142 got transplants from deceased donors and 1114 received transplants from living donors. Secondly, a patient may receive a kidney from a living donor who can be a family member, a relative, or a friend of the patient. In this case the patient and the donor are called a patient-donor pair, and the patient is a paired patient and the donor a paired donor. But the patient may not be compatible

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with the donor and therefore is unable to use the kidney directly because of blood or tissue incompatibility. Thirdly, although most people have one more kidney than they need, it is almost universally illegal to buy or sell a kidney. A central issue here is how to design an effective mechanism to enable as many patients as possible to receive a suitable kidney transplant, within existing medical, legal and social constraints.

The operation of a suitable kidney transplant must satisfy several essential constraints as follows. The first are two medical constraints: the patient must be both blood-compatible and tissue-compatible with the donor. The second is the incentive constraint. This will not be a problem when a patient receives a kidney from an altruistic deceased or living donor. The issue arises when it involves patient-donor pairs. If a paired patient is incompatible with her paired donor, she need to exchange one kidney for another. Then the order of implementing kidney transplant becomes crucial to incentive-compatible exchange. We illustrate this point by an example. Suppose there are two patient-donor pairs. The first paired patient is compatible with the second paired donor while the second paired patient is compatible with the first paired donor. If the first patient first receives kidney transplant from the second donor, there is a possibility that the first donor may regret and renege her promise because one cannot force her to donate her kidney to the second patient. To avoid this moral hazard, exchanges between the two pairs must be carried out simultaneously. The third is the capacity constraint which is caused by the second constraint. Because transplants need to be performed simultaneously, it means that such operations must take place in the same hospital or hospitals in close proximity to each other. Just two-way exchange or two pair exchange already requires four simultaneous surgical treatments. Obviously, in practice there is a limit to the number of possible kidney transplants in each hospital. It is therefore desirable to have short chains or cycles of exchange.

Kidney exchange has been previously studied by medical researchers (see Rapaport 1986, Ross et al. 1997, Ross and Woodle 2000, Zenios, Woodle, and Ross 2001, etc). Roth, Sönmez and Ünver (2004) initialized economic analysis of kidney exchange and transformed it into a fertile area of economic research. They examined a model of kidney exchange in which there are many patient-donor pairs. Each patient has strict preferences over compatible kidneys, her paired kidney and the wait-list option. They proposed an exchange mechanism-the top trading cycles and chains (TTCC) mechanism- a generalization of the top trading cycle procedure from Shapley and Scarf (1974) for a housing market model that achieves efficiency and incentive compatibility; see also Abdulkadiroğlu and Sönmez (1999) for a related mechanism. In this case cycles and chains could be long. Roth, Sönmez and Ünver (2007) considered a simpler but more practical model where patients are indifferent between compatible kidneys and prefer compatible kidneys to incompatible ones. Their model consists of many incompatible patient-donor pairs. They demonstrate that allowing three-ways as well as two-ways could significantly increase the number of possible exchanges, and 4-ways are sufficient to capture all potential gains of exchange. Roth, Sönmez and Ünver (2007) and Saidman et al. (2006) provide computational results on real and simulated patient data to show significant efficiency gains from two-way and three-way exchanges.

In this paper we consider a general and also practical model of kidney exchange. The model consists of compatible patient-donor pairs, incompatible patient-donor pairs, (altruistic) single donors (decreased or living), and patients on the waiting list. Our aim is to explore how many kidney transplants can be possibly arranged within the same medical, incentive, and capacity constraints as those used in Roth, Sönmez and Ünver (2007). In their model, if a patient is compatible with her donor, then a transplant will take place just between the pair. So in their model, exchanges are carried out among only incompatible patient-donor pairs. In contrast, in the current paper we also allow compatible patient-donor pairs to participate in exchange with incompatible patient-donor pairs, if necessary, in order to enable more patients to receive transplants and thus save more lives. Let us show a case in point. Suppose that there are three blood-incompatible patient-donor pairs (O, AB), two compatible patient-donor pairs $(AB, O)^c$, and one tissue-incompatible pair $(AB, O)^i$. For each pair, its first component indicates the patient's blood type, its second is the donor's blood type, and its superscripts c and i stand for tissue-compatible and tissue-incompatible, respectively. If compatible patient-donor pairs do not exchange with incompatible patient-donor pairs, only four patients will receive kidney plants, i.e. two $(AB, O)^c$ and one two-way exchange $(O, AB) - (AB, O)^i$. In contrast, if we allow compatible pairs to exchange with incompatible pairs, six patients will receive kidney transplants, i.e., three two-way exchanges $(O, AB) - (AB, O)^i$, $(O, AB) - (AB, O)^c$ and $(O, AB) - (AB, O)^c$. In this way, two more patients will get kidney transplants and be saved. Ross and Woodle (2000) suggest to introduce the inclusion of compatible pairs in kidney exchange with incompatible ones and Roth, Sönmez and Ünver (2005b, p. 377) also indicate this potential.

We will establish several basic results for this general and practical model, going beyond and improving considerably those of Roth, Sönmez and Ünver (2007). Briefly speaking, in each case of k-way exchange, $k = 2, 3, 4$, we derive a tight upper bound (in fact an explicit formula) of the possible number of feasible kidney transplants and propose a sequential matching procedure to achieve this upper bound. We find that two or three-way cycles and chains of exchange can substantially increase the number of feasible transplants, and at most four-way cycles and chains are sufficient to achieve full potential gains of exchange. In particular, allowing compatible patient-donor pairs to participate in exchange with incompatible pairs will considerably enhance efficiency of kidney exchange, which means many more patients can receive transplants and their lives could be saved. It will be shown in Section 3.1 that this benefit becomes very obvious and significant even just under two-way cycles and chains of exchange. This benefit becomes even more substantial as the pool of patients and donors gets large, or single donors are also allowed to participate in exchange with all compatible or incompatible pairs. We prove that in every-way (2-way, 3-way or 4-way) of exchange, each cycle contains at most two blood-type compatible pairs and each chain comprises at most one blood-type compatible pair. Moreover, we discuss a more general model of type-compatible exchanges with patient-donor pairs, single donors and patients on the waiting list and demonstrate that the maximal size of exchange to achieve efficiency equals the number of total types.

As our basic model is quite practical and general, our analysis will become inevitably much more involved and more difficult due to a large number of combinatorial cases caused by the presence of compatible or incompatible patient-donor pairs, single donors and patients on the waiting list.

To test theory and explore its policy implications, we provide substantial simulation results. Simulations are carried out based on two real life data sets from the USA national patient and donor characteristics from 1993 to 2002 and from 1995 to 2016, respectively. The first period from 1993 to 2002 is the same as that used by Roth, Sönmez and Ünver (2007), and Saidman et al. (2006), except that in our new data set we add more relevant information including the distribution of compatible patient-donor pairs and single donors, which is not used in their models. Compared with the first time slot data, the second time slot data from 1995 to 2016 contains more accurate information on tissue-type incompatibility. We run Monte-Carlo simulations of 5000 random population constructions for 25, 50, 100, 150 and 200 incompatible patient-donor pairs, and run also Monte-Carlo simulations of 500 random population constructions for 300 and 400, respectively, with their corresponding compatible patient-donor pairs and single donors (and patients on the waiting list who need no simulation as they are populous) based on the 1993-2002 data set and the 1995-2016 data set. By comparison, Roth, Sönmez and Ünver (2007) have done Monte-Carlo simulations of 500 random population constructions for 25, 50, and 100 incompatible patient-donor pairs based on the 1993-2002 data set, by using two-, three-, higher- and unrestricted-way of exchange, and Saidman et al. (2006) have tested the case of 25 and 100 incompatible patient-donor pairs.

In our simulations we use only two-way chains or cycles of exchange. For the same population, in comparison with Roth, Sönmez and Ünver (2007) whose mechanism will be simply called *the exclusive (exchange) mechanism*, our mechanism (called *the first degree inclusive (exchange) mechanism*) of allowing compatible pairs to exchange with incompatible pairs can have at least 10% net increase of feasible kidney transplants and our mechanism (called *the second degree inclusive (exchange) mechanism*) of allowing compatible pairs and single donors to exchange with incompatible pairs can have at least 30% net increase of feasible kidney transplants. For instance, for the 1993-2002 data set, if a population has 100 incompatible patient-donor pairs, the population will have 22 compatible patient-donor pairs and 39 single donors, the exclusive mechanism will enable 49 incompatible paired patients to get feasible transplants whereas the first degree inclusive mechanism will increase this number to 64 and the second degree inclusive mechanism will raise it to 89. For the 1995-2016 data set, if a population has 100 incompatible patient-donor pairs, the population will have 20 compatible patient-donor pairs and 36 single donors, the exclusive mechanism will enable 34 incompatible paired patients to get feasible transplants whereas the first degree inclusive mechanism will increase this number to 46 and the second degree inclusive mechanism will raise it to 69.

Major findings from our simulations are briefly stated here. Firstly, our simulations clearly indicate that as the number of incompatible patient-donor pairs in the population reaches 100, the slope of matching rates (in percentage) of incompatible paired patients getting transplants becomes almost flat, albeit upwards, (which implies efficiency of exchange becomes asymptotically constant). This is somewhat surprising and has an important and novel policy implication: *Kidney exchange can be decentralized* in the sense that in a country with a relatively large population, separate kidney exchange programs can be established in several major regions, not just one centralized program for the entire country. Secondly, we find that the actual maximal number of kidney exchanges is surprisingly close to the predicted number given by our derived formulae. Thirdly, we find that as the size of the population gets larger, the predictive power of our theory becomes increasingly better. Fourthly, our results show that because the 1995-2016 data set contains more precise information on the physical characteristics of the population, this will improve the quality of feasible kidney transplants but at the same time reduce the number of feasible transplants roughly by 20%. Fifthly, we find that two-way exchange can reap most benefits of exchange and will play an even more important role in gaining benefits of exchange as the size of the population increases.

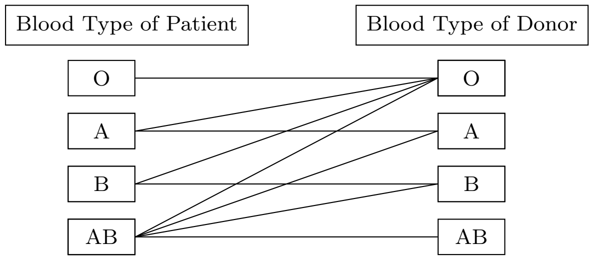
We conclude this introductory section by briefly reviewing several other related papers. Roth, Sönmez and Ünver (2005a) consider a kidney exchange model in which the size of kidney exchanges is restricted to two patient-donor pairs and patients are indifferent compatible kidneys. They propose both deterministic and stochastic efficient and strategy-proof mechanisms. The deterministic ones can accommodate certain priority structure while the stochastic ones exhibit a distributive justice property. Yilmaz (2011) proposes an egalitarian mechanism that uses two-way exchanges and list exchanges. A list exchange means that an incompatible paired donor gives a kidney to a patient on the waiting list and in return the incompatible paired patient gets a priority on the waiting list. Sönmez and Ünver (2014) study a model consisting of compatible pairs and incompatible pairs under two-way exchange. They examine the structure of Pareto-efficient matchings and show that all such matchings have the same number of kidney transplants for patients. They find a novel application of the well-known Gallai-Edmonds decomposition in kidney exchange. Ausubel and Morrill (2014) observe that incentive compatibility for kidney exchange requires only kidney donation to occur no later than the associated kidney receipt. They show that sequential exchanges can also increase the number of beneficial exchanges. Andersson and Kratz (2016) examine efficient kidney exchanges under a refined structure of blood type compatibility. In their model, every patient prefers a fully acceptable donor to any donor who is not fully acceptable, and yet prefers an acceptable donor to any unacceptable donor. In a related development, Ünver (2010) studies efficient kidney exchanges in a dynamic environment in which agents arrive according to a stochastic Poisson process. We refer to Sönmez and Ünver (2013) for a survey on the subject and references therein contained.

This paper is organised as follows. The model and basic concepts are introduced in Section 2. Maximal numbers of transplants from two, three, and four-way exchanges are derived in Section 3. A general n-way exchange model is discussed in Section 4. Simulations are presented in Section 5 and conclusion is given in Section 6. Most of proofs are deferred to the appendix.

2 The Model

Kidney exchanges involve patients and donors. A kidney can be transplanted from a willing donor to a patient if the donor's kidney is compatible to the patient both in blood type and tissue type. There are four blood types, A, B, AB, and O. A patient of O type can receive a kidney only from a donor of O type, a patient of A type can receive a kidney from a donor of A or O type, a patient of B type can receive a kidney from a donor of B or O type, while a patient of AB type can receive a kidney from a donor of any blood type. Blood-compatibility is shown in Figure 1. Another medical test concerns tissue. Tissue-compatibility is determined by six HLA (human leukocyte antigen) proteins (three from the father and another three from the mother). If the potential recipient shows antibodies against HLA in the donor kidney called a positive crossmatch, then the donor kidney cannot be transplanted to the patient. Unlike blood-compatibility, tissue-compatibility does not require exact HLA match between a patient and a donor. Moreover, in reality, the percentage of tissue-incompatibility is also very low; see Zenios, Woodle and Ross (2001).

Figure 1: Blood-type compatibility between patients and donors.



Formally our kidney exchange model consists of a set D^S of single donors, a set of patients P^W on the waiting list (on TWL in short) and a set PD of patient-donor pairs. Single donors could be altruistic cadavers or living people. Patients on TWL are also called single patients. A patient-donor pair describes a designated patient and a living donor who is willing to give a kidney to the patient or to exchange a kidney with another kidney for the designated patient. A patient (donor) in a patient-donor pair will be called a paired patient (donor). Patients are indifferent between compatible kidneys, indifferent between incompatible kidneys, and prefer compatible kidneys to incompatible ones. In reality there is always a large pool of patients on the waiting list so that such patients can be found to match compatibly with any given kidney. This will be a part of our model. Our primary objective is to enable as many patients as possible to receive compatible kidneys, i.e., to achieve a maximal number of feasible kidney transplants between patients and donors.

It is natural to bring compatible patient-donor pairs and single donors into exchange with incompatible pairs as more patients can be benefited from their involvement. In practiced single donors play a significant role. For instance, the Organ Donation and Transplantation Activity Report from NHS in 2014 shows that the number of living donors in UK from 2013 to 2014 is 1114, meanwhile the number of total kidney donors in USA is 16,526 including 11,195 deceased donors and 5,331 living donors according to OPTN/SRTR 2012 Annual Data Report.

In our paper, the symbol (X, Y) indicates a pair of a patient with blood type X and a donor with blood type Y, and $(X, Y)^i$ ($(X, Y)^c$) means a pair of patient and donor who are tissue-incompatible (tissue-compatible). Furthermore, we use $\#X^d$ to denote the number of single donors with blood-type X, $\#Y^p$ the number of patients on the waiting list with blood-type Y, and $\#(X, Y)$ the number of patient-donor pairs with blood-type X for patients and blood-type Y for donors. For any real number k, $\lfloor k \rfloor$ stands for the largest integer no bigger than k.

An outcome of the kidney exchange problem is a *matching* of kidneys (i.e., donors)/the waiting list option to patients such that each paired patient is either assigned a compatible kidney (i.e., donor) or stays with his paired donor, each patient on the waiting list is either assigned a compatible kidney (i.e., donor) or stays put, and no kidney (i.e., donor) is assigned to more than one patient. A matching μ is *efficient or maximal* if there exists no other matching ν such that $|\nu| > |\mu|$ where $|\mu|$ is the number of possible kidney transplants for the matching μ.

A matching can be made through several ways of exchange between patients and donors. A *two-way cycle exchange* involves two patient-donor pairs in which each patient is compatible with the other patient's donor. For instance, we have two patient-donor pairs (A, B) and (B, A) and use $(A, B) - (B, A)$ to indicate a two-way cycle exchange in which blood-type A patient in first pair receives the kidney from blood-type A paired donor in second pair and blood-type B patient in second pair can receive the kidney from blood-type B paired donor in first pair. A *three-way cycle exchange* involves three patient-donor pairs in which the patient in the first pair is compatible with the donor in the second pair, the patient in the second pair is compatible with the donor in the third pair, and the patient in the third pair is compatible with the donor in the first pair. An example consists of three pairs (X, Z), (Z, Y), and (Y, X), and the three-way cycle exchange is given by $(X, Z) - (Z, Y) - (Y, X)$ in which each patient receives a compatible kidney. Similarly we can define a four-way cycle exchange.

We also need to use chain exchanges. A *one-way chain exchange* involves a single donor, denoted by X^d, and a compatible patient, denoted by Y^p, on the waiting list. We write this exchange as $X^d - Y^p$. A *two-way chain exchange* is a chain $X^d - (X, Y) - Y^p$ in which the patient of blood-type X in the pair receives the kidney from the single donor X^d, and the patient Y^p on the waiting list receives the kidney from the donor in the pair. A *three-way chain exchange* is a chain $X^d - (X, Y) - (Y, Z) - Z^p$ in which the single donor X^d gives her compatible kidney to the patient X in the first pair, the donor Y in the first pair gives hers to the patient Y in the second pair, and the donor Z in the second pair gives hers to the patient Z^p in waiting. Four-chain exchanges can be defined analogously. For a given positive integer k, we say that a matching μ is *k-efficient* if there exists no other matching ν such that $|\nu| > |\mu|$ when the maximum size of kidney exchanges is no more than k-way cycles or chains of exchange. In the following *when we say a k-way exchange, it can be an l-way cycle or chain of exchange for any* $1 \leq l \leq k$.

To derive an analytical expression for the maximum number of feasible transplants among the whole kidney exchange pool, we impose the following three basic assumptions.

Assumption 2.1 (Upper Bound Assumption): *Every patient on the waiting list is tissue-compatible with every blood-type compatible donor and every paired patient is tissue-compatible with a blood-type compatible single donor or paired donor of any other paired patient.*

This assumption can be seen as a generalization of Assumption 1 of Roth, Sönmez and

Ünver (2007, p. 831). With evolving clinical practice, the significance of HLA matching has diminished (Su et al. 2004). To decide whether a person can donate a kidney or not, the level of HLA level does not play a central role. This is consistent with the practical evidence from OPTN & SRTR annual data report in 2012 that most of transplanted patients have HLA mistakes with donors.

Assumption 2.2 $\#(A, B) > \#(B, A)$.

Terasaki, Gjertson, and Cecka (1998) and OPTN & SRTR annual data report in 2012 have provided statistical evidence for this assumption that the number of pairs (A, B) is greater than the number of pairs (B, A). This assumption is used as Assumption 3 in Roth, Sönmez and Ünver (2007, p. 834).

Assumption 2.3 *Let (X, Y) denote a blood-compatible type from (A, A), (B, B), (AB, AB), (O, O), (A, O), (B, O), (AB, O), (AB, A) and (AB, B). There exists either no pair of type (X, Y) or at least one tissue-compatible pair of type (X, Y).*

This assumption can be easily satisfied for a relatively large population and generalizes Assumption 4 of Roth, Sönmez and Ünver (2007, p. 834).

For a relatively large population, due to blood-compatibility constraints, there will be likely higher demand for kidneys of type O than type A or B, and higher demand for kidneys of type A or B than type AB. As a result, pairs of type (O, A), (O, B), (O, AB), (A, AB), or (B, AB) are on the long side of the exchange and will have to wait longer for a feasible exchange than pairs of other types. Their opposite blood-type compatible but tissue-type incompatible pairs are on the short side. This is used as their Assumption 2 of Roth, Sönmez and Ünver (2007, p. 832). Our model will dispense with this assumption and can handle cases that violate or satisfy this assumption.

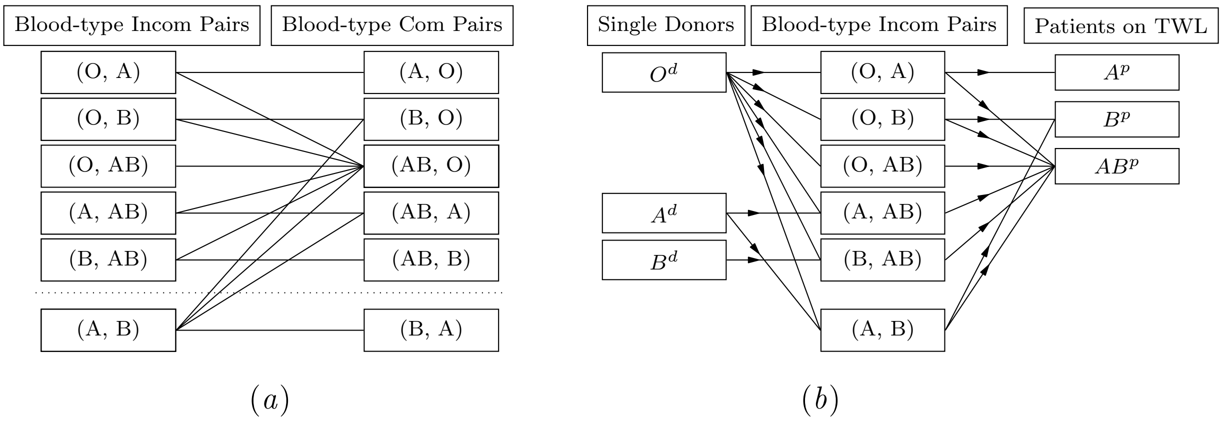
3 Efficient Kidney Exchange

In this section we will derive a maximum number of feasible kidney transplants, when one-way, two-way, three-way, or four-way cycles or chains of exchange are used.

3.1 Two-Way Exchange

Recall that to distinguish blood-type compatible but tissue-type incompatible pairs and compatible pairs, we use $(X, Y)^i$ to denote the first group and $(X, Y)^c$ to denote the second group. Obviously $\#(X, Y) = \#(X, Y)^i + \#(X, Y)^c$. In the following, the notation $(A, B)-$

Figure 2: Two-way cycles (a) and chains (b) of exchange.



$(C,D)/(X,Y)$ means that $(A,B)-(C,D)$ and/or $(A,B)-(X,Y)$, and $(A,B)/(C,D)-(X,Y)$ means that $(A,B)-(X,Y)$ and/or $(C,D)-(X,Y)$.

Figure 2 shows several basic two-way cycles and chains of exchange but do not include pairs (X,X). In Figure 2(a) the right column above the dot line represents blood-type compatible pairs while the left column above the dot line stands for the blood-type incompatible pairs. By Assumption 2.3 all tissue incompatible pairs of type $(X,Y)^i$ on the right side can be matched by two-way cycle $(X,Y)^i-(X,Y)^i$ or two-way cycle $(X,Y)^i-(X,Y)^c$. The problem becomes how to take full advantage of blood-type compatible pairs and single donors to match a maximum number of blood-type incompatible pairs because blood-type incompatible pairs cannot match with each other in two-way cycles.

A cell in the left column linking a cell in the right column means a two-way cycle, for instance, $(O,A)-(A,O)$ and $(O,A)-(AB,O)$. In Figure 2(b) a cell in the left column linking a cell in the middle column linking a cell in the right column implies a two-way chain, for instance, $O^d-(O,A)-A^p$, $O^d-(O,B)-B^p$ and $O^d-(O,B)-AB^p$. Using this idea we propose a sequential matching procedure to find a maximal number of (feasible) transplants when at most two-way cycles or chains of exchange will be used. We call it a *sequential 2-way matching procedure*. In the following two-way, three-way, or four-way matching procedures, whenever cycles or chains of exchange are going to be made, priority is given to incompatible pairs.

A Sequential Two-Way Matching Procedure

Step 1: Make a maximum number of two-way cycles of exchange $(A,A)^i-(A,A)^i$. Then make a maximum number of two-way cycles of exchange $(A,A)^i-(A,A)^c$ if any. Carry out transplants for the remaining pairs $(A,A)^c$. Repeat the same process for each type (B,B), (O,O), (AB,AB), respectively.

Step 2: Make a maximum number of two-way cycles of exchange $(O,A)-(A,O)^i$, $(O,B)-(B,O)^i$, $(O,AB)-(AB,O)^i$, $(A,AB)-(AB,A)^i$, $(B,AB)-(AB,B)^i$, and $(A,B)-(B,A)$, respectively.

Step 3: Make a maximum number of two-way cycles or chains of exchange $(O,A)-(A,O)^c$, $(O,B)-(B,O)^c$, $(A,AB)-(AB,A)^c$, $(B,AB)-(AB,B)^c$, $A^d-(A,B)-AB^p/B^p$, $A^d-(A,AB)-AB^p$, and $B^d-(B,AB)-AB^p$, respectively. Match a maximum number of two-way cycles $(B,O)^c-(A,B)$, $(AB,A)^c-(A,B)$, $(B,O)^i-(A,B)$, $(AB,A)^i-(A,B)$ and two-way chain $A^d-(A,B)-Y^p$.

Step 4: Make a maximal number of two-way cycles of exchange

$$(AB,O)^c/(AB,O)^i-(O,A)/(O,B)/(O,AB)/(A,AB)/(B,AB)/(A,B),$$

respectively. And then match a maximum number of single donors O^d with the remaining pairs $(O,A)/(O,B)/(O,AB)/(A,AB)/(B,AB)/(A,B)$, respectively.

Step 5: Match a maximum number of the remaining single donors O^d, A^d, B^d, AB^d with any remaining single patients O^p, A^p, B^p, AB^p. Match a maximum number of two-way cycles of exchange $(A,O)^i-(A,O)^i$. Then make a maximum number of two-way cycles of exchange $(A,O)^i-(A,O)^c$ if any. Repeat the same process for each type $(B,O)^i$, $(AB,O)^i$, $(AB,A)^i$, $(AB,B)^i$. Match any remaining paired patients from compatible patient-donor pairs with their own paired donors.

The following example will be used to show how each matching procedure assigns compatible kidneys to patients and how efficiency will be improved as more ways of exchange are permitted.

Example 3.1 *There are 32 incompatible patient-donor pairs consisting of three incompatible pairs of type $(AB,AB)^i$, five pairs of type (O,A), one pairs of type (O,B), one pair of type (O,AB), two pairs of type (A,AB), seven pairs of type (B,AB), seven pairs of type (A,B), one incompatible pair of each type of $(A,O)^i$, $(B,O)^i$, $(AB,O)^i$, $(AB,A)^i$, $(AB,B)^i$ and (B,A); three compatible patient-donor pairs consisting of one compatible pair of each type $(AB,AB)^c$, $(AB,O)^c$ and $(A,O)^c$; and five single donors consisting of three single donors of type A^d, one single donor of type B^d and one single donor of type AB^d, and a large number of single patients.*

Observe that in the example there are in total 35 patient-donor pairs including 32 incompatible pairs and three compatible ones and many single patients. Table 1 shows that when the sequential two-way kidney exchange procedure is implemented, 24 paired patients and 5 single patients can receive kidney transplants and all three compatible pairs

Table 1: The illustration of the sequential two-way matching procedure.

Steps	Number of Cycles or Chains	Cycles or Chains	Number of Remaining Pairs and Donors
Step 1	2	$(AB,AB)^i-(AB,AB)^i$ $(AB,AB)^i-(AB,AB)^c$	
Step 2	1	$(O,A)-(A,O)^i$	$4\ (O,A)$
	1	$(O,B)-(B,O)^i$	
	1	$(O,AB)-(AB,O)^i$	
	1	$(A,AB)-(AB,A)^i$	(A,AB)
	1	$(B,AB)-(AB,B)^i$	$6\ (B,AB)$
	1	$(A,B)-(B,A)$	$6\ (A,B)$
Step 3	1	$(O,A)-(A,O)^c$	$3\ (O,A)$
	1	$A^d-(A,AB)-AB^p$	$2\ A^d$
	1	$B^d-(B,AB)-AB^p$	$5\ (B,AB)$
	2	$A^d-(A,B)-B^p/AB^p$	$4\ (A,B)$
Step 4	1	$(AB,O)^c-(B,AB)$	$4\ (B,AB)$
Step 5 (End)	1	AB^d-AB^p	

Note that we can randomly pick kidney exchanges from cycles $(AB,O)^c-(O,A)/(O,B)/(B,AB)$ and chains $O^d-(O,A)/(O,B)/(B,AB)-Y^p$ in Step 4.

are involved in kidney exchange with incompatible pairs. Four pairs of type (B,AB), three pairs of type (O,A) and four pairs of type $(A,B)/(A,AB)$ stay put. In Table 1, Step 1 has two cycles, i.e., $(AB,AB)^i-(AB,AB)^i$ and $(AB,AB)^i-(AB,AB)^c$.

We have the following easy observation.

