## University of Vork



## Discussion Papers in Economics

No. 16/14

## A Theory of Marriage with Mutually Consented Divorces

Ning Sun and Zaifu Yang

Department of Economics and Related Studies
University of York
Heslington
York, YO10 5DD

# A Theory of Marriage with Mutually Consented Divorces ${ }^{1}$ 

Ning Sun ${ }^{2}$ and Zaifu Yang ${ }^{3}$


#### Abstract

We study a general model of marriage in which there are finitely many singles (unmarried men or women) and married couples. Singles wish to marry. Married couples can divorce and thus remarry as long as both parties will not be made worse off than they maintain the status quo. This is called mutual consent divorce. We examine the problem of how to make marriages between men and women as well as possible by allowing mutual consent divorce. We show the existence of a nonempty core of marriage matchings and also propose a finite procedure for finding a core matching. The procedure is a novel blend of modifications of two celebrated algorithms: the deferred acceptance procedure of Gale and Shapley (1962) and the top trading cycle method from Shapley and Scarf (1974).


Keywords: Marriage, core, stability, mutual consent divorce, partial commitment.

JEL classification: C71, C78, J12.

## 1 Introduction

Marriage is undoubtedly one of the most important human relationships and the cornerstone of most families. In this paper taking the classic model of Gale and Shapley (1962) as a starting point, we examine a natural and practical marriage model which has not been explored previously. Consider a monogamy and free society where there are finitely many singles (unmarried men or women), and married couples. Taking all matters into account, every man has preferences over the women and every woman has preferences over the men. Any man and any woman who both agree to marry each other can proceed to do so, and any man or woman can also withhold their consent and stay single. This inalienable right of pursuing happiness is a widely accepted principle and will be used in the paper as well. The major difficulty of the current model lies in

[^0]the fact that married couples may want to divorce for a variety of reasons and remarry but could not do so freely or irresponsibly. A married person may wish to divorce because her/his circle of friends and acquaintances has become larger and thus she/he may find a more attractive person, or simply because as time goes by, her/his preference has changed considerably and consequently she/he may not like her/his partner as much as before.

Marriage means a solemn commitment in almost every society from the east to the west, from Christianity to Islam, from Buddhism to Judaism. This is clearly reflected on marriage vows and promises. For instance, in China a bride and her bridegroom will say to each other: "I want to hold your hand and with you I will grow old. And I will love you until the seas dry up and the rocks crumble." In the UK, if a wedding ceremony takes place in a church, a bride and her bridegroom will vow to each other: "I take you to be my husband/wife, to have and to hold from this day forward; for better, for worse, for richer, for poorer, in sickness and in health, in sorrow and in joy, to love and to cherish, till death us do part." It is therefore no wonder that in most countries, whether by law or by tradition, divorce is seen negatively or sadly as the impasse of marriage and a breach of promise and is generally discouraged. Divorce is permitted for good reasons but could be a Marathon painful legal battle for the concerned parties if one side of the marriage is unwilling to consent, ${ }^{4}$ implying divorce will make him/her worse off than keeping the marriage. Because marriage is protected by law and seen as a serious commitment and divorce as a failure or an undesirable outcome of marriage, it is plausible to assume that married couples are partially committed in the sense that divorce is possible only if doing so does not hurt either of the two. ${ }^{5}$ Partial commitment can be viewed both as a normative criterion and as approximately a positive description of the spirit of law or practice on marriage and divorce. It is an alternative description of the mutual consent divorce in the current paper or in the context of divorce laws.

In the environment where everyone not only possesses the fundamental right of pursuing happiness but also has an obligation of partial commitment, is there any solution to the intriguing problem of how to match men and women as well as possible? The notion of core brings marital bliss! It is one of the most fundamental solution concepts for stability and efficiency in game

[^1]theory and economics and can be traced back to the idea of contract curve of Edgeworth (1881). This solution has been widely used for general exchange economies and for both transferable and nontransferable utility games; see Gilles (1953), Debreu and Scarf (1963), Scarf (1967), Shapley (1971), Shapley and Shubik (1971), Shapley and Scarf (1974), Quinzii (1984), and Predtetchinski and Herings (2004) amongst others. The current marriage matching problem can be formulated as a nontransferable utility game. Suppose that $\mu^{0}$ is the initial marriage matching. That is, if $\mu^{0}(m)=w$ and $\mu^{0}(w)=m$, we say that man $m$ and woman $w$ are initially married to each other; and if $\mu^{0}(m)=m$ and $\mu^{0}(w)=w$, we say that man $m$ and woman $w$ are initially unmarried. In this environment, a coalition of men and women is permissible if a married person $x$ is in the coalition, then her/his partner $\mu^{0}(x)$ is also in the coalition. Given a marriage matching $\mu$ of all men and women, we say that a permissible coalition (strongly) improves upon the matching $\mu$ if all members in the coalition can arrange a new marriage matching amongst themselves to make at least one (every) member in the coalition strictly better off and none in the coalition worse off. A marriage matching $\mu$ is in the strict core (in the core) if it cannot be (strongly) improved upon by any permissible coalition and every divorced person likes her/his partner at $\mu$ at least as much as hers $/$ his at $\mu^{0}$.

Now it will be helpful to illustrate the above ideas by an example. Consider a society of three men $m_{0}, m_{1}$, and $m_{2}$, and two women $w_{1}$ and $w_{2} . m_{1}$ has married $w_{1}$ and $m_{2}$ has married $w_{2} . m_{0}$ prefers $w_{1}$ to himself; $m_{1}$ prefers $w_{2}$ to $w_{1}$, to himself; $m_{2}$ prefers $w_{1}$ to $w_{2}$, to himself; $w_{1}$ prefers $m_{0}$ to $m_{2}$, to $m_{1}$, to herself; and $w_{2}$ prefers $m_{1}$ to $m_{2}$, to herself. If married couples are partially committed, the two couples will get divorced and remarried. The new marriage matching makes $m_{1}$ marry $w_{2}$, $m_{2}$ marry $w_{1}$, and $m_{0}$ remain single. This is a strict core matching! Not a single individual gets worse off. The initial two couples get divorced and remarried and become strictly better off! If married couples have no commitment at all and therefore can act as if they are singles, then the unique stable marriage matching in the sense of Gale and Shapley (1962) makes $m_{0}$ marry $w_{1}, m_{1}$ marry $w_{2}$, and $m_{2}$ stay single. In this case $m_{2}$ is forced to divorce and gets worse off.

A natural question is whether the core in the marriage model under consideration is always nonempty or not. We will give an affirmative answer to this question. It will be shown that if all individuals have strict preferences, the strict core is not empty, and that if indifference in preferences is allowed, the core is not empty but the strict core may not be necessarily nonempty. A large part of this paper is devoted to developing a procedure that will always find a (strict)
core marriage matching in finite steps. This procedure is a novel blend of modifications of two celebrated methods: the deferred acceptance (DA) procedure of Gale and Shapley (1962) and the top trading cycle (TTC) method of Shapley and Scarf (1974). It should be noted that neither of the two methods alone will be sufficient for our model but a proper mixture of the two will work perfectly. In our procedure chains and cycles in which men and women appear alternatively and reveal their favourite choice and also their willingness to accept will be utilized for producing a (strict) core marriage matching. Cycles and chains will be found by our modified TTC method. Cycles will form part of a core matching and people involved in the core will leave. When a chain is produced, then the modified DA procedure will run and find another part of a core matching, and people involved in the core will leave. The same process will be repeatedly applied to the remaining people until none is left. In this manner, a (strict) core matching can be found in finite rounds. We also examine a variety of properties of the (strict) core and other solutions. For instance, we demonstrate that there does not exist any optimal core matching for men in the sense that every man likes it as least as well as any other core matching, nor any optimal core matching for women in the sense that every woman likes it as least as well as any other core matching. This is in contrast to the model of Gale and Shapley (1962).

We conclude this introductory section by briefly reviewing closely related studies and relevant ones. Our model can be seen as a substantial and practical generalization of the marriage model of Gale and Shapley (1962). They consider a marriage matching model in which there are finitely unmarried men and women. Each man has preferences over the women and each woman has preferences over the men. Each person wants to find a best partner of the opposite sex. In this model, a stable marriage matching is a matching that will not be blocked by any individual or any pair of man and woman. The set of stable matchings coincides with the core of the corresponding nontransferable utility game of the marriage problem. Gale and Shapley develop a method called the deferred acceptance procedure that always finds a stable matching in finite time. The second most closely related study is the marriage model of Becker (1973, 1974, 1981). Becker formulates the problem as a transferable utility assignment market or game of Koopmans and Beckmann (1957) and Shapley and Shubik (1972) and explores its economic implications. The third most related study is the housing market model of Shapley and Scarf (1974). In their model each individual is endowed with a house and nothing else and has preferences over the houses. They prove that the problem is a balanced nontransferable utility game and thus has a nonempty core by invoking the core existence theorem of Scarf (1967) for balanced games. They also give a
constructive proof through a procedure called the top trading cycle method which they attribute to David Gale.