Lemma 3.2 *Assume that the kidney exchange model satisfies the Assumptions 2.1 and 2.3. Let μ be a 2-efficient matching. Then in μ every cycle contains at most two blood-type compatible pairs and every chain contains at most one blood-type compatible pair.*

Proof. It follows immediately from Figure 2 and the description of the above matching procedure. □

Proposition 3.3 *Assume that the kidney exchange model obeys the Assumptions 2.1, 2.2, and 2.3. Then the matching μ obtained from the above mechanism is 2-efficient and the maximum number of transplants through two-way exchanges is*

$$\begin{aligned}&\#(A,O)+\#(B,O)+\#(AB,O)+\#(AB,A)+\#(AB,B)+\#(B,A)\\&+\#(A,A)+\#(B,B)+\#(O,O)+\#(AB,AB)\\&+\#A^d+\#B^d+\#AB^d+\#O^d\\&+\min\{N_1,N_2,N_3,N_4,N_5,N_6,N_7,N_8,N_9,N_{10},N_{11},N_{12},N_{13},N_{14},N_{15},N_{16},N_{17}\}\end{aligned}$$

where

$$\begin{aligned}
N_1 &= \#(O,A)+\#(O,B)+\#(O,AB)+\#(A,AB)+\#(A,B)+\#(B,AB)\\
N_2 &= \#(O,A)+\#(O,B)+\#O^d+\#(AB,O)+\#A^d+\#(AB,A)+\#(B,AB)\\
&\quad +\#(A,B)\\
N_3 &= \#(O,A)+\#(O,B)+\#O^d+\#(AB,O)+\#(A,AB)+\#(A,B)+\#(B,AB)\\
N_4 &= \#(O,A)+\#(O,B)+\#O^d+\#(AB,O)+\#A^d+\#(AB,A)+\#B^d\\
&\quad +\#(AB,B)+\#(A,B)\\
N_5 &= \#(O,A)+\#(O,B)+\#O^d+\#(AB,O)+\#(A,AB)+\#(A,B)\\
&\quad +\#B^d+\#(AB,B)\\
N_6 &= \#(A,O)+\#(O,B)+\#O^d+\#(AB,O)+\#A^d+\#(AB,A)\\
&\quad +\#(B,AB)\ +\#(A,B)\\
N_7 &= \#(A,O)+\#(O,B)+\#O^d+\#(AB,O)+\#(A,AB)+\#(A,B)+\#(B,AB)\\
N_8 &= \#(A,O)+\#(O,B)+\#O^d+\#(AB,O)+\#A^d+\#(AB,A)+\#B^d\\
&\quad +\#(AB,B)+\#(A,B)\\
N_9 &= \#(A,O)+\#(O,B)+\#O^d+\#(AB,O)+\#(A,AB)+\#(A,B)\\
&\quad +\#B^d+\#(AB,B)\\
N_{10} &= \#(O,A)+\#(B,O)+\#O^d+\#(AB,O)+\#A^d+\#(AB,A)\\
&\quad +\#(B,AB)+\#(B,A)\\
N_{11} &= \#(O,A)+\#(B,O)+\#O^d+\#(AB,O)+\#(A,AB)+\#(A,B)\\
&\quad +\#(B,AB)\\
N_{12} &= \#(O,A)+\#(B,O)+\#O^d+\#(AB,O)+\#A^d+\#(AB,A)+\#B^d\\
&\quad +\#(AB,B)+\#(B,A)\\
N_{13} &= \#(O,A)+\#(B,O)+\#O^d+\#(AB,O)+\#(A,AB)+\#(A,B)\\
&\quad +\#B^d+\#(AB,B)\\
N_{14} &= \#(A,O)+\#(B,O)+\#O^d+\#(AB,O)+\#A^d+\#(AB,A)\\
&\quad +\#(B,AB)+\#(B,A)\\
N_{15} &= \#(A,O)+\#(B,O)+\#O^d+\#(AB,O)+\#(A,AB)+\#(A,B)+\#(B,AB)\\
N_{16} &= \#(A,O)+\#(B,O)+\#O^d+\#(AB,O)+\#A^d+\#(AB,A)\\
&\quad +\#B^d+\#(AB,B)+\#(B,A)\\
N_{17} &= \#(A,O)+\#(B,O)+\#O^d+\#(AB,O)+\#(A,AB)+\#(A,B)\\
&\quad +\#B^d+\#(AB,B)
\end{aligned}$$

Proof. Under Assumptions 2.1 to 2.3, all blood-type compatible but tissue-type incompatible pairs and pairs of type (B,A) can be matched through two-way cycles. All compatible pairs can be matched because even if paired patients from compatible pairs are not involved into two-way cycles, they can receive their own donors. All pairs of types (A,A), (B,B), (O,O), (AB,AB) can be also matched in two-way cycles. As long as a kidney can be allocated to a patient in waiting, we can always find a compatible patient in waiting because of the large population of patients in waiting. Hence, the maximal number of transplantations for patients in waiting, paired patients from blood-type compatible pairs and paired patients from pairs of type (B, A) is

$$\begin{aligned}&\#(A,O)+\#(B,O)+\#(AB,O)+\#(AB,A)+\#(AB,B)\\&+\#(B,A)+\#(A,A)+\#(B,B)+\#(AB,AB)+\#(O,O)\\&+\#A^d+\#B^d+\#AB^d+\#O^d\end{aligned}$$

Next, let N be the maximum number of transplants for blood-type incompatible paired patients of types (O, A), (O, B), (O, AB), (A, AB), (B, AB), (A, B). The number of two-way cycles $(A, B) - (B, A)$ is bounded by $\#(B, A)$ by Assumption 2.2. The number of two-way cycles $(O, A) - (A, O)$ is bounded by $\min\{\#(O, A), \#(A, O)\}$. Similarly, the number of two-way cycles $(O, B) - (B, O)$ is bounded by $\min\{\#(O, B), \#(B, O)\}$; the number of two-way cycles and chains $(AB, A) - (A, AB)$, $A^d - (A, AB) - Y^p$ is bounded by $\min\{\#A^d + \#(AB, A), \#(A, AB)\}$; the number of two-way cycles and chains $(AB, A) - (A, B)$, $A^d - (A, B) - Y^p$, $(B, O) - (A, B)$ is bounded by $\min\{\#A^d + \#(AB, A) - \min\{\#A^d + \#(AB, A), \#(A, AB)\} + \#(B, O) - \min\{\#(O, B), \#(B, O)\}, \#(A, B) - \#(B, A)\}$; the number of two-way cycles and chains $(AB, B) - (B, AB)$, $B^d - (B, AB) - AB^p$ is bounded by $\min\{\#B^d + \#(AB, B), \#(B, AB)\}$; and the number of two-way cycles and chains

$$(AB,O)-(O,A)/(O,B)/(O,AB)/(A,B)/(A,AB)/(B,AB), \text{ and,}$$

$$O^d-(O,A)/(O,B)/(O,AB)/(A,B)/(A,AB)/(B,AB)-Y^w$$

is bounded either by $\#O^d + \#(AB, O)$ or all blood-type incompatible paired patients are matched. Therefore, we have either

$$\begin{aligned}N \leq\ &\#(B,A)+\min\{\#(O,A),\#(A,O)\}+\min\{\#(O,B),\#(B,O)\}+\\&\min\{\#A^d+\#(AB,A),\#(A,AB)\}+\\&\min\{\#A^d+\#(AB,A)-\min\{\#A^d+\#(AB,A),\#(A,AB)\}\\&+\#(B,O)-\min\{\#(O,B),\#(B,O)\},\#(A,B)-\#(B,A)\}+\\&\min\{\#B^d+\#(AB,B),\#(B,AB)\}+\\&\min\{\#B^d+\#(AB,B),\#(B,AB)\}+\#O^d+\#(AB,O)\end{aligned}$$

or

$$N \leq \#(O,A)+\#(O,B)+\#(O,AB)+\#(A,AB)+\#(A,B)+\#(B,AB).$$

The expressions can be rewritten as follows

$N \leq \min\{N_1, N_2, N_3, N_4, N_5, N_6, N_7, N_8, N_9, N_{10}, N_{11}, N_{12}, N_{13}, N_{14}, N_{15}, N_{16}, N_{17}\}$ and hence the maximum number of transplants can be reached is:

$N = \min\{N_1, N_2, N_3, N_4, N_5, N_6, N_7, N_8, N_9, N_{10}, N_{11}, N_{12}, N_{13}, N_{14}, N_{15}, N_{16}, N_{17}\}$.

We now prove that the sequential matching procedure achieves the maximum number of kidney transplants.

Since for every one/two-way chains, we can always find a compatible single patient, the number of transplantations for single patients equals $\#A^d + \#B^d + \#AB^d + \#O^d$.

By Assumption 2.3, all pairs of type $(A, A)^i$, $(B, B)^i$, $(O, O)^i$, $(AB, AB)^i$ can be matched through two-way cycles in Step 1. By Assumption 2.3, all remaining blood-type compatible but tissue-type incompatible pairs $(A, O)^i$, $(B, O)^i$, $(AB, O)^i$, $(AB, A)^i$, $(AB, B)^i$ can be matched through two-way cycles in Step 5. All compatible pairs $(A, O)^c$, $(B, O)^c$, $(AB, O)^c$, $(AB, A)^c$, $(AB, B)^c$, $(A, A)^c$, $(B, B)^c$, $(O, O)^c$, $(AB, AB)^c$ can be matched either through two-way cycles or doing transplantations with their own donors. Moreover, by Assumption 2.2, all pairs of type (B, A) can be matched through two-way cycle $(A, B) - (B, A)$ in Step 2 so that the remaining number of pairs of type (A, B) is $\#(A, B) - \#(B, A)$. Hence, the number of transplants for compatible pairs, blood-type compatible pairs and pairs of type (B, A) in the procedure is

$$\begin{aligned}
&\#(A, O) + \#(B, O) + \#(AB, O) + \#(AB, A) + \#(AB, B) \\
&+\#(B, A) + \#(A, A) + \#(B, B) + \#(AB, AB) + \#(O, O)
\end{aligned}$$

Next, we prove that the maximum number of transplants for blood-type incompatible pairs of types (O, A), (O, B), (O, AB), (A, AB), (B, AB), (A, B) can be achieved in the procedure.

Denote X_1 as the number of blood-type incompatible paired patients from pairs of types (O, A), (O, B), (O, AB), (A, AB), (B, AB), (A, B) involved in Step 2 so that

$$X_1 = \#(B, A) + e_1 + e_2 + e_3 + e_4 + e_5$$

where

$$\begin{aligned}
e_1 &= \min\{\#(O, A), \#(A, O)^i\} \\
e_2 &= \min\{\#(O, B), \#(B, O)^i\} \\
e_3 &= \min\{\#(O, AB), \#(AB, O)^i\} \\
e_4 &= \min\{\#(A, AB), \#(AB, A)^i\} \\
e_5 &= \min\{\#(B, AB), \#(AB, B)^i\}
\end{aligned}$$

Denote X_2 as the number of blood-type incompatible paired patients from pairs of types (O, A), (O, B), (O, AB), (A, AB), (B, AB), (A, B) involved in Step 3 so that

$$X_2 = \quad a_1 + a_2 + b_1 + b_2 + b_3$$

where

$$\begin{aligned}
a_1 = &\quad \min\{\#(O, A) - e_1, \#(A, O)^c\} \\
a_2 = &\quad \min\{\#(O, B) - e_2, \#(B, O)^c\} \\
b_1 = &\quad \min\{\#A^d + \#(AB, A)^c, \#(A, AB) - e_4\} \\
b_2 = &\quad \min\{\#B^d + \#(AB, B)^c, \#(B, AB) - e_5\} \\
b_3 = &\quad \min\{\#A^d + \#(AB, A)^c + \#(AB, A)^i - e_4 - b_1 + \#(B, O)^c \\
&\quad +\#(B, O)^i - e_2 - b_2, \#(A, B) - \#(B, A)\}
\end{aligned}$$

Denote X_3 as the number of blood-type incompatible paired patients from pairs of types (O,A),(O,B), (O,AB), (A,AB), (B,AB), (A,B) involved in Step 4 so that

$$\begin{aligned} X_3 = \quad & \min\{\#O^d + \#(AB,O)^c + \#(AB,O)^i - e_3, \#(O,A) - e_1 - a_1 \\ & + \#(O,B) - e_2 - a_2 + \#(O,AB) - e_3 + \#(A,AB) - e_4 - b_1 \\ & + \#(B,AB) - e_5 - b_2 + \#(A,B) - \#(B,A) - b_3\} \end{aligned}$$

Therefore, the total number of transplants for paired patients from pairs of types (O,A), (O,B), (O,AB), (A,AB), (B,AB), (A,B) in the procedure is $X = X_1 + X_2 + X_3$; one may refer to Tables from A1 to A15 in Supplement A of Cheng and Yang (2017) for detail. Then the equation can be rewritten as follows:

$X = \min\{N_1, N_2, N_3, N_4, N_5, N_6, N_7, N_8, N_9, N_{10}, N_{11}, N_{12}, N_{13}, N_{14}, N_{15}, N_{16}, N_{17}\}$.

Therefore, the total number of transplants can be achieved in the mechanism is that

$$\begin{aligned} & \#(A,O) + \#(B,O) + \#(AB,O) + \#(AB,A) + \#(AB,B) \\ & + \#(B,A) + \#(A,A) + \#(B,B) + \#(AB,AB) + \#(O,O) \\ & + \#A^d + \#B^d + \#AB^d + \#O^d \\ & + \min\{N_1, N_2, N_3, N_4, N_5, N_6, N_7, N_8, N_9, N_{10}, N_{11}, N_{12}, N_{13}, N_{14}, N_{15}, N_{16}, N_{17}\} \end{aligned}$$

We proved that every matching produced by the procedure achieves the maximum number of transplants in the pool and hence the procedure is 2-efficient. □

Now we compare the lower bound of the number in Proposition 3.3 with the case in which incompatible patient-donor pairs, compatible patient-donor pairs, and patients in waiting and single donors are treated separately under two-way exchange. We consider the most common situation that the number of blood-type incompatible pairs of each type: $\#(O,A)$, $\#(O,B)$, $\#(O,AB)$, $\#(A,AB)$, and $\#(B,AB)$, is at least as large as the number of its opposite blood-type compatible but tissue-type incompatible pairs: $\#(A,O)^i$, $\#(B,O)^i$, $\#(AB,O)^i$, $\#(AB,A)^i$, and $\#(AB,B)^i$ respectively. We can do similar comparison for other situations. Hence, the maximum number of feasible transplants for the group of incompatible patient-donor pairs under two-way cycles is

$$\begin{aligned} & 2(\#(A,O)^i + \#(B,O)^i + \#(AB,O)^i + \#(AB,A)^i + \#(AB,B)^i) \\ & + 2\#(B,A) + 2(\lfloor \tfrac{\#(A,A)^i}{2} \rfloor + \lfloor \tfrac{\#(B,B)^i}{2} \rfloor + \lfloor \tfrac{\#(AB,AB)^i}{2} \rfloor + \lfloor \tfrac{\#(O,O)^i}{2} \rfloor) \end{aligned}$$

The maximum number of transplants for patients on the waiting list under one/two-way chains equals $(\#A^d + \#B^d + \#AB^d + \#O^d)$ because the number of patients on the waiting list exceeds the number of single donors so that a single donor can always find a compatible patient on the waiting list to donate. The maximum number of transplants for the group of compatible patient-donor pairs equals $\#(A,O)^c + \#(B,O)^c + \#(AB,O)^c + \#(AB,A)^c + \#(AB,B)^c + \#(A,A)^c + \#(B,B)^c + \#(O,O)^c + \#(AB,AB)^c$ because every patient in a compatible pair can receive the kidney from its own paired donor.

Since for any blood-type compatible pair of type (X,Y), we have $\#(X,Y) = \#(X,Y)^i + \#(X,Y)^c$, the maximum number of transplants in the whole pool becomes

$$\begin{aligned}
&\#(A,O) + \#(B,O) + \#(AB,O) + \#(AB,A) + \#(AB,B) \\
&+\#(A,O)^i + \#(B,O)^i + \#(AB,O)^i + \#(AB,A)^i + \#(AB,B)^i \\
&+2\#(B,A) + 2(\lfloor \tfrac{\#(A,A)^i}{2} \rfloor + \lfloor \tfrac{\#(B,B)^i}{2} \rfloor + \lfloor \tfrac{\#(AB,AB)^i}{2} \rfloor + \lfloor \tfrac{\#(O,O)^i}{2} \rfloor) \\
&+\#(A,A)^c + \#(B,B)^c + \#(AB,AB)^c + \#(O,O)^c \\
&+\#A^d + \#B^d + \#AB^d + \#O^d
\end{aligned}$$

We compare the above number with the lower bound of the number in Proposition 3.3 and obtain

$$\begin{aligned}
&\min\{N_1, N_2, N_3, N_4, N_5, N_6, N_7, N_8, N_9, N_{10}, N_{11}, N_{12}, N_{13}, N_{14}, N_{15}, N_{16}, N_{17}\} \\
&-(\#(A,O)^i + \#(B,O)^i + \#(AB,O)^i + \#(AB,A)^i + \#(AB,B)^i + \#(B,A)) \\
&+\#(A,A) + \#(B,B) + \#(AB,AB) + \#(O,O) \\
&-(\#(A,A)^c + \#(B,B)^c + \#(AB,AB)^c + \#(O,O)^c) \\
&-2(\lfloor \tfrac{\#(A,A)^i}{2} \rfloor + \lfloor \tfrac{\#(B,B)^i}{2} \rfloor + \lfloor \tfrac{\#(AB,AB)^i}{2} \rfloor + \lfloor \tfrac{\#(O,O)^i}{2} \rfloor) \\
&\geq 0
\end{aligned}$$

This shows the benefits of allowing compatible patient-donor pairs to join incompatible pairs for exchange and adding two-way chain exchange.

3.2 Three-Way Exchange

To improve potential gains of exchange, three-way cycles and three-way chains of exchange can be explored.

Figures 3 and 4 show all possible three-way cycles and chains under Assumptions 2.1, 2.2 and 2.3. Note that these figures do not include two-way exchanges. Recall that blood-compatible pairs can always be matched by Assumption 2.3. To have more transplants we can make the best use of every blood-compatible pair to match with a blood-incompatible pair. As a result, three-way cycles can be formed.

We first consider some beneficial three-way cycles or chains with two blood-incompatible pairs. Under three-way exchanges, blood-compatible pair (AB,O) (the right column) can involve not one but two blood-type incompatible pairs through 4 three-way cycles $(AB,O)-(O,A)-(A,AB)$, $(AB,O)-(O,A)-(A,B)$, $(AB,O)-(A,B)-(B,AB)$ and $(AB,O)-(O,B)-(B,AB)$. For blood-compatible pair (B,O), we have just one three-way cycle $(B,O)-(O,A)-(A,B)$. For blood-compatible pair (AB,A), we have also just one three-way cycle $(AB,A)-(A,B)-(B,AB)$. Similarly, we can use single donors to match with two blood-incompatible pairs and patients on the waiting list. Consequently, three-way chains can be generated. With one-way and two-way chains, each single donor can trade with at most one blood-incompatible pair. If three-way chains are allowed, single donor O^d can trade with two blood-incompatible pairs through three-way chains $O^d-(O,A)-(A,AB)-AB^p$, $O^d-(O,A)-(A,B)-B^p$, $O^d-(A,B)-(B,AB)-AB^p$ and $O^d-(O,B)-(B,AB)-AB^p$. Moreover, if there is any (A,B) left, type (A,B) can bring an extra blood-incompatible pair into chains through three-way chains $A^d-(A,B)-(B,AB)-AB^p$.

We now consider some beneficial three-way cycles or chains with one pair (B,A) or with one blood-incompatible pair. Observe that (B,A) pairs are on the short side by Assumption 2.2. These pairs can be very beneficial in the following situations: Firstly, there are pairs or singles, (A,O), (O,B), $A^d/(AB,A)$, and (B,AB). In this case, we cannot match blood-incompatible pairs (O,B) and (B,AB) in a two-way cycle. But if we break two-way cycle $(A,B)-(B,A)$, we can make three-way cycles $(A,O)-(O,B)-(B,A)$ and $(AB,A)-(A,B)-(B,AB)$ and thus increase the number of transplants. Also three-way cycles $(A,O)-(O,B)-(B,A)$ and chains $A^d-(A,B)-(B,AB)-AB^p$ can yield more transplants. Secondly, there are pairs or singles, (A,AB), (B,O), $B^d/(AB,B)$, and (O,A). In this case, we cannot match blood-incompatible pairs (O,A) and (A,AB) in a two-way cycle, but we can make three-way cycles $(B,O)-(O,A)-(A,B)$ and $(AB,B)-(B,A)-(A,AB)$ and increase the number of transplants. Also three-way cycles $(B,O)-(O,A)-(A,B)$ and three-way chains $B^d-(B,A)-(A,AB)-AB^p$ can bring more transplants.

Furthermore, it is easy to see that $(AB,A)-(A,O)$, or $(AB,B)-(B,O)$ can make a three-way cycle of exchange with any pair (X,Y), and that $A^d-(A,O)$ or $B^d-(B,O)$ can yield a three-way chain with any pair (X,Y). In particular, when there are pairs (B,AB), (O,B) and (O,AB), it is impossible to use them in two-way exchange but it is easy to combine them with $(AB,A)-(A,O)$ to yield three-way exchange $(AB,A)-(A,O)-(B,AB)/(O,B)/(O,AB)$. Similarly, we can make three-way exchanges $A^d-(A,O)-(B,AB)/(O,B)/(O,AB)-Y^p$, $(AB,B)-(B,O)-(A,AB)/(O,A)/(O,AB)$ and $B^d-(B,O)-(A,AB)/(O,A)/(O,AB)-Y^p$.

An efficient sequential three-way matching procedure is introduced below.

A Sequential Three-Way Matching Procedure

Step 1: Match a maximum number of pairs (A,A), (B,B), (O,O), (AB,AB) through two-way.

Step 2: (Take full advantage of three-way cycles and chains starting with A^d, (AB,A) and (B,O)) The number of pairs (A,B) should not exceed $\#(A,B)-\#(B,A)$ in this step.

- Match a maximum number of three-way cycles $(AB,A)-(A,B)-(B,AB)$ and chains $A^d-(A,B)-(B,AB)-AB^p$, where the available number of pairs

Figure 3: Three-way cycles (a) and chains (b) of exchange with two blood-incompatible pairs.

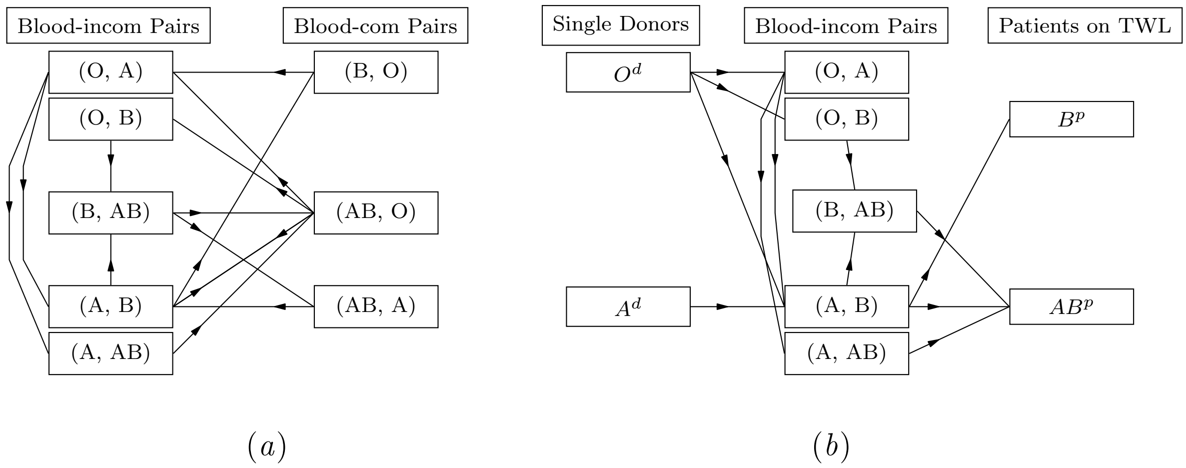
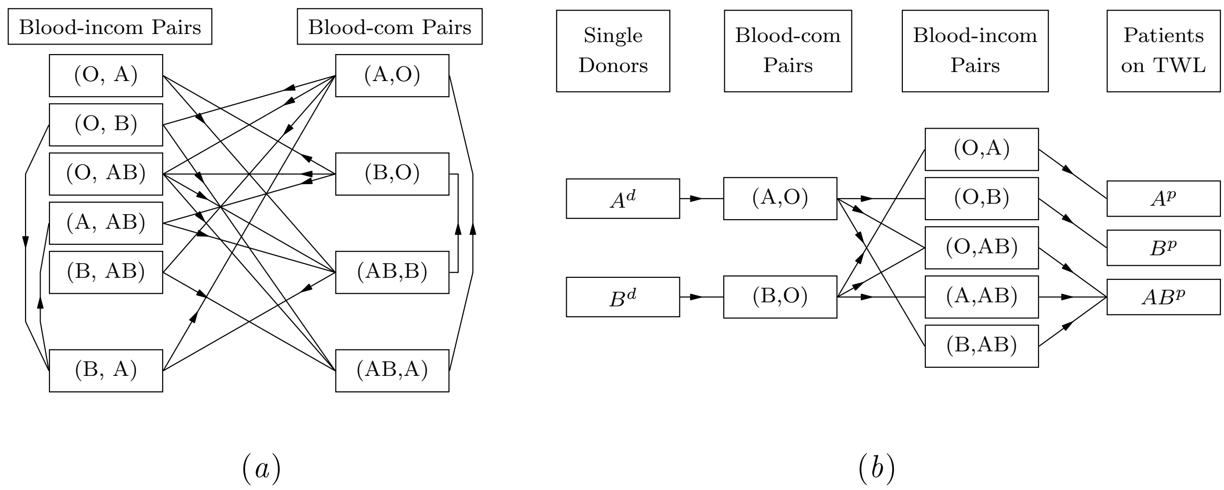


Figure 4: Three-way cycles (a) and chains (b) of exchange with (B, A) and three-way cycles of exchange with one blood-incompatible pair.



(AB, A) and single donors A^d in this step is $\#A^d + \#(AB, A) - \min\{\#A^d + \#(AB, A), \#(A, AB)\}$.

- Match a maximum number of three-way cycles $(B, O) - (O, A) - (A, B)$, where the available number of pairs (O, A) in this step is $\#(O, A) - \min\{\#(O, A), \#(A, O)\}$.
- Match a maximum number of three-way cycles $(AB, A) - (A, B) - (B, AB)$ and chains $A^d - (A, B) - (B, AB) - AB^p$.
- Match a maximum number of three-way cycles $(B, O) - (O, A) - (A, B)$.

Step 3: (Take full advantage of pairs (B, A)) Denote $\#(X, Y)^r$ as the number of all currently available pairs of any type (O, A), (O, B), (A, AB), (B, AB), (A, O), (B, O), (AB, A) and (AB, B) and $\#X^{dr}$ as the number of currently available single donors of any type A^d and B^d. Match the following three-way cycles and chains.

Step 3.1: Make a maximum number of three-way cycles $(A, O) - (O, B) - (B, A)$, $(AB, A) - (A, B) - (B, AB)$ and chains $A^d - (A, B) - (B, AB) - AB^p$, subject to the following constraints: the number of three-way cycles $(A, O) - (O, B) - (B, A)$ should equal the total number of three-way cycles $(AB, A) - (A, B) - (B, AB)$ and chains $A^d - (A, B) - (B, AB) - AB^p$, and the number of pairs (A, B) used in this step should not exceed the number of currently available pairs (B, A), the number of pairs $(O, B)/(A, O)$ used in this step should not exceed $\#(X, Y)^r - \min\{\#(X, Y)^r, \#(Y, X)^r\}$, the number of pairs (AB, A) and single donors A^d used in this step should not exceed $\#A^{dr} + \#(AB, A)^r - \min\{\#A^{dr} + \#(AB, A)^r, \#(A, AB)^r\}$, and the number of pairs (B, AB) used in this step should not exceed $\#(B, AB)^r - \min\{B^{dr} + \#(AB, B)^r, \#(B, AB)^r\}$.

Step 3.2: Make a maximum number of three-way cycles $(B, O) - (O, A) - (A, B)$, $(AB, B) - (B, A) - (A, AB)$ and chains $B^d - (B, A) - (A, AB) - AB^p$, subject to the following constraints: the number of three-way cycles $(B, O) - (O, A) - (A, B)$ should equal the total number of three-way cycles $(AB, B) - (B, A) - (A, AB)$ and chains $B^d - (B, A) - (A, AB) - AB^p$, and the number of pairs (A, B) used in this step should not exceed the number of currently available pairs (B, A), the number of pairs $(B, O)/(O, A)$ used in this step should not exceed $\#(X, Y)^r - \min\{\#(X, Y)^r, \#(Y, X)^r\}$, the number of pairs (AB, B) and single donors B^d used in this step should not exceed $\#B^{dr} + \#(AB, B)^r - \min\{\#B^{dr} + \#(AB, B)^r, \#(B, AB)^r\}$, and the number of pairs (A, AB) used in this step should not exceed $\#(A, AB)^r - \min\{\#A^{dr} + \#(AB, A)^r, \#(A, AB)^r\}$.

Step 3.3: If there is at least one pair of each type (A, O), (O, B) and (B, AB) which are left from the previous Step 3.1. Then, make a maximum number of three-way cycles $(A,O)-(O,B)-(B,A)$, $(AB,A)-(A,B)-(B,AB)$ and chains $A^d-(A,B)-(B,AB)-AB^p$, subject to the following constraints: the number of three-way cycles $(A,O)-(O,B)-(B,A)$ should equal the total number of three-way cycles $(AB,A)-(A,B)-(B,AB)$ and chains $A^d-(A,B)-(B,AB)-AB^p$, and the number of pairs (A,B) used in this step should not exceed the number of currently available pairs (B,A), the number of pairs (O,B) used in this step should not exceed $\#(O,B)^r - \min\{\#(O,B)^r, \#(B,O)^r\}$, the number of pairs (A,O) used in this step should not exceed $\min\{\#(A,O)^r, \#(O,A)^r\}$, and the number of pairs (B,AB) used in this step should not exceed $\#(B,AB)^r - \min\{B^{dr} + \#(AB,B)^r, \#(B,AB)^r\}$.

Step 3.4: If there is at least one pair of each type (O,A), (A,AB) and $B^d/(AB,B)$ which are left from the previous Step 3.2. Then, make a maximum number of three-way cycles $(B,O)-(O,A)-(A,B)$, $(AB,B)-(B,A)-(A,AB)$ and chains $B^d-(B,A)-(A,AB)-AB^p$, subject to the following constraints: the number of three-way cycles $(B,O)-(O,A)-(A,B)$ should equal the total number of three-way cycles $(AB,B)-(B,A)-(A,AB)$ and chains $B^d-(B,A)-(A,AB)-AB^p$, and the number of pairs (A,B) used in this step should not exceed the number of currently available pairs (B,A), the number of pairs (O,A) used in this step should not exceed $\#(O,A)^r - \min\{\#(O,A)^r, \#(A,O)^r\}$, the number of pairs (AB,B) and single donors B^d used in this step should not exceed $\#B^{dr} + \#(AB,B)^r - \min\{\#B^{dr} + \#(AB,B)^r, \#(B,AB)^r\}$, and the number of pairs (A,AB) used in this step should not exceed $\#(A,AB)^r - \min\{\#A^{dr} + \#(AB,A)^r, \#(A,AB)^r\}$.

Step 3.5: If there exists at least one pair of each type (B,AB), (O,B) and $A^d/(AB,A)$ which are left from the previous Step 3.1. Then, make a maximum number of three-way cycles $(A,O)-(O,B)-(B,A)$, $(AB,A)-(A,B)-(B,AB)$ and chains $A^d-(A,B)-(B,AB)-AB^p$, subject to the following constraints: the number of three-way cycles $(A,O)-(O,B)-(B,A)$ should equal the total number of three-way cycles $(AB,A)-(A,B)-(B,AB)$ and chains $A^d-(A,B)-(B,AB)-AB^p$, and the number of pairs (A,B) used in this step should not exceed the number of currently available pairs (B,A), the number of pairs (O,B) used in this step should not exceed $\#(O,B)^r - \min\{\#(O,B)^r, \#(B,O)^r\}$, the number of pairs (AB,A) and single donors A^d used in this step should not exceed $\#A^{dr} + \#(AB,A)^r - \min\{\#A^{dr} + \#(AB,A)^r, \#(A,AB)\}$, and the number of pairs (B,AB) used in this step should not exceed $\#(B,AB)^r - \min\{B^{dr} + \#(AB,B)^r, \#(B,AB)^r\}$.

Step 3.6: If there exists at least one pair of each type (B,O), (O,A) and (A,AB)

which are left form the previous Step 3.2. Then, make a maximum number of three-way cycles $(B,O)-(O,A)-(A,B)$, $(AB,B)-(B,A)-(A,AB)$ and chains $B^d-(B,A)-(A,AB)-AB^p$, subject to the following constraints: the number of three-way cycles $(B,O)-(O,A)-(A,B)$ should equal the total number of three-way cycles $(AB,B)-(B,A)-(A,AB)$ and chains $B^d-(B,A)-(A,AB)-AB^p$, and the number of pairs (A,B) used in this step should not exceed the number of currently available pairs (B,A), the number of pairs $(B,O)/(O,A)$ used in this step should not exceed $\#(X,Y)^r - \min\{\#(X,Y)^r, \#(Y,X)^r\}$, and the number of pairs (A,AB) used in this step should not exceed $\#(A,AB)^r - \min\{\#A^{dr} + \#(AB,A)^r, \#(A,AB)^r\}$.

Step 4: Match the following two-way cycles and two-way chains:

- Match a maximum number of the remaining pairs (A,O) with pairs (O,A). Match a maximum number of the remaining pairs (B,O) with pairs (O,B). Match a maximum number of the remaining pairs (A,B) with pairs (B,A). Match a maximum number of the remaining pairs (AB,A) and single donors A^d with pairs (A,AB), and match a maximum number of the remaining pairs (AB,B) and single donors B^d with pairs (B,AB).
- Match a maximum number of the remaining pairs (AB,A), (B,O) and single donors A^d with the remaining pairs (A,B), where the available number of pairs (B,O) in this step is $\#(B,O)^r - \min\{\#B^{dr} + \#(AB,B)^r, \#(B,O)^r\}$ and the available number of pairs (AB,A) and single donors A^d is $\#A^{dr} + \#(AB,A)^r - \min\{\#A^{dr} + \#(AB,A)^r, \#(A,O)^r\}$.