The studies of Gale and Shapley (1962) and Shapley and Scarf (1974) have been very influential in the development of auction, matching and market design. Dubins and Freedman (1981) investigate the strategic issue in the Gale-Shapley marriage model and prove that in the face of the DA procedure it is optimal for every man (woman) to act truthfully as the proposer if women (men) are honest as seconders. Crawford and Knoer (1981) generalize the DA procedure as a price adjustment process to the assignment market mentioned above which allows transferable utilities between buyers and sellers. Kelso and Crawford (1982) extend this multi-item price adjustment process to a more general setting where every bidder (firm) can demand multiple items (workers) and prove their process converges to a competitive equilibrium provided that every bidder (firm) views all items (workers) as substitutes. ${ }^{6}$ Roth (1984) reveals that the DA procedure had in fact been in practical use since 1951 for the assignment of medical interns to hospitals in the United States. The book of Roth and Sotomayor (1990) is a landmark monograph on the Gale-Shapley marriage matching model and two-sided matching models in general. Ma (1994) establishes an axiomatic characterization of the TTC method. Abdulkadiroğlu and Sönmez (1999) extend the TTC method to the house allocation problem such as college dormitories or subsidized public houses with existing tenants. ${ }^{7}$ Chung (2000) identifies the condition of no odd rings for the existence of stable roommate matching which subsumes the Gale-Shapley marriage model. Abdulkadiroğlu and Sönmez (2003) adapt both the DA procedure and the TTC method to the context of school choice. ${ }^{8}$ Roth, Sönmez and Ünver (2004) modify the DA procedure for efficient kidney exchange. Ostrovsky (2008) introduces a general vertical supply chain model and proposes an important generalization of the DA procedure to find a stable matching. ${ }^{9}$ Hitsch, Hortaşu and Ariely (2010) explore a novel dataset obtained from a major online dating service to estimate mate preferences. They use the DA procedure to predict stable matches and find the predicted matches very similar to the actual matches. Kojima and Manea (2010) propose several axioms

[^2]for the DA procedure. Kojima (2011) shows that in the matching problem between schools and students there exists a robustly stable mechanism if and only if the priority structure of schools is acyclic. Voena (2015) conducts an empirical study of how divorce laws in the United States affect couples' intertemporal choices and well-being.

This paper is organised as follows. Section 2 introduces the model and basic concepts. Section 3 establishes the main results. Section 4 discusses the case of preferences with indifference and other issues. The paper concludes in Section 5. Most of the proofs are given in the Appendix of the paper.

## 2 The Model

We study a new and general marriage matching model with existing married couples that goes beyond the celebrated model of Gale and Shapley (1962). Consider a society where there are finitely many unmarried men and women, and married couples. Unmarried men and women are looking for opportunities to marry, and married couples may not be stable and can break up (and thus form new marriages) if the concerned parties agree to do so. More specifically, there are two finite sets of men and women. Some of them are married and other unmarried. Let $M=\left\{m_{1}, \cdots, m_{n}\right\}$ be the set of all men, $W=\left\{w_{1}, \cdots, w_{m}\right\}$ the set of all women, and $A=M \cup W$ the set of all men and women. Every individual tries to find a best partner for herself or himself. An outcome of such endeavours is a set of marriages and will be denoted by a matching. Formally, a matching $\mu$ is a mapping from the set of all men and all women to itself such that $\mu(\mu(x))=x$ for every person $x$, and $\mu(x) \neq x$ implies that $\{x, \mu(x)\}$ must be a pair of man and woman, which will be called a couple. If $\mu(x) \neq x, \mu(x)$ and $x$ are called their respective partner. If $\mu(x)=x, x$ is said to be self-matched or a single. A matching $\mu$ can be written as a collection of couples and singles at $\mu$.

Let $\mu^{0}$ denote the initial marriage matching. A person $x \in A$ is said to be unmarried (married) if $\mu^{0}(x)=x\left(\mu^{0}(x) \neq x\right)$. If $x \in M$ and $x \neq \mu^{0}(x)$, then $x$ is called the husband of $\mu^{0}(x)$ and $\mu^{0}(x)$ the wife of $x$. Each man has strict preferences over the women and himself, and each woman has strict preferences over the men and herself. ${ }^{10} \mathrm{~A}$ man $m$ 's strict preferences can be presented by a list on $W \cup\{m\}$, for instance,

$$
\succ_{m}: w_{1}, w_{2}, m, w_{3}, w_{4}=\mu^{0}(m) .
$$

[^3]A woman $w$ 's strict preferences can be presented by a list on $M \cup\{w\}$, for instance,

$$
\succ_{w}: m_{1}, m_{2}=\mu^{0}(w), m_{3}, w .
$$

For any couple $\{m, w\}$ with $\mu^{0}(m)=w$, if $m \succ_{m} w$ and $w \succ_{w} m$, they can be effectively treated as two single man and woman, because this couple will automatically dissolve the marriage. Thus, in this paper we assume that such couples do not exist in the model. This means that in our model for any couple $\{m, w\}$ with $\mu^{0}(m)=w, m \succ_{m} w$ implies $m \succ_{w} w$ and $w \succ_{w} m$ implies $w \succ_{m} m$. Let $\mathcal{M}=\left(A, \mu^{0}, \succ\right)$ or $\left(M, W, \mu^{0}, \succ\right)$ represent the marriage model where the symbol $\succ=\left(\succ_{x}, x \in A\right)$ stands for the preference profile of all agents.

In many countries, by law or by custom, a couple can negate their marital relation if the concerned two parties consent to do so. The man or woman in a couple is divorceable if none of the two parties gets worse off by breaking up their marriage and they agree to nullify their marriage. Such couples are said to be partially committed, and such a divorce will be called $a$ mutual consent divorce. We say that a woman $w$ is acceptable to a man $m$ if $w \succeq_{m} \mu^{0}(m)$, and is strongly acceptable to a man $m$ if $w \succeq_{m} \mu^{0}(m)$ and $w \succeq_{m} m$. In other words, if a man weakly prefers a woman to his own partner at $\mu^{0}$, then that woman is acceptable to the man. If a man likes a woman at least as well as his partner at $\mu^{0}$ and also as well as remaining single, then that woman is strongly acceptable to the man. Similarly, we say that a man $m$ is acceptable to a woman $w$ if $m \succeq_{w} \mu^{0}(w)$, and is strongly acceptable to a woman $w$ if $m \succeq_{w} \mu^{0}(w)$ and $m \succeq_{w} w$. We say that a matching $\mu$ is proper or weakly individually rational if $\mu(x) \succeq_{x} \mu^{0}(x)$ for every person $x$, and is individually rational if $\mu(x) \succeq_{x} \mu^{0}(x)$ for every person $x$ and further $\mu(x) \succeq_{x} x$ for every person $x$ with $\mu(x) \neq \mu^{0}(x)$ (such a person $x$ is said to be rematched). In other word, a matching $\mu$ is proper if each person has an acceptable partner at $\mu$. Clearly, the initial matching $\mu^{0}$ is proper. A matching $\mu$ is individually rational if it is proper and moreover each rematched person has a strongly acceptable partner. Observe that if a matching $\mu$ is not proper, then some couple at $\mu^{0}$ must be broken at $\mu$ and at least one party becomes worse off, meaning that the mutual consent divorce regime is not respected. We will therefore focus on proper matchings and may omit the term "proper" when no confusion arises. Clearly, the problem of marriage matching is a typical case of competition, cooperation and commitment.

One of the most widely used and most important solution concepts in the environment of competition and cooperation is the notion of core; see for instance Scarf (1967), Shapley (1971), Shapley and Shubik (1971), Shapley and Scarf (1974), Quinzii (1984), and Predtetchinski and Herings (2004). As a prime concept of strategic equilibrium it achieves Pareto efficiency, has
an intuitive explanatory power and is robust against the threat of any coalition deviation. This concept can be adapted to the current marriage matching setting as follows. A nonempty subset $S$ of the set $A$ of men and women is called a coalition. $A$ itself is called the grand coalition. A coalition $S$ is permissible if $x \in S$ implies $\mu^{0}(x) \in S$. This means that when a married person at $\mu^{0}$ contemplates divorcing, she/he should not do so unilaterally but collectively with her/his partner. In the current setting, only permissible coalitions will be considered because such coalitions ensure the two parties of every couple at $\mu^{0}$ to act in accordance with the mutual consent divorce regime. We say that a coalition $S$ improves upon a matching $\mu$ of the grand coalition $A$ if there exists a matching $\tau$ amongst men and women from the coalition alone $S$ such that everyone $x$ in $S$ weakly prefers $\tau(x)$ to $\mu(x)$ and at least one person $y \in S$ prefers $\tau(y)$ to $\mu(y)$. A coalition $S$ strongly improves upon a matching $\mu$ if there exists a matching $\tau$ amongst men and women from the coalition alone such that every person $x$ in $S$ prefers $\tau(x)$ to $\mu(x)$.

Definition 1 A proper matching $\mu$ is in the strict core and is called a strict core matching if it cannot be improved upon by any permissible coalition. It is in the core if it cannot be strongly improved upon by any permissible coalition.

We also say a proper matching $\mu$ is
1A. improved upon by a chain, if there is a sequence of distinct people $\left(x_{1}, x_{2}, \cdots, x_{K}\right)(K \geq 2)$ such that $\mu^{0}\left(x_{k}\right) \in\left\{x_{k-1}, x_{k+1}\right\}$ and $z_{k} \succeq_{x_{k}} \mu\left(x_{k}\right)$ for all $k=1, \cdots, K$, and $z_{k} \succ_{x_{k}} \mu\left(x_{k}\right)$ for some $k=1, \cdots, K$, where $x_{0}=x_{1}, x_{K+1}=x_{K}$, and $z_{k} \in\left\{x_{k-1}, x_{k+1}\right\} \backslash\left\{\mu^{0}\left(x_{k}\right)\right\}$;

1B. improved upon by a cycle, if there is a sequence of an even number of distinct people $\left(x_{1}, y_{1}, x_{2}, y_{2}, \cdots, x_{K}, y_{K}\right)(K \geq 2)$ such that: (i) $\mu^{0}\left(y_{k}\right)=x_{k+1}$ for all $k=1,2, \cdots, K$ (where $x_{K+1}$ denotes $x_{1}$ ); and (ii) $y_{k} \succeq_{x_{k}} \mu\left(x_{k}\right)$ and $x_{k} \succeq_{y_{k}} \mu\left(y_{k}\right)$ for all $k=1, \cdots, K$, $y_{k} \succ_{x_{k}} \mu\left(x_{k}\right)$ or $x_{k} \succ_{y_{k}} \mu\left(y_{k}\right)$ for some $k=1, \cdots, K$.

By 1A, a sequence $\left(x_{1}, x_{2}, \cdots, x_{K-1}, x_{K}\right)$ of $K$ distinct people is a chain if $x_{k}$ is a man (woman), $k=1,2, \cdots, K-1$, then $x_{k+1}$ must be a woman (man), and if $\mu^{0}\left(x_{1}\right)=x_{1}$, then $\mu^{0}\left(x_{2}\right)=x_{3}, \mu^{0}\left(x_{4}\right)=x_{5}, \cdots, \mu^{0}\left(x_{K-1}\right)=x_{K}$ for $K$ being odd, $\mu^{0}\left(x_{K-2}\right)=x_{K-1}, \mu^{0}\left(x_{K}\right)=$ $x_{K}$ for $K$ being even, or if $\mu^{0}\left(x_{1}\right)=x_{2}$, then $\mu^{0}\left(x_{3}\right)=x_{4}, \cdots, \mu^{0}\left(x_{K-1}\right)=x_{K}$ for $K$ being even, $\mu^{0}\left(x_{K-2}\right)=x_{K-1}, \mu^{0}\left(x_{K}\right)=x_{K}$ for $K$ being odd. $x_{k}$ and $x_{k+1}$ are said to be adjacent, $k=1, \cdots, K-1$, and $x_{1}$ and $x_{K}$ are called end members while the rest in between are called intermediate members. Observe that along the chain sex changes alternatively, the two end

$$
\begin{aligned}
& y_{1} \frac{\mu_{0}}{\mu_{2}} x_{2} \frac{\tau}{\mu_{0}} x_{3} \frac{\tau}{\mu_{0}} \cdots \cdots \frac{}{\mu_{0}} x_{K-1} \frac{\tau}{\mu_{0}} y_{K-1} \frac{\mu_{0}}{\mu_{0}} x_{K} \\
& y_{1} \frac{\tau}{\mu_{0}} x_{3} \frac{\tau}{\mu_{2}} \cdots \cdots \frac{\tau}{\mu_{0}} y_{K-1} \frac{\tau}{\mu_{K}} y_{K} \\
& x_{1} \frac{\tau}{\mu_{0}} y_{1} \frac{\tau}{\mu_{0}} x_{2} \frac{\tau}{\mu_{0}} y_{2-1} \frac{\tau}{\mu_{0}} y_{K-1} \frac{}{\mu_{0}} x_{K} \\
& x_{1} \frac{\tau}{\mu_{0}} y_{1} \frac{\tau}{\mu_{0}} y_{2} \frac{\tau \cdots \frac{\tau}{\mu_{0}} y_{K-1} \frac{\mu_{0}}{} x_{K} \frac{\tau}{\tau} y_{K}}{}
\end{aligned}
$$

Figure 1: An illustration of chain improvements upon a matching.
members $x_{1}$ and/or $x_{K}$ could be single at $\mu^{0}$ but all intermediate members are married at $\mu^{0}$. Clearly, if we reverse the chain into $\left(x_{K}, x_{K-1}, \cdots, x_{1}\right)$, this new sequence is also a chain. The family $\left\{x_{1}, x_{2}, \cdots, x_{K}\right\}$ from the two chains is a permissible coalition.

Observe that if the length $K$ of the chain $\left(x_{1}, x_{2}, \cdots, x_{K}\right)$ is odd, exactly one end member must be a single at $\mu^{0}$, and that if $K$ is even, either the two end members are single at $\mu^{0}$ or are both married at $\mu^{0}$. The chain $\left(x_{1}, x_{2}, \cdots, x_{K}\right)$ improving upon the proper matching $\mu$ means that if the end member $x_{1}$ is a single at $\mu^{0}$, then matching $x_{1}$ to $x_{2}, x_{3}$ to $x_{4}, \cdots$, and $x_{K-1}$ to $x_{K}$ for an even $K$, or $x_{K-2}$ to $x_{K-1}$, and $x_{K}$ to $x_{K}$ for an odd $K$ will make no member in the coalition $\left\{x_{1}, x_{2}, \cdots, x_{K}\right\}$ worse off and at least one member strictly better off than they are at $\mu$, or if the end member $x_{1}$ is married at $\mu^{0}\left(\mu^{0}\left(x_{1}\right) \neq x_{1}\right)$, then matching $x_{1}$ to $x_{1}, x_{2}$ to $x_{3}, x_{4}$ to $x_{5}, \cdots, x_{K-2}$ to $x_{K-1}$, and $x_{K}$ to $x_{K}$ for an even $K$, or $x_{K-3}$ to $x_{K-2}$ and $x_{K-1}$ to $x_{K}$ for an odd $K$ will make no member in the coalition $\left\{x_{1}, x_{2}, \cdots, x_{K}\right\}$ worse off and at least one member strictly better off than they are at $\mu$. In particular, if a proper matching $\mu$ is improved upon by a chain $\left(x_{1}, x_{2}\right)$ (i.e., $K=2$ ), then $x_{1}$ and $x_{2}$ must be single at $\mu^{0}$. In Figure 1 we illustrate the four possible cases where a matching is improved upon by a chain.

By 1B, all members $x_{1}, y_{1}, x_{2}, y_{2}, \cdots, x_{K}$ and $y_{K}$ can be split into two groups $\left\{x_{1}, x_{2}, \cdots, x_{K}\right\}$ and $\left\{y_{1}, y_{2}, \cdots, y_{K}\right\}$ of opposite sex and form a cycle so that $x_{k}$ and $y_{k}, k=1,2, \cdots, K$, and $x_{1}$ and $y_{K}$ are next to each other. At $\mu^{0}, y_{1}$ is married to $x_{2}$, and $y_{2}$ to $x_{3}, \cdots$, and $y_{K}$ to $x_{1}$. Moreover, if we rematch $x_{k}$ to $y_{k}$ for every $k$, we can make all members in the permissible coalition $\left\{x_{1}, y_{1}, \cdots, x_{K}, y_{K}\right\}$ better off than they are at $\mu$. Unlike the first case there are no end members and all members are intermediate. $y_{k-1}$ and $x_{k}$ are adjacent, and $x_{k}$ and $y_{k}$ are adjacent so are $x_{1}$ and $y_{K}$. Figure 2 shows the case where a matching is improved upon by a cycle.

From the above discussion, we see that the matching $\mu$ is improved upon by a chain or a cycle through a new matching $\tau$ as constructed above, which is a matching amongst men and


Figure 2: An illustration of a cycle improvement upon a matching.
women from the chain or cycle. From the graphic point of view, the chain or cycle can be easily understood from the combination of the matchings $\mu^{0}$ and $\tau$ as follows: Let $S$ be the coalition formed by all the members from the chain or cycle. We use $S$ as the set of nodes in a bipartite graph where there is an edge between a man in $S$ and a woman $w \in S$ if and only if $m$ and $w$ are matched at $\mu^{0}$ or $\tau$.