Step 5: Match a maximum number of the following three-way cycles and chains:

- Three-way cycles $(AB,O)-(O,A)-(A,AB)$ and chains $O^d-(O,A)-(A,AB)-AB^p$.
- Three-way cycles $(AB,O)-(O,B)-(B,AB)$ and chains $O^d-(O,B)-(B,AB)-AB^p$.
- Three-way cycles $(AB,O)-(O,A)-(A,B)$ and chains $O^d-(O,A)-(A,B)-Y^p$.
- Three-way cycles $(AB,O)-(A,B)-(B,AB)$ and chains $O^d-(A,B)-(B,AB)-AB^p$.

Step 6: Match a maximum number of the remaining single donors O^d and pairs (AB,O) with the remaining pairs (O,A), (O,B), (O,AB), (A,AB), (B,AB) and (A,B). Match a maximum number of the combinations of $(AB,A)-(A,O)$ and

$(AB, B) - (B, O)$ with remaining pairs (O, A), (O, B), (O, AB), (A, AB), (B, AB) and (A, B). Match a maximum number of the combinations of $A^d - (A, O)$ and $B^d - (B, O)$ with remaining pairs (O, A), (O, B), (O, AB), (A, AB), (B, AB) and (A, B) and patients on TWL.

Step 7: Match a maximum number of remaining blood-compatible but tissue-incompatible pair $(A, O)^i$ through two-way cycles $(A, O)^i - (A, O)^i$. If there is one remaining pair $(A, O)^i$, match the pair $(A, O)^i$ with $(A, O)^c$. Apply the same procedure to any remaining pair $(B, O)^i$, $(AB, O)^i$, $(AB, A)^i$ and $(AB, B)^i$. Match a maximum number of remaining single donors O^d, A^d, B^d, AB^d with any remaining single patients O^p, A^p, B^p, AB^p; match any paired patients from compatible pairs with their own paired donors.

We use Example 3.1 to demonstrate the sequential three-way matching procedure and compare it with the previous procedure.

Table 2: The illustration of the sequential three-way matching procedure.

Steps	Number of Cycles or Chains	Cycles or Chains	Number of Remaining Pairs and Donors
Step 1	1	$(AB, AB)^i - (AB, AB)^i$	
	1	$(AB, AB)^i - (AB, AB)^c$	
Step 2	1	$(AB, A)^i - (A, B) - (B, AB)$	6 (B, AB), 6 (A, B)
	1	$A^d - (A, B) - (B, AB) - AB^p$	2 A^d, 5 (B, AB), 5 (A, B)
	1	$(B, O)^i - (O, A) - (A, B)$	4 (O, A), 4 (A, B)
	2	$A^d - (A, B) - (B, AB) - AB^p$	3 (B, AB), 2 (A, B)
Step 4	1	$(O, A) - (A, O)^i$	3 (O, A)
	1	$(O, A) - (A, O)^c$	2 (O, A)
	1	$(A, B) - (B, A)$	(A, B)
	1	$(B, AB) - (AB, B)^i$	2 (B, AB)
	1	$B^d - (B, AB) - AB^p$	(B, AB)
Step 5	1	$(AB, O)^i - (O, A) - (A, AB)$	(O, A), (A, AB)
	1	$(AB, O)^c - (O, A) - (A, AB)$	
Step 6 (End)	1	$AB^d - AB^p$	

Table 2 shows that if we use the sequential three-way matching procedure, 31 paired patients and five single patients will receive kidney transplants and four pairs of type (B, AB), (A, B), (O, B) and (O, AB) stay put. Compared with the previous two-way matching procedures, the three-way procedure increases the maximum number of kidney transplants by seven.

Lemma 3.4 *Assume that the kidney exchange model satisfies the Assumptions 2.1 and 2.3. Then every* 3*-efficient matching* μ *can be transformed to another* 3*-efficient matching in which every cycle contains at most two blood-type compatible pairs and every chain contains at most one blood-type compatible pair.*

Proof. Consider any given 3-efficient matching μ as stated in the lemma. If μ consists only of cycles with no more than two blood-type compatible pairs and chains with no more than one blood-type compatible pair, we are done. Suppose to the contrary that μ contains a cycle with more than two blood-type compatible pairs or a chain with more than one blood-type compatible pair. We only need to consider the case of three-way cycles or chains. We will show that a three-way cycle with three blood-type compatible pairs can be decomposed into three single blood-compatible pairs and a three-way chain with two blood-compatible pairs can be decomposed into two single blood-compatible pairs and a one-way chain in which the single donor donates its kidney to a patient on the waiting list. Then, we will show that the all pairs which are decomposed from cycles and chains can be matched.

Because a blood-type compatible and tissue-type compatible pair can directly do transplant, all blood-type compatible and tissue-type compatible pairs can do transplants separately. Let $\mathcal{D}$ be the set of all blood-type compatible but tissue-type incompatible pairs in a three-way cycle or chain under consideration. Let $(X,Y)^i$ present the type of a blood-type compatible but tissue-type incompatible pair. If there exists two or more pairs of type $(X,Y)^i$, we can have a two-way cycle among them $(X,Y)^i-(X,Y)^i$. Therefore, at most one pair of type $(X,Y)^i$ left after the process. By Assumption 2.3, there exists at least one blood-type and tissue-type compatible pair of type $(X,Y)^c$. If the compatible pair $(X,Y)^c$ does not involve in any cycle or chain, then we can match the remaining pair $(X,Y)^i$ with pair $(X,Y)^c$. Otherwise, the compatible pair $(X,Y)^c$ involves in a cycle consisting of no more than two blood-type compatible pairs or a chain consisting of no more than one blood-type compatible pair. Then we can use pair $(X,Y)^i$ instead of $(X,Y)^c$ based on Assumption 2.1 and pair $(X,Y)^c$ do transplant directly. Therefore, all remaining pairs of type $(X,Y)^i$ can be matched. □

Proposition 3.5 *Assume that the kidney exchange model satisfies the Assumptions 2.1, 2.2, and 2.3. Then the matching μ generated by the above procedure is* 3*-efficient and the maximum number of transplants through at most three-way exchanges is*

$$\begin{aligned}
&\#(A,O)+\#(B,O)+\#(AB,O)+\#(AB,A)+\#(AB,B)+\#(B,A)\\
&+\#(A,A)+\#(B,B)+\#(O,O)+\#(AB,AB)\\
&+\#A^d+\#B^d+\#AB^d+\#O^d\\
&+\min\{N_1,N_2,N_3,N_4,N_5,N_6,N_7,N_8,N_9,N_{10},N_{11},N_{12},N_{13},N_{14},N_{15},N_{16},N_{17}\}
\end{aligned}$$

where

$$\begin{aligned}
N_1 &= \#(O,A)+\#(O,B)+\#(O,AB)+\#(A,AB)+\#(A,B)+\#(B,AB)\\
N_2 &= \#(O,A)+\#(O,B)+\#O^d+\#(AB,O)+\#(A,AB)+\#(A,B)+\#(B,AB)\\
&\quad +\#B^d+\#(AB,B)+\#(A,O)
\end{aligned}$$

$$
\begin{aligned}
N_3 \quad &= \#(O,A) + \#(O,B) + \#O^d + \#(AB,O) + \#A^d + \#(AB,A) + \#(A,B)\\
&\quad + \#B^d + \#(AB,B)\\
N_4 \quad &= \#(A,O) + \#(O,B) + 2\#O^d + 2\#(AB,O) + \#A^d + \#(AB,A) + 2\#(A,B)\\
&\quad + 2\#B^d + 2\#(AB,B) - \#(B,A)\\
N_5 \quad &= \#(A,O) + \#(O,B) + 2\#O^d + 2\#(AB,O) + \#A^d + \#(AB,A) + 2\#(A,B)\\
&\quad + \#B^d + \#(AB,B)\\
N_6 \quad &= \#(A,O) + \#(O,B) + \#O^d + \#(AB,O) + \#A^d + \#(AB,A) + \#(A,B)\\
&\quad + \#B^d + \#(AB,B) + \#(A,AB) + \#(B,O)\\
N_7 \quad &= \#(A,O) + \#(O,B) + \#O^d + \#(AB,O) + \#(A,AB) + 2\#(A,B)\\
&\quad + \#B^d + \#(AB,B) - \#(B,A)\\
N_8 \quad &= \#(O,A) + \#(B,O) + 2\#O^d + 2\#(AB,O) + 2\#A^d + 2\#(AB,A)\\
&\quad + \#B^d + \#(AB,B) + \#(B,A)\\
N_9 \quad &= \#(O,A) + \#(B,O) + 2\#O^d + 2\#(AB,O) + \#A^d + \#(AB,A) + \#(A,B)\\
&\quad + \#B^d + \#(AB,B) + \#(B,A)\\
N_{10} \quad &= \#(O,A) + \#(B,O) + \#O^d + \#(AB,O) + \#A^d + \#(AB,A) + \#(B,AB)\\
&\quad + \#(B,A)\\
N_{11} \quad &= \#(A,O) + 2\#(B,O) + 2\#O^d + 2\#(AB,O) + \#A^d + \#(AB,A) + \#(B,AB)\\
&\quad + \#(B,A)\\
N_{12} \quad &= \#(A,O) + 2\#(B,O) + 2\#O^d + 2\#(AB,O) + 2\#A^d + 2\#(AB,A) + \#(AB,B)\\
&\quad + \#B^d + \#(B,A)\\
N_{13} \quad &= \#(A,O) + \#(B,O) + 2\#O^d + 2\#(AB,O) + \#A^d + \#(AB,A) + \#(A,B)\\
&\quad + \#(B,AB) + \#(B,A)\\
N_{14} \quad &= 2\#(A,O) + \#(B,O) + 2\#O^d + 2\#(AB,O) + \#(A,AB) + 2\#(A,B)\\
&\quad + \#(AB,B) + \#B^d - \#(B,A)\\
N_{15} \quad &= \#(A,O) + \#(B,O) + 2\#O^d + 2\#(AB,O) + \#A^d + \#(AB,A) + \#(A,B)\\
&\quad + \#B^d + \#(AB,B)\\
N_{16} \quad &= \#(A,O) + \#(B,O) + 2\#O^d + 2\#(AB,O) + \#(A,AB) + 2\#(A,B)\\
&\quad + \#B^d + \#(AB,B)\\
N_{17} \quad &= \#(A,O) + \#(B,O) + \#O^d + \#(AB,O) + \#(A,AB) + \#(A,B) + \#(B,AB)
\end{aligned}
$$

The proof is deferred to the appendix.

3.3 Four-Way Exchange

If four-way cycles and chains of exchange can be used, more kidney transplants will be made possible. Figures 5 and 6 show all four-way cycles and chains of exchange but do not include two- or three-way exchange.

In this case we have a four-way cycle with three blood-incompatible pairs $(AB,O) - (O,A) - (A,B) - (B,AB)$, a four-way chain with three blood-incompatible pairs $O^d -$

Figure 5: Four-way cycles (a) and chains (b) of exchange with three blood-incompatible pairs.

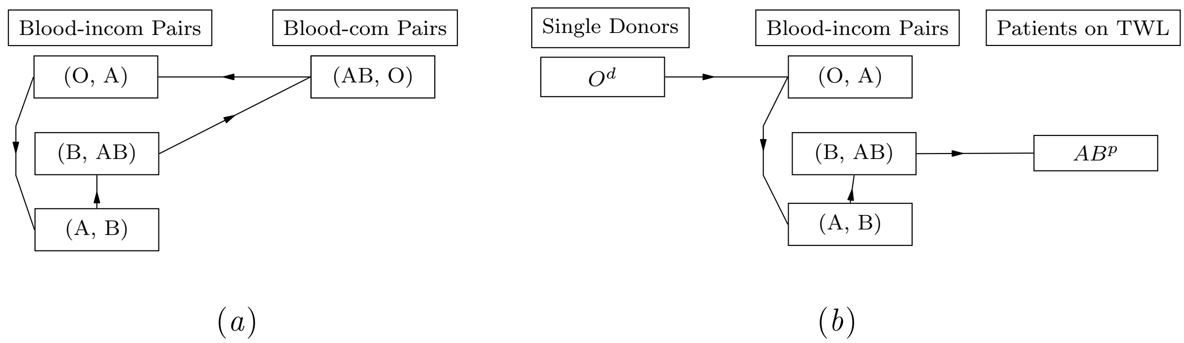
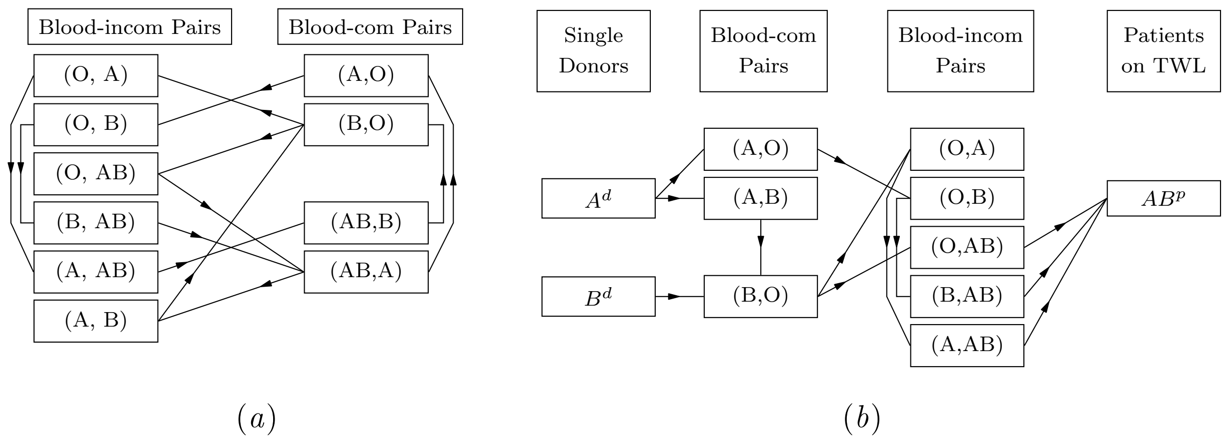


Figure 6: Four-way cycles (a) and chains (b) of exchange with two blood-incompatible pairs.



$(O, A) - (A, B) - (B, AB) - AB^p$, two four-way cycles with two blood-compatible pairs $(AB, A)-(A, O)-(O, B)-(B, AB)$ and $(AB, B)-(B, O)-(O, A)-(A, AB)$, two four-way chains with one blood-compatible pair $A^d - (A, O) - (O, B) - (B, AB)$ and $B^d - (B, O) - (O, A) - (A, AB)$, one four-way cycle $(AB, A) - (A, B) - (B, O) - (X, Y)$ and one four-way chain $A^d - (A, B) - (B, O) - (X, Y) - Z^p$, where (X, Y) is any pair and Z^p is any single patient.

An efficient sequential matching procedure under four-way exchange is proposed and described as follows.

A Sequential Four-Way Matching Procedure

Step 1: Match a maximum number of pairs (A, A), (B, B), (O, O), (AB, AB) through two-way exchange.

Step 2: (Take full advantage of three-way cycles and chains starting with A^d, (AB, A)

and (B, O)) The number of pairs (A, B) should not exceed $\#(A, B) - \#(B, A)$ in this step.

- Match a maximum number of three-way cycles $(AB, A) - (A, B) - (B, AB)$ and chains $A^d - (A, B) - (B, AB) - AB^p$, where the available number of pair (AB, A) and single donors A^d in this step is $\#A^d + \#(AB, A) - \min\{\#A^d + \#(AB, A), \#(A, AB)\}$.
- Match a maximum number of three-way cycles $(B, O) - (O, A) - (A, B)$, where the available number of pairs (O, A) in this step is $\#(O, A) - \min\{\#(O, A), \#(A, O)\}$.
- Match a maximum number of three-way cycles $(AB, A) - (A, B) - (B, AB)$ and chains $A^d - (A, B) - (B, AB) - AB^p$.
- Match a maximum number of three-way cycles $(B, O) - (O, A) - (A, B)$.

Step 3: (Take full advantage of four-way cycles and chains with the combinations) Denote $\#(X, Y)^r$ as the number of all currently available pairs of any type (O, A), (O, B), (A, AB), (B, AB), (A, O), (B, O), (AB, A), (AB, B) and (A, B). Denote $\#Y^{dr}$ as the number of all currently available single donors of any type A^d and B^d. Match the following four-way cycles and chains.

Step 3.1: Make a maximum number of four-way cycles $(AB, A) - (A, O) - (O, B) - (B, AB)$ and four-way chains $A^d - (A, O) - (O, B) - (B, AB) - AB^p$, subject to the following constraints: the number of pairs $(O, B)/(A, O)$ used in this step should not exceed $\#(X, Y)^r - \min\{\#(X, Y)^r, \#(Y, X)^r\}$, the number of pairs (AB, A) and single donors A^d used in this step should not exceed $\#A^{dr} + \#(AB, A)^r - \min\{\#A^{dr} + \#(AB, A)^r, \#(A, AB)^r\}$, and the number of pairs (B, AB) used in this step should not exceed $\#(B, AB)^r - \min\{B^{dr} + \#(AB, B)^r, \#(B, AB)^r\}$.

Step 3.2: Make a maximum number of four-way cycles $(AB, B) - (B, O) - (O, A) - (A, AB)$ and four-way chains $B^d - (B, O) - (O, A) - (A, AB) - AB^p$, subject to the following constraints: the number of pairs $(B, O)/(O, A)$ used in this step should not exceed $\#(X, Y)^r - \min\{\#(X, Y)^r, \#(Y, X)^r\}$, the number of pairs (AB, B) and single donors B^d used in this step should not exceed $\#B^{dr} + \#(AB, B)^r - \min\{\#B^{dr} + \#(AB, B)^r, \#(B, AB)^r\}$, and the number of pairs (A, AB) used in this step should not exceed $\#(A, AB)^r - \min\{\#A^{dr} + \#(AB, A)^r, \#(A, AB)^r\}$.

Step 3.3: If there exists at least one pair of each type (A, O), (O, B) and (B, AB) which are left from the previous Step 3.1. Then, make a maximum number of four-way cycles $(AB, A) - (A, O) - (O, B) - (B, AB)$ and four-way chains $A^d - (A, O) - (O, B) - (B, AB) - AB^p$, subject to the following constraints: the number of pairs $(O, B)/(A, O)$ used in this step should not exceed $\#(X, Y)^r - \min\{\#(X, Y)^r, \#(Y, X)^r\}$, and the number of pairs (B, AB) used in this step should not exceed $\#(B, AB)^r - \min\{B^{dr} + \#(AB, B)^r, \#(B, AB)^r\}$.

Step 3.4: If there exists at least one pair of each type (O, A), (A, AB) and $B^d/(AB, B)$ which are left from the previous Step 3.2. Then, make a maximum number of four-way cycles $(AB, B) - (B, O) - (O, A) - (A, AB)$ and four-way chains $B^d - (B, O) - (O, A) - (A, AB) - AB^p$, subject to the following constraints: the number of pairs (O, A) used in this step should not exceed $\#(O, A)^r - \min\{\#(O, A)^r, \#(A, O)^r\}$, the number of pairs (AB, B) and single donors B^d used in this step should not exceed $\#B^{dr} + \#(AB, B)^r - \min\{\#B^{dr} + \#(AB, B)^r, \#(B, AB)^r\}$, and the number of pairs (A, AB) used in this step should not exceed $\#(A, AB)^r - \min\{\#A^{dr} + \#(AB, A)^r, \#(A, AB)^r\}$.

Step 3.5: If there exists at least one remaining pair of each type (B, AB), (O, B) and $A^d/(AB, A)$ which are left from the previous Step 3.1. Then, make a maximum number of four-way cycles $(AB, A) - (A, O) - (O, B) - (B, AB)$ and four-way chains $A^d - (A, O) - (O, B) - (B, AB) - AB^p$, subject to the following constraints: the number of pairs (O, B) used in this step should not exceed $\#(O, B)^r - \min\{\#(O, B)^r, \#(B, O)^r\}$, the number of pairs (AB, A) and single donors A^d used in this step should not exceed $\#A^{dr} + \#(AB, A)^r - \min\{\#A^{dr} + \#(AB, A)^r, \#(A, AB)\}$, and the number of pairs (B, AB) used in this step should not exceed $\#(B, AB)^r - \min\{B^{dr} + \#(AB, B)^r, \#(B, AB)^r\}$.

Step 3.6: If there exists at least one pair of each type (B, O), (O, A) and (A, AB) which are left from the previous Step 3.2. Then, make a maximum number of four-way cycles $(AB, B) - (B, O) - (O, A) - (A, AB)$ and four-way chains $B^d - (B, O) - (O, A) - (A, AB) - AB^p$, subject to the following constraints: the number of pairs $(B, O)/(O, A)$ used in this step should not exceed $\#(X, Y)^r - \min\{\#(X, Y)^r, \#(Y, X)^r\}$, and the number of pairs (A, AB) used in this step should not exceed $\#(A, AB)^r - \min\{\#A^{dr} + \#(AB, A)^r, \#(A, AB)^r\}$.

Step 4: Do the following two-way cycles and two-way chains:

- Match a maximum number of the remaining pairs (A, O) with pairs (O, A). Match a maximum number of the remaining pairs (B, O) with pairs (O, B). Match a maximum number of the remaining pairs (A, B) with pairs (B, A). Match a maximum number of the remaining pairs (AB, A) and single donors

A^d with pairs (A, AB), and match a maximum number of the remaining pairs (AB, B) and single donors B^d with pairs (B, AB).

– Match a maximum number of the remaining pairs (AB, A), (B, O) and single donors A^d with the remaining pairs (A, B), where the available number of pairs (B, O) in this step is

$$\begin{aligned}&\#(B,O)^r - \min\{\#B^{dr} + \#(AB,B)^r, \#(B,O)^r\}\\&- \min\{\#A^{dr} + \#(AB,A)^r - \min\{\#A^{dr} + \#(AB,A)^r, \#(A,O)^r\},\\&\#(B,O)^r - \min\{\#B^{dr} + \#(AB,B)^r, \#(B,O)^r\}, \#(A,B)^r\}\end{aligned}$$

and the available number of pairs (AB, A) and single donors A^d is

$$\begin{aligned}&\#A^{dr} + \#(AB,A)^r - \min\{\#A^{dr} + \#(AB,A)^r, \#(A,O)^r\}\\&- \min\{\#A^{dr} + \#(AB,A)^r - \min\{\#A^{dr} + \#(AB,A)^r, \#(A,O)^r\},\\&\#(B,O)^r - \min\{\#B^{dr} + \#(AB,B)^r, \#(B,O)^r\}, \#(A,B)^r\}\end{aligned}$$

Step 5: Match a maximum number of the following cycles and chains:

– Four-way cycles $(AB, O) - (O, A) - (A, B) - (B, AB)$ and chains $O^d - (O, A) - (A, B) - (B, AB)$.
– Three-way cycles $(AB, O)-(O, A)-(A, AB)$ and chains $O^d-(O, A)-(A, AB)-AB^p$.
– Three-way cycles $(AB, O)-(O, B)-(B, AB)$ and chains $O^d-(O, B)-(B, AB)-AB^p$.
– Three-way cycles $(AB, O)-(O, A)-(A, B)$ and chains $O^d-(O, A)-(A, B)-Y^p$.
– Three-way cycles $(AB, O)-(A, B)-(B, AB)$ and chains $O^d-(A, B)-(B, AB)-AB^p$.

Step 6: Match a maximum number of the remaining single donors O^d and pairs (AB, O) with the remaining pairs (O, A), (O, B), (O, AB), (A, AB), (B, AB) and (A, B). Match a maximum number of the combinations of $(AB, A)-(A, O)$, $(AB, B)-(B, O)$ and $(AB, A) - (A, B) - (B, O)$ with remaining pairs (O, A), (O, B), (O, AB), (A, AB), (B, AB) and (A, B). Match a maximum number of the combinations of $A^d - (A, O)$, $B^d - (B, O)$ and $A^d - (A, B) - (B, O)$ with remaining pairs (O, A), (O, B), (O, AB), (A, AB), (B, AB), (A, B) and patients on TWL.

Step 7: Match a maximum number of remaining blood-compatible but tissue-incompatible pairs $(A, O)^i$ through two-way cycles $(A, O)^i - (A, O)^i$. If there is one remaining pair

$(A,O)^i$, match the pair $(A,O)^i$ with $(A,O)^c$. Apply the same procedure to any remaining pair $(B,O)^i$, $(AB,O)^i$, $(AB,A)^i$ and $(AB,B)^i$. Match a maximum number of remaining single donors O^d, A^d, B^d, AB^d with any remaining single patients O^p, A^p, B^p, AB^p; match any paired patients from compatible pairs with their own paired donors.

Example 3.1 will be used to show the performance of the four-way matching procedure. The process and outcome generated by the procedure is shown in Table 3. One can see that if the sequential four-way matching procedure is used, 32 paired patients and five single patients will receive kidney transplants and three pairs of type (A,AB), (O,B) and (O,AB) will be left. In comparison with the three-way procedure, the four-way procedure increases the maximum number of kidney transplants by one.

Lemma 3.6 *Assume that the kidney exchange model satisfies the Assumptions 2.1 and 2.3. Then every* 4*-efficient matching* μ *can be transformed to another* 4*-efficient matching in which every cycle contains at most two blood-type compatible pairs and every chain contains at most one blood-type compatible pair.*

Its proof is given in the appendix.

Table 3: The illustration of the sequential four-way matching procedure

Steps	Number of Cycles or Chains	Cycles or Chains	Number of Remaining Pairs and Donors
Step 1	1	$(AB,AB)^i-(AB,AB)^i$	
	1	$(AB,AB)^i-(AB,AB)^c$	
Step 2	1	$(AB,A)^i-(A,B)-(B,AB)$	6 (B,AB), 6 (A,B)
	1	$A^d-(A,B)-(B,AB)-AB^p$	2 A^d, 5 (B,AB), 5 (A,B)
	1	$(B,O)^i-(O,A)-(A,B)$	4 (O,A), 4 (A,B)
	2	$A^d-(A,B)-(B,AB)-AB^p$	3 (B,AB), 2 (A,B)
Step 4	1	$(O,A)-(A,O)^i$	3 (O,A)
	1	$(O,A)-(A,O)^c$	2 (O,A)
	1	$(A,B)-(B,A)$	(A,B)
	1	$(B,AB)-(AB,B)^i$	2 (B,AB)
	1	$B^d-(B,AB)-AB^p$	(B,AB)
Step 5	1	$(AB,O)^i-(O,A)-(A,B)-(B,AB)$	(O,A)
	1	$(AB,O)^c-(O,A)-(A,AB)$	(A,AB)
Step 6 (End)	1	AB^d-AB^p	

Proposition 3.7 *Assume that the kidney exchange model obeys the Assumptions 2.1, 2.2, and 2.3. Then the matching* μ *from the above procedure is* 4*-efficient and the maximum*

number of transplants through four-way exchanges equals

$$\begin{aligned}
&\#(A,O)+\#(B,O)+\#(AB,O)+\#(AB,A)+\#(AB,B)\\
&+\#(B,A)+\#(A,A)+\#(B,B)+\#(O,O)+\#(AB,AB)\\
&+\#A^d+\#B^d+\#AB^d+\#O^d\\
&+\min\{N_1,N_2,N_3,N_4,N_5,N_6,N_7,N_8,N_9,N_{10},N_{11}\}
\end{aligned}$$

where

$$\begin{aligned}
N_1 &= \#(O,A)+\#(O,B)+\#(O,AB)+\#(A,AB)+\#(A,B)+\#(B,AB)\\
N_2 &= \#(O,A)+\#(O,B)+\#O^d+\#(AB,O)+\#A^d+\#(AB,A)+\#(A,B)\\
&\quad+\#B^d+\#(AB,B)\\
N_3 &= \#(A,O)+\#(O,B)+2\#O^d+2\#(AB,O)+\#A^d+\#(AB,A)+2\#(A,B)\\
&\quad+2\#B^d+2\#(AB,B)-\#(B,A)\\
N_4 &= \#(A,O)+\#(O,B)+\#O^d+\#(AB,O)+\#(A,AB)+2\#(A,B)\\
&\quad+\#B^d+\#(AB,B)-\#(B,A)\\
N_5 &= \#(O,A)+\#(B,O)+2\#O^d+2\#(AB,O)+2\#A^d+2\#(AB,A)\\
&\quad+\#B^d+\#(AB,B)+\#(B,A)\\
N_6 &= \#(O,A)+\#(B,O)+\#O^d+\#(AB,O)+\#A^d+\#(AB,A)+\#(B,AB)\\
&\quad+\#(B,A)\\
N_7 &= \#(A,O)+2\#(B,O)+3\#O^d+3\#(AB,O)+2\#A^d+2\#(AB,A)\\
&\quad+\#(AB,B)+\#B^d+\#(B,A)\\
N_8 &= \#(A,O)+2\#(B,O)+2\#O^d+2\#(AB,O)+\#A^d+\#(AB,A)+\#(B,AB)\\
&\quad+\#(B,A)\\
N_9 &= 2\#(A,O)+\#(B,O)+2\#O^d+2\#(AB,O)+\#(A,AB)+2\#(A,B)\\
&\quad+\#(AB,B)+\#B^d-\#(B,A)\\
N_{10} &= \#(A,O)+\#(B,O)+2\#O^d+2\#(AB,O)+\#A^d+\#(AB,A)+\#(A,B)\\
&\quad+\#B^d+\#(AB,B)\\
N_{11} &= \#(A,O)+\#(B,O)+\#O^d+\#(AB,O)+\#(A,AB)+\#(A,B)+\#(B,AB)
\end{aligned}$$

The proof is given in the appendix.

4 Multi-Way Cycles and Chains of Exchange

In the previous sections we have focused on two-way, three-way, and four-way cycles and chains of exchange and derived the upper bounds of the possible number of kidney transplants under those given assumptions. In the current section, we consider a more general model of kidney exchange and show that under similar conditions, five or higher-way cycles and chains of exchange even if available will not further increase the number of feasible kidney transplants. In other words, four or less-way exchanges are sufficient to capture all the potential gains of kidney exchange.

Our general model consists of pairs, single donors and patients on the waiting list. We also call a patient on the waiting list *a single patient.* Each pair i has a patient P_i^p and a donor D_i^p. Each single patient is denoted by P_i^s and each single (decreased or altruistic) donor is denoted as D_i^s.

Let $\mathcal{B}$ be the family of primary types such as blood shared by patients and donors with $|\mathcal{B}| = n > 2$. In other words, all patients and donors have their types X in $\in \mathcal{B}$. For any given two primary types $X, Y \in \mathcal{B}$, $X \succeq Y$ means that agent of type X is primary type compatible with agent of type Y. In the context of kidney exchange, a patient of type Y is blood-type compatible with a donor of type X. Following Roth, Sönmez and Ünver (2007), we assume that the compatibility relation $\succeq$ for primary types satisfies reflexivity, asymmetry and transitivity properties:

1. (Reflexivity) $X \succeq X$ for any $X \in \mathcal{B}$,
2. (Asymmetry) $X \succeq Y$ and $X \neq Y \Rightarrow Y \not\succeq X$ for any $X, Y \in \mathcal{B}$, and
3. (Transitivity) $X \succeq Y$ and $Y \succeq Z \Rightarrow X \succeq Z$ for any $X, Y \in \mathcal{B}$.