Lemma 1 If a proper matching $\mu$ is improved upon by a permissible coalition $S$, it must be improved upon by a chain or by a cycle.

Proof: By definition, there exists a matching $\tau$ amongst people from the coalition $S$ alone such that every person $x$ in $S$ weakly prefers $\tau(x)$ to $\mu(x)$ and at least one person $y \in S$ prefers $\tau(y)$ to $\mu(y)$. Using the matchings $\mu^{0}$ and $\tau$, we can define an undirected bipartite graph $G=(S, E)$ on $S$ by setting $E=\left\{\{m, w\} \mid m \in M \cap S, w \in W \cap S, m=\mu^{0}(w)\right.$ or $\left.m=\tau(w)\right\}$. That is, there is an edge between a man $m \in S$ and a woman $w \in S$ if and only if, they are matched under $\mu^{0}$ or under $\tau$. Choose any person $y \in S$ such that $\tau(y) \succ_{y} \mu(y) \succeq_{y} \mu^{0}(y)$. Note that in this graph $G$ each vertex's degree is less than or equal to 2 . Let $G^{\prime}$ denote the component (the maximal connected subgraph) of $G$ which contains $y$. Then, $G^{\prime}$ must be a chain or a cycle. If $G^{\prime}$ is a chain, it is easy to check that $\mu$ is improved upon by chain $G^{\prime}$. Otherwise, i.e., $G^{\prime}$ is a cycle, then we can also check that matching $\mu$ is proved upon by chain $G^{\prime}$.

By Lemma 1, we further have the following result.
Lemma 2 A proper matching $\mu$ is in the strict core if it cannot be improved upon by any chain or any cycle.

Now we turn to discuss other relevant solution concepts. The notion of stability is originally introduced in Gale and Shapley (1962). A matching $\mu$ is blocked by an individual $x$ if this person
prefers being single to its partner $\mu(x)$, i.e., $x \succ_{x} \mu(x)$. A matching $\mu$ is blocked by a pair of man $m$ and woman $w$ if they are not partner at $\mu$ but prefer each other to their own partner, i.e., $w \succ_{m} \mu(m)$ and $m \succ_{w} \mu(w)$. A matching $\mu$ is stable if it is not blocked by any individual or any pair of man and woman. It is well-known that the family of stable matchings coincides with the core of the marriage matching problem in Gale and Shapley (1962). At this point, one may wonder if it is still appropriate to apply this conventional solution to the current model where existing couples are partially committed. Here we reproduce the example discussed in the introduction to show that Gale and Shapley's notion of stability is incompatible with the mutual consent divorce regime.

Example 1 There are a single man $m_{0}$ and two existing couples $\left\{m_{1}, w_{1}\right\}$ and $\left\{m_{2}, w_{2}\right\}$. The preferences of each individual are given by:

$$
\begin{array}{llllllll}
\succ_{m_{0}}: & w_{1}, & m_{0} & \succ_{w_{1}}: & m_{0}, & m_{2}, & m_{1}, & w_{1} \\
\succ_{m_{1}}: & w_{2}, & w_{1}, & m_{1} & \succ_{w_{2}}: & m_{1}, & m_{2}, & w_{2} \\
\succ_{m_{2}}: & w_{1}, & w_{2}, & m_{2} & & & &
\end{array}
$$

To this example, if we apply Gale and Shapley's solution directly, then $m_{0}$ marries $w_{1}$ and $m_{1}$ marries $w_{2}$, and $m_{2}$ stays single. This is a unique stable matching in the Gale and Shapley sense but $m_{2}$ is forced to divorce and made worse off. However, by respecting the mutual consent principle this example has a unique strict core in which $m_{1}$ marries $w_{2}$ and $m_{2}$ marries $w_{1}$ and $m_{0}$ remains single. The initial two couples get divorced and remarried and become better off.

A person $x \in A$ is said to be free if $\mu^{0}(x) \succeq_{\mu^{0}(x)} x$. In other words, that $x$ is free means that this person can divorce and remarry without any constraint from her/his partner $\mu^{0}(x)$ because $\mu^{0}(x)$ would be happy to end the marriage with $x$. Obviously every single at $\mu^{0}$ is free. Since by hypothesis for any married couple $\{m, w\}$ with $\mu^{0}(m)=w, m \succ_{m} w$ implies $m \succ_{w} w$ and $w \succ_{w} m$ implies $w \succ_{m} m$, we have $\mu^{0}(x) \succeq_{x} x$ for every free person $x$. Clearly, the partner $\mu^{0}(x)$ of a free person $x$ cannot be free. For a given matching $\mu$, a person $x$ is free under $\mu$ if $\mu(x)=\mu^{0}(x) \succeq_{\mu^{0}(x)} x$ or $\mu(x) \neq \mu^{0}(x)$. Note that every rematched person $x$, i.e., $\mu(x) \neq \mu^{0}(x)$, is free under $\mu$. This implies that each person $x$ is partially committed only to its partner at $\mu^{0}$.

A proper matching $\mu$ is said to be
2A. blocked by a chain, if there is a sequence of an even number of distinct people ( $x_{1}, y_{1}, x_{2}$, $\left.y_{2}, \cdots, x_{K}, y_{K}\right)(K \geq 1)$ such that (i) $x_{1}$ and $y_{K}$ are both free under $\mu$, (ii) $x_{k}$ and $y_{k}$ are mutually strongly acceptable, $y_{k} \succ_{x_{k}} \mu\left(x_{k}\right)$ and $x_{k} \succ_{y_{k}} \mu\left(y_{k}\right)$, for all $k=1, \cdots, K$, and (iii) $\mu^{0}\left(y_{k}\right)=x_{k+1}$ for all $k=1,2, \cdots, K-1 ;$

2B. blocked by a cycle, if there is a sequence of an even number of distinct people ( $x_{1}, y_{1}, x_{2}$, $\left.y_{2}, \cdots, x_{K}, y_{K}\right)(K \geq 2)$ such that $x_{k}$ and $y_{k}$ are mutually strongly acceptable, $y_{k} \succ_{x_{k}} \mu\left(x_{k}\right)$ and $x_{k} \succ_{y_{k}} \mu\left(y_{k}\right)$, and $\mu^{0}\left(y_{k}\right)=x_{k+1}$, for all $k=1,2, \cdots, K$ (where $x_{K+1}$ denotes $x_{1}$ ).

A blocking chain $\left(x_{1}, y_{1}, x_{2}, y_{2}, \cdots, x_{K}, y_{K}\right)(K \geq 1)$ has an even number of different people with either $x_{k}, k=1,2, \cdots, K$ being men (women) or exclusively $y_{k}, k=1,2, \cdots, K$ being women (men). Observe that if $\mu^{0}\left(x_{1}\right)$ or $\mu^{0}\left(y_{K}\right)$ is not in $\left\{x_{1}, y_{1}, \cdots, x_{K}, y_{K}\right\}$, the coalition $\left\{x_{1}, y_{1}, \cdots, x_{K}, y_{K}\right\}$ is not permissible. We also allow $\mu^{0}\left(y_{K}\right)=x_{1}$ in which case the coalition $\left\{x_{i}, y_{i}, i=1,2, \cdots, K\right\}$ is permissible and $\mu\left(y_{K}\right) \neq \mu^{0}\left(y_{K}\right)$. If $\mu^{0}\left(y_{K}\right) \neq x_{1}$, it satisfies either $\mu^{0}\left(x_{1}\right)=x_{1}$ or $\mu^{0}\left(x_{1}\right) \notin\left\{x_{1}, y_{1}, \cdots, x_{K}, y_{K}\right\}$, and either $\mu^{0}\left(y_{K}\right)=y_{K}$ or $\mu^{0}\left(y_{K}\right) \notin$ $\left\{x_{1}, y_{1}, \cdots, x_{K}, y_{K}\right\}$. The chain can block the matching $\mu$ because matching $x_{k}$ to $y_{k}, k=$ $1,2, \cdots, K$ makes all $x_{k}$ and $y_{k}, k=1,2, \cdots, K$ better off than being at $\mu$, and at least as well as being at $\mu^{0}$ and being single. This means that every member $x_{k}$ would choose $y_{k}$ if $x_{k}$ faces members $\mu\left(x_{k}\right), \mu^{0}\left(x_{k}\right), y_{k}$, and $x_{k}$; and every member $y_{k}$ would choose $x_{k}$ if $y_{k}$ faces $\mu\left(y_{k}\right)$, $\mu^{0}\left(y_{k}\right), x_{k}$ and $y_{k}$. This notion of blocking chains is a special case of chain blocks introduced by Ostrovsky (2008), who extends the concept of stability of Gale and Shapley (1962) for a supply chain network model.

The cycle contains an even number of different people with either $x_{i}, i=1,2, \cdots, K$ being men (women) or exclusively $y_{i}, i=1,2, \cdots, K$ being women (men). The coalition $\left\{x_{1}, y_{1}, \cdots, x_{K}, y_{K}\right\}$ is permissible and can block the matching $\mu$ because pairing $x_{i}$ and $y_{i}, i=1,2, \cdots, K$ makes all $x_{i}$ and $y_{i}, i=1,2, \cdots, K$ better off than being at $\mu$, and at least as well as being at $\mu^{0}$ and being single.

A blocking chain is also a blocking cycle if $\mu^{0}\left(y_{K}\right)=x_{1}$, and a blocking cycle becomes a blocking chain if two members in the cycle are married to each other at $\mu^{0}$ but free under $\mu$. Notice that if a matching is blocked by a cycle, it is strongly improved upon by the cycle. Similar to the discussion of improvement upon by a chain or cycle we can also construct a graph to see how a blocking chain or cycle emerges.

Observe that every member in a blocking chain has incentive to deviate from the matching $\mu$ and the end members $x_{1}$ and $y_{K}$ can initiate the deviation because the two have both incentive and freedom to make proposals to $y_{1}$ and $x_{K}$ respectively as $x_{1}$ and $y_{K}$ are free under $\mu$. In this sense $\mu$ is highly unstable. Every member in a blocking cycle has also incentive to deviate from the matching $\mu$, but this blocking cycle may be hard to form because unlike a blocking chain we do not assume that there is any free member in the cycle who can initiate a deviation.

Furthermore, it is not difficult to see that both blocking chains and blocking cycles comply with the mutual consent divorce regime.

Definition 2 A matching $\mu$ is weakly stable if it is individually rational and it is not blocked by any chain.

Definition 3 A matching $\mu$ is stable if it is individually rational and it is not blocked by any chain or cycle.

It is obvious that a stable matching must be weakly stable. The following example demonstrates that a stable matching may not be in the strict core and a strict core matching may not be weakly stable.

Example 2 Consider a society with two married couples $\left\{m_{1}, w_{1}\right\}$ and $\left\{m_{2}, w_{2}\right\}$. The preferences of each individual are given by

$$
\begin{array}{llllllll}
\succ_{m_{1}}: & w_{2}, & w_{1}, & m_{1} & \succ_{w_{1}}: & w_{1}, & m_{2}, & m_{1} \\
\succ_{m_{2}}: & w_{1}, & w_{2}, & m_{2} & \succ_{w_{2}}: & w_{2}, & m_{1}, & m_{2}
\end{array}
$$

In this example the existing matching $\mu^{0}$ is stable, but is not in the strict core, because the grand coalition can improve upon $\mu^{0}$. The matching $\mu$ defined by $\mu\left(m_{1}\right)=w_{2}$ and $\mu\left(m_{2}\right)=w_{1}$ is in the strict core, but it is not individually rational and therefore not weakly stable, because $w_{1}$ prefers being single to $m_{2}$ and $m_{1}$.

We now turn to Example 1. In this example, there are two proper matchings

$$
\mu^{0}=\left(\begin{array}{ccc}
m_{0} & m_{1} & m_{2} \\
m_{0} & w_{1} & w_{2}
\end{array}\right) \quad \text { and } \quad \mu^{1}=\left(\begin{array}{ccc}
m_{0} & m_{1} & m_{2} \\
m_{0} & w_{2} & w_{1}
\end{array}\right)
$$

There is no stable matching, because $\mu^{0}$ is blocked by the cycle ( $m_{1}, w_{2}, m_{2}, w_{1}$ ) and $\mu^{1}$ is blocked by the chain $\left(m_{0}, w_{1}\right) . \mu^{0}$ is weakly stable but neither stable nor in the strict core, because it can be improved upon and blocked by the cycle $\left(m_{1}, w_{2}, m_{2}, w_{1}\right) . \mu^{1}$ is in the strict core and is not weakly stable, because $\mu^{1}$ can be blocked by the chain $\left\{m_{0}, w_{1}\right\}$.

Let $C(\mathcal{M}), S C(\mathcal{M}), S(\mathcal{M})$, and $W S(\mathcal{M})$ be the core, the strict core, the set of stable matchings, and the set of weakly stable matchings of the model $\mathcal{M}=\left(A, \mu^{0}, \succ\right)$, respectively. Examples 1 and 2 above indicate that we may have $S C(\mathcal{M}) \backslash W S(\mathcal{M}) \neq \emptyset$ and $S(\mathcal{M}) \backslash S C(\mathcal{M}) \neq \emptyset$ and $S(\mathcal{M})=\emptyset$. In general we have

Proposition 1 For the marriage model $\mathcal{M}=\left(A, \mu^{0}, \succ\right)$, its core $C(\mathcal{M})$ does not intersect with those weakly stable matchings which are not stable, i.e., $C(\mathcal{M}) \cap(W S(\mathcal{M}) \backslash S(\mathcal{M}))=\emptyset$.

## 3 The Main Results

We are now ready to present the first major existence theorem of this paper.
Theorem 1 The strict core of the marriage model $\mathcal{M}=\left(A, \mu^{0}, \succ\right)$ is not empty.
The proof of this result will be deferred to the appendix of the paper and is an inductive one. The proof itself is not only interesting in its own right but also yields a procedure for finding a strict core matching in finite time. In the rest of this section we will discuss this procedure in detail. The procedure turns out to be a blend of modifications of two celebrated algorithms: the deferred acceptance procedure of Gale and Shapley (1962) and the top trading cycle procedure of Shapley and Scarf (1974). It will be shown that neither of the two alone suffices to serve our purpose.

We first introduce the following modification of the original deferred acceptance procedure of Gale and Shapley (1962). Kojima and Manea (2010) have axiomatised the DA procedure.

## The Men Proposing Deferred Acceptance (MP-DA) Procedure

- At the beginning, every free man proposes to the woman who is his most preferred amongst all his strongly acceptable women. Each woman rejects the proposal of any man who is not her partner at $\mu^{0}$ nor strongly acceptable to her. Any woman receiving more than one proposal rejects all but her most preferred of these and keeps him engaged, with the constraint that if she receives a proposal from her husband at $\mu^{0}$, she should treat him as her unique favourite man.
- At any step, every man who has just become a free man ${ }^{11}$ or who was rejected at the previous step proposes to his remaining favourite woman, as long as there is a strongly acceptable woman to whom he has not yet proposed. Each woman receiving new proposals rejects the proposal of any man who is not strongly acceptable to her, and also rejects all but her most preferred amongst those men who have just proposed to her and are strongly acceptable to her, as well as any man she may have kept engaged from the previous step, with the constraint that if she receives a proposal from her husband at $\mu^{0}$, she should view her husband as her unique favorite man.

[^4]- The process stops as soon as there is no new proposal from any man.

To facilitate a good understanding of the procedure we run it through an example.

Example 3 Consider a marriage model with four initial couples $\left\{m_{1}, w_{1}\right\},\left\{m_{2}, w_{2}\right\},\left\{m_{3}, w_{3}\right\}$ and $\left\{m_{4}, w_{4}\right\}$, two single men $m_{5}, m_{6}$, and two single women $w_{5}, w_{6}$. Their preferences are given by

$$
\left.\begin{array}{llllllll}
\succ_{m_{1}}: & w_{2}, & w_{1}, & m_{1} & \succ_{w_{1}}: & m_{5}, & m_{2}, & m_{1}, \\
\succ_{m_{2}}: & w_{1}, & w_{2}, & m_{2} & & \succ_{w_{2}}: & m_{1}, & m_{2}, \\
\succ_{2}
\end{array}\right]
$$

First round: Only $m_{5}$ and $m_{6}$ are free men. $m_{5}$ proposes to $w_{1}$ and $m_{6}$ proposes to $w_{6} ; w_{1}$ keeps $m_{5}$ engaged and $w_{6}$ keeps $m_{6}$ engaged. Now $m_{1}$ becomes a free man.

Second round: $m_{1}$ proposes to $w_{2} . w_{2}$ keeps $m_{1}$ engaged. Now $m_{2}$ becomes free.
Third round: $m_{2}$ proposes to $w_{1} . w_{1}$ rejects $m_{2}$ and keeps $m_{5}$ engaged.
Fourth round: $m_{2}$ proposes to $w_{2}$. Observe that $w_{2}$ has to reject $m_{1}$ and keeps $m_{2}$ engaged because $m_{2}$ is her husband at $\mu^{0}$.

Fifth round: $m_{1}$ proposes to $w_{1} . w_{1}$ rejects $m_{5}$ and keeps $m_{1}$ engaged.
Sixth round: $m_{5}$ proposes to $w_{6} . w_{6}$ rejects $m_{5}$ and keeps $m_{6}$ engaged.
Seventh round: The procedure terminates with a weakly stable matching (not a core matching) given by

$$
\mu=\left(\begin{array}{ccccccc}
m_{1} & m_{2} & m_{3} & m_{4} & m_{6} & m_{5} & w_{5} \\
w_{1} & w_{2} & w_{3} & w_{4} & w_{6} & m_{5} & w_{5}
\end{array}\right) .
$$

Analogously one can introduce the woman proposing deferred acceptance (WP-DA) procedure.