Blood-type compatibility possess the properties of reflexivity, asymmetry and transitivity.

Let $\mathcal{C}$ be the family of secondary types such as tissue shared by patients and donors with $|\mathcal{C}| = n \geq 2$. For any given two secondary types $Z, W \in \mathcal{C}$, $Z \sim W$ means that agent of type Z is secondary type compatible with agent of type W. In the context of kidney exchange, a patient of type Z is tissue-type compatible with a donor of type W. We assume that the compatibility relation $\sim$ for secondary types satisfies symmetry and intransitivity properties:

I. (Symmetry) $Z \sim W \Rightarrow W \sim Z$ for any $Z, W \in \mathcal{C}$, and

II. (Intransitivity) $Z \sim W$ and $W \sim L \not\Rightarrow Z \sim L$ for any $Z, W, L \in \mathcal{C}$.

Tissue-type compatibility possess the properties of symmetry and intransitivity.

An agent of primary type $X \in \mathcal{B}$ and secondary type $Z \in \mathcal{C}$ is compatible with an agent of primary type $Y \in \mathcal{B}$ and secondary type $W \in \mathcal{C}$ if and only if $X \succeq Y$ and $Z \sim W$. In the context of kidney exchange, a patient of type $Y \in \mathcal{B}$ and $W \in \mathcal{C}$ can accept a kidney from a donor of type $X \in \mathcal{B}$ and $Z \in \mathcal{C}$.

Because the compatibility of secondary types is symmetric and intransitive, we use symbol i to stand for $\nsim$ and symbol c to stand for $\sim$. Let $(X, Y)^t$ describe a pair which has a patient of primary type $X \in \mathcal{B}$ and a donor of primary type $Y \in \mathcal{B}$ and the compatibility relation of secondary types between the patient and the doctor is $t \in \{i, c\}$. Therefore, we have four categories for pairs:

1. $(X, Y)^i$ for any $X, Y \in \mathcal{B}$, and $Y \not\succeq X$,

2. $(X, Y)^c$ for any $X, Y \in \mathcal{B}$, and $Y \not\succeq X$,

3. $(X, Y)^i$ for any $X, Y \in \mathcal{B}$, and $Y \succeq X$,

4. $(X, Y)^c$ for any $X, Y \in \mathcal{B}$, and $Y \succeq X$.

In this model, category 4 demonstrates compatible pairs and the other three categories cover incompatible pairs. To simplify the notation, we write incompatible pairs from categories 1 and 2 as (X, Y) for which donors are primary type incompatible with patients, i.e., $Y \not\succeq X$.

We can describe a three-way cycle as

$$E = ((P_1^p, D_1^p), (P_2^p, D_2^p), (P_3^p, D_3^p)),$$

which means that the paired donor D_1^p is matched with the paired patient P_2^p, the paired donor D_2^p is matched with the paired patient P_3^p, and the paired donor D_3^p is matched with the paired patient P_1^p. Any size cycle can be defined similarly. A cycle E is *feasible* if the type of each donor in E is compatible with the type of patient who is matched with the donor. Also, we can describe a three-way chain as

$$C = (D_1^s, (P_1^p, D_1^p), (P_2^p, D_2^p), P_1^s),$$

in which the single donor D_1^s is matched with the paired patient P_1^p, the paired donor D_1^p is matched with the paired patient P_2^p, and the paired donor D_2^p is matched with the single patient P_1^s. Any size chain can be defined in a similar way. A chain C is *feasible* if the type of every donor in C is compatible with the type of patient who is matched with the donor.

We can recast the Assumptions 2.1 and 2.3 into the present model, respectively.

Assumption 4.1 *Every single agent of primary type $X \in \mathcal{B}$ and secondary type $Z \in \mathcal{C}$ is $Z \sim W$ with every agent of type $Y \in \mathcal{B}$ and $W \in \mathcal{C}$ who is $Y \succeq X$. Every agent in a pair of type $X \in \mathcal{B}$ and $Z \in \mathcal{C}$ is $Z \sim W$ with every agent other than agents in the pair of type $Y \in \mathcal{B}$ and $W \in \mathcal{C}$ who is $Y \succeq X$.*

Assumption 4.2 *Let $X, Y \in \mathcal{B}$ be such that $Y \succeq X$. There exists either no pair of type (X, Y) or at least one pair of type $(X, Y)^c$.*

When the compatibility relation of primary type satisfies reflexivity, asymmetry and transitivity, the compatibility relation of secondary type satisfies symmetry and intransitivity, and the Assumption 4.1 for all agents, the Assumptions 4.2 for paired agents are satisfied, a maximal size exchange in the model can be achieved through no more than n-way cycles and n-way chains. The next two results generalize those of Roth, Sönmez and Ünver (2007, p. 837) to the setting which allows patients on the waiting list and single donors and need to use both cycles and chains of exchange.

Theorem 4.3 (n-way exchange suffices): *Assume that the Assumption 4.1 and 4.2 hold. Let μ be any maximal matching in the sense that any size of kidney exchanges is permitted in the matching. Then there exists a maximal matching ν which contains at most n-way cycles and chains of exchange but has the same number of patients matched with compatible donors as in the matching μ.*

The proof of this theorem is given in the appendix. The following is an immediate consequence of the theorem.

Corollary 4.4 (Four-way exchange suffices in kidney exchange): *Consider a kidney exchange model under the Assumptions 2.1 and 2.3. Let μ be any maximal matching without any restriction on the size of exchange. Then there exists a maximal matching ν which contains at most four-way exchanges but has the same number of patients who can benefit from exchanges as in the matching μ.*

5 Simulations Based on the USA Data

In this section, we use two data sets from the U.S. Organ Procurement and Transplantation Network (OPTN) and the Scientific Registry of Transplant Recipients (SRTR) from 1993 to 2002 and from 1995 to 2016, respectively,[3] to generate simulated data reflecting the characteristics of the population involved and to test how well our theoretical results can predict. Although the simulated population which is almost identical or very close to the real life situation may not fully meet the simplifying assumptions made for the model, we find that the predicted maximum number of transplants given by our derived formulas is surprisingly close to the number of transplants that can be actually realized.

5.1 Data Construction

Data is collected for two time slots. The first time slot data is from 1993 to 2002 and is shown in Table 5, and the second time slot data is from 1995 to 2016 and is shown in Table 6. These data sets illustrate the national characteristics of the USA population involved in kidney exchanges. The first period data from 1993 to 2002 is largely similar to

[3]They are retrieved from http://optn.transplant.hrsa.gov/data/view-data-reports/national-data.

those used by Roth, Sönmez and Ünver (2007), and Saidman et al. (2006), except that in our new data set we include more relevant information like the distribution of compatible patient-donor pairs and single donors, which are not used in Roth, Sönmez and Ünver (2007), and Saidman et al. (2006).

5.1.1 Patient-Donor Pairs and Single Donors Construction

Following Roth, Sönmez and Ünver (2007), to avoid the complications of possible impact of genetics on immunological incompatibilities we exclude all blood-related incompatible patient-donor pairs in all our samples.

In the first time slot from 1993 to 2002, we use the same characteristics of incompatible pairs as that of Roth, Sönmez and Ünver (2007) but add the blood-type characteristics for compatible patients, compatible donors and single donors; see Table 5. The second time slot data from 1995 to 2016 contains more detailed information about characteristics of the population. Compared to three levels of PRA (Percent Reactive Antibody) of patients from the data of the first time slot, five levels of PRA called CPRA (Calculated Percent Reactive Antibody) are provided in the data of the second time slot. The second time slot data contains also the information of compatible paired patient gender, compatible paired patient CPRA types and the blood-type information of incompatible paired donor; see Table 6.

It is important to point out that in the OPTN/SRTR annual report there is no clear information about the number of incompatible patient-donor pairs. Following Roth, Sönmez and Ünver (2007) we use newly-added patients on the waiting list every year as approximately incompatible paired patients and the blood-type distribution of donors whose kidneys have been transplanted as the blood-type distribution of incompatible paired donors. The information on single donors is collected from the data of deceased donors in each year.

Because there exist a large number of patients on the waiting list, we can always find a patient who is compatible with any given kidney. Hence, we do not need to simulate any data for patients on the waiting list.

5.1.2 Tissue-type Incompatibility

Tissue-type compatibility is the second condition for kidney transplants. In our simulations of the first time slot from 1993 to 2002, we adopt the same method as used by Roth, Sönmez and Ünver (2007) such that patients are divided into three groups based on the difficult level of tissue-type compatible with a random donor. In the first group called Low PRA group, patients are tissue-type incompatible with less than 10 percent of the population. The second group called Medium PRA contains patients who are tissue-type incompatible with 10-80 percent of the population. And, the third one called High PRA has patients who have a tissue-type incompatibility problem with more than 80 percent of the population. We use the following categories as used by Roth, Sönmez and Ünver (2007):

1. In Low PRA group, each patient is tissue-type incompatible with 5 percent of the population,

2. In Medium PRA group, each patient is tissue-type incompatible with 45 percent of the population, and

3. In High PRA group, each patient is tissue-type incompatible with 90 percent of the population.

In our simulations for the second time slot from 1995 to 2016, CPRA index is used to check whether a patient is sensitive or not according to OPTN/SRTR database. Five levels are calculated in CPRA index, which are 0, 1-19, 20-79, 80-97, and 98-100. If a patient CPRA equals 0, it means the patient has no PRA problem with potential donors; 1-19 means the patient has 1 percent to 19 percent to have problem with potential donors and so on. In this simulation, we divide patients into five groups based on the difficult levels of tissue-type compatibility with a random donor. Based on the CPRA data, we use the following five groups:

1. In 0 CPRA group, each paired patient is tissue-type incompatible with 0 percent of the population;

2. In 1-19 CPRA group, each paired patient is tissue-type incompatible with 9.5 percent of the population;

3. In 20-79 CPRA group, each paired patient is tissue-type incompatible with 50 percent of the population;

4. In 80-97 CPRA group, each paired patient is tissue-type incompatible with 88 percent of the population;

5. In 98-100 CPRA group, each paired patient is tissue-type incompatible with 99 percent of the population;

Because the data from 1995 to 2016 contains more detailed information on the tissue-type compatibility of patients and donors, it provides more accurate information than the first time slot data does. This has important implications: it will yield better results as shown in the subsequent section.

According to Zenios, Woodle and Ross (2001), a female patient is more likely to have a positive corssmatch with her husband. For instance, when positive crossmatch probability is 11.1 percent between random pairs, it becomes 33.3 percent between female patients and their donor husbands. Hence, when a patient is female and her potential donor is her husband, we adjust the probability of tissue-type incompatibility between them by using the formulas

$$PRA^* = 100 - 0.75(100 - PRA) \quad and \quad CPRA^* = 100 - 0.75(100 - CPRA).$$

5.2 Simulations

We generate a Monte-Carlo simulation size of 5,000 random population constructions for five population sizes of 25, 50, 100, 150 and 200 incompatible patient-donor pairs together with the corresponding population sizes of compatible patient-donor pairs and single donors according to the population distributions given by Table 5 for the period of 1993 to 2002 and by Table 6 for the period of 1995 to 2016, respectively. In addition we do a Monte-Carlo simulation size of 500 random population constructions for two big population sizes of 300 and 400 incompatible patient-donor pairs. Note that for these big population sizes we only generate 500 instead of 5,000 random population constructions in order to save time as it involves a relatively large and computationally difficult integer programming problem.

In our simulations we use the Kuhn-Munkres Algorithm (Kuhn, 1955) to find the maximal number of incompatible paired patients who can actually receive a compatible kidney when the exclusive exchange mechanism, the first degree inclusive mechanism and the second degree inclusive mechanism are applied respectively. This maximal number will be simply called *simulation*. We compare these numbers with those predicted by the formula given by Proposition 3.3 to see how close or far the actual maximal number of kidney transplants can be from the predicted number based on the formula in Proposition 3.3. As said earlier, we only use two-way exchanges in all simulations. Following Roth, Sönmez and Ünver (2007), we make use of two types of upper bounds:

Upper Bound 1. This is the number given by the formula in Proposition 3.3 for the simulated population sample of 25, 50, 100, 150, 200, 300, and 400 incompatible patient-donor pairs.

Upper Bound 2. For each simulated population sample, there may exist some patients who cannot find a compatible donor in the simulated population. We exclude those hopeless patients from the sample and compute the number given by the formula in Proposition 3.3 for the remaining population. This number is called the Upper Bound 2 and clearly gives a more accurate upper bound for the number of feasible transplants that can be realized.

For each population size of 25, 50, 100, 150, and 200 incompatible patient-donor pairs, we generate 5000 random samples and calculate the average of all 5000 simulations, upperbound 1's and upperbound 2's. For each population size of 300 and 400 incompatible patient-donor pairs, we generate 500 random samples and calculate the average of all 500 simulations, upperbound 1's and upperbound 2's. All results are collected in Tables 7 and 8 for the period of 1993-2002 and the period of 1995-2016, respectively.

5.3 Discussion of the Simulation Results

The simulation results from Tables 7, 8, 9, and 10 indicate that

1. the simulation results are very close to the theoretical bounds predicted by the formula in Proposition 3.3. Note that all our simulated population samples contain tissue-type incompatibilities, whereas Proposition 3.3 basically assumes away the issue of tissue-type incompatibility.

2. when both compatible patient-donor pairs and single donors participate in kidney exchanges, efficiency of exchange increases significantly.

3. increasing the size of the population can help the theory predict better.

4. two-way exchanges can achieve most of the potential gains from exchange. Even more so if the size of population gets bigger.

5. when the number of incompatible patient-donor pairs exceeds a certain threshold, say, 100, efficiency of exchange becomes almost a constant. This strongly suggests that it is possible to decentralize kidney exchanges in a number of places with a relatively large size of population.

6. more accurate information can improve the quality of transplants and at the same time reduce the matching rate. This will be explained in the following subsection.

Before explaining the above points in detail, we introduce two performance measures. We first define the deviation of each simulation with upper bound 1 and upper bound 2 by

$$\frac{\text{upper bound i} - \text{simulation}}{\text{upper bound i}}, \quad \text{i=1, 2}$$

All deviations are given in Table 9. It is clear that as the size of the population increases, the deviation becomes smaller.

We next define the matching rate for each case of feasible transplants for incompatible paired patients over the number of incompatible patient-donor pairs under each exchange mechanism by

$$\frac{\text{the number of feasible transplants for incompatible paired patients}}{\text{the number of all incompatible paired patients}}$$

All matching rates are collected in Table 10 and shown in Figure 7. It is clear that as the size of the population increases, the matching rate increases.

Points 1 and 3 can be seen from Table 9. For the two data sets, the table indicates that the deviation becomes smaller as the size of the population increases, and that the 2nd degree inclusive mechanism performs better than the 1st degree inclusive mechanism which outperforms the exclusive exchange mechanism. Look at the case of the 1993-2002 data set. For 25 incompatible pairs, the deviations for upper bound 1 under the exclusive exchange mechanism, the 1st degree inclusive mechanism and the 2nd degree inclusive mechanism are

27%, 19% and 10%, respectively; and the corresponding deviations for upper bound 2 are 7%, 8% and 6%, respectively. For 100 incompatible pairs, the deviations for upper bound 1 under the exclusive exchange mechanism, the 1st degree inclusive mechanism and the 2nd degree inclusive mechanism are 12%, 6% and 2%, respectively; and the corresponding deviations for upper bound 2 are 6%, 4% and 2%, respectively. For 200 incompatible pairs, the deviations for upper bound 1 under the exclusive exchange mechanism, the 1st degree inclusive mechanism and the 2nd degree inclusive mechanism are 6%, 2% and 0.7%, respectively; and the corresponding deviations for upper bound 2 are 4%, 2% and 0.7%, respectively. This suggests that increasing the size of the population can make the theory predict better. These observations hold true also for the 1995-2016 data set.

Points 2 and 4 become quite obvious if we compare our Table 7 with Table 2 of Roth, Sönmez and Ünver (2007, p.841) for the data of the same period of 1993-2002. For instance, for a population of 25 incompatible patient-donor pairs, in their Table 2 under two-, three-, ..., unlimited-way exchange, their mechanism gives 11.992 feasible transplants, whereas our Table 7 shows that under two-way exchange, our 1st degree inclusive mechanism gives 12.838 feasible transplants and the 2nd degree inclusive mechanism yields 19.59 feasible transplants.

Finally we turn to Point 5. Figure 7 demonstrates that overall the slope of matching rate is upward and when the number of incompatible patient-donor pairs is below 100-a kind of threshold, the slope is relatively steep, and after 100, the slope becomes almost flat albeit upward, i.e., efficiency of exchange is nearly a constant. This may have important policy implications: Kidney exchanges could be *decentralized.* Any country with a large population like USA can have several separate kidney exchange programs spread across the country where each program covers a sufficient number of patients and donors, say, no less than 100 of incompatible patient-donor pairs. This can be very important in practice, as the life of kidneys from decreased donors is short and shortening travelling time can be extremely helpful.

5.3.1 An Explanation of the Matching Rate on the Second Dataset

In this subsection we explain why the matching rate in the 1995-2016 dataset (the second time slot) is lower than in the 1993-2002 dataset (the first time slot). In our simulations, we first draw a population of n incompatible pairs from the pool. Each incompatible pair is either blood-type incompatible or tissue-type compatible or both. When a compatible pair is drawn, we put the compatible pair back to the pool and keep drawing pairs from the pool until the population of n incompatible pairs is generated.

From the information given in Tables 5 and 6, we can calculate the percentage of incompatible pairs in the pool.

We give an example of the calculation by using the first group of each time slot. 89.24 percent of patients have no tissue type problem (CPRA=0) in the second time slot while 70.19 percent of patients have a low PRA value of 5 percent in the first time slot. Therefore, the percentages of drawing incompatible pairs from this group in the first and second time slots are given as follows, respectively:

(Low PRA): $5\% + 95\% * (7/16) = 0.05 + 0.415625 = 0.465625$

(0): $(7/16) = 0.4375$.

When an incompatible paired patient is tissue-type compatible with a paired donor, the patient is blood-type incompatible with the donor. We have seven types of blood-type incompatible pairs (O, A), (O, B), (O, AB), (A, B), (B, A), (A, AB) and (B, AB). From the theoretical part, we can see that the blood-type incompatible pairs are difficult to find compatible pairs because they cannot match with each other except $(A, B) - (B, A)$, especially among incompatible pairs.

Table 4: The percentage of incompatible pairs in the pool

Groups from 1992-2003	The rate of tissue type incompatible pairs (%)	The rate of blood-type incompatible but tissue type compatible pairs (%)	The rate of incompatible pairs (%)
Low PRA	0.05	0.415625	0.465625
Medium PRA	0.45	0.240625	0.690625
High PRA	0.9	0.04375	0.94375
Average	0.213385	0.34414	0.5575
Groups from 1995-2016	The rate of tissue type incompatible pairs (%)	The rate of blood-type incompatible but tissue type compatible pairs (%)	The rate of incompatible pairs (%)
0	0	0.4375	0.4375
1-19	0.095	0.39593	0.49093
20-79	0.5	0.21875	0.71875
80-97	0.88	0.0525	0.9325
98-100	0.99	0.004375	0.994375
Average	0.05658	0.41274	0.46933

We can see that blood-type incompatible pairs account for 61.729 (0.34414/0.5575) percent of the total incompatible pairs in the first time slot. While blood-type incompatible pairs account for 87.9423 (0.41274/0.46933) percent of the total incompatible pairs in the second time slot, which is 26.2133 percent higher than that of the first time slot. This means that the number of blood-type incompatible pairs from the second time slot is larger than those from the first time slot.

On the other hand, the number of blood-type compatible but tissue type incompatible pairs $(0.213385 * (9/16) = 0.12)$ in the first time slot is larger than that in the second one

$(0.05658*(9/16)=0.032)$. Since blood-type incompatible pairs cannot be matched except $(A,B)-(B,A)$ with each other, it will be more difficult for incompatible paired patients to be matched in the second time slot than in the first time slot. This shows why the matching rate in the second time slot is lower than that in the second time slot.

6 Conclusion

The current study has been motivated by two major issues concerning kidney exchange. The first one is very practical and concerns the engineering aspect of conducting efficient kidney exchanges in a real life environment. In this environment, there are many compatible patient-donor pairs, incompatible patient-donor pairs, patients on the waiting list, and single donors who are altruistic living or cadaver donors, and kidney exchanges can be done mostly by two-way, occasionally by three-way, and rarely by four-way. We have examined how to design kidney exchange procedures in this practical environment so that a maximal number of patients can receive compatible kidneys. The second one is more theoretical and concerns the derivation of a precise upper bound of a possible number of patients who can benefit from two-way, three-way, and four-way exchanges, respectively.

Our model is very practical and general, as it reflects a typical real life kidney exchange environment and includes incompatible patient-donor pairs, compatible patient-donor pairs, patients on the waiting list, and single donors who can be altruistic living donors or decreased donors. A salient feature of the current model is to allow compatible patient-donor pairs and single donors to participate in kidney exchange with incompatible patient-donor pairs. In this way, the number of incompatible paired patients who can receive compatible kidneys will be increased considerably and is directly propositional to the size of compatible paired donors and single donors, and therefore more lives can be saved.

For this general model we have derived a precise maximum number of patients who can possibly receive compatible kidneys under two-way, three-way, and four-way exchanges respectively, although the analysis has become more difficult and more complicated. In each case (two-, three-, or four-way exchange), we develop a procedure by which kidney exchange should be conducted to enable a maximal number of patients to receive compatible kidneys. It is shown that even for this general model at most four-way cycles or chains will be sufficient to accomplish all potential gains of kidney exchange, and that in every efficient exchange, each cycle contains at most two blood-type compatible pairs and each chain contains at most one blood-type compatible pair. We have also provided substantial simulation results based on the USA national patient data for the period 1993-2002 and the period 1995-2016. Our results shed new insights into the kidney exchange problem and are stated as follows.

Our results are fully consistent with those found in Roth, Sönmez and Ünver (2007), when kidney exchanges are carried out among only incompatible pairs. However, in our model when compatible patient-donor pairs are allowed to exchange with incompatible patient-donor pairs, the number of incompatible paired patients who can receive compatible kidneys increases considerably; and this number will increase significantly when both compatible patient-donor pairs and single donors participate in exchange with incompatible pairs. Our theory can predict surprisingly well in the sense that the actual maximal number of feasible kidney transplants is very close to the predicated number given by our derived formula. As the size of the population increases, the predictive power of our theory becomes stronger; two-way exchange can accomplish most of the potential gains of exchange. If the population is large enough, it is sufficient to use two-way exchange to clear all incompatible pairs. Our results have a novel and significant policy implication: kidney exchange can be decentralized in the sense that in a country with a large population, several separate kidney exchange programs can be established across the country, not just one centralized program for the entire country. In the course of our study it becomes clear to us that at the current stage it is very difficult to conduct simulations with a population size of 500 incompatible patient-donor pairs, as it involves a quite large and difficult integer programming problem. We expect to report simulation results in the near future by also making use of three-, four- or higher-way cycles and chains of exchange.

We hope the current study will be useful in helping design practical kidney exchange program and stimulate further research.

Table 5: Patient-donor pair and single donor distributions used in simulations based on OPTN/SRTR database from 1993 to 2002, retrieved from https://optn.transplant.hrsa.gov/data/view-data-reports/national-data.

Incompatible paired patient blood type	**Percent**
O	48.14
A	33.73
B	14.28
AB	3.85
Patient gender	**Percent**
Female	40.9
Male	59.1
Relationship of patient-donor pair	**Percent**
Spouse	48.97
Other	51.03
PRA types	**Percent**
Low PRA	70.19
Medium PRA	20.00
High PRA	9.81
Compatible paired patient blood type	**Percent**
O	45.12
A	38.54
B	12.64
AB	3.7
Compatible paired donor blood type	**Percent**
O	63.74
A	27.12
B	8.08
AB	1.06
Single donor blood type	**Percent**
O	47.31
A	38.14
B	11.16
AB	3.39
Transplant ratio by donor types	**Percent**
Single Donors	39.83
Paired Donors	22.77

Table 6: Patient-donor pair and single donor distributions used in simulations based on OPTN/SRTR database from 1995 to 2016, retrieved from https://optn.transplant.hrsa.gov/data/view-data-reports/national-data.

Incompatible paired patient blood type	**Percent**	**S.D.**
O	48.46	0.0032
A	33.22	0.0047
B	14.48	0.0028
AB	3.84	0.0011
Incompatible paired patient gender	**Percent**	**S.D.**
Female	40.1	0.0117
Male	59.9	0.0117
Incompatible paired patient CPRA type	**Percent**	**S.D.**
0	89.24	0.0145
1-19	2.79	0.0071
20-79	4.64	0.005
80-97	2.03	0.001
98-100	1.3	0.002
Compatible paired patient blood type	**Percent**	**S.D.**
O	44.71	0.0092
A	38.47	0.0075
B	12.99	0.0044
AB	3.83	0.0029
Compatible paired patient gender	**Percent**	**S.D.**
Female	39.95	0.0204
Male	60.05	0.0204
Compatible paired patient CPRA type	**Percent**	**S.D.**
0	73.11	0.0241
1-19	9.43	0.0154
20-79	12.82	0.0084
80-97	3.38	0.0041
98-100	1.26	0.0025
Relationship of patient-donor pair	**Percent**	**S.D.**
Spouse	35.8	0.1201
Other	64.2	0.1201
Incompatible paired donor blood type	**Percent**	**S.D.**
O	55.3	0.0122
A	32.46	0.0081
B	9.9	0.0041
AB	2.34	0.0022
Compatible paired donor blood type	**Percent**	**S.D.**
O	64.66	0.011
A	26.45	0.0074
B	7.91	0.0044
AB	0.98	0.0021
Single donor blood type	**Percent**	**S.D.**
O	47.59	0.0068
A	37.41	0.0084
B	11.57	0.0055
AB	3.43	0.0026
Transplant ratio by donor type	**Percent**	**S.D.**
Single Donors	36.02	0.0398
Paired Donors	19.9	0.039

Table 7: Simulation results about average maximal number of incompatible paired patients actually receiving transplants and average predicted number by the formula based on the 1993-2002 data.

Population Size of Incompatible Pairs	Method	Number of incompatible paired patients getting transplants: The Exclusive Exchange Mechanism	The First Degree Inclusive Exchange Mechanism	The Second Degree Inclusive Exchange Mechanism
n=25	Simulation	8.9992 (3.3465)	12.8388 (3.36736)	19.5904 (3.1966)
	Upper Bound 1	12.4444 (3.62319)	15.8782 (3.55402)	21.919 (3.0039)
	Upper Bound 2	9.7012 (3.69614)	14.0782 (3.59381)	20.964 (3.02684)
n=50	Simulation	21.7872 (5.04759)	29.599 (5.17304)	42.8134 (4.77275)
	Upper Bound 1	27.0408 (5.16082)	33.5676 (5.31818)	45.413 (4.45821)
	Upper Bound 2	23.7656 (5.47378)	31.9192 (5.4182)	44.8486 (4.41678)
n=100	Simulation	49.8772 (7.36965)	64.2164 (7.4473)	89.8862 (6.9542)
	Upper Bound 1	56.7104 (7.36069)	68.614 (7.58903)	92.2014 (6.59551)
	Upper Bound 2	53.4844 (7.70327)	67.4584 (7.6945)	92.0746 (6.57535)
n=150	Simulation	78.9256 (9.29992)	100.014 (9.42842)	137.567 (8.63815)
	Upper Bound 1	86.692 (9.1035)	104.442 (9.48898)	139.417 (8.33299)
	Upper Bound 2	83.6704 (9.54597)	103.647 (9.58259)	139.383 (8.31955)
n=200	Simulation	108.716 (10.7764)	135.571 (10.9588)	184.819 (10.3357)
	Upper Bound 1	116.799 (10.5688)	139.742 (11.0306)	186.254 (10.1569)
	Upper Bound 2	114.232 (10.9591)	139.168 (11.0965)	186.245 (10.1546)
n=300	Simulation	170.54 (13.8317)	208.974 (13.8698)	280.91 (13.4347)
	Upper Bound 1	178.668 (13.6163)	212.676 (14.0197)	281.688 (13.4062)
	Upper Bound 2	176.948 (13.9028)	212.404 (14.0379)	281.688 (13.4062)
n=400	Simulation	231.628 (15.1099)	281.492 (15.1398)	375.198 (15.2474)
	Upper Bound 1	239.524 (14.592)	284.636 (15.0674)	375.65 (15.2176)
	Upper Bound 2	238.36 (14.8267)	284.466 (15.1155)	375.65 (15.2176)

Table 8: Simulation results about average maximal number of incompatible paired patients actually receiving transplants and average predicted number by the formula based on the 1995-2016 data.

Population Size of Incompatible Pairs	Method	Number of incompatible paired patients getting transplants: The Exclusive Exchange Mechanism	The First Degree Inclusive Exchange Mechanism	The Second Degree Inclusive Exchange Mechanism
n=25	Simulation	6.6844 (3.02308)	9.6722 (3.16884)	16.1756 (3.39085)
	Upper Bound 1	8.3772 (3.29944)	11.3094 (3.4135)	17.6964 (3.55437)
	Upper Bound 2	6.832 (3.12092)	10.0494 (3.30825)	16.7298 (3.50851)
n=50	Simulation	15.008 (4.5394)	21.5734 (4.71549)	33.8482 (4.9819)
	Upper Bound 1	18.5984 (4.79534)	24.1956 (4.9522)	36.1456 (5.1907)
	Upper Bound 2	16.0188 (4.75009)	22.3364 (4.88135)	34.7956 (5.1391)
n=100	Simulation	34.496 (6.8107)	46.3272 (7.05924)	69.7068 (7.42242)
	Upper Bound 1	39.6832 (6.96165)	50.2572 (7.27533)	73.0118 (7.62722)
	Upper Bound 2	35.8428 (7.01817)	47.6532 (7.22253)	71.1594 (7.55584)
n=150	Simulation	54.2632 (8.65407)	71.5348 (8.9778)	105.994 (9.24828)
	Upper Bound 1	60.934 (8.82225)	76.3784 (9.16827)	110.046 (9.4449)
	Upper Bound 2	56.2608 (8.89426)	73.313 (9.17326)	107.991 (9.44535)
n=200	Simulation	74.134 (10.0771)	96.6472 (10.4245)	142.411 (10.6297)
	Upper Bound 1	81.8596 (10.1768)	102.143 (10.691)	146.966 (10.8525)
	Upper Bound 2	76.5832 (10.2132)	98.7708 (10.659)	144.941 (10.8549)
n=300	Simulation	114.904 (11.8696)	147.89 (12.4127)	215.976 (13.0876)
	Upper Bound 1	124.292 (11.9257)	154.37 (12.6063)	221.19 (13.282)
	Upper Bound 2	118.272 (12.0396)	150.724 (112.6819)	219.358 (13.3162)
n=400	Simulation	155.572 (13.846)	198.49 (14.1654)	288.54 (14.4048)
	Upper Bound 1	166.024 (13.8123)	205.776 (14.4168)	294.304 (14.7397)
	Upper Bound 2	159.384 (13.8554)	202.008 (14.3976)	292.802 (14.7343)

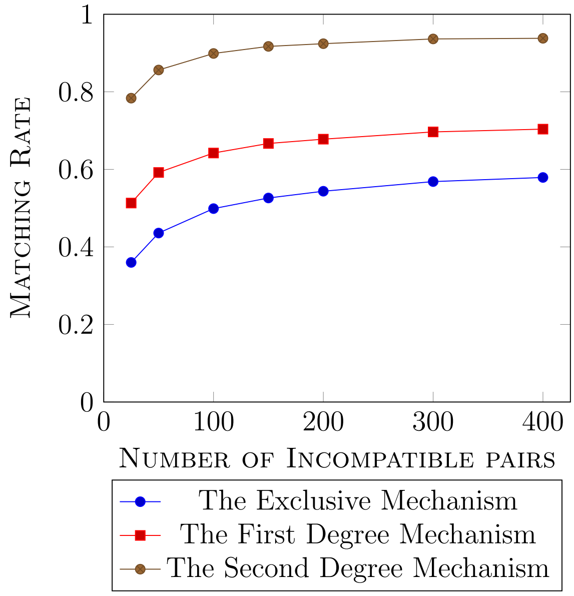
Table 9: Deviation from upper bounds 1 and 2 in simulation based on the 1993-2002 data and 1995-2016 data.