Lemma 3 The MP-DA procedure finds a weakly stable matching in finitely many steps.

We now turn to present a modification of the original top trading cycle method of Shapley and Scarf (1974) which will be an integral part of our hybrid procedure. Ma (1994) has given
an axiomatic characterization of the TTC method. Notice that in the following procedure each man makes choice amongst his strongly acceptable women, while each woman concerns only her acceptable men.

## The Man Proposing Top Trading Cycle (MP-TTC) Procedure

- At the first step, every man proposes to his most preferred amongst all his strongly acceptable women and every woman rejects the proposal of any man who is unacceptable to her.
- At any step, any man who was rejected at the previous step proposes to his next choice, as long as there remains a strongly acceptable woman to whom he has not yet proposed. Each woman receiving new proposals rejects the proposal from any man unacceptable to her.
- The process stops as soon as there is no new proposal from any man.

Analogously one can define the women proposing top trading cycle (WP-TTC) procedure.
We say that a single person $x$ is isolated if $\mu(x)=x$ for any proper matching $\mu$, and that a married couple $\{m, w\}$ is isolated if $\mu(m)=w$ for any proper matching $\mu$. Note that if a single man $m$ is rejected by all his strongly acceptable women in the MP-TTC procedure, then he is an isolated man. And, if a married man $m$ has proposed to his wife $\mu^{0}(m)$ in the MP-TTC procedure, then $\left\{m, \mu^{0}(m)\right\}$ is an isolated couple.

A sequence of an even number of distinct people ( $\left.\bar{m}_{1}, \bar{w}_{1}, \bar{m}_{2}, \bar{w}_{2}, \cdots, \bar{m}_{K}, \bar{w}_{K}\right)(K \geq 2)$ is called an MP-TTC cycle, if for each $k=1, \cdots, K, \bar{m}_{k}$ is acceptable to $\bar{w}_{k}, \bar{w}_{k}$ is the favorite strongly acceptable woman of $\bar{m}_{k}$ amongst those women who accept him, and $\mu^{0}\left(\bar{w}_{k}\right)=\bar{m}_{k+1}$, where $\bar{m}_{K+1}=\bar{m}_{1}$. Similarly, a sequence of an even number of distinct people ( $\bar{w}_{1}, \bar{m}_{1}, \bar{w}_{2}, \bar{m}_{2}, \cdots, \bar{w}_{K}, \bar{m}_{K}$ ) $(K \geq 2)$ is called a WP-TTC cycle, if for each $k=1, \cdots, K, \bar{w}_{k}$ is acceptable to $\bar{m}_{k}, \bar{m}_{k}$ is the favorite strongly acceptable man of $\bar{w}_{k}$ amongst those men who accept her, and $\mu^{0}\left(\bar{m}_{k}\right)=\bar{w}_{k+1}$, where $\bar{w}_{K+1}=\bar{w}_{1}$. A sequences of an even number of distinct people ( $\bar{m}_{1}, \bar{w}_{1}, \bar{m}_{2}, \bar{w}_{2}, \cdots, \bar{m}_{K}, \bar{w}_{K}$ ) ( $K \geq 1$ ) is called an MP-TTC chain, if it satisfies that (i) $\bar{m}_{k}$ is acceptable to $\bar{w}_{k}, \bar{w}_{k}$ is the strongly acceptable favorite woman of $\bar{m}_{k}$ amongst those women who accept him, for all $k=1,2, \cdots, K$; (ii) both $\bar{m}_{1}$ and $\bar{w}_{K}$ are free, and $\mu^{0}\left(\bar{w}_{k}\right)=\bar{m}_{k+1}$ for all $k=1,2, \cdots, K-1$. An MP-TTC chain $\left(\bar{m}_{1}, \bar{w}_{1}, \bar{m}_{2}, \bar{w}_{2}, \cdots, \bar{m}_{K}, \bar{w}_{K}\right)$ is further called a minimum MP-TTC chain if every man $\bar{m}_{k}$
$(k=2,3, \cdots, K)$ except $\bar{m}_{1}$ is not free. Similarly, we can define a WP-TTC chain and a minimum WP-TTC chain.

We illustrate the MP-TTC procedure and the WP-TTC procedure through Example 3.

First round: In the MP-TTC procedure, $m_{1}$ proposes to $w_{2}, m_{2}$ proposes to $w_{1}, m_{3}$ proposes to $w_{5}, m_{4}$ proposes to $w_{3}, m_{5}$ proposes to $w_{1}$, and $m_{6}$ proposes to $w_{6}$.

Second round: The MP-TTC procedure stops with the MP-TTC cycle $\left(m_{1}, w_{2}, m_{2}, w_{1}\right)$, yielding the matching

$$
\mu^{1}=\left(\begin{array}{llllllllll}
m_{1} & m_{2} & m_{3} & m_{4} & m_{5} & m_{6} & w_{3} & w_{4} & w_{5} & w_{6} \\
w_{2} & w_{1} & m_{3} & m_{4} & m_{5} & m_{6} & w_{3} & w_{4} & w_{5} & w_{6}
\end{array}\right)
$$

which is not in the core.
First round: In the WP-TTC procedure $w_{1}$ proposes to $m_{5}$, $w_{2}$ proposes to $m_{1}, w_{3}$ proposes to $m_{4}, w_{4}$ proposes to $m_{3}, w_{5}$ proposes to $m_{3}$, and $w_{6}$ proposes to $m_{6}$.

Second round: The WP-TTC procedure terminates with the WP-TTC cycle $\left(w_{4}, m_{3}, w_{3}, m_{4}\right)$, yielding the matching

$$
\mu^{2}=\left(\begin{array}{llllllllll}
m_{3} & m_{4} & m_{1} & m_{2} & m_{5} & m_{6} & w_{1} & w_{2} & w_{5} & w_{6} \\
w_{4} & w_{3} & m_{1} & m_{2} & m_{5} & m_{6} & w_{1} & w_{2} & w_{5} & w_{6}
\end{array}\right)
$$

which is not a core matching.

Observe that isolated people, isolated couples, and MP-TTC (WP-TTC) cycles are disjoint from each other. In a minimum MP-TTC chain, each pair $\bar{m}_{k}$ and $\bar{w}_{k}$ are mutually strong acceptable. And so every minimum MP-TTC chain is a blocking chain of the original matching $\mu^{0}$. Concerning isolated persons, MP-TTC or WP-TTC cycles, MP-TTC or WP-TTC chains, we have the following results.

## Lemma 4

1. Every MP-TTC chain contains a minimum MP-TTC chain, and every WP-TTC chain contains a minimum WP-TTC chain.
2. Every minimum $M P$ - or WP-TTC chain is a blocking chain of the original matching $\mu^{0}$.
3. If there is an MP-TTC chain or a WP-TTC chain, the outcome yielded by the MP-DA or $W P-D A$ procedure will not coincide with $\mu^{0}$.

Proposition 2 Implementing the MP-TTC and WP-TTC procedures each at most once yields at least one of the following results:

1. an isolated person or an isolated couple;
2. an MP-TTC cycle or a WP-TTC cycle;
3. an MP-TTC chain or a WP-TTC chain.

From Proposition 2 and Lemma 4, we have the following result.

Proposition 3 In the third case of Proposition 2, implementing the MP-DA or WP-DA procedure must generate a weakly stable matching $\mu$ different from the initial matching $\mu^{0}$.

We can now give the basic idea of our inductive proof of Theorem 1. Consider the marriage model $\mathcal{M}$. Clearly the theorem holds true trivially if the model contains only one agent. Suppose that the result is true for any group of agents. We need to consider two cases.

Case 1: If there is any isolated single, or any isolated couple, or any MP-TTC cycle, or any WP-cycle, we match every woman in each WP-TTC cycle to her favorite man on the cycle and every man in each MP-TTC cycle to his favorite woman on the cycle. These isolated singles, isolated couples, and newly formed couples form a matching $\tau^{1}$. Remove them from the model. The remaining agents in the model have a strict core matching $\tau^{2}$ by hypothesis. We can show that the union of the two matchings $\tau^{1}$ and $\tau^{2}$ is a strict core matching for the model $\mathcal{M}$.

Case 2: If Case 1 does not happen, there must be a weakly stable matching $\tau \neq \mu^{0}$. Make new couples for those $x$ satisfying $\tau(x) \neq \mu^{0}(x)$ from $\tau$ and new singles $x$ satisfying $\tau(x)=x \neq \mu^{0}(x)$ from $\tau$. These new couples and new singles form a matching $\tau^{1}$. Take them out of the model. The remaining agents in the model have a strict core matching $\tau^{2}$. Again, we can show that the union of the two matchings $\tau^{1}$ and $\tau^{2}$ is a strict core matching for the model $\mathcal{M}$.