Data from 1993-2002				
Population Size of Incompatible Pairs	Method	Deviation Value		
		The Exclusive Exchange Mechanism	The First Degree Inclusive Exchange Mechanism	The Second Degree Inclusive Exchange Mechanism
n=25	Upper Bound 1	0.2768	0.1914	0.1062
	Upper Bound 2	0.0724	0.088	0.0655
n=50	Upper Bound 1	0.1943	0.1182	0.0572
	Upper Bound 2	0.0832	0.0727	0.04538
n=100	Upper Bound 1	0.1205	0.0641	0.0251
	Upper Bound 2	0.0674	0.0481	0.0238
n=150	Upper Bound 1	0.0896	0.0424	0.0133
	Upper Bound 2	0.0567	0.0351	0.013
n=200	Upper Bound 1	0.0692	0.0299	0.0077
	Upper Bound 2	0.0483	0.0258	0.00766
n=300	Upper Bound 1	0.04549	0.0174	0.00276
	Upper Bound 2	0.03621	0.01615	0.00276
n=400	Upper Bound 1	0.03296	0.011	0.0012
	Upper Bound 2	0.02824	0.0104	0.0012
Data from 1995-2016				
Population Size of Incompatible Pairs	Method	Deviation Value		
		The Exclusive Exchange Mechanism	The First Degree Inclusive Exchange Mechanism	The Second Degree Inclusive Exchange Mechanism
n=25	Upper Bound 1	0.2021	0.1448	0.0859
	Upper Bound 2	0.0216	0.0375	0.03312
n=50	Upper Bound 1	0.1665	0.108375	0.06356
	Upper Bound 2	0.032337	0.03416	0.02723
n=100	Upper Bound 1	0.1307	0.07819	0.04526
	Upper Bound 2	0.037575	0.028622	0.02041
n=150	Upper Bound 1	0.1094	0.063415	0.03682
	Upper Bound 2	0.0355	0.02425	0.01849
n=200	Upper Bound 1	0.09437	0.0538	0.03099
	Upper Bound 2	0.03198	0.0215	0.01745
n=300	Upper Bound 1	0.075532	0.041977	0.02357
	Upper Bound 2	0.0284767	0.0188	0.015418
n=400	Upper Bound 1	0.062954	0.03541	0.01958
	Upper Bound 2	0.02391	0.017415	0.01455

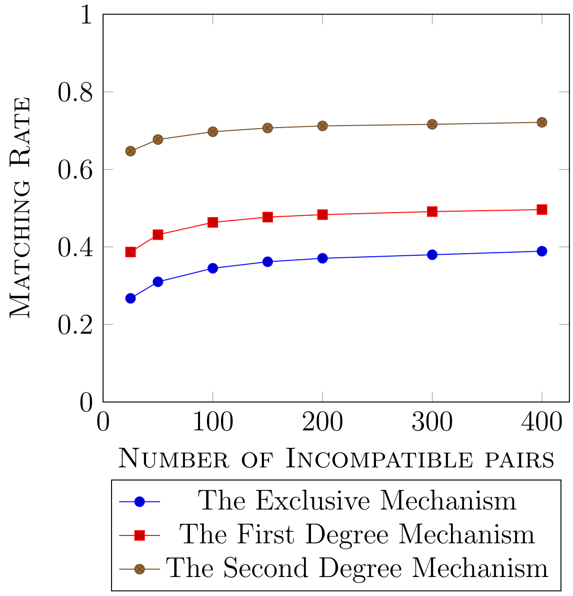
Table 10: Matching rates of incompatible paired patients in simulation based on the 1993-2002 data and 1995-2016 data.

Data from 1993-2002				
Population Size of Incompatible Pairs	Method	Matching Rate		
		The Exclusive Exchange Mechanism	The First Degree Inclusive Exchange Mechanism	The Second Degree Inclusive Exchange Mechanism
n=25	Simulation	0.35997	0.51355	0.78362
	Upper Bound 1	0.49778	0.63513	0.87676
	Upper Bound 2	0.38805	0.56313	0.83856
n=50	Simulation	0.43574	0.59198	0.85627
	Upper Bound 1	0.54096	0.67135	0.90826
	Upper Bound 2	0.47531	0.63838	0.89697
n=100	Simulation	0.49877	0.64216	0.89886
	Upper Bound 1	0.5671	0.68614	0.92201
	Upper Bound 2	0.53484	0.67458	0.92075
n=150	Simulation	0.52617	0.66676	0.91711
	Upper Bound 1	0.57795	0.69628	0.92945
	Upper Bound 2	0.5578	0.69098	0.92945
n=200	Simulation	0.54358	0.67786	0.9241
	Upper Bound 1	0.584	0.69871	0.93127
	Upper Bound 2	0.57116	0.69584	0.93122
n=300	Simulation	0.5685	0.69658	0.936
	Upper Bound 1	0.59556	0.70892	0.93896
	Upper Bound 2	0.5898	0.708	0.93896
n=400	Simulation	0.57907	0.70373	0.93799
	Upper Bound 1	0.59881	0.71159	0.9391
	Upper Bound 2	0.5959	0.7116	0.9391
Data from 1995-2016				
Population Size of Incompatible Pairs	Method	Matching Rate		
		The Exclusive Exchange Mechanism	The First Degree Inclusive Exchange Mechanism	The Second Degree Inclusive Exchange Mechanism
n=25	Simulation	0.267376	0.386888	0.647024
	Upper Bound 1	0.335088	0.452376	0.707856
	Upper Bound 2	0.27328	0.401976	0.669192
n=50	Simulation	0.310016	0.431468	0.676964
	Upper Bound 1	0.371968	0.483912	0.722912
	Upper Bound 2	0.320376	0.446728	0.695912
n=100	Simulation	0.34496	0.463272	0.697068
	Upper Bound 1	0.396832	0.502572	0.730118
	Upper Bound 2	0.358428	0.476532	0.711594
n=150	Simulation	0.36175	0.476869	0.7066
	Upper Bound 1	0.406226	0.509189	0.73364
	Upper Bound 2	0.375072	0.48875	0.71994
n=200	Simulation	0.37067	0.483236	0.712055
	Upper Bound 1	0.409298	0.510715	0.73483
	Upper Bound 2	0.382916	0.493854	0.724705
n=300	Simulation	0.383	0.49296	0.71992
	Upper Bound 1	0.4143	0.51456	0.7373
	Upper Bound 2	0.39424	0.5024	0.73119
n=400	Simulation	0.38893	0.496225	0.72135
	Upper Bound 1	0.41506	0.51444	0.73576
	Upper Bound 2	0.39846	0.50502	0.7320

Figure 7: Matching rates of incompatible paired patients based on the 1993-2002 data (a) and based on the 1995-2016 data (b).



(a)



(b)

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The Appendix

Proof of Proposition 3.5: Under Assumption 2.1 and 2.3, all blood-type compatible pairs but tissue-type incompatible pairs (A, A), (B, B), (AB, AB), (O, O), (A, O), (B, O), (AB, O), (AB, A), (AB, B) can be matched through two-way and three-way cycles. Under Assumptions 2.1 and 2.2, all pairs of type (B, A) can be matched through two-way cycles. All compatible pairs can be matched because even if paired patients from compatible pairs are not involved into two-way cycles, they can receive their own donors. As long as a kidney can be allocated to the waiting list, we can always find a compatible patient in waiting list because of the large population of patients on the waiting list. Hence, the maximal number of transplantations for patients on the waiting list, paired patients from blood-type compatible pairs and paired patients from pairs of type (B, A) is:

$$\begin{aligned}
&\#(A,O) + \#(B,O) + \#(AB,O) + \#(AB,A) + \#(AB,B) \\
&+\#(B,A) + \#(A,A) + \#(B,B) + \#(AB,AB) + \#(O,O) \\
&+\#A^d + \#B^d + \#AB^d + \#O^d
\end{aligned}$$

Let N be the maximum number of transplants for blood-type incompatible paired patients of types (O, A), (O, B), (O, AB), (A, AB), (B, AB), (A, B). We first consider three-way cycles with two blood-incompatible pairs. We can match three-way cycles starting with pairs (O, AB) and three-way chains starting with single donors O^d latter because a pair (O, AB) or a single donor O^d can match with any patient. We now consider three-way cycles $(AB, A)-(A, B)-(B, AB)$, $(B, O)-(O, A)-(A, B)$ and chains $A^d-(A, B)-(B, AB)-AB^p$. By Assumption 2.2, all pairs (B, A) can be matched and hence the number of remaining pairs (A, B) is $\#(A, B) - \#(B, A)$. To make full advantage of those three-way cycles and chains, we need avoid the over-match problems. Consider the process that match a maximum number of three-way cycles $(AB, A) - (A, B) - (B, AB)$ and then match a maximum number of three-way cycles $(B, O) - (O, A) - (A, B)$. If all pair (A, B) are matched in three-way cycles $(AB, A)-(A, B)-(B, AB)$, we will lose efficiency when there are sufficient pairs (B, O) to make two-way cycles $(B, O)-(O, B)$ but insufficient pairs (AB, A) to make two-way cycles $(AB, A) - (A, AB)$. The method is to restrict the number of three-way cycles $(AB, A) - (A, B) - (B, AB)$ by $\#A^d + \#(AB, A) - \min\{\#A^d + \#(AB, A), \#(A, AB)\}$ and then release. After matching three-way cycle $(AB, A) - (A, B) - (B, AB)$ under restriction, we match three-way cycle $(B, O) - (O, A) - (A, B)$. When there are remaining pair (A, O), pair (B, AB), pair (AB, A) and pair (O, B), we will lose efficiency because one more pair can be matched by separating a three-way cycle $(B, O) - (O, A) - (A, B)$ into a two-way cycle $(B, O) - (O, B)$, a two-way cycle $(A, O) - (O, A)$ and a three-way cycle $(AB, A) - (A, B) - (B, AB)$ with remaining pairs. The method is to restrict the number of cycle $(B, O) - (O, A) - (A, B)$ by $\#(O, A) - \min\{\#(A, O), \#(O, A)\}$ and then release.

Therefore, the procedure of taking full advantage of three-way cycles $(AB, A) - (A, B) - (B, AB)$ and $(B, O) - (O, A) - (A, B)$ is as follows:

Process 1: The number of pairs (A, B) in this process should not exceed $\#(A, B) - \#(B, A)$. Match a maximum number of three-way cycles $(AB, A) - (A, B) - (B, AB)$ and chains $A^d - (A, B) - (B, AB) - AB^p$, where the available number of (AB, A) and A^d is $\min\{\#A^d + \#(AB, A), \#(A, AB)\}$. Match a maximum number of three-way cycles $(B, O) - (O, A) - (A, B)$, where the available number of (O, A) is $\#(O, A) - \min\{\#(A, O), \#(O, A)\}$.

Process 2: Match a maximum number of three-way cycles $(AB, A) - (A, B) - (B, AB)$ and chains $A^d - (A, B) - (B, AB) - AB^p$. Match a maximum number of three-way cycles $(B, O) - (O, A) - (A, B)$.

The number of transplants for blood-type incompatible paired patients of types (O, A), (O, B), (O, AB), (A, AB), (B, AB), (A, B) in the procedure is $2 * g_1 + 2 * g_2 + 2 * g_3 + 2 * g_4$, where

$$\begin{aligned}
g_1 = \;& \min\{\#A^d + \#(AB, A) - \min\{\#A^d + \#(AB, A), \#(A, AB)\}, \#(A, B) - \#(B, A), \\
& \#(B, AB)\} \\
g_2 = \;& \min\{\#(B, O), \#(O, A) - \min\{\#(A, O), \#(O, A)\}, \#(A, B) - \#(B, A) - g_1\} \\
g_3 = \;& \min\{\#A^d + \#(AB, A) - g_1, \#(A, B) - \#(B, A) - g_1 - g_2, \#(B, AB) - b_1\} \\
g_4 = \;& \min\{\#(B, O) - g_2, \#(O, A) - g_2, \#(A, B) - \#(B, A) - g_1 - g_2 - g_3\}
\end{aligned}$$

After the procedure, sixteen situations occur when we match remaining (O, A) with (A, O), (O, B) with remaining (B, O), (A, AB) with remaining pair (AB, A) and single donor A^d, and remaining (B, AB) with pair (AB, B) and single donor B^d.

(1) When (O, A), (O, B), (A, AB), (B, AB) remaining, we have $\min\{\#(A, O), \#(O, A)\} = \#(A, O)$, $g_1 = \#A^d + \#(AB, A) - \min\{\#A^d + \#(AB, A), \#(A, AB)\}$, $g_2 = \min\{\#(B, O), \#(A, B) - \#(B, A) - g_1\}$, $g_3 = \min\{\#A^d + \#(AB, A) - g_1, \#(A, B) - \#(B, A) - g_1 - g_2\}$ and $g_4 = 0$. There is no potential gains from three-way cycles and chains with one pair (B, A) or with one blood-incompatible side because all blood-type compatible pairs are matched. Therefore, we first take full of advantage of three-way cycles and chains starting from single donor O^d and pairs of type (AB, O) and then match remaining pairs with single donor O^d and pairs of type (AB, O). The maximum number of transplants in situation (1) is:

$$\begin{aligned}
N = \;& 2 * g_1 + 2 * g_2 + 2 * g_3 + w_1 + w_2 + w_3 + w_4 + 2 * g_5 + 2 * g_6 + 2 * g_7 + 2 * g_8 + w_5 \\
\textit{where} & \\
w_1 = \;& \#(A, O) \\
w_2 = \;& \#B^d + \#(AB, B) \\
w_3 = \;& \#A^d + \#(AB, A) - g_1 - g_3
\end{aligned}$$

$$
\begin{aligned}
w_4 = \;& \#(B,O) - g_2 \\
g_5 = \;& \min\{\#O^d + \#(AB,O), \#(O,A) - g_2 - w_1, \#(A,AB) - w_3\} \\
g_6 = \;& \min\{\#O^d + \#(AB,O) - g_5, \#(O,B) - w_4, \#(B,AB) - g_1 - g_3 - w_2\} \\
g_7 = \;& \min\{\#O^d + \#(AB,O) - g_5 - g_6, \#(O,A) - g_2 - w_1 - g_5, \\
& \#(A,B) - \#(B,A) - g_1 - g_2 - g_3\} \\
g_8 = \;& \min\{\#O^d + \#(AB,O) - g_5 - g_6 - g_7, \#(A,B) - \#(B,A) - g_1 - g_2 - g_3 - g_7, \\
& \#(B,AB) - g_1 - g_3 - w_2 - g_6\} \\
w_5 = \;& \min\{\#O^d + \#(AB,O) - g_5 - g_6 - g_7 - g_8, \#(O,A) + \#(O,B) + \#(O,AB) \\
& + \#(A,AB) + \#(A,B) - \#(B,A) + \#(B,AB) - 2 * g_1 - 2 * g_2 - 2 * g_3 \\
& - w_1 - w_2 - w_3 - w_4 - g_5 - g_6 - g_7 - g_8\}
\end{aligned}
$$

The maximum number of feasible transplants can be rewritten as:

$$
\begin{aligned}
N = \;& 2 * g_1 + 2 * g_2 + 2 * g_3 + w_1 + w_2 + w_3 + w_4 + 2 * g_5 + 2 * g_6 + 2 * g_7 + 2 * g_8 + w_5 \\
= \;& \min\{N_1, N_3, N_6, N_7, N_{10}, N_{12}, N_{15}, N_{17}\}.
\end{aligned}
$$

One may refer to Tables from B1 to B4 in Supplement B of Cheng and Yang (2017).

(2) When (A,O), (B,O), (A,AB), (B,AB) remaining, we have $g_1 = \#A^d + \#(AB,A) - \min\{\#A^d + \#(AB,A), \#(A,AB)\}$, $g_2 = \#(O,A) - \min\{\#(A,O), \#(O,A)\}$, $g_3 = \min\{\#A^d + \#(AB,A) - g_1, \#(A,B) - \#(B,A) - g_1 - g_2\}$ and $g_4 = \min\{\#(O,A) - g_2, \#(A,B) - \#(B,A) - g_1 - g_2 - g_3\}$. There is no potential gains from three-way cycles and chains with one pair (B,A), $(A,O)-(O,B)-(B,A)$, $(AB,B)-(B,A)-(A,AB)$, $B^d-(B,A)-(A,AB)-AB^p$ because there is no pair (O,B), pair (AB,B) and single donor B^d left. Moreover, there is no potential gains from the combinations $(AB,A)-(A,O)$, $(AB,B)-(B,O)$, $A^d-(A,O)$ and $B^d-(B,O)$ because there is no pair (AB,A), pair (AB,B), single donor A^d and single donor B^d left. Since there are remaining pair (B,O), we can match remaining pair (A,B) with (B,O). Then, do the same matching process as situation (1). Because there is no remaining pair (O,A) and (O,B), we have $g_5 = g_6 = g_7 = 0$. The maximum number of transplants is:

$$
\begin{aligned}
N = \;& 2 * g_1 + 2 * g_2 + 2 * g_3 + 2 * g_4 + w_1 + w_2 + w_3 + w_4 + s_1 + 2 * g_8 + w_5 \\
where & \\
s_1 = \;& \min\{\#(B,O) - g_2 - g_4 - w_4, \#(A,B) - \#(B,A) - g_1 - g_2 - g_3 - g_4\} \\
g_8 = \;& \min\{\#O^d + \#(AB,O), \#(A,B) - \#(B,A) - g_1 - g_2 - g_3 - g_4 - s_1, \\
& \#(B,AB) - g_1 - g_3 - w_2\} \\
w_5 = \;& \min\{\#O^d + \#(AB,O) - g_8, \#(O,A) + \#(O,B) + \#(O,AB) + \#(A,AB) \\
& + \#(A,B) - \#(B,A) + \#(B,AB) - 2 * g_1 - 2 * g_2 - 2 * g_3 - 2 * g_4 - w_1 \\
& - w_2 - w_3 - w_4 - s_1 - g_8\}
\end{aligned}
$$

The maximum number of transplants N can be rewritten as

$$
\begin{aligned}
N = \;& 2 * g_1 + 2 * g_2 + 2 * g_3 + 2 * g_4 + w_1 + w_2 + w_3 + w_4 + s_1 + 2 * g_8 + w_5 \\
= \;& \min\{N_1, N_3, N_8, N_{10}\}.
\end{aligned}
$$

One may refer to Tables from B5 to B6 in Supplement B for detail.

(3) When (O,A), (O,B), $A^d/(AB,A)$, $B^d/(AB,B)$ remaining, we have $\min\{\#A^d + \#(AB,A), \#(A,AB)\} = \#(A,AB)\}$, $\min\{\#(A,O), \#(O,A)\} = \#(A,O)$, $g_1 = \min\{\#(A,B) - \#(B,A), \#(B,AB)\}\}$, $g_2 = \min\{\#(B,O), \#(A,B) - \#(B,A) - g_1\}$, $g_3 = 0$ and $g_4 = 0$. There is no potential gains from three-way cycles and chains with one pair (B,A), $(A,O)-(O,B)-(B,A)$, $(AB,B)-(B,A)-(A,AB)$, $B^d-(B,A)-(A,AB)-AB^p$ because there is no pair (A,O) and pair (A,AB) left. Moreover, there is no potential gains from the combinations $(AB,A)-(A,O)$, $(AB,B)-(B,O)$, $A^d-(A,O)$ and $B^d-(B,O)$ because there is no pair (A,O) and pair (B,O) left. Because there is no remaining pair (A,AB) and (B,AB), we have $g_5 = g_6 = g_8 = 0$. Because we have pair (AB,A) and single donor A^d remaining, we can first match remaining (A,B) with remaining pair (AB,A) and single donor A^d and then process the same procedure as situation (1) and the number of transplants N is

$$
\begin{aligned}
N = \quad & 2*g_1 + 2*g_2 + w_1 + w_2 + w_3 + w_4 + s_1 + 2*g_7 + w_5 \\
where \quad & \\
s_1 = \quad & \min\{\#A^d + \#(AB,A) - g_1 - w_3, \#(A,B) - \#(B,A) - g_1 - g_2\} \\
g_7 = \quad & \min\{\#O^d + \#(AB,O), \#(O,A) - g_2 - w_1, \#(A,B) - \#(B,A) - g_1 - g_2 - s_1\} \\
w_5 = \quad & \min\{\#O^d + \#(AB,O) - g_7, \#(O,A) + \#(O,B) + \#(O,AB) + \#(A,AB) \\
& +\#(A,B) - \#(B,A) + \#(B,AB) - 2*g_1 - 2*g_2 - w_1 - w_2 - w_3 \\
& -w_4 - g_7 - s_1\}
\end{aligned}
$$

The maximum number of transplants can be rewritten as:

$$N = 2*g_1 + 2*g_2 + w_1 + w_2 + w_3 + w_4 + s_1 + 2*g_7 + w_5 = \min\{N_1, N_{10}, N_{11}, N_{17}\}$$

One may refer to Table B7 in Supplement B for detail.

(4) When (A,O), (B,O), $A^d/(AB,A)$, $B^d/(AB,B)$ remaining, we have $\min\{\#A^d + \#(AB,A), \#(A,AB)\} = \#(A,AB)\}$, $g_1 = \min\{\#(A,B) - \#(B,A), \#(B,AB)\}$, $g_2 = \#(O,A) - \min\{\#(A,O), \#(O,A)\}$, $g_3 = 0$ and $g_4 = \min\{\#(O,A) - g_2, \#(A,B) - \#(B,A) - g_1 - g_2 - g_3\}$. There is no potential gains from three-way cycles and chains with one pair (B,A), $(A,O)-(O,B)-(B,A)$, $(AB,B)-(B,A)-(A,AB)$, $B^d-(B,A)-(A,AB)-AB^p$ because there is no (O,B) and (A,AB) left. Because there is no remaining (O,A), (O,B), (A,AB) and (B,AB), there is no potential gains from three-way cycles and chains starting from single donor O^d and pairs of type (AB,O). That is, $g_5 = g_6 = g_7 = g_8 = 0$. Since there is remaining pair (B,O), pair (A,O), pair (AB,A), pair (AB,B), single donor A^d and B^d, we can match the combinations of $(AB,A)-(A,O)$, $A^d-(A,O)$, $(AB,B)-(B,O)$ and $B^d-(B,O)$ to any pair, and match pairs (AB,A) and (B,O) with remaining pair (A,B). To take full advantage of the combinations, we first reserve the maximum number of the combinations and then match remaining pairs (AB,A), (B,O) and single donor A^d

with pair (A, B). Then, match remaining pairs with the combinations, single donor O^d and pairs of type (AB, O). The maximum number of transplants in situation (4) is:

$$\begin{aligned}
N = \quad & 2 * g_1 + 2 * g_2 + 2 * g_4 + w_1 + w_2 + w_3 + w_4 + s_1 + w_5 \\
where \quad & \\
c_2 = \quad & \min\{\#A^d + \#(AB, A) - g_1 - w_3, \#(A, O) - w_1\} \\
c_3 = \quad & \min\{\#B^d + \#(AB, B) - w_2, \#(B, O) - g_2 - g_4 - w_4\} \\
s_1 = \quad & \min\{\#A^d + \#(AB, A) - g_1 - w_3 - c_2 + \#(B, O) - g_2 - g_4 - w_4 - c_3, \\
& \#(A, B) - \#(B, A) - g_1 - g_2 - g_4\} \\
w_5 = \quad & \min\{\#O^d + \#(AB, O) + c_2 + c_3, \#(O, A) + \#(O, B) + \#(O, AB) + \#(A, AB) \\
& + \#(A, B) - \#(B, A) + \#(B, AB) - 2 * g_1 - 2 * g_2 - 2 * g_4 - w_1 \\
& - w_2 - w_3 - w_4 - s_1\}
\end{aligned}$$

The maximum number of transplants can be rewritten as:

$$N = 2 * g_1 + 2 * g_2 + 2 * g_4 + w_1 + w_2 + w_3 + w_4 + s_1 + w_5 = \min\{N_1, N_2, N_3, N_7, N_{10}, N_{17}\}$$

One may refer to Tables from B8 to B11 in Supplement B for detail.

(5) When (A, O), (B, O), $A^d/(AB, A)$, (B, AB) remaining, we have $\min\{\#A^d + \#(AB, A), \#(A, AB)\} = \#(A, AB)\}$, $\min\{\#(A, O), \#(O, A)\} = \#(O, A)$, $g_1 = \#(A, B) - \#(B, A)$ and $g_2 = g_3 = g_4 = 0$. There is no potential gains from three-way cycles and chains with one pair (B, A), $(A, O) - (O, B) - (B, A)$, $(AB, B) - (B, A) - (A, AB)$, $B^d - (B, A) - (A, AB) - AB^p$ because there is no pair (O, B) and pair (A, AB) left. Since we have taken full advantage of three-way cycles $(B, O) - (O, A) - (A, B)$, $(AB, A) - (A, B) - (B, AB)$, $A^d - (A, B) - (B, AB) - AB^p$, all surplus pair (A, B) $(\#(A, B) - \#(B, A))$ are matched in the procedure. Therefore, there is no potential gains for $(AB, A) - (A, B) - (B, AB)$ by breaking up a two-way cycle $(A, B) - (B, A)$. Because there is no remaining (O, A), (O, B) and (A, B), there is no potential gains from three-way cycles and chains starting with single donor O^d and pairs of type (AB, O). That is, $g_5 = g_6 = g_7 = g_8 = 0$. Because all pair (A, B) are matched, we have $s_1 = 0$. Since there is remaining pair (A, O), pair (B, O), pair (AB, A) and single donor A^d, we can match the combinations $(AB, A) - (A, O)$ and $A^d - (A, O)$ to any pair. To take full advantage of pair (B, O), pair (AB, A) and single donor A^d, we do the same process as situation (4). The maximum number of transplants is

$$N = 2 * g_1 + w_1 + w_2 + w_3 + w_4 + w_5 = \min\{N_1, N_3, N_7\}$$

One may refer to Table B12 in Supplement B for detail.

(6) When (A, O), (B, O), (A, AB), $B^d/(AB, B)$ remaining, we have $g_1 = \#A^d + \#(AB, A) - \min\{\#A^d + \#(AB, A), \#(A, AB)\}$, $g_2 = \#(O, A) - \min\{\#(A, O), \#(O, A)\}$,

$g_3 = \min\{\#A^d + \#(AB, A) - g_1, \#(A, B) - \#(B, A) - g_1 - g_2, \#(B, AB) - b_1\}$ and $g_4 = \min\{\#(O, A) - g_2, \#(A, B) - \#(B, A) - g_1 - g_2 - g_3\}$. By Assumption 2.2, all pairs (B, A) can be matched in two-way cycles $(A, B) - (B, A)$. There is no potential gains from three-way cycles and chains with one pair (B, A), $(A, O) - (O, B) - (B, A)$, $(AB, B) - (B, A) - (A, AB)$, $B^d - (B, A) - (A, AB) - AB^p$ because there is no pair (O, A), (O, B) left. There is also no potential gains from three-way cycle $(AB, B) - (B, A) - (A, AB)$ and/or three-way chain $B^d - (B, A) - (A, AB) - AB^p$ by breaking up two-way cycle $(A, B) - (B, A)$ because there is no pair (O, A) left. Because there is no remaining (O, A), (O, B) and (B, AB), there is no potential gains from three-way cycles and chains starting from single donor O^d and pairs of type (AB, O). That is, $g_5 = g_6 = g_7 = g_8 = 0$. There is potential gains from the combinations $(AB, B) - (B, O)$, $B^d - (B, O)$ and two-way cycles $(B, O) - (A, B)$. To take full advantage of the combinations, we do the same process as situation (4). The maximum number of transplants is

$$N = 2 * g_1 + 2 * g_2 + 2 * g_3 + 2 * g_4 + w_1 + w_2 + w_3 + w_4 + s_1 + w_5 = \min\{N_1, N_3, N_{10}\}$$

One may refer to Tables from B13 to B18 in Supplement B for detail.

(7) When (A, O), (O, B), $A^d/(AB, A)$, $B^d/(AB, B)$ remaining, we have $\min\{\#A^d + \#(AB, A), \#(A, AB)\} = \#(A, AB)\}$, $g_1 = \min\{\#(A, B) - \#(B, A), \#(B, AB)\}$, $g_2 = \#(O, A) - \min\{\#(A, O), \#(O, A)\}$, $g_3 = 0$ and $g_4 = \min\{\#(B, O) - g_2, \#(O, A) - g_2, \#(A, B) - \#(B, A) - g_1 - g_2 - g_3\}$. There is no potential gains from three-way cycles and chains with one pair (B, A), $(AB, B) - (B, A) - (A, AB)$, $B^d - (B, A) - (A, AB) - AB^p$ because there is no pair (A, AB) left. Based on Assumption 2.2, all pairs (B, A) can be matched by two-way cycle $(A, B) - (B, A)$. There is no potential gains for $(A, O) - (O, B) - (B, A)$ by breaking up two-way cycle $(A, B) - (B, A)$. Because there is no remaining (O, A), (A, AB) and (B, AB), there is no potential gains from three-way cycles and chains starting from single donor O^d and pairs of type (AB, O). That is, $g_5 = g_6 = g_7 = g_8 = 0$. Since there is remaining pair (A, O), pair (AB, A) and single donor A^d, we can match the combinations of $(AB, A) - (A, O)$ and $A^d - (A, O)$ to any pair and match remaining pair (AB, A) with remaining pair (A, B). To take full advantage of pair (A, O), pair (AB, A) and single donor A^d, we do the same process as situation (4). The maximum number of transplants is

$$N = 2 * g_1 + 2 * g_2 + 2 * g_4 + w_1 + w_2 + w_3 + w_4 + s_1 + w_5 = \min\{N_1, N_{10}, N_{17}\}$$

One may refer to Tables B19 and B20 in Supplement B for detail.

(8) When (O, A), (B, O), $A^d/(AB, A)$, $B^d/(AB, B)$ remaining, we have $\min\{\#(A, O), \#(O, A)\} = \#(A, O)$, $\min\{\#A^d + \#(AB, A), \#(A, AB)\} = \#(A, AB)$, $g_1 = \min\{\#(A, B) - \#(B, A), \#(B, AB)\}$, $g_2 = \#(A, B) - \#(B, A) - g_1$ and $g_3 = g_4 = 0$. There is no potential gains from three-way cycles and chains with one pair (B, A), $(A, O) - (O, B) - (B, A)$,

$(AB, B)-(B, A)-(A, AB)$, $B^d-(B, A)-(A, AB)-AB^p$ because there is no pair (O, B) and pair (A, AB) left. Since all surplus pair (A, B) $(\#(A, B)-\#(B, A))$ are matched in the procedure, there is no potential gains for $(B, O)-(O, A)-(A, B)$ by breaking up a two-way cycle $(A, B)-(B, A)$. Because there is no remaining pair (O, B), pair (A, AB), pair (A, B) and pair (B, AB), there is no potential gains from three-way cycles and chains starting from single donor O^d and pairs of type (AB, O), and two-way cycles $(AB, A)-(A, B)$, $(A^d-(A, B)-Y^p$ and $(B, O)-(A, B)$. That is, $s_1=0$ and $g_5=g_6=g_7=g_8=0$. Since there is remaining pair (B, O), pair (AB, B) and single donor B^d, we can match the combinations $(AB, B)-(B, O)$ and $B^d-(B, O)$ to any pair. To take full advantage of the combinations, we do the same process as situation (4). The maximum number of transplants is

$$N=2*g_1+2*g_2+w_1+w_2+w_3+w_4+w_5=\min\{N_1, N_7, N_{17}\}$$

One may refer to Table B21 in Supplement B for detail.

(9) When (A, O), (O, B), (A, AB), $B^d/(AB, B)$ remaining, we have $g_1=\#A^d+\#(AB, A)-\min\{\#A^d+\#(AB, A), \#(A, AB)\}$, $g_2=\#(O, A)-\min\{\#(A, O), \#(O, A)\}$, $g_3=\min\{\#A^d+\#(AB, A)-g_1, \#(A, B)-\#(B, A)-g_1-g_2, \#(B, AB)-b_1\}$ and $g_4=\min\{\#(B, O)-g_2, \#(O, A)-g_2, \#(A, B)-\#(B, A)-g_1-g_2-g_3\}$. Based on Assumption 2.2, all (B, A) can be matched by two-way cycles $(A, B)-(B, A)$. There is no potential gains from three-way cycles and chains with one pair (B, A), $(A, O)-(O, B)-(B, A)$, $(AB, B)-(B, A)-(A, AB)$, $B^d-(B, A)-(A, AB)-AB^p$ by breaking up two-way cycle $(A, B)-(B, A)$. Because there is no remaining pair (O, A) and pair (B, AB), there is no potential gains from three-way cycles and chains starting from single donor O^d and pairs of type (AB, O). That is, $g_5=g_6=g_7=g_8=0$. Since there is no remaining pair (B, O), pair (AB, A) and single donor A^d, there is no beneficial from the combinations and two-way cycles $(B, O)-(A, B)$, $(AB, A)-(A, B)$ and chain $A^d-(A, B)-Y^p$. We do the same process as situation (4) with $c_2=c_3=s_1=0$. The maximum number of transplants is

$$N=2*g_1+2*g_2+2*g_3+2*g_4+w_1+w_2+w_3+w_4+w_5=\min\{N_1, N_{10}\}$$

One may refer to Tables from B22 to B25 in Supplement B for detail.

(10) When (O, A), (B, O), $A^d/(AB, A)$, (B, AB) remaining, we have $\min\{\#(A, O), \#(O, A)\}=\#(A, O)$, $\min\{\#A^d+\#(AB, A), \#(A, AB)\}=\#(A, AB)$, $g_1=\#(A, B)-\#(B, A)$ and $g_2=g_3=g_4=0$. Since all surplus pair (A, B) $(\#(A, B)-\#(B, A))$ are matched in the procedure, there is no potential gains from three-way cycle $(B, O)-(O, A)-(A, B)$, $(AB, A)-(A, B)-(B, AB)$ and chain $A^d-(A, B)-(B, AB)-AB^p$ by breaking up a two-way cycle $(A, B)-(B, A)$. Because there is no remaining pair (O, B), pair (A, B) and pair (A, AB), there is no potential gains from three-way cycles and chains starting

from single donor O^d and pairs of type (AB, O). That is, $g_5 = g_6 = g_7 = g_8 = 0$. Because all pair (A, B) are matched, we have $s_1 = 0$. We can do the same process as situation (4). The maximum number of transplants is

$$N = 2 * g_1 + w_1 + w_2 + w_3 + w_4 + w_5 = \min\{N_1, N_7\}$$

One may refer to Table B26 in Supplement B for detail.

(11) When (A, O), (O, B), $A^d/(AB, A)$, (B, AB) remaining, we have $\min\{\#(A, O), \#(O, A)\} = \#(O, A)$, $\min\{\#A^d + \#(AB, A), \#(A, AB)\} = \#(A, AB)$, $g_1 = \#(A, B) - \#(B, A)$ and $g_2 = g_3 = g_4 = 0$. Because no remaining pair (O, A), (A, B), (A, AB) is left, there is no potential gains from three-way cycles $(AB, O) - (O, A) - (A, AB)$, $(AB, O) - (O, A) - (A, B)$, $(AB, O) - (A, B) - (B, AB)$ and chains $O^d - (O, A) - (A, AB) - AB^p$, $O^d - (O, A) - (A, B) - Y^p$, $O^d - (A, B) - (B, AB) - AB^p$. That is, $g_5 = g_7 = g_8 = 0$. There is potential gains from three-way cycles $(A, O) - (O, B) - (B, A)$, $(AB, A) - (A, B) - (B, AB)$ and chains $A^d - (A, B) - (B, AB)$ by breaking two-way cycle $(A, B) - (B, A)$ because two more blood-type incompatible pairs of types (O, B) and (B, AB) can be matched in this case. Since all surplus pairs (A, B) $(\#(A, B) - \#(B, A))$ are matched in Step 1, the number of remaining (A, B) equals to $\#(B, A)$. Therefore, to take full advantage of pairs (B, A) and (A, B), we match the maximum number of $(A, O) - (O, B) - (B, A)$, $(AB, A) - (A, B) - (B, AB)$ and chain $A^d - (A, B) - (B, AB) - AB^p$ bounded by the number of remaining pairs (A, O), (O, B), $A^d/(AB, A)$, (B, AB) and (B, A). If all remaining pairs (AB, A) and single donors A^d are matched, there is potential gains from three-way cycles $(A, O) - (O, B) - (B, A)$, $(AB, A) - (A, B) - (B, AB)$ and chain $A^d - (A, B) - (B, AB) - AB^p$ by breaking two-way cycle $(AB, A) - (A, AB)$ and chain $A^d - (A, AB) - AB^p$ because one more pair can be matched in this case. Similarly, if all remaining pairs (A, O) are matched, there is potential gains from three-way cycles $(A, O) - (O, B) - (B, A)$, $(AB, A) - (A, B) - (B, AB)$ and chain $A^d - (A, B) - (B, AB) - AB^p$ by breaking two-way cycle $(A, O) - (O, A)$.

Since there is remaining pair (A, O), pair (AB, A) and single donor A^d, we can match the combinations $(AB, A) - (A, O)$ and $A^d - (A, O)$ to any pair. Since there is no reaming (A, B), there is no potential gains by matching remaining pair (AB, A) and single donor A^d with remaining pair (A, B). Therefore, the maximum number of transplants in situation (11) is

$$\begin{aligned}
N = \quad & 2 * g_1 + w_1 + w_2 + w_3 + w_4 + 2 * u_1 + v_1 + v_2 + 2 * g_6 + w_5 \\
where \quad & \\
u_1 = \quad & \min\{\#(A, O) - w_1, \#(O, B) - w_4, \#A^d + \#(AB, A) - g_1 - w_3, \\
& \#(B, AB) - g_1 - w_2, \#(B, A)\} \\
v_1 = \quad & \min\{\#A^d + \#(AB, A) - g_1 - u_1, \#(A, O) - w_1 - u_1, \#(O, B) - w_4 - u_1, \\
& \#(B, AB) - g_1 - w_2 - u_1, \#(B, A) - u_1\}
\end{aligned}$$

$$
\begin{aligned}
v_2 = \;& \min\{\#(A,O)-u_1, \#A^d+\#(AB,A)-g_1-u_1, \#(O,B)-w_4-u_1, \\
& \#(B,AB)-g_1-w_2-u_1, \#(B,A)-u_1\} \\
c_2 = \;& \min\{\#A^d+\#(AB,A)-w_3-u_1-v_2, \#(A,O)-w_1-u_1-v_1\} \\
g_6 = \;& \min\{\#O^d+\#(AB,O), \#(O,B)-w_4-u_1-v_1-v_2, \#(B,AB)-g_1-u_1-w_2 \\
& -v_1-v_2\} \\
w_5 = \;& \min\{\#O^d+\#(AB,O)+c_2-g_6, \#(O,A)+\#(O,B)+\#(O,AB)+\#(A,AB) \\
& +\#(A,B)-\#(B,A)+\#(B,AB)-2*g_1-w_1-w_2-w_3-w_4-2*u_1 \\
& -v_1-v_2-2*g_6\}
\end{aligned}
$$

The maximum number of transplants is

$$
\begin{aligned}
N = \;& 2*g_1+w_1+w_2+w_3+w_4+2*u_1+v_1+v_2+2*g_6+w_5 \\
= \;& \min\{N_1, N_3, N_7, N_8, N_9, N_{10}, N_{14}, N_{15}, N_{16}, N_{17}\}
\end{aligned}
$$

One may refer to Tables from B27 to B30 in Supplement B for detail.

(12) When (O,A), (O,B), $A^d/(AB,A)$, (B,AB) remaining, we have $\min\{\#(A,O), \#(O,A)\} = \#(A,O)$, $\min\{\#A^d+\#(AB,A), \#(A,AB)\} = \#(A,AB)$, $g_1 = \#(A,B)-\#(B,A)$ and $g_2 = g_3 = g_4 = 0$. Because no remaining pair (A,B), (A,AB) is left, there is no potential gains from three-way cycles $(AB,O)-(O,A)-(A,AB)$, $(AB,O)-(O,A)-(A,B)$, $(AB,O)-(A,B)-(B,AB)$ and chains $O^d-(O,A)-(A,AB)-AB^p$, $O^d-(O,A)-(A,B)-Y^p$, $O^d-(A,B)-(B,AB)-AB^p$. That is, $g_5 = g_7 = g_8 = 0$. Because all pairs (A,O) and pairs(B,O) are matched, there is no potential gains from the combinations. Since there is no reaming (A,B), there is no potential gains by matching remaining pair (AB,A) and single donor A^d with remaining pair (A,B). There is potential gains from three-way cycle $(A,O)-(O,B)-(B,A)$, three-way cycle $(AB,A)-(A,B)-(B,AB)$ and chain $A^d-(A,B)-(B,AB)-AB^p$ by breaking two-way cycle $(A,O)-(O,A)$ because one more pair can be matched in this case. That is, $v_2 \neq 0$. We can do the same process in situation (11). The maximum number of transplants is

$$
N = 2*g_1+w_1+w_2+w_3+w_4+v_2+2*g_6+w_5 = \min\{N_1, N_7, N_{14}, N_{15}, N_{16}, N_{17}\}
$$

One may refer to Table B31 in Supplement B for detail.

(13) When (A,O), (O,B), (A,AB), (B,AB) remaining, we have $g_1 = \#A^d+\#(AB,A)-\min\{\#A^d+\#(AB,A), \#(A,AB)\}$, $g_2 = \#(O,A)-\min\{\#(A,O), \#(O,A)\}$, $g_3 = \min\{\#A^d+\#(AB,A)-g_1, \#(A,B)-\#(B,A)-g_1-g_2\}$ and $g_4 = \min\{\#(O,A)-g_2, \#(A,B)-\#(B,A)-g_1-g_2-g_3\}$. Because no remaining pair (O,A) is left, there is no potential gains from three-way cycles $(AB,O)-(O,A)-(A,AB)$, $(AB,O)-(O,A)-(A,B)$ and chains $O^d-(O,A)-(A,AB)-AB^p$, $O^d-(O,A)-(A,B)-Y^p$. That is, $g_5 = g_7 = 0$. Because all pairs (AB,A), (AB,B) and single donors A^d, B^d are matched, there is no potential gains from the combinations. Since no reaming pair (B,O), pair

(AB,B) and single donor A^d is left, there is no potential gains by matching remaining pair (AB,A), pair (B,O) and single donor A^d with remaining pair (A,B). There is potential gains from three-way cycle $(A,O)-(O,B)-(B,A)$, $(AB,A)-(A,B)-(B,AB)$ and chain $A^d-(A,B)-(B,AB)-AB^p$ by breaking two-way cycle $(AB,A)-(A,AB)$ and chain $A^d-(A,AB)-AB^p$ because one more pair can be matched in this case. That is, $v_1 \neq 0$. To take full advantage of three-way cycles and chains, we first match $(A,O)-(O,B)-(B,A)$, $(AB,A)-(A,B)-(B,AB)$ and chain $A^d-(A,B)-(B,AB)-AB^p$, and match three-way cycles $(AB,O)-(O,B)-(B,AB)$, $(AB,O)-(A,B)-(B,AB)$ and chains $O^d-(O,B)-(B,AB)-AB^p$, $O^d-(A,B)-(B,AB)-AB^p$ if any. Then, we match remaining pairs with pair (AB,O) and single donor O^d. Therefore, the maximum number of transplants in situation (13) is

$$\begin{aligned}
N = \quad & 2*g_1+2*g_2+2*g_3+2*g_4+w_1+w_2+w_3+w_4+v_1+2*g_6+2*g_8+w_5 \\
where \quad & \\
g_8 = \quad & \min\{\#O^d+\#(AB,O)-g_6, \#(A,B)-\#(B,A)-g_1-g_2-g_3-g_4, \\
& \#(B,AB)-g_1-g_3-w_2-v_1-g_6\} \\
w_5 = \quad & \min\{\#O^d+\#(AB,O)-g_6-g_8, \#(O,A)+\#(O,B)+\#(O,AB)+\#(A,AB) \\
& +\#(A,B)-\#(B,A)+\#(B,AB)-2*g_1-2*g_2-2*g_3-2*g_4 \\
& -w_1-w_2-w_3-w_4-v_1-2*g_6-2*g_8\}
\end{aligned}$$

The maximum number of transplants is

$$\begin{aligned}
N = \quad & 2*g_1+2*g_2+2*g_3+2*g_4+w_1+w_2+w_3+w_4+v_1+2*g_6+2*g_8+w_5 \\
= \quad & \min\{N_1, N_3, N_8, N_9, N_{10}, N_{15}\}
\end{aligned}$$

One may refer to Tables from B32 to B37 in Supplement B for detail.

(14) When (O,A), (B,O), (A,AB), $B^d/(AB,B)$ remaining, we have $\min\{\#(A,O), \#(O,A)\} = \#(A,O)$, $g_1 = \#A^d+\#(AB,A)-\min\{\#A^d+\#(AB,A), \#(A,AB)\}$, $g_2 = \#(A,B)-\#(B,A)-g_1$ and $g_3 = g_4 = 0$. Because no remaining pair (O,B), (A,B), (B,AB) is left, there is no potential gains from three-way cycles $(AB,O)-(O,B)-(B,AB)$, $(AB,O)-(O,A)-(A,B)$, $(AB,O)-(A,B)-(B,AB)$ and chains $O^d-(O,B)-(B,AB)-AB^p$, $O^d-(O,A)-(A,B)-Y^p$, $O^d-(A,B)-(B,AB)-AB^p$. That is, $g_6 = g_7 = g_8 = 0$. Since no remaining pair $A^d/(AB,A)$, there is no potential gains from two-way cycle $(AB,A)-(A,B)$ and chain $A^d-(A,B)-Y^p$. There is potential gains from three-way cycle $(B,O)-(O,A)-(A,B)$, $(AB,B)-(B,A)-(A,AB)$ and chain $B^d-(B,A)-(A,AB)-AB^p$ by breaking two-way cycle $(A,B)-(B,A)$ because two more blood-type incompatible pairs of types (O,A) and (A,AB) can be matched in this case. Since all surplus pairs (A,B) $(\#(A,B)-\#(B,A))$ are matched in Step 1, the number of remaining (A,B) equals to $\#(B,A)$. Therefore, we take full advantage of (B,A), (A,B) and match the maximum number of $(B,O)-(O,A)-(A,B)$, $(AB,B)-(B,A)-(A,AB)$

and chain $B^d-(B,A)-(A,AB)-AB^p$ bounded by the number of remaining pairs (O,A), (B,O), (A,AB), $B^d/(AB,B)$ and (B,A). If all remaining pairs (AB,B) and single donors B^d are matched, there is potential gains from three-way cycles $(B,O)-(O,A)-(A,B)$, $(AB,B)-(B,A)-(A,AB)$ and chain $B^d-(B,A)-(A,AB)-AB^p$ by breaking two-way cycle $(AB,B)-(B,AB)$ and chain $B^d-(B,AB)-AB^p$ because one more pair can be matched in this case. Similarly, if all remaining pairs (B,O) are matched, there is potential gains from three-way cycles $(B,O)-(O,A)-(A,B)$, $(AB,B)-(B,A)-(A,AB)$ and chain $B^d-(B,A)-(A,AB)-AB^p$ by breaking two-way cycle $(B,O)-(O,B)$.

Since there is remaining pair (B,O), pair (AB,B) and single donor B^d, we can match the combinations of $(AB,B)-(B,O)$ and $B^d-(B,O)$ to any pair. Therefore, the maximum number of transplants in situation (14) is

$$\begin{aligned}
N = \quad & 2*g_1+2*g_2+w_1+w_2+w_3+w_4+2*u_2+v_3+v_4+2*g_5+w_5 \\
where \quad & \\
u_2 = \quad & \min\{\#(B,O)-g_2-w_4, \#(O,A)-g_2-w_1, \#(A,AB)-w_3, \\
& \#B^d+\#(AB,B)-w_2, \#(B,A)\} \\
v_3 = \quad & \min\{\#B^d+\#(AB,B)-u_2, \#(B,O)-g_2-w_4-u_2, \#(O,A)-g_2-w_1-u_2, \\
& \#(A,AB)-w_3-u_2, \#(B,A)-u_2\} \\
v_4 = \quad & \min\{\#(B,O)-g_2-u_2, \#B^d+\#(AB,B)-w_2-u_2, \#(O,A)-g_2-w_1-u_2, \\
& \#(A,AB)-w_3-u_2, \#(B,A)-u_2\} \\
c_3 = \quad & \min\{\#B^d+\#(AB,B)-w_2-u_2-v_4, \#(B,O)-w_4-u_2-v_3\} \\
g_5 = \quad & \min\{\#O^d+\#(AB,O), \#(O,A)-g_2-w_1-u_2-v_3-v_4, \#(A,AB)-w_3-u_2 \\
& -v_3-v_4\} \\
w_5 = \quad & \min\{\#O^d+\#(AB,O)+c_3-g_5, \#(O,A)+\#(O,B)+\#(O,AB)+\#(A,AB) \\
& +\#(A,B)-\#(B,A)+\#(B,AB)-2*g_1-2*g_2-w_1-w_2-w_3-w_4 \\
& -2*u_2-v_3-v_4-2*g_5\}
\end{aligned}$$

The maximum number of transplants is

$$\begin{aligned}
N = \quad & 2*g_1+2*g_2+w_1+w_2+w_3+w_4+2*u_2+v_3+v_4+2*g_5+w_5 \\
= \quad & \min\{N_1, N_3, N_4, N_5, N_7, N_{10}, N_{11}, N_{13}, N_{15}, N_{17}\}
\end{aligned}$$

One may refer to Tables from B38 to B41 in Supplement B for detail.

(15) When (O,A), (B,O), (A,AB), (B,AB) remaining, we have $\min\{\#(A,O), \#(O,A)\} = \#(A,O)$, $g_1 = \#A^d+\#(AB,A)-\min\{\#A^d+\#(AB,A), \#(A,AB)\}$, $g_2 = \#(A,B)-\#(B,A)-g_1$ and $g_3 = g_4 = 0$. Because no remaining pair (O,B), (A,B) is left, there is no potential gains from three-way cycles $(AB,O)-(O,B)-(B,AB)$, $(AB,O)-(O,A)-(A,B)$, $(AB,O)-(A,B)-(B,AB)$ and chains $O^d-(O,B)-(B,AB)-AB^p$, $O^d-(O,A)-(A,B)-Y^p$, $O^d-(A,B)-(B,AB)-AB^p$. That is, $g_6 = g_7 = g_8 = 0$. Because all pairs (AB,A), (AB,B) and single donors A^d, B^d are matched, there is no potential gains

from the combinations. Since no remaining pair $A^d/(AB, A)$, there is no potential gains from two-way cycle $(AB, A) - (A, B)$ and chain $A^d - (A, B) - Y^p$. There is potential gains from three-way cycle $(B, O) - (O, A) - (A, B)$, $(AB, B) - (B, A) - (A, AB)$ and chain $B^d - (B, A) - (A, AB) - AB^p$ by breaking two-way cycle $(AB, B) - (B, AB)$ and chain $B^d - (B, AB) - AB^p$ because one more pair can be matched in this case. That is, $v_3 \neq 0$. We can do the same process in situation (14). The maximum number of transplants is

$$N = 2 * g_1 + 2 * g_2 + w_1 + w_2 + w_3 + w_4 + v_3 + 2 * g_5 + w_5 = \min\{N_1, N_3, N_4, N_5, N_7, N_{15}\}$$

One may refer to Table B42 in Supplement B for detail.

(16) When (O, A), (O, B), (A, AB), $B^d/(AB, B)$ remaining, we have $\min\{\#(A, O), \#(O, A)\} = \#(A, O)$, $g_1 = \#A^d + \#(AB, A) - \min\{\#A^d + \#(AB, A), \#(A, AB)\}$, $g_2 = \min\{\#(B, O), \#(A, B) - \#(B, A) - g_1\}$, $g_3 = \min\{\#A^d + \#(AB, A) - g_1, \#(A, B) - \#(B, A) - g_1 - g_2, \#(B, AB) - b_1\}$ and $g_4 = 0$. Because no remaining pair (B, AB) is left, there is no potential gains from three-way cycles $(AB, O) - (O, B) - (B, AB)$, $(AB, O) - (A, B) - (B, AB)$ and chains $O^d - (O, B) - (B, AB) - AB^p$, $O^d - (A, B) - (B, AB) - AB^p$. That is, $g_6 = g_8 = 0$. Because all pairs (B, O) and (A, O) are matched, there is no potential gains from the combinations. Since all pair $A^d/(AB, A)$ and (B, O) are matched, there is no potential gains from two-way cycles $(AB, A) - (A, B)$, $(B, O) - (A, B)$ and chain $A^d - (A, B) - Y^p$. There is potential gains from three-way cycle $(B, O) - (O, A) - (A, B)$, $(AB, B) - (B, A) - (A, AB)$ and chain $B^d - (B, A) - (A, AB) - AB^p$ by breaking two-way cycle $(B, O) - (O, B)$ because one more pair can be matched in this case. That is, $v_4 \neq 0$. To take full advantage of three-way cycles and chains, we first match $(B, O) - (O, A) - (A, B)$, $(AB, B) - (B, A) - (A, AB)$ and chain $B^d - (B, A) - (A, AB) - AB^p$, and match three-way cycles $(AB, O) - (O, A) - (A, AB)$, $(AB, O) - (O, A) - (A, B)$ and chains $O^d - (O, A) - (A, AB) - AB^p$, $O^d - (O, A) - (A, B) - Y^p$ if any. Then, we match remaining pairs with pair (AB, O) and single donor O^d. Therefore, the maximum number of transplants in situation (16) is

$$\begin{aligned}
N = \quad & 2 * g_1 + 2 * g_2 + 2 * g_3 + w_1 + w_2 + w_3 + w_4 + v_4 + 2 * g_5 + 2 * g_7 + w_5 \\
where \quad & \\
g_7 = \quad & \min\{\#O^d + \#(AB, O) - g_5, \#(O, A) - g_2 - w_1 - v_4 - g_5, \\
& \#(A, B) - \#(B, A) - g_1 - g_2 - g_3\} \\
w_5 = \quad & \min\{\#O^d + \#(AB, O) - g_5 - g_7, \#(O, A) + \#(O, B) + \#(O, AB) + \#(A, AB) \\
& +\#(A, B) - \#(B, A) + \#(B, AB) - 2 * g_1 - 2 * g_2 - 2 * g_3 - w_1 - w_2 - w_3 \\
& -w_4 - v_4 - 2 * g_5 - 2 * g_7\}
\end{aligned}$$

The maximum number of transplants is

$$\begin{aligned}
N = \quad & 2 * g_1 + 2 * g_2 + 2 * g_3 + w_1 + w_2 + w_3 + w_4 + v_4 + 2 * g_5 + 2 * g_7 + w_5 \\
= \quad & \min\{N_1, N_{10}, N_{11}, N_{13}, N_{15}, N_{17}\}
\end{aligned}$$

One may refer to Tables from B43 to B46 in Supplement B for detail. Combining cases (1) to (16), we have proved that the maximum number of transplants for blood-type incompatible paired patients of types (O, A), (O, B), (O, AB), (A, AB), (B, AB), (A, B) is

$$N = \min\{N_1, N_2, N_3, N_4, N_5, N_6, N_7, N_8, N_9, N_{10}, N_{11}, N_{12}, N_{13}, N_{14}, N_{15}, N_{16}, N_{17}\}$$

We now prove that the sequential mechanism is 3-efficient which achieves the maximum number of transplants in the pool by exploring every possible route. Figures 3 and 4 show all possible three-way cycles and chains can occur in the mechanism.

Since for every one/two-way chains, we can always find a compatible patient on the waiting list in the mechanism, the number of transplantations for patients on the waiting list through one/two/three-way chains in the mechanism equals to $\#A^d + \#B^d + \#AB^d + \#O^d$. Based on Assumption 2.3, all pairs of type $(A, A)^i$, $(B, B)^i$, $(O, O)^i$, $(AB, AB)^i$ can be matched through two-way in Step 1. Based on Assumption 2.3, all pairs of type $(A, O)^i$, $(B, O)^i$, $(AB, O)^i$, $(AB, A)^i$, $(AB, B)^i$ can be matched through two-way and three-way cycles from Step 2 to Step 7 in the mechanism. All compatible pairs $(A, O)^c$, $(B, O)^c$, $(AB, O)^c$, $(AB, A)^c$, $(AB, B)^c$, $(A, A)^c$, $(B, B)^c$, $(O, O)^c$, $(AB, AB)^c$ can be matched either through two-way/three-way cycles or doing transplantations with their own donors. Moreover, under Assumption 2.2, all pairs of type (B, A) can be matched through two-way cycle $(A, B) - (B, A)$ or three-way cycle $(AB, B) - (B, A) - (A, AB)$, $(A, O) - (O, B) - (B, A)$ or three-way chain $B^d - (B, A) - (A, AB) - AB^p$ in Step 3 and Step 4. Hence, the number of transplantations for compatible pairs, blood-type compatible pairs and pairs of type (B, A) in the mechanism is

$$\begin{aligned}&\#(A, O) + \#(B, O) + \#(AB, O) + \#(AB, A) + \#(AB, B)\\&+\#(B, A) + \#(A, A) + \#(B, B) + \#(AB, AB) + \#(O, O)\end{aligned}$$

Next, we prove that the maximum number of transplants for blood-type incompatible pairs of types (O, A), (O, B), (O, AB), (A, AB), (B, AB), (A, B) can be achieved in the mechanism.