As shown above, the MP-TTC, WP-TTC, and MP-DA or WP-DA procedures can find isolated singles, or isolated couples, or MP-TTC cycles, or WP-TTC cycles, or weakly stable matchings. The inductive proof above in fact indicates how to make use of these procedures to find a strict core matching of the current marriage matching problem. This is summarized in the following hybrid procedure.

## The Hybrid Procedure of TTC and DA for Finding a Strict Core Matching

Step 0. Consider any given marriage matching model in which there are many unmarried men and women, and married couples.

Step 1. Implement the MP-TTC procedure to all the remaining men and women. If neither an isolated person nor an isolated couple nor an MP-TTC cycle is found, go to Step 2. Otherwise, all isolated people and isolated couples should leave. If there is an MP-TTC cycle, then match every man of the cycle to his favorite woman on the cycle and ask the newly formed couples to leave. If all people have gone, go to Step 4. Otherwise, return to Step 1.

Step 2. Implement the WP-TTC procedure to the remaining people. ${ }^{12}$ If neither an isolated person nor an isolated couple nor a WP-TTC cycle is found, go to Step 3. Otherwise, all isolated people and isolated couples should leave. If there is a WP-TTC cycle, then match every woman in each WP-TTC cycle to her favorite man on the cycle and ask the newly formed couples to leave. If all people have left, go to Step 4. Otherwise, return to Step 1.

Step 3. Implement the MP-DA procedure to the remaining people, and find a weakly stable matching $\tau$. Make couples for those $x$ satisfying $\tau(x) \neq \mu^{0}(x)$ from this weakly stable matching and ask all the newly formed couples and new singles $x$ satisfying $\tau(x)=x \neq \mu^{0}(x)$ to leave. If all people have gone, go to Step 4. Otherwise, return to Step 1.

Step 4. Stop.

We use again Example 3 to demonstrate that all MP-TTC, WP-TTC and DA procedures have to be used in order to find a strict core matching. The hybrid procedure runs as follows:

First round: The MP-TTC procedure is implemented and finds an MP-TTC cycle $\tau^{1}=$ $\left(m_{1}, w_{2}, m_{2}, w_{1}\right)$. We form new couples $\left\{m_{1}, w_{2}\right\}$ and $\left\{m_{2}, w_{2}\right\}$ and remove them from the model.

Second round: The WP-TTC procedure is implemented and finds a WP-TTC cycle $\tau^{2}=$ $\left(w_{4}, m_{3}, w_{3}, m_{4}\right)$. New couples $\left\{m_{3}, w_{4}\right\}$ and $\left\{m_{4}, w_{3}\right\}$ are formed and removed.

Third round: The MP-DA procedure is implemented and finds a weakly stable matching $\tau^{3}=\left\{\left\{m_{5}, m_{5}\right\},\left\{w_{5}, w_{5}\right\},\left\{m_{6}, w_{6}\right\}\right\}$. A new couple $\left\{m_{6}, w_{6}\right\}$ is made and removed.

Fourth round: The MP-TTC is run and finds two isolated single persons $m_{5}$ and $w_{5}$. Remove these two people and stop the procedure.

[^5]Finally a strict core matching (in fact, the unique one) is found and given by

$$
\mu=\left(\begin{array}{ccccccc}
m_{1} & m_{2} & m_{3} & m_{4} & m_{6} & m_{5} & w_{5} \\
w_{2} & w_{1} & w_{4} & w_{3} & w_{6} & m_{5} & w_{5}
\end{array}\right) .
$$

The hybrid procedure can be seen as a process of rematching and removing men and women in the sense that forming new couples and making singles is rematching while asking them to leave is removing.

Theorem 2 The hybrid procedure always finds a strict core matching of the marriage problem in finite time.

Proof: The hybrid procedure of TTC and DA is a process of removing and rematching men and women. Suppose all people are removed in $T$ rounds. For each $t=1,2, \cdots, T$, let $A_{t}$ denote the set of all removed people at round $t$ and $A^{(t)}$ denote the set of all remaining people at the end of round $t$. We then see that each $A_{t}$ and each $A^{(t)}$ is a permissible coalition, and especially $A^{(T)}=\emptyset$. Let $\mathcal{M}^{(t)}=\mathcal{M}\left(A^{(t-1)}\right)$ denote the sub-model composed of people in $A^{(t-1)}$, where $A^{(0)}=A$ and $\mathcal{M}^{(1)}=\mathcal{M}$. Note that every person $x \in A_{t}$ belongs to one of the following cases:
(1) $x$ is an isolated person in $\mathcal{M}^{(t)}$;
(2) $x$ is a married person of an isolated couple in $\mathcal{M}^{(t)}$;
(3) $x$ is in an MP-TTC (or WP-TTC) cycle of $\mathcal{M}^{(t)}$;
(4) there is a weakly stable matching $\tau_{t}$ of $\mathcal{M}^{(t)}$ such that $\tau_{t}(x) \neq \mu^{0}(x)$ for all $x \in A_{t}$.

Thus, we can define a matching $\rho_{t}$ on $A_{t}$ as follows:
(1) $\rho_{t}(x)=x$, if $x$ is an isolated person in $\mathcal{M}^{(t)}$;
(2) $\rho_{t}(x)=\mu^{0}(x)$, if $x$ is in an isolated couple;
(3) $\rho_{t}(x)\left(\neq \mu^{0}(x)\right)$ is the adjacent person next to $x$ in the cycle, if $x$ is in an MP-TTC (or WP-TTC) cycle;
(4) $\rho_{t}(x)=\tau_{t}(x)$ for each $x \in A_{t}$, otherwise.

At Step 1 and Step 2 of the hybrid procedure, we may remove more than one isolated person, isolated couple, and MP-TTC (or WP-TTC) cycle simultaneously. However, for simplicity, at

Step 1 and Step 2 we can just remove one isolated person, or one isolated couple, or one MPTTC (or WP-TTC) cycle. Recall that $A_{T}=A^{(T-1)}$. Thus, if $A_{T}$ is removed at Step 1 or Step $2, \mu_{T}=\rho_{T}$ is clearly a core matching of $\mathcal{M}^{(T)}$. Otherwise, $A_{T}$ is removed at Step 3. That is, at Step 3 we find a weakly stable matching $\mu_{T}=\rho_{T}=\tau_{T}$ of $\mathcal{M}^{(T)}$ such that $\tau_{T}(x) \neq \mu^{0}(x)$ for each $x \in A^{(T-1)}$. Thus, the proper matching $\mu_{T}$ is a standard Gale-Shapley stable matching of $\mathcal{M}^{(T)}$, and so is in the strict core $\mathcal{M}^{(T)}$.

Let us define a matching $\mu_{T-1}$ of $\mathcal{M}^{(T-1)}$ as follows:

$$
\mu_{T-1}(x)= \begin{cases}\rho_{T-1}(x) & \text { if } x=A_{T-1} \\ \mu_{T}(x) & \text { if } x \in A^{(T-1)}\end{cases}
$$

Then, by the proof of Theorem 1, we can show that $\mu_{T-1}$ is a strict core matching of $\mathcal{M}^{(T-1)}$. Iteratively, for each $t=T-1, T-2, \cdots, 1$, we can show that the matching $\mu_{t}$ defined by

$$
\mu_{t}(x)= \begin{cases}\rho_{t}(x) & \text { if } x=A_{t} \\ \mu_{t+1}(x) & \text { if } x \in A^{(t)}\end{cases}
$$

is a strict core matching of $\mathcal{M}^{(t)}$. As a result, the hybrid procedure finds a strict core matching $\mu_{1}$ of $\mathcal{M}=\mathcal{M}^{(1)}$.

## 4 General Preferences with Indifference and Other Issues

In the previous sections we have assumed that every man or woman has strict preferences. We are now going to relax this assumption by allowing indifference in every person's preferences. In this more general environment, we can establish the following existence of a nonempty core.

Theorem 3 The core of the marriage model $\left(A, \mu^{0}, \succeq\right)$ is not empty under preferences with indifference.

Proof: If an agent is indifferent between several choices, we use a tie-breaking rule as long as it does not affect the part of her strict preferences. In this way we generate a new model $\left(A, \mu^{0}, \succ\right)$ with strict preferences. By Theorem 1, the new model has a nonempty strict core. Take any strict core matching $\mu$. We will show $\mu$ is a core matching of the original model. Suppose that $\mu$ is not a core matching. Then $\mu$ must be strictly improved upon by a permissible coalition. Clearly, $\mu$ must be improved upon by the same permissible coalition with respect to the strict preference profile $\succ$, contradicting that $\mu$ is a strict core of the model $\left(A, \mu^{0}, \succ\right)$.

The following result shows that the tie-breaking rule has an effect on those individuals who may be indifferent between their partners at $\mu^{0}$ and others.