Denote X_2 as the number of blood-type incompatible paired patients from pairs of types (O, A), (O, B), (O, AB), (A, AB), (B, AB), (A, B) involved in Step 2 so that

$$\begin{aligned}X_2 = \;& 2 * b_1 + 2 * b_2 + 2 * b_3 + 2 * b_{21}\\ \textit{where}\;&\\ b_1 = \;& \min\{\#A^d + \#(AB, A) - \min\{\#A^d + \#(AB, A), \#(A, AB)\},\\ & \#(A, B) - \#(B, A), \#(B, AB)\}\\ b_2 = \;& \min\{\#(B, O), \#(O, A) - \min\{\#(A, O), \#(O, A)\}, \#(A, B) - \#(B, A) - b_1\}\\ b_3 = \;& \min\{\#A^d + \#(AB, A) - b_1, \#(A, B) - \#(B, A) - b_1 - b_2, \#(B, AB) - b_1\}\\ b_{21} = \;& \min\{\#(B, O) - b_2, \#(O, A) - b_2, \#(A, B) - \#(B, A) - b_1 - b_2 - b_3\}\end{aligned}$$

Denote X_3 as the number of blood-type incompatible paired patients from pairs of types (O,A), (O,B), (O,AB), (A,AB), (B,AB), (A,B) involved in Step 3 so that

$$\begin{aligned}
X_3 = \quad & 2*e_1 + 2*e_2 + 2*f_2 + 2*f_3 + 2*f_4 + 2*f_5 \\
where & \\
e_1 = \quad & \min\{\#(A,O) - a_1, \#(B,A), \#A^d + \#(AB,A) - b_1 - b_3 - a_1, \\
& \#(O,B) - a_4, \#(B,AB) - b_1 - b_3 - a_2\} \\
e_2 = \quad & \min\{\#B^d + \#(AB,B) - a_2, \#(A,AB) - a_3, \#(B,O) - b_2 - a_4 - b_{21}, \\
& \#(O,A) - b_2 - b_{21} - a_1, \#(B,A)\} \\
f_2 = \quad & \min\{\#(A,O) - a_1 - e_1, \#(B,A) - e_1 - e_2, \#A^d + \#(AB,A) - b_1 - b_3 - e_1, \\
& \#(O,B) - a_4 - e_1, \#(B,AB) - b_1 - b_3 - a_2 - e_1\} \\
f_3 = \quad & \min\{\#B^d + \#(AB,B) - a_2 - e_2, \#(A,AB) - a_3 - e_2, \#(B,O) - b_2 - e_2 - b_{21}, \\
& \#(O,A) - b_2 - b_{21} - a_1 - e_2, \#(B,A) - e_1 - e_2\} \\
f_4 = \quad & \min\{\#(A,O) - e_1, \#(B,A) - e_1 - e_2, \#A^d + \#(AB,A) - b_1 - b_3 - e_1 - a_3, \\
& \#(O,B) - a_4 - e_1, \#(B,AB) - b_1 - b_3 - a_2 - e_1\} \\
f_5 = \quad & \min\{\#B^d + \#(AB,B) - e_2, \#(A,AB) - a_3 - e_2, \#(B,O) - b_2 - e_2 - a_4 - b_{21}, \\
& \#(O,A) - b_2 - b_{21} - a_1 - e_2, \#(B,A) - e_1 - e_2\} \\
where & \\
a_1 = \quad & \min\{\#(A,O), \#(O,A) - b_2 - b_{21}\} \\
a_2 = \quad & \min\{\#B^d + \#(AB,B), \#(B,AB) - b_1 - b_3\} \\
a_3 = \quad & \min\{\#A^d + \#(AB,A) - b_1 - b_3, \#(A,AB)\} \\
a_4 = \quad & \min\{\#(B,O) - b_2 - b_{21}, \#(O,B)\}
\end{aligned}$$

Denote X_4 as the number of blood-type incompatible paired patients from pairs of types (O,A), (O,B), (O,AB), (A,AB), (B,AB), (A,B) involved in Step 4 so that

$$\begin{aligned}
X_4 = \quad & \#(B,A) + p_1 + p_2 + p_3 + p_4 + a_8 \\
where & \\
p_1 = \quad & a_1 - f_4 \\
p_2 = \quad & a_2 - f_5 \\
p_3 = \quad & a_3 - f_2 \\
p_4 = \quad & a_4 - f_3 \\
a_8 = \quad & \min\{\#A^d + \#(AB,A) - b_1 - b_3 - e_1 - f_2 - f_4 - p_3 - c_2 + \#(B,O) - b_2 - b_{21} \\
& -e_2 - f_3 - f_5 - p_4 - c_3, \#(A,B) - \#(B,A) - b_1 - b_2 - b_3 - b_{21}\} \\
where & \\
c_2 = \quad & \min\{\#A^d + \#(AB,A) - b_1 - b_3 - e_1 - f_2 - f_4 - p_3, \#(A,O) - e_1 - f_2 - f_4 - p_1\} \\
c_3 = \quad & \min\{\#(B,O) - b_2 - b_{21} - e_2 - f_3 - f_5 - p_4, \#B^d + \#(AB,B) - e_2 - f_3 - f_5 - p_2\}
\end{aligned}$$

Denote X_5 as the number of blood-type incompatible paired patients from pairs of types

(O,A), (O,B), (O,AB), (A,AB), (B,AB), (A,B) involved in Step 5 so that

$$\begin{aligned}
X_5 = \quad & 2*b_4 + 2*b_5 + 2*b_6 + 2*b_7 \\
\textit{where} \quad & \\
b_4 = \quad & \min\{\#O^d + \#(AB,O), \#(O,A) - b_2 - b_{21} - e_2 - f_3 - f_5 - p_1, \\
& \#(A,AB) - e_2 - f_3 - f_5 - p_3\} \\
b_5 = \quad & \min\{\#O^d + \#(AB,O) - b_4, \#(O,B) - e_1 - f_2 - f_4 - p_4, \\
& \#(B,AB) - b_1 - b_3 - e_1 - f_2 - f_4 - p_2\} \\
b_6 = \quad & \min\{\#O^d + \#(AB,O) - b_4 - b_5, \#(O,A) - b_2 - b_{21} - e_2 - f_3 - f_5 - p_1 - b_4, \\
& \#(A,B) - \#(B,A) - b_1 - b_2 - b_3 - b_{21} - a_8\} \\
b_7 = \quad & \min\{\#O^d + \#(AB,O) - b_4 - b_5 - b_6, \#(A,B) - \#(B,A) - b_1 - b_2 \\
& -b_3 - b_{21} - a_8 - b_6, \#(B,AB) - b_1 - b_3 - e_1 - f_2 - f_4 - p_2 - b_5\}
\end{aligned}$$

Denote X_6 as the number of blood-type incompatible paired patients from pairs of types (O,A), (O,B), (O,AB), (A,AB), (B,AB), (A,B) involved in Step 6 so that

$$\begin{aligned}
X_6 = a_9 = \quad & \min\{\#O^d + \#(AB,O) - b_4 - b_5 - b_6 - b_7 + c_2 + c_3, \#(O,A) \\
& +\#(O,B) + \#(O,AB) + \#(A,AB) + \#(A,B) + \#(B,AB) - 2*b_1 \\
& -2*b_2 - 2*b_3 - 2*b_{21} - 2*e_1 - 2*e_2 - 2*f_2 - 2*f_3 - 2*f_4 \\
& -2*f_5 - p_1 - p_2 - p_3 - p_4 - a_8 - 2*b_4 - 2*b_5 - 2*b_6 - 2*b_7\}
\end{aligned}$$

Therefore, the total number of transplants for paired patients from pairs of types (O,A), (O,B), (O,AB), (A,AB), (B,AB), (A,B) in the mechanism is $X = X_2+X_3+X_4+X_5+X_6$. The equation can be rewritten as follows:

$$X = \min\{N_1, N_2, N_3, N_4, N_5, N_6, N_7, N_8, N_9, N_{10}, N_{11}, N_{12}, N_{13}, N_{14}, N_{15}, N_{16}, N_{17}\}$$

One may refer to Tables from B47 to B82 in Supplement B for detail. Therefore, the total number of transplants can be achieved in the mechanism is that

$$\begin{aligned}
& \#(A,O) + \#(B,O) + \#(AB,O) + \#(AB,A) + \#(AB,B) \\
& +\#(B,A) + \#(A,A) + \#(B,B) + \#(AB,AB) + \#(O,O) \\
& +\#A^d + \#B^d + \#AB^d + \#O^d \\
& + \min\{N_1, N_2, N_3, N_4, N_5, N_6, N_7, N_8, N_9, N_{10}, N_{11}, N_{12}, N_{13}, N_{14}, N_{15}, N_{16}, N_{17}\}
\end{aligned}$$

Therefore, we proved that every matching produced by the mechanism achieves the maximum number of transplants in the pool and hence the mechanism is 3-efficient.

□

Proof of Lemma 3.6: Consider any given 4-efficient matching μ as stated in the lemma. If μ consists only of cycles with no more than two blood-type compatible pairs and chains with no more than one blood-type compatible pair, we are done. Suppose to the contrary that μ contains a cycle with more than two blood-type compatible pairs or a chain with more than one blood-type compatible pair. We only need to consider the case of four-way cycles or chains, as the case of three-way cycles or chains is reduced to that of Lemma 3.4.

A four-way cycle may consist of either four or three blood-type compatible pairs and a four-way chain may consist of either three or two blood-type compatible pairs. Such a cycle or chain can be decomposed into small cycles or chains as follows:

a. In a four-way cycle with four blood-compatible pairs, we can decompose it into four single blood-compatible pairs.

b. In a four-way chain with three blood-type compatible pairs, we can decompose it into three single blood-compatible pairs and a one-way chain in which the single donor gives its kidney to a patient on the waiting list.

c. A four-way cycle with three blood-compatible pairs can be split into a three-way cycle with two blood-compatible pairs and one blood-compatible pair.

Let $(P_1, D_1) - (P_2, D_2) - (P_3, D_3) - (P_4, D_4)$ denote a four-way cycle with three blood-compatible pairs. If there exists pair (X, X) from types (A, A), (B, B), (O, O) and $AB, AB)$, then the four-way cycle can be separated into a three-way cycle and a blood-type compatible pair (X, X). If there exists pair of type (AB, O) in the cycle, then the four-way cycle can be separated into a three-way cycle and a blood-type compatible pair because pair (AB, O) is compatible with any pair by Assumption 2.1. If there exists pairs of types (AB, A) and (A, O) in the cycle, then the cycle can be separated into a three-way starting with $(AB, A) - (A, O)$ and a blood-type compatible pair because the combination of $(AB, A) - (A, O)$ is compatible with any pair. Similar to pairs of types (AB, B) and (B, O).

Now we consider other four-way cycles without pairs (AB, O), (A, A), (B, B), (O, O), (AB, AB) and the combinations of $(AB, A) - (A, O)$ and $(AB, B) - (B, O)$. In a four-way cycle $(A, O) - (P_2, D_2) - (P_3, D_3) - (P_4, D_4)$, two cases occur. In the first case, there exists one (B, O) and hence there is no (AB, O), (AB, A), (AB, B), (A, A), (B, B), (O, O), (AB, AB) in the cycle. Therefore, the cycle have either two pairs (A, O) or two pairs (B, O), then the four-way can be separated into a three-way cycle and one blood-type compatible pair. In the second case, there exists pair (AB, B) and hence there is no (AB, O), (AB, A), (B, O), (A, A), (B, B), (O, O), (AB, AB) in the cycle. Therefore, the cycle have either two pairs (A, O) or two pairs (AB, B), then the four-way can be separated into a three-way cycle and one blood-type compatible pair. The similar proof can be applied to four-way cycle $(B, O) - (P_2, D_2) - (P_3, D_3) - (P_4, D_4)$, $(AB, A) - (P_2, D_2) - (P_3, D_3) - (P_4, D_4)$ and $(AB, B) - (P_2, D_2) - (P_3, D_3) - (P_4, D_4)$.

d. A four-way chain with two blood-type compatible pair can be decomposed into two-way or three-way cycles with at most two blood-compatible pairs or chains with at most one blood-compatible pair.

Let $X^d - (P_1, D_1) - (P_2, D_2) - (P_3, D_3) - Y^p$ denote a four-way chain with two blood-compatible pairs. Firstly, from the previous proof, if there exists either pair (AB, O), or pairs (AB, A) and (A, O), or pairs (AB, B) and (B, O), the chain can be separated into a three-way cycle and a one-way chain such that the single donor donates its kidney to a patient on the waiting list. Secondly, if there exist a single donor O^d in the chain, the four-way chain can be separated into a three-way chain and a blood-type compatible pair because single donor O^d is compatible with any patient by Assumption 2.1. If there exists single donor A^d and pair (A, O) in the chain, then the chain can be separated into a three-way starting with $A^d - (A, O)$ and a blood-type compatible pair because the combination of $A^d - (A, O)$ is compatible with any pair. Similar to single donor B^d and pair (B, O). Thirdly, if there exists pair (X, X) from types (A, A), (B, B), (O, O) and (AB, AB), then the four-way chain can be separated into a three-way chain and a blood-type compatible pair (X, X). Fourthly, we consider the situation that there exists a blood-type compatible pair (AB, D) in the four-way chain. If pair (P_1, D_1) is type (AB, D), then the chain can be divided into a three-way cycle $(P_1, D_1) - (P_2, D_2) - (P_3, D_3)$ and a one-way chain $X^d - Y^p$ because patient of type AB can receive kidney from any donor by Assumption 2.1. Similarly, if pair (P_2, D_2) is type (AB, D), then the chain can be divided into either a three-way chain $X^d - (P_1, D_1) - (P_2, D_2) - Y^p$ and a blood-type compatible pair (P_3, D_3) if pair (P_1, D_1) is blood-type incompatible pair; or a two-way cycle $(P_2, D_2) - (P_3, D_3)$ and two-way chain $X^d - (P_1, D_1) - Y^p$ if pair (P_3, D_3) is blood-type incompatible pair. If pair (P_3, D_3) is type (AB, D), the chain can be separated into a three-way chain $X^d - (P_1, D_1) - (P_2, D_2) - Y^p$ and a blood-type compatible pair (P_3, D_3).

Now we consider other four-way chains without single donor O^d, pairs (AB, O), (A, A), (B, B), (O, O), (AB, AB), (AB, A), (AB, B), (A, A), (B, B), (O, O), $AB, AB)$ and the combinations of $A^d/(AB, A) - (A, O)$, $B^d/(AB, B) - (B, O)$. Hence, a four-way chain $A^d - (P_2, D_2) - (P_3, D_3) - (P_4, D_4)$ has two pairs of type (B, O) and can be separated into a three-way chain and a blood-type compatible pair (B, O). A four-way chain $B^d - (P_2, D_2) - (P_3, D_3) - (P_4, D_4)$ has two pairs of type (A, O) and can be separated into a three-way chain and a blood-type compatible pair (A, O). In a four-way chain $AB^d - (P_2, D_2) - (P_3, D_3) - (P_4, D_4)$, we have pair (P_2, D_2) of type (AB, D_2) because donor AB^d can only donate to patient of type AB. Hence, the chain can be separated into a three-way cycle starting with (P_2, D_2) and a one-way chain.

Therefore, every cycle and chain under consideration can be decomposed into a smaller cycle or chain or a blood-type compatible pair. Then, we will show that the all pairs which are decomposed from cycles and chains can be matched. Because a blood-type compatible and tissue-type compatible pair can directly do the transplant, all blood-type compatible and tissue-type compatible pairs can do the transplants separately. Let $\mathcal{D}$ be the set of all blood-type compatible but tissue-type incompatible pairs in a cycle or chain under consideration. Let $(X,Y)^i$ present the type of a blood-type compatible but tissue-type incompatible pair. If there exists two or more pairs of type $(X,Y)^i$, we can have a two-way cycle among them $(X,Y)^i-(X,Y)^i$. Therefore, at most one pair of type $(X,Y)^i$ left after the process. By Assumption 2.3, there exists at least one blood-type and tissue-type compatible pair of type $(X,Y)^c$. If the compatible pair $(X,Y)^c$ does not involve in any cycle or chain, then we can match the remaining pair $(X,Y)^i$ with pair $(X,Y)^c$. Otherwise, the compatible pair $(X,Y)^c$ involves in a cycle consisting of no more than two blood-type compatible pairs or a chain consisting of no more than one blood-type compatible pair. Then we can use pair $(X,Y)^i$ instead of $(X,Y)^c$ based on Assumption 2.1 and pair $(X,Y)^c$ do transplant directly. Therefore, all remaining pairs of type $(X,Y)^i$ can be matched.

□

Proof of Proposition 3.7: The proof is similar to one in the case of three-way cycles and chains. Let N be the maximum number of transplants for blood-type incompatible paired patients of types (O,A), (O,B), (O,AB), (A,AB), (B,AB), (A,B) when four-way cycles and chains are considered. We will prove that

$$N = \min\{N_1,N_2,N_3,N_4,N_5,N_6,N_7,N_8,N_9,N_{10},N_{11}\}$$

Sixteen situations are considered. We only discuss the potential gains from four-way cycles and chains because the analysis of cycles and chains other than four-way exchange in each situation is similar to one in the case of three-way cycles and chains.

(1) When (O,A), (O,B), (A,AB), (B,AB) remaining, we have $\min\{\#(A,O),\#(O,A)\} = \#(A,O)$, $g_1 = \#A^d+\#(AB,A)-\min\{\#A^d+\#(AB,A),\#(A,AB)\}$, $g_2 = \min\{\#(B,O), \#(A,B)-\#(B,A)-g_1\}$, $g_3 = \min\{\#A^d+\#(AB,A)-g_1, \#(A,B)-\#(B,A)-g_1-g_2\}$ and $g_4 = 0$. There is potential gains from four-way cycle $(AB,O)-(O,A)-(A,B)-(B,AB)$ and chain $O^d-(O,A)-(A,B)-(B,AB)-AB^p$ if any pair (A,B) remains. Because all remaining blood type compatible pairs are matched, there is no potential benefit from the combinations $(AB,A)-(A,O)$, $(AB,B)-(B,O)$, $(AB,A)-(A,B)-(B,O)$ and $A^d-(A,B)-(B,O)$. Therefore, we first take full of advantage of four-way cycle$(AB,O)-(O,A)-(A,B)-(B,AB)$ and chain $O^d-(O,A)-(A,B)-(B,AB)-AB^p$ and the maximum number of transplants in situation (1) is:

$$\begin{aligned}
N = \;& 2*g_1+2*g_2+2*g_3+w_1+w_2+w_3+w_4+3*d_1+2*g_5+2*g_6+2*g_7 \\
& +2*g_8+w_5 \\
\textit{where} & \\
d_1 = \;& \min\{\#O^d+\#(AB,O),\#(O,A)-g_2-w_1,\#(A,B)-\#(B,A)-g_1-g_2-g_3, \\
& \#(B,AB)-g_1-g_3-w_2\}
\end{aligned}$$

$$
\begin{aligned}
g_5 = \;& \min\{\#O^d + \#(AB,O) - d_1, \#(O,A) - d_1 - g_2 - w_1, \#(A,AB) - w_3\} \\
g_6 = \;& \min\{\#O^d + \#(AB,O) - d_1 - g_5, \#(O,B) - w_4, \#(B,AB) - g_1 - g_3 - w_2 - d_1\} \\
g_7 = \;& \min\{\#O^d + \#(AB,O) - d_1 - g_5 - g_6, \#(O,A) - d_1 - g_2 - w_1 - g_5, \\
& \#(A,B) - \#(B,A) - g_1 - g_2 - g_3 - d_1\} \\
g_8 = \;& \min\{\#O^d + \#(AB,O) - d_1 - g_5 - g_6 - g_7, \#(A,B) - \#(B,A) - g_1 - g_2 - g_3 \\
& -d_1 - g_7, \#(B,AB) - g_1 - g_3 - w_2 - g_6 - d_1\} \\
w_5 = \;& \min\{\#O^d + \#(AB,O) - d_1 - g_5 - g_6 - g_7 - g_8, \#(O,A) + \#(O,B) + \#(O,AB) \\
& +\#(A,AB) + \#(A,B) - \#(B,A) + \#(B,AB) - 2 * g_1 - 2 * g_2 - 2 * g_3 \\
& -w_1 - w_2 - w_3 - w_4 - 3 * d_1 - 2 * g_5 - 2 * g_6 - 2 * g_7 - 2 * g_8\}
\end{aligned}
$$

The maximum number of transplants can be rewritten as:

$$
\begin{aligned}
N = \;& 2 * g_1 + 2 * g_2 + 2 * g_3 + w_1 + w_2 + w_3 + w_4 + 3 * d_1 + 2 * g_5 + 2 * g_6 + 2 * g_7 \\
& +2 * g_8 + w_5 = \min\{N_1, N_2, N_4, N_5, N_6, N_7, , N_8, N_{10}, N_{11}\}.
\end{aligned}
$$

One may refer to Tables from C1 to C4 in Supplement C of Cheng and Yang (2017) for detail.

(2) When (A,O), (B,O), (A,AB), (B,AB) remaining, we have $g_1 = \#A^d + \#(AB,A) - \min\{\#A^d + \#(AB,A), \#(A,AB)\}$, $g_2 = \#(O,A) - \min\{\#(A,O), \#(O,A)\}$, $g_3 = \min\{\#A^d + \#(AB,A) - g_1, \#(A,B) - \#(B,A) - g_1 - g_2\}$ and $g_4 = \min\{\#(O,A) - g_2, \#(A,B) - \#(B,A) - g_1 - g_2 - g_3\}$. There is no potential benefits from four-way cycle and chain starting from the combinations $(AB,A) - (A,O)$, $(AB,B) - (B,O)$, $A^d - (A,O)$, $B^d - (B,O)$ because all pair (O,A) and pair (O,B) are matched. There is no potential benefits from four-way cycle and chain starting from the combinations $(AB,A) - (A,B) - (B,O)$ and $A^d - (A,B) - (B,O)$ because all pair (AB,A) and pair (AB,B), single donor A^d and single donor B^d are matched. There is no potential gains from four-way cycles and chains $(AB,O) - (O,A) - (A,B) - (B,AB)$ and chain $O^d - (O,A) - (A,B) - (B,AB) - AB^p$ because all pair (O,A) are matched. Therefore, the maximum number of transplants is the same as that of three-way cycles and chains in situation (2):

$$
N = \min\{N_5, N_2, N_1, N_6\}.
$$

(3) When (O,A), (O,B), $A^d/(AB,A)$, $B^d/(AB,B)$ remaining, we have $\min\{\#A^d + \#(AB,A), \#(A,AB)\} = \#(A,AB)\}$, $\min\{\#(A,O), \#(O,A)\} = \#(A,O)$, $g_1 = \min\{\#(A,B) - \#(B,A), \#(B,AB)\}\}$, $g_2 = \min\{\#(B,O), \#(A,B) - \#(B,A) - g_1\}$, $g_3 = 0$ and $g_4 = 0$. There is no potential benefits from four-way cycle and chain starting from the combinations $(AB,A) - (A,O)$, $(AB,B) - (B,O)$, $A^d - (A,O)$, $B^d - (B,O)$ because all pair (A,AB) and pair (B,AB) are matched. There is no potential benefits from the combinations $(AB,A) - (A,B) - (B,O)$ and $A^d - (A,B) - (B,O)$ because all pair (B,O) are matched. There is no potential gains from four-way cycles and chains $(AB,O) - (O,A) - (A,B) - (B,AB)$

and chain $O^d - (O,A) - (A,B) - (B,AB) - AB^p$ because all pair (B,AB) are matched. Therefore, the maximum number of transplants is the same as that of three-way cycles and chains in situation (3):

$$N = \min\{N_1, N_6, N_8, N_{11}\}$$

(4) When (A,O), (B,O), $A^d/(AB,A)$, $B^d/(AB,B)$ remaining, we have $\min\{\#A^d + \#(AB,A), \#(A,AB)\} = \#(A,AB)\}$, $g_1 = \min\{\#(A,B) - \#(B,A), \#(B,AB)\}$, $g_2 = \#(O,A) - \min\{\#(A,O), \#(O,A)\}$, $g_3 = 0$ and $g_4 = \min\{\#(O,A) - g_2, \#(A,B) - \#(B,A) - g_1 - g_2 - g_3\}$. There is no potential gains from four-way cycles and chains $(AB,O) - (O,A) - (A,B) - (B,AB)$ and chain $O^d - (O,A) - (A,B) - (B,AB) - AB^p$ because all pair (B,AB) are matched. There is no potential benefits from four-way cycle and chain starting from the combinations $(AB,A) - (A,O)$, $(AB,B) - (B,O)$, $A^d - (A,O)$, $B^d - (B,O)$ because all pair (A,AB) and pair (B,AB) are matched. Because there are remaining pair (AB,A), pair (B,O) and single donor A^d, there is potential gains from the combinations $(AB,A) - (A,B) - (B,O)$ and $A^d - (A,B) - (B,O)$. To take full advantage of the combinations, based on the three-way process in situation (4), we first reserve the maximum number of the combinations and then match two-way cycles $(AB,A) - (A,B)$, $(B,O) - (A,B)$ and chain $A^d - (A,B) - Y^p$. The maximum number of transplants in situation (4) is:

$$\begin{aligned}
N = \ & 2*g_1 + 2*g_2 + 2*g_4 + w_1 + w_2 + w_3 + w_4 + c_4 + s_1 + w_5 \\
where & \\
c_2 = \ & \min\{\#A^d + \#(AB,A) - g_1 - w_3, \#(A,O) - w_1\} \\
c_3 = \ & \min\{\#B^d + \#(AB,B) - w_2, \#(B,O) - g_2 - g_4 - w_4\} \\
c_4 = \ & \min\{\#A^d + \#(AB,A) - g_1 - w_3 - c_2, \#(B,O) - g_2 - g_4 - w_4 - c_3, \\
& \#(A,B) - \#(B,A) - g_1 - g_2 - g_4\} \\
s_1 = \ & \min\{\#A^d + \#(AB,A) - g_1 - w_3 - c_2 - c_4 + \#(B,O) - g_2 - g_4 - w_4 - c_3 - c_4, \\
& \#(A,B) - \#(B,A) - g_1 - g_2 - g_4 - c_4\} \\
w_5 = \ & \min\{\#O^d + \#(AB,O) + c_2 + c_3 + c_4, \#(O,A) + \#(O,B) + \#(O,AB) + \#(A,AB) \\
& + \#(A,B) - \#(B,A) + \#(B,AB) - 2*g_1 - 2*g_2 - 2*g_4 - w_1 \\
& - w_2 - w_3 - w_4 - s_1 - c_4\}
\end{aligned}$$

The maximum number of transplants can be rewritten as:

$$N = 2*g_1 + 2*g_2 + 2*g_4 + w_1 + w_2 + w_3 + w_4 + c_4 + s_1 + w_5 = \min\{N_1, N_2, N_4, N_6, N_{11}\}$$

One may refer to Tables from C5 to C8 in Supplement C for detail.

(5) When (A,O), (B,O), $A^d/(AB,A)$, (B,AB) remaining, we have $\min\{\#A^d + \#(AB,A), \#(A,AB)\} = \#(A,AB)\}$, $\min\{\#(A,O), \#(O,A)\} = \#(O,A)$, $g_1 = \#(A,B) - \#(B,A)$

and $g_2 = g_3 = g_4 = 0$. There is no potential gains from four-way cycles and chains $(AB,O)-(O,A)-(A,B)-(B,AB)$ and chain $O^d-(O,A)-(A,B)-(B,AB)-AB^p$ because all pair (O,A) are matched. There is no potential benefits from four-way cycle and chain starting from the combinations $(AB,A)-(A,O)$, $(AB,B)-(B,O)$, $A^d-(A,O)$, $B^d-(B,O)$ because all pair (A,AB), pair (O,A) and pair (O,B) are matched. There is no potential gains from the combinations $(AB,A)-(A,B)-(B,O)$ and $A^d-(A,B)-(B,O)$ because all pair (A,B) are matched. Therefore, the maximum number of transplants is the same as that of three-way cycles and chains in situation (5):

$$N = \min\{N_1, N_2, N_4\}$$

(6) When (A,O), (B,O), (A,AB), $B^d/(AB,B)$ remaining, we have $g_1 = \#A^d + \#(AB,A) - \min\{\#A^d + \#(AB,A), \#(A,AB)\}$, $g_2 = \#(O,A) - \min\{\#(A,O), \#(O,A)\}$, $g_3 = \min\{\#A^d + \#(AB,A) - g_1, \#(A,B) - \#(B,A) - g_1 - g_2, \#(B,AB) - b_1\}$ and $g_4 = \min\{\#(O,A) - g_2, \#(A,B) - \#(B,A) - g_1 - g_2 - g_3\}$. There is no potential gains from four-way cycles and chains $(AB,O)-(O,A)-(A,B)-(B,AB)$ and chain $O^d-(O,A)-(A,B)-(B,AB)-AB^p$ because all pair (O,A) are matched. There is no potential benefits from four-way cycle and chain starting from the combinations $(AB,A)-(A,O)$, $(AB,B)-(B,O)$, $A^d-(A,O)$, $B^d-(B,O)$ because all pair (O,A) and pair (O,B) are matched. There is no potential gains from the combinations $(AB,A)-(A,B)-(B,O)$ and $A^d-(A,B)-(B,O)$ because all pair (AB,A) and single donor A^d are matched. Therefore, there is no potential gains from four-way cycles and chains and the maximum number of transplants is the same as that of three-way cycles and chains in situation (6):

$$N = \min\{N_1, N_2, N_6\}$$

(7) When (A,O), (O,B), $A^d/(AB,A)$, $B^d/(AB,B)$ remaining, we have $\min\{\#A^d + \#(AB,A), \#(A,AB)\} = \#(A,AB)\}$, $g_1 = \min\{\#(A,B) - \#(B,A), \#(B,AB)\}$, $g_2 = \#(O,A) - \min\{\#(A,O), \#(O,A)\}$, $g_3 = 0$ and $g_4 = \min\{\#(B,O) - g_2, \#(O,A) - g_2, \#(A,B) - \#(B,A) - g_1 - g_2 - g_3\}$. There is no potential gains from four-way cycles and chains $(AB,O)-(O,A)-(A,B)-(B,AB)$ and chain $O^d-(O,A)-(A,B)-(B,AB)-AB^p$ because all pair (O,A) are matched. There is no potential gains from four-way cycles and chains starting with the combinations $(AB,A)-(A,O)$ and $A^d-(A,O)$ because all pair (A,AB) and pair (B,AB) are matched. There is no potential gains from four-way cycles and chains starting with the combinations $(AB,A)-(A,B)-(B,O)$ and $A^d-(A,B)-(B,O)$ because all pair (B,O) are matched. Therefore, the maximum number of transplants is the same as that of three-way cycles and chains in situation (7)

$$N = \min\{N_1, N_6, N_{11}\}$$

(8) When (O, A), (B, O), $A^d/(AB, A)$, $B^d/(AB, B)$ remaining, we have $\min\{\#(A, O), \#(O, A)\} = \#(A, O)$, $\min\{\#A^d + \#(AB, A), \#(A, AB)\} = \#(A, AB)$, $g_1 = \min\{\#(A, B) - \#(B, A), \#(B, AB)\}$, $g_2 = \#(A, B) - \#(B, A) - g_1$ and $g_3 = g_4 = 0$. There is no potential gains from four-way cycles and chains $(AB, O) - (O, A) - (A, B) - (B, AB)$ and chain $O^d - (O, A) - (A, B) - (B, AB) - AB^p$ because all pair (B, AB) are matched. There is no potential gains from four-way cycles and chains starting with the combinations $(AB, B) - (B, O)$ and $B^d - (B, O)$ because all pair (A, AB) and pair (B, AB) are matched. There is no potential gains from four-way cycles and chains starting with the combinations $(AB, A) - (A, B) - (B, O)$ and $A^d - (A, B) - (B, O)$ because all pair (A, B) are matched. Therefore, the maximum number of transplants is the same as that of three-way cycles and chains in situation (8)

$$N = \min\{N_1, N_4, N_{11}\}$$

(9) When (A, O), (O, B), (A, AB), $B^d/(AB, B)$ remaining, we have $g_1 = \#A^d + \#(AB, A) - \min\{\#A^d + \#(AB, A), \#(A, AB)\}$, $g_2 = \#(O, A) - \min\{\#(A, O), \#(O, A)\}$, $g_3 = \min\{\#A^d + \#(AB, A) - g_1, \#(A, B) - \#(B, A) - g_1 - g_2, \#(B, AB) - b_1\}$ and $g_4 = \min\{\#(B, O) - g_2, \#(O, A) - g_2, \#(A, B) - \#(B, A) - g_1 - g_2 - g_3\}$. There is no potential gains from four-way cycles and chains $(AB, O) - (O, A) - (A, B) - (B, AB)$ and chain $O^d - (O, A) - (A, B) - (B, AB) - AB^p$ because all pair (O, A) are matched. There is no potential benefits from four-way cycle and chain starting from the combinations $(AB, A) - (A, O)$, $(AB, B) - (B, O)$, $A^d - (A, O)$, $B^d - (B, O)$ because all pair (A, AB), pair (O, A) and pair (B, AB) are matched. Therefore, the maximum number of transplants is the same as that of three-way cycles and chains in situation (9)

$$N = \min\{N_1, N_6\}$$

(10) When (O, A), (B, O), $A^d/(AB, A)$, (B, AB) remaining, we have $\min\{\#(A, O), \#(O, A)\} = \#(A, O)$, $\min\{\#A^d + \#(AB, A), \#(A, AB)\} = \#(A, AB)$, $g_1 = \#(A, B) - \#(B, A)$ and $g_2 = g_3 = g_4 = 0$. There is no potential gains from four-way cycles and chains $(AB, O) - (O, A) - (A, B) - (B, AB)$ and chain $O^d - (O, A) - (A, B) - (B, AB) - AB^p$ because all pair (A, B) are matched. There is no potential gains from four-way cycles and chains starting from the combinations $(AB, B) - (B, O)$, $B^d - (B, O)$, $(AB, A) - (A, O)$ and $A^d - (A, O)$ because all pair (A, O), pair (AB, B) and single donor B^d are matched. There is no potential gains from four-way cycles and chains starting from the combinations $(AB, A) - (A, B) - (B, O)$ and $A^d - (A, B) - (B, O)$ because all pair (A, B) are matched. Therefore, the maximum number of transplants is the same as that of three-way cycles and chains in situation (10)

$$N = \min\{N_1, N_4\}$$

(11) When (A,O), (O,B), $A^d/(AB,A)$, (B,AB) remaining, we have $\min\{\#(A,O), \#(O,A)\} = \#(O,A)$, $\min\{\#A^d + \#(AB,A), \#(A,AB)\} = \#(A,AB)$, $g_1 = \#(A,B) - \#(B,A)$ and $g_2 = g_3 = g_4 = 0$. There is no potential gains from four-way cycles and chains $(AB,O)-(O,A)-(A,B)-(B,AB)$ and chain $O^d-(O,A)-(A,B)-(B,AB)-AB^p$ because all pair (O,A) and pair (A,B) are matched. There is no potential gains from four-way cycles and chains starting by the combinations $(AB,A)-(A,B)-(B,O)$ and $A^d-(A,B)-(B,O)$ because all pair (A,B) and pair (B,O) are matched. There is potential gains from four-way cycle $(AB,A)-(A,O)-(O,B)-(B,AB)$ and chain $A^d-(A,O)-(O,B)-(B,AB)-AB^p$. To take full advantages, we match the maximum number of four-way cycle $(AB,A) - (A,O) - (O,B) - (B,AB)$ and chain $A^d - (A,O) - (O,B) - (B,AB) - AB^p$ bounded by the number of remaining pairs (A,O), (O,B), $A^d/(AB,A)$ and (B,AB). If all remaining pairs (AB,A) and single donors A^d are matched, there is potential gains from four-way cycle $(AB,A) - (A,O) - (O,B) - (B,AB)$ and chain $A^d - (A,O) - (O,B) - (B,AB)$ by breaking two-way cycle $(AB,A) - (A,AB)$ and chain $A^d - (A,AB)$ because one more pair can be matched in this case. Similarly, if all remaining pairs (A,O) are matched, there is potential gains from three-way cycles $(A,O)-(O,B)-(B,A)$, $(AB,A)-(A,B)-(B,AB)$ and chain $A^d - (A,B) - (B,AB) - AB^p$ by breaking two-way cycle $(A,O) - (O,A)$.

Therefore, the maximum number of transplants in situation (11) is

$$N = 2 * g_1 + w_1 + w_2 + w_3 + w_4 + 2 * u_1 + v_1 + v_2 + 2 * g_6 + w_5$$

$$\begin{aligned}
&where\\
u_1 = \;& \min\{\#(A,O) - w_1, \#(O,B) - w_4, \#A^d + \#(AB,A) - g_1 - w_3,\\
& \#(B,AB) - g_1 - w_2\}\\
v_1 = \;& \min\{\#A^d + \#(AB,A) - g_1 - u_1, \#(A,O) - w_1 - u_1, \#(O,B) - w_4 - u_1,\\
& \#(B,AB) - g_1 - w_2 - u_1\}\\
v_2 = \;& \min\{\#(A,O) - u_1, \#A^d + \#(AB,A) - g_1 - u_1, \#(O,B) - w_4 - u_1,\\
& \#(B,AB) - g_1 - w_2 - u_1\}\\
c_2 = \;& \min\{\#A^d + \#(AB,A) - w_3 - u_1 - v_2, \#(A,O) - w_1 - u_1 - v_1\}\\
g_6 = \;& \min\{\#O^d + \#(AB,O), \#(O,B) - w_4 - u_1 - v_1 - v_2, \#(B,AB) - g_1 - u_1 - w_2\\
& -v_1 - v_2\}\\
w_5 = \;& \min\{\#O^d + \#(AB,O) + c_2 - g_6, \#(O,A) + \#(O,B) + \#(O,AB) + \#(A,AB)\\
& +\#(A,B) - \#(B,A) + \#(B,AB) - 2 * g_1 - w_1 - w_2 - w_3 - w_4 - 2 * u_1\\
& -v_1 - v_2 - 2 * g_6\}
\end{aligned}$$

The maximum number of transplants is

$$\begin{aligned}
N = \;& 2 * g_1 + w_1 + w_2 + w_3 + w_4 + 2 * u_1 + v_1 + v_2 + 2 * g_6 + w_5\\
= \;& \min\{N_1, N_2, N_4, N_5, N_6, N_9, N_{10}, N_{11}\}
\end{aligned}$$

One may refer to Tables C9 and C10 in Supplement C for detail.

(12) When (O,A), (O,B), $A^d/(AB,A)$, (B,AB) remaining, we have $\min\{\#(A,O),\#(O,A)\} = \#(A,O)$, $\min\{\#A^d + \#(AB,A), \#(A,AB)\} = \#(A,AB)$, $g_1 = \#(A,B) - \#(B,A)$ and $g_2 = g_3 = g_4 = 0$. There is no potential gains from four-way cycles and chains $(AB,O)-(O,A)-(A,B)-(B,AB)$ and chain $O^d-(O,A)-(A,B)-(B,AB)-AB^p$ because all pair (A,B) are matched. There is potential gains from four-way cycle $(AB,A)-(A,O)-(O,B)-(B,AB)$ and chain $A^d-(A,O)-(O,B)-(B,AB)-AB^p$ by breaking two-way cycle $(A,O)-(O,A)$ because one more pair can be matched in this case. That is, $v_2 \neq 0$. We can do the same process in situation (11). The maximum number of transplants is

$$N = 2*g_1 + w_1 + w_2 + w_3 + w_4 + v_2 + 2*g_6 + w_5 = \min\{N_1, N_4, N_9, N_{10}, N_{11}\}$$

One may refer to Table C11 in Supplement C for detail.

(13) When (A,O), (O,B), (A,AB), (B,AB) remaining, we have $g_1 = \#A^d + \#(AB,A) - \min\{\#A^d + \#(AB,A), \#(A,AB)\}$, $g_2 = \#(O,A) - \min\{\#(A,O), \#(O,A)\}$, $g_3 = \min\{\#A^d + \#(AB,A) - g_1, \#(A,B) - \#(B,A) - g_1 - g_2\}$ and $g_4 = \min\{\#(O,A) - g_2, \#(A,B) - \#(B,A) - g_1 - g_2 - g_3\}$. There is no potential gains from four-way cycles and chains $(AB,O)-(O,A)-(A,B)-(B,AB)$ and chain $O^d-(O,A)-(A,B)-(B,AB)-AB^p$ because all pair (O,A) are matched. There is potential gains from four-way cycle $(AB,A)-(A,O)-(O,B)-(B,AB)$ and chain $A^d-(A,O)-(O,B)-(B,AB)-AB^p$ by breaking two-way cycle $(AB,A)-(A,AB)$ and chain $A^d-(A,AB)-AB^p$ because one more pair can be matched in this case. That is, $v_1 \neq 0$. To take full advantage of four-way cycles and chains, we can first do the same process in situation (11) and then match three-way cycles $(AB,O)-(O,B)-(B,AB)$, $(AB,O)-(A,B)-(B,AB)$ and chains $O^d-(O,B)-(B,AB)-AB^p$, $O^d-(A,B)-(B,AB)-AB^p$ if any before matching remaining pairs with pair (AB,O) and single donor O^d. Therefore, the maximum number of transplants in situation (13) is

$$\begin{aligned}
N = \quad & 2*g_1 + 2*g_2 + 2*g_3 + 2*g_4 + w_1 + w_2 + w_3 + w_4 + v_1 + 2*g_6 + 2*g_8 + w_5 \\
where & \\
g_8 = \quad & \min\{\#O^d + \#(AB,O) - g_6, \#(A,B) - \#(B,A) - g_1 - g_2 - g_3 - g_4, \\
& \#(B,AB) - g_1 - g_3 - w_2 - v_1 - g_6\} \\
w_5 = \quad & \min\{\#O^d + \#(AB,O) - g_6 - g_8, \#(O,A) + \#(O,B) + \#(O,AB) + \#(A,AB) \\
& +\#(A,B) - \#(B,A) + \#(B,AB) - 2*g_1 - 2*g_2 - 2*g_3 - 2*g_4 \\
& -w_1 - w_2 - w_3 - w_4 - v_1 - 2*g_6 - 2*g_8\}
\end{aligned}$$

The maximum number of transplants is

$$\begin{aligned}
N = \quad & 2*g_1 + 2*g_2 + 2*g_3 + 2*g_4 + w_1 + w_2 + w_3 + w_4 + v_1 + 2*g_6 + 2*g_8 + w_5 \\
= \quad & \min\{N_1, N_2, N_5, N_6, N_{10}\}
\end{aligned}$$

One may refer to Tables from C12 to C17 in Supplement C for detail.

(14) When (O,A), (B,O), (A,AB), $B^d/(AB,B)$ remaining, we have $\min\{\#(A,O), \#(O,A)\} = \#(A,O)$, $g_1 = \#A^d + \#(AB,A) - \min\{\#A^d + \#(AB,A), \#(A,AB)\}$, $g_2 = \#(A,B) - \#(B,A) - g_1$ and $g_3 = g_4 = 0$. There is no potential gains from four-way cycles and chains $(AB,O)-(O,A)-(A,B)-(B,AB)$ and chain $O^d-(O,A)-(A,B)-(B,AB)-AB^p$ because all pair (A,B) are matched. There is potential gains from four-way cycle $(AB,B)-(B,O)-(O,A)-(A,AB)$ and chain $B^d-(B,O)-(O,A)-(A,AB)-AB^p$ because two more blood-type incompatible pairs of types (O,A) and (A,AB) can be matched in this case. Therefore, we take full advantage of (B,A), (A,B) and match the maximum number of four-way cycle $(AB,B)-(B,O)-(O,A)-(A,AB)$ and chain $B^d-(B,O)-(O,A)-(A,AB)-AB^p$ bounded by the number of remaining pairs (O,A), (B,O), (A,AB) and $B^d/(AB,B)$. If all remaining pairs (AB,B) and single donors B^d are matched, there is potential gains from four-way cycle $(AB,B)-(B,O)-(O,A)-(A,AB)$ and chain $B^d-(B,O)-(O,A)-(A,AB)-AB^p$ by breaking two-way cycle $(AB,B)-(B,AB)$ and chain $B^d-(B,AB)-AB^p$ because one more pair can be matched in this case. Similarly, if all remaining pairs (B,O) are matched, there is potential gains from four-way cycle $(AB,B)-(B,O)-(O,A)-(A,AB)$ and chain $B^d-(B,O)-(O,A)-(A,AB)-AB^p$ by breaking two-way cycle $(B,O)-(O,B)$. Therefore, the maximum number of transplants in situation (14) is

$$\begin{aligned}
N = \quad & 2*g_1+2*g_2+w_1+w_2+w_3+w_4+2*u_2+v_3+v_4+2*g_5+w_5 \\
where \quad & \\
u_2 = \quad & \min\{\#(B,O)-g_2-w_4, \#(O,A)-g_2-w_1, \#(A,AB)-w_3, \\
& \#B^d+\#(AB,B)-w_2\} \\
v_3 = \quad & \min\{\#B^d+\#(AB,B)-u_2, \#(B,O)-g_2-w_4-u_2, \#(O,A)-g_2-w_1-u_2, \\
& \#(A,AB)-w_3-u_2\} \\
v_4 = \quad & \min\{\#(B,O)-g_2-u_2, \#B^d+\#(AB,B)-w_2-u_2, \#(O,A)-g_2-w_1-u_2, \\
& \#(A,AB)-w_3-u_2\} \\
c_3 = \quad & \min\{\#B^d+\#(AB,B)-w_2-u_2-v_4, \#(B,O)-w_4-u_2-v_3\} \\
g_5 = \quad & \min\{\#O^d+\#(AB,O), \#(O,A)-g_2-w_1-u_2-v_3-v_4, \#(A,AB)-w_3-u_2 \\
& -v_3-v_4\} \\
w_5 = \quad & \min\{\#O^d+\#(AB,O)+c_3-g_5, \#(O,A)+\#(O,B)+\#(O,AB)+\#(A,AB) \\
& +\#(A,B)-\#(B,A)+\#(B,AB)-2*g_1-2*g_2-w_1-w_2-w_3-w_4 \\
& -2*u_2-v_3-v_4-2*g_5\}
\end{aligned}$$

The maximum number of transplants is

$$\begin{aligned}
N = \quad & 2*g_1+2*g_2+w_1+w_2+w_3+w_4+2*u_2+v_3+v_4+2*g_5+w_5 \\
= \quad & \min\{N_1, N_2, N_3, N_4, N_6, N_8, N_{10}, N_{11}\}
\end{aligned}$$

One may refer to Tables C18 and C19 in Supplement C for detail.

(15) When (O,A), (B,O), (A,AB), (B,AB) remaining, we have $\min\{\#(A,O),\#(O,A)\} = \#(A,O)$, $g_1 = \#A^d + \#(AB,A) - \min\{\#A^d + \#(AB,A), \#(A,AB)\}$, $g_2 = \#(A,B) - \#(B,A) - g_1$ and $g_3 = g_4 = 0$. There is no potential gains from four-way cycles and chains $(AB,O)-(O,A)-(A,B)-(B,AB)$ and chain $O^d-(O,A)-(A,B)-(B,AB)-AB^p$ since all pair (A,B) are matched. There is potential gains from four-way cycle $(AB,B)-(B,O)-(O,A)-(A,AB)$ and chain $B^d-(B,O)-(O,A)-(A,AB)-AB^p$ by breaking two-way cycle $(AB,B)-(B,AB)$ and chain $B^d-(B,AB)-AB^p$ because one more pair can be matched in this case. That is, $v_3 \neq 0$. We can do the same process in situation (14). The maximum number of transplants is

$$
\begin{aligned}
N = \quad & 2*g_1 + 2*g_2 + w_1 + w_2 + w_3 + w_4 + v_3 + 2*g_5 + w_5 \\
= \quad & \min\{N_1, N_2, N_3, N_4, N_{10}\}
\end{aligned}
$$

One may refer to Table C20 in Supplement C for detail.

(16) When (O,A), (O,B), (A,AB), $B^d/(AB,B)$ remaining, we have $\min\{\#(A,O), \#(O,A)\} = \#(A,O)$, $g_1 = \#A^d + \#(AB,A) - \min\{\#A^d + \#(AB,A), \#(A,AB)\}$, $g_2 = \min\{\#(B,O), \#(A,B) - \#(B,A) - g_1\}$, $g_3 = \min\{\#A^d + \#(AB,A) - g_1, \#(A,B) - \#(B,A) - g_1 - g_2, \#(B,AB) - b_1\}$ and $g_4 = 0$. There is no potential gains from four-way cycles and chains $(AB,O)-(O,A)-(A,B)-(B,AB)$ and chain $O^d-(O,A)-(A,B)-(B,AB)-AB^p$ because all pair (B,AB) are matched. There is potential gains from four-way cycle $(AB,B)-(B,O)-(O,A)-(A,AB)$ and chain $B^d-(B,O)-(O,A)-(A,AB)-AB^p$ by breaking two-way cycle $(B,O)-(O,B)$ because one more pair can be matched in this case. That is, $v_4 \neq 0$. To take full advantage of three-way cycles and chains, we can first do the same process in situation (14) and then match three-way cycles $(AB,O)-(O,A)-(A,AB)$, $(AB,O)-(O,A)-(A,B)$ and chains $O^d-(O,A)-(A,AB)-AB^p$, $O^d-(O,A)-(A,B)-Y^p$ if any before matching remaining pairs with pair (AB,O) and single donor O^d. Therefore, the maximum number of transplants in situation (16) is

$$
\begin{aligned}
N = \quad & 2*g_1 + 2*g_2 + 2*g_3 + w_1 + w_2 + w_3 + w_4 + v_4 + 2*g_5 + 2*g_7 + w_5 \\
where & \\
g_7 = \quad & \min\{\#O^d + \#(AB,O) - g_5, \#(O,A) - g_2 - w_1 - v_4 - g_5, \\
& \#(A,B) - \#(B,A) - g_1 - g_2 - g_3\} \\
w_5 = \quad & \min\{\#O^d + \#(AB,O) - g_5 - g_7, \#(O,A) + \#(O,B) + \#(O,AB) + \#(A,AB) \\
& +\#(A,B) - \#(B,A) + \#(B,AB) - 2*g_1 - 2*g_2 - 2*g_3 - w_1 - w_2 - w_3 \\
& -w_4 - v_4 - 2*g_5 - 2*g_7\}
\end{aligned}
$$

The maximum number of transplants is

$$
\begin{aligned}
N = \quad & 2*g_1 + 2*g_2 + 2*g_3 + w_1 + w_2 + w_3 + w_4 + v_4 + 2*g_5 + 2*g_7 + w_5 \\
= \quad & \min\{N_1, N_6, N_8, N_{10}, N_{11}\}
\end{aligned}
$$

One may refer to Tables from C21 to C24 in Supplement C for detail. Combining cases (1) to (16), we have proved that the maximum number of transplants for blood-type incompatible paired patients of types (O, A), (O, B), (O, AB), (A, AB), (B, AB), (A, B) is

$$N = \min\{N_1, N_2, N_3, N_4, N_5, N_6, N_7, N_8, N_9, N_{10}, N_{11}\}$$

We now prove that the sequential mechanism is 4-efficient which achieves the maximum number of transplants in the pool. The same as the mechanism when three-way cycles and chains are allowed, the number of transplantations for compatible pairs, blood-type compatible pairs and pairs of type $(B - A)$ in the mechanism is

$$\begin{aligned}&\#(A, O) + \#(B, O) + \#(AB, O) + \#(AB, A) + \#(AB, B)\\&+\#(B, A) + \#(A, A) + \#(B, B) + \#(AB, AB) + \#(O, O)\end{aligned}$$

Next, we prove that the maximum number of transplants for blood-type incompatible pairs of types (O, A), (O, B), (O, AB), (A, AB), (B, AB), (A, B) can be achieved in the mechanism by exploring every possible route.

Denote X_2 as the number of blood-type incompatible paired patients from pairs of types (O, A), (O, B), (O, AB), (A, AB), (B, AB), (A, B) involved in Step 2 so that

$$\begin{aligned}X_2 =\ & 2 * b_1 + 2 * b_2 + 2 * b_3 + 2 * b_{21}\\ where &\end{aligned}$$

$$\begin{aligned}b_1 =\ & \min\{\#A^d + \#(AB, A) - \min\{\#A^d + \#(AB, A), \#(A, AB)\},\\ & \#(A, B) - \#(B, A), \#(B, AB)\}\\ b_2 =\ & \min\{\#(B, O), \#(O, A) - \min\{\#(A, O), \#(O, A)\}, \#(A, B) - \#(B, A) - b_1\}\\ b_3 =\ & \min\{\#A^d + \#(AB, A) - b_1, \#(A, B) - \#(B, A) - b_1 - b_2, \#(B, AB) - b_1\}\\ b_{21} =\ & \min\{\#(B, O) - b_2, \#(O, A) - b_2, \#(A, B) - \#(B, A) - b_1 - b_2 - b_3\}\end{aligned}$$

Denote X_3 as the number of blood-type incompatible paired patients from pairs of types (O, A), (O, B), (O, AB), (A, AB), (B, AB), (A, B) involved in Step 3 so that

$$\begin{aligned}X_3 =\ & 2 * e_1 + 2 * e_2 + 2 * f_2 + 2 * f_3 + 2 * f_4 + 2 * f_5\\ where &\\ e_1 =\ & \min\{\#(A, O) - a_1, \#A^d + \#(AB, A) - b_1 - b_3 - a_1, \#(O, B) - a_4,\\ & \#(B, AB) - b_1 - b_3 - a_2\}\\ e_2 =\ & \min\{\#B^d + \#(AB, B) - a_2, \#(A, AB) - a_3, \#(B, O) - b_2 - a_4,\\ & \#(O, A) - b_2 - a_1\}\\ f_2 =\ & \min\{\#(A, O) - a_1 - e_1, \#A^d + \#(AB, A) - b_1 - b_3 - e_1,\\ & \#(O, B) - a_4 - e_1, \#(B, AB) - b_1 - b_3 - a_2 - e_1\}\end{aligned}$$

$$
\begin{aligned}
f_3 = \quad & \min\{\#B^d + \#(AB,B) - a_2 - e_2, \#(A,AB) - a_3 - e_2, \#(B,O) - b_2 - e_2, \\
& \#(O,A) - b_2 - a_1 - e_2\} \\
f_4 = \quad & \min\{\#(A,O) - e_1, \#A^d + \#(AB,A) - b_1 - b_3 - e_1 - a_3, \\
& \#(O,B) - a_4 - e_1, \#(B,AB) - b_1 - b_3 - a_2 - e_1\} \\
f_5 = \quad & \min\{\#B^d + \#(AB,B) - e_2, \#(A,AB) - a_3 - e_2, \#(B,O) - b_2 - b_{21} - e_2 \\
& -a_4, \#(O,A) - b_2 - b_{21} - a_1 - e_2\} \\
\textit{where} & \\
a_1 = \quad & \min\{\#(A,O), \#(O,A) - b_2 - b_{21}\} \\
a_2 = \quad & \min\{\#B^d + \#(AB,B), \#(B,AB) - b_1 - b_3\} \\
a_3 = \quad & \min\{\#A^d + \#(AB,A) - b_1 - b_3, \#(A,AB)\} \\
a_4 = \quad & \min\{\#(B,O) - b_2 - b_{21}, \#(O,B)\}
\end{aligned}
$$

In Step 4, all remaining pairs (B,A) are matched with pair (A,B). Denote X_4 as the number of blood-type incompatible paired patients from pairs of types (O,A), (O,B), (O,AB), (A,AB), (B,AB), (A,B) involved in Step 4 so that

$$
\begin{aligned}
X_4 = \quad & \#(B,A) + p_1 + p_2 + p_3 + p_4 + a_8 + c_4 \\
\textit{where} & \\
p_1 = \quad & a_1 - f_4 \\
p_2 = \quad & a_2 - f_5 \\
p_3 = \quad & a_3 - f_2 \\
p_4 = \quad & a4 - f_3 \\
a_8 = \quad & \min\{\#A^d + \#(AB,A) - b_1 - b_3 - e_1 - f_2 - f_4 - p_3 - c_2 - c_4 \\
& +\#(B,O) - b_2 - e_2 - f_3 - f_5 - p_4 - c_3 - c_4, \#(A,B) - \#(B,A) \\
& -b_1 - b_2 - b_3 - b_{21} - c_4\} \\
\textit{where} & \\
c_2 = \quad & \min\{\#A^d + \#(AB,A) - b_1 - b_3 - e_1 - f_2 - f_4 - p_3, \#(A,O) - e_1 - f_2 \\
& -f_4 - p_1\} \\
c_3 = \quad & \min\{\#(B,O) - b_2 - b_{21} - e_2 - f_3 - f_5 - p_4, \#B^d + \#(AB,B) - e_2 - f_3 \\
& -f_5 - p_2\} \\
c_4 = \quad & \min\{\#A^d + \#(AB,A) - b_1 - b_3 - e_1 - f_2 - f_4 - p_3 - c_2, \#(B,O) - b_2 \\
& -b_{21} - e_2 - f_3 - f_5 - p_4 - c_3, \#(A,B) - \#(B,A) - b_1 - b_2 - b_3 - b_{21}\}
\end{aligned}
$$

Denote X_5 as the number of blood-type incompatible paired patients from pairs of types (O,A), (O,B), (O,AB), (A,AB), (B,AB), (A,B) involved in Step 5 so that

$$
\begin{aligned}
X_5 = \quad & 3 * d_1 + 2 * b_4 + 2 * b_5 + 2 * b_6 + 2 * b_7 \\
\textit{where} & \\
d_1 = \quad & \min\{\#O^d + \#(AB,O), \#(O,A) - b_2 - b_{21} - e_2 - f_3 - f_5 - p_1, \#(A,B) \\
& -\#(B,A) - b_1 - b_2 - b_3 - b_{21} - a_8, \#(A,AB) - e_2 - f_3 - f_5 - p_3\}
\end{aligned}
$$

$$
\begin{aligned}
b_4 = \;& \min\{\#O^d + \#(AB,O) - d_1, \#(O,A) - b_2 - b_{21} - e_2 - f_3 - f_5 - p_1 - d_1, \\
& \#(A,AB) - e_2 - f_3 - f_5 - p_3 - d_1\} \\
b_5 = \;& \min\{\#O^d + \#(AB,O) - b_4 - d_1, \#(O,B) - e_1 - f_2 - f_4 - p_4, \\
& \#(B,AB) - b_1 - b_3 - e_1 - f_2 - f_4 - p_2\} \\
b_6 = \;& \min\{\#O^d + \#(AB,O) - b_4 - b_5 - d_1, \#(O,A) - b_2 - b_{21} - e_2 - f_3 - f_5 \\
& -p_1 - b_4 - d_1, \#(A,B) - \#(B,A) - b_1 - b_2 - b_3 - b_{21} - a_8 - d_1\} \\
b_7 = \;& \min\{\#O^d + \#(AB,O) - b_4 - b_5 - b_6 - d_1, \#(A,B) - \#(B,A) - b_1 - b_2 \\
& -b_3 - b_{21} - a_8 - b_6 - d_1, \#(B,AB) - b_1 - b_3 - e_1 - f_2 - f_4 - p_2 - b_5\}
\end{aligned}
$$

Denote X_6 as the number of blood-type incompatible paired patients from pairs of types (O,A), (O,B), (O,AB), (A,AB), (B,AB), (A,B) involved in Step 6 so that

$$
\begin{aligned}
X_6 = a_9 = \;& \min\{\#O^d + \#(AB,O) - b_4 - b_5 - b_6 - b_7 + c_2 + c_3 + c_4, \#(O,A) \\
& +\#(O,B) + \#(O,AB) + \#(A,AB) + \#(A,B) + \#(B,AB) - 2*b_1 \\
& -2*b_2 - 2*b_3 - 2*e_1 - 2*e_2 - 2*f_2 - 2*f_3 - 2*f_4 - 2*f_5 \\
& -p_1 - p_2 - p_3 - p_4 - a_8 - 3*d_1 - 2*b_4 - 2*b_5 - 2*b_6 - 2*b_7 - b_{21} - b_{21}\}
\end{aligned}
$$

Therefore, the total number of transplants for paired patients from pairs of types (O,A), (O,B), (O,AB), (A,AB), (B,AB), (A,B) in the mechanism is $X = X_2+X_3+X_4+X_5+X_6$. The equation can be rewritten as follows:

$$X = \min\{N_1, N_2, N_3, N_4, N_5, N_6, N_7, N_8, N_9, N_{10}, N_{11}\}$$

One may refer to Tables from C25 to C60 in Supplement C for detail. Therefore, the total number of transplants can be achieved in the mechanism is that

$$
\begin{aligned}
& \#(A,O) + \#(B,O) + \#(AB,O) + \#(AB,A) + \#(AB,B) \\
& +\#(B,A) + \#(A,A) + \#(B,B) + \#(AB,AB) + \#(O,O) \\
& +\#A^d + \#B^d + \#AB^d + \#O^d \\
& + \min\{N_1, N_2, N_3, N_4, N_5, N_6, N_7, N_8, N_9, N_{10}, N_{11}\}
\end{aligned}
$$

Therefore, we proved that every matching produced by the mechanism achieves the maximum number of transplants in the pool and hence the mechanism is 4-efficient.

□

Proof of Corollary 4.4: Consider an efficient matching μ for a population stated in the theorem. If the maximal matching μ only consists of n-way cycles and chains, or smaller cycles and chains, we are done. Otherwise, we will prove that there exists a matching ν which consists of at most n-way cycles or chains can match the same set of receiving agents as matching μ.

We will prove the theorem for the case in which the largest exchanges (cycles or chains) in matching μ is (n+1)-way. The same proof can be applied to show that for any maximal matching in which the largest exchanges is m-way where $m > (n+1)$, there exists another matching which matches the same set of receiving agent through (m-1)-way or smaller exchanges. Then, repeat the same proof process to obtain the desired result.

Three cases may occur in the matching μ, which are both cycle and chain have (n+1)-way or either cycle or chain has (n+1)-way. We will prove the most complicated case such that matching μ consisting of both (n+1)-way cycles and (n+1)-way chains. Then, other two situations can be automatically proved.

Let

$$E^0 = ((P_1^p, D_1^p), (P_2^p, D_2^p), (P_3^p, D_3^p), ..., (P_n^p, D_n^p), (P_{n+1}^p, D_{n+1}^p))$$
$$C^0 = (D_1^s, (P_1^p, D_1^p), (P_2^p, D_2^p), ..., (P_n^p, D_n^p), P_1^s)$$

be any (n+1)-way cycle and chain respectively in matching μ. We will prove that all receiving agents in these two exchanges can be matched via smaller exchanges without changing the set of pairs that are matched.

Since we have only n types, there are at least two receiving agents in cycle E^0 who have the same type. Pick any two such receiving agent. We have two cases to consider.

Case 1. The two receiving agents are not matched together.

Suppose these receiving agents are P_1^p and P_n^p in cycle E^0. The receiving agent P_1^p is matched with donating agent D_{n+1}^p and the receiving agent P_n^p is matched with donating agent D_{n-1}^p. Since agents P_1^p and P_n^p have the same type, donating agents D_{n-1}^p and D_{n+1}^p are compatible with the two receiving agent P_1^p and P_n^p. Hence, the (n+1)-way cycle can be divided into two smaller cycles as follows.

$$E_1^1 = ((P_1^p, D_1^p), (P_2^p, D_2^p), (P_3^p, D_3^p), ..., (P_{n-1}^p, D_{n-1}^p)),\ E_2^1 = ((P_n^p, D_n^p), (P_{n+1}^p, D_{n+1}^p))$$

Suppose these receiving agents are P_1^p and P_1^s in chain C^0. The receiving agent P_1^p is matched with donating agent D_1^s and the receiving agent P_1^s is matched with donating agent D_n^p. Since agents P_1^p and P_1^s have the same type, donating agents D_1^s and D_n^p are compatible with both the two agents. Hence, the (n+1)-way chain C^0 can be divided into one cycle and one chain as follows.

$$C_2^1 = (D_1^s, P_1^s),\ E_2^1 = ((P_1^p, D_1^p), (P_2^p, D_2^p), ..., (P_n^p, D_n^p))$$

Case 2. The two receiving agents are matched together. Suppose agents P_1^p and P_2^p have the same type.

Under cycle E^0, since agent P_1^p is matched with donating agent D_{n+1}, donating agent D_{n+1} is compatible with the receiving agent P_2^p. Hence, the following n-way exchange is feasible.

$$E_1^2 = ((P_2^p, D_2^p), (P_3^p, D_3^p), ..., (P_{n-1}^p, D_{n-1}^p))$$

Under chain C^0, since agent P_1^p is matched with donating agent D_1^s, donating agent D_1^s is compatible with receiving agent P_2^p. Hence, the following n-way chain is feasible.

$$C_2^2 = (D_1^s, (P_2^p, D_2^p), ..., (P_{n-1}^p, D_{n-1}^p), (P_n^p, D_n^p), P_1^s)$$

Now, we will prove that the remaining pair (P_1^p, D_1^p) can be matched in an exchange without affecting pairs that are matched under μ. Because of the Assumption 2.1, we directly use "type" to present the primary type. Let pair (P_1^p, D_1^p) be of type $(X, Y)^t$ where $t \in \{i, c\}$, and hence receiving agent P_2^p is type X. Since donating agent D_1^p of type Y is compatible with receiving agent P_2^p, we have $Y \succeq X$. Therefore, pair of type (X, Y) is primary type compatible pair.

Let $\mathcal{A}$ be the set of $n+1$-way cycles and $n+1$-way chains in case 2. From previous proof, every cycle can be separated into a n-way cycle and one remaining primary type compatible pair and every chain can be separated into a n-way chain and one remaining primary type compatible pair. Let $\mathcal{D}$ be the set of remaining primary type compatible pairs in $\mathcal{A}$. Then, we have $Y \succeq X$. If remaining pairs are compatible, we can do transplants directly. Otherwise, let $(X, Y)^i$ present the type of a primary type compatible but secondary type incompatible pair. If there exists two or more pairs of type $(X, Y)^i$, we can match them by two-way cycles $(X, Y)^i - (X, Y)^i$. Therefore, at most one pair of type $(X, Y)^i$ left. By Assumption 4.2, there exists at least one pair of type $(X, Y)^c$. If the pair $(X, Y)^c$ does not involve in a cycle or a chain, we can match the remaining pair $(X, Y)^i$ with pair $(X, Y)^c$. Otherwise, pair $(X, Y)^c$ involves in a cycle or chain no larger than n-way, then the remaining pair $(X, Y)^i$ can replace the position of pair $(X, Y)^c$ and pair $(X, Y)^c$ can do the transplant straightforwardly.

□