Proposition 4 For any agent $x$ who is different between her partner $\mu^{0}(x)$ and others, if the tie-breaking rule puts her partner $\mu^{0}(x)$ before any other whom she ranks equally as her partner, she will be strictly better off in any core matching $\mu$ than her status quo if she is rematched, i.e., $\mu(x) \neq \mu^{0}(x)$.

The following simple example indicates that indifference in preferences may fail the existence of a nonempty strict core.

Example 4 Consider a society with two single men $m_{1}$ and $m_{2}$ and two single women $w_{1}$ and $w_{2}$. Their preferences are given by

$$
\begin{array}{llllll}
\succ_{m_{1}}: & w_{1}, & w_{2}, & m_{1} & \succ_{w_{1}}: & {\left[m_{1}, m_{2}\right],} \\
\succ_{1} \\
\succ_{m_{2}}: & w_{1}, & w_{2}, & m_{2} & \succ_{w_{2}}: & {\left[m_{1}, m_{2}\right],} \\
w_{2}
\end{array}
$$

In this example both $w_{1}$ and $w_{2}$ are indifferent between $m_{1}$ and $m_{2}$. There are two core matchings

$$
\mu^{1}=\left(\begin{array}{cc}
m_{1} & m_{2} \\
w_{1} & w_{2}
\end{array}\right) \quad \text { and } \quad \mu^{2}=\left(\begin{array}{cc}
m_{1} & m_{2} \\
w_{2} & w_{1}
\end{array}\right)
$$

But neither of the matchings is a strict core matching, because for instance $\left\{m_{2}, w_{1}\right\}$ can improve upon $\mu^{1}$.

However, the following easy observation says that for the Gale-Shapley marriage model its core and its strict core are equal under strict preferences.

Proposition 5 The core coincides with the strict core for the Gale-Shapley marriage model under strict preferences.

The above result fails to be true for the current marriage model with existing couples. For the current model, the core in general can be strictly larger than the strict core even under strict preferences.

Example 5 Consider a society with three existing couples $\left\{m_{1}, w_{1}\right\},\left\{m_{2}, w_{2}\right\}$ and $\left\{m_{3}, w_{3}\right\}$. All agents have strict preferences as follows:

$$
\begin{array}{lllllllll}
\succ_{m_{1}}: & w_{2}, & w_{1}, & m_{1} & \succ_{w_{1}}: & m_{3}, & m_{2}, & m_{1}, & w_{1} \\
\succ_{m_{2}}: & w_{3}, & w_{1}, & w_{2}, & m_{2} & \succ_{w_{2}}: & m_{1}, & m_{2}, & w_{2} \\
\succ_{m_{3}}: & w_{1}, & w_{3}, & m_{3} & \succ_{w_{3}}: & m_{2}, & m_{3}, & w_{3}
\end{array}
$$

In this example there are two core matchings

$$
\mu^{1}=\left(\begin{array}{ccc}
m_{1} & m_{2} & m_{3} \\
w_{2} & w_{1} & w_{3}
\end{array}\right) \quad \text { and } \quad \mu^{2}=\left(\begin{array}{ccc}
m_{1} & m_{2} & m_{3} \\
w_{2} & w_{3} & w_{1}
\end{array}\right)
$$

amongst which $\mu^{2}$ is a strict core matching.
A well-known feature of the Gale-Shapley marriage model under strict preferences is the lattice structure of stable marriage matchings; see e.g., Roth and Sotomayor (1990). However, the current model with existing couples does not possess this lattice property as shown next.

Example 6 Consider a society with an initial matching $\mu^{0}$ of three existing couples $\left\{m_{1}, w_{1}\right\}$, $\left\{m_{2}, w_{2}\right\}$ and $\left\{m_{3}, w_{3}\right\}$. All agents have strict preferences as follows:

$$
\begin{array}{lllllllll}
\succ_{m_{1}}: & w_{2}, & w_{1}, & m_{1} & \succ_{w_{1}}: & m_{2}, & m_{1}, & w_{1} \\
\succ_{m_{2}}: & w_{1}, & w_{3}, & w_{2}, & m_{2} & \succ_{w_{2}}: & m_{3}, & m_{1}, & m_{2}, \\
\succ_{m_{3}}: & w_{2}, & w_{3}, & m_{3} & \succ_{w_{3}}: & m_{2}, & m_{3}, & w_{3}
\end{array}
$$

In this example core and strict core coincide. There are only two (strict) core matchings

$$
\mu^{1}=\left(\begin{array}{ccc}
m_{1} & m_{2} & m_{3} \\
w_{2} & w_{1} & w_{3}
\end{array}\right) \quad \text { and } \quad \mu^{2}=\left(\begin{array}{ccc}
m_{1} & m_{2} & m_{3} \\
w_{1} & w_{3} & w_{2}
\end{array}\right) .
$$

In fact, $\mu^{1}$ can be found by the hybrid procedure and $\mu^{2}$ can be also found by the procedure by running first the WP-TTC procedure and then the MP-TTC procedure. Note that men $m_{1}$, $m_{2}$ and woman $w_{1}$ prefer $\mu^{1}$ to $\mu^{2}$, while women $w_{2}, w_{3}$ and man $m_{3}$ prefer $\mu^{2}$ to $\mu^{1}$. Clearly, there is no (strict) core matching which is preferred by all men or all women to another strict matching. This demonstrates that in our setting the (strict) core does not exhibit the lattice structure.

In this example, $\mu^{0}, \mu^{1}$ and $\mu^{3}$ are the only weakly stable matchings and amongst them $\mu^{1}$ and $\mu^{2}$ are stable. The lattice property does not hold either.

## 5 Conclusion

We conclude by summarizing the main contributions of this paper. We introduce a general marriage matching problem in which there are finitely many unmarried men and women, and married men and women. By allowing existing couples to divorce and remarry this model goes beyond the celebrated model of Gale and Shapley (1962) in which there are only unmarried men and women. This is a novel and natural generalization of Gale and Shapely's model and makes the model more practical and closer to the reality. Each person wishes to find a best possible
partner of the opposite sex to marry. We analyze the model under the mutual consent divorce regime in the sense that a married couple can divorce and thus remarry if divorce can make the concerned two parties at least as well as they maintain the status quo. The central problem is how to make marriages between men and women as well as possible. To tackle this problem, we adapt the fundamental solution of core to the current setting. Given the initial marriage matching $\mu^{0}$, a coalition of men and women is permissible if the coalition contains both $x$ and its partner $\mu^{0}(x)$ for every member $x$ of the coalition. $\mu^{0}(x)=x$ is allowed, i.e., $x$ is a single. A matching $\mu$ is in the strict core (in the core) if it cannot be (strongly) improved upon by any permissible coalition and every person $x$ likes its partner $\mu(x)$ at least as much as its partner $\mu^{0}(x)$.

We have shown that the model has a nonempty strict core if every person has strict preferences and that it has a nonempty core if indifference is allowed in everyone's preferences. An iterative procedure is proposed that can always find a (strict) core marriage matching in finite time. This procedure is a mixture of modifications of two famous algorithms: the deferred acceptance procedure of Gale and Shapley (1962) and the top trading cycle method of Shapley and Scarf (1974). We also demonstrate that neither the deferred acceptance procedure nor the top trading cycle method guarantees to find a core matching in the current model. Following Gale and Shapley (1962) and Ostrovsky (2008) we also introduce the notion of stable matching in the current setting. We establish the existence of a weakly stable matching. However, stable matchings may not exist in general nor are stronger than (strict) core matchings. Unlike the model of Gale and Shapley (1962), due to the existing couples the (strict) core in the current model does not exhibit the lattice structure and therefore does not have the polarization of interests between men and women. We also discuss other similarities and differences between (strict) core matchings and (weakly) stable matchings.

## The Appendix

Proof of Lemma 3: The deferred acceptance procedure must stop in finite time because there is only a finite number of men and women, and no man proposes more than once to any woman. The outcome that it produces is a matching, because each man is engaged at any step to at most one woman, and each woman is engaged at any step to at most one man. Furthermore, this matching is individually rational, because no man or woman is ever engaged to a new but not strongly acceptable partner. We will show that this outcome $\mu$ is a weakly stable matching.

Suppose this individually rational matching $\mu$ is blocked by a chain ( $\bar{m}_{1}, \bar{w}_{1}, \bar{m}_{2}, \bar{w}_{2}, \cdots, \bar{m}_{K}, \bar{w}_{K}$ ) $(K \geq 1)$. By the definition of blocking chain, we see that man $\bar{m}_{1}$ is free under $\mu, \bar{m}_{1}$ and $\bar{w}_{1}$ are mutually strongly acceptable, $\bar{w}_{1} \succ_{\bar{m}_{1}} \mu\left(\bar{m}_{1}\right)$, and $\bar{m}_{1} \succ_{\bar{w}_{1}} \mu\left(\bar{w}_{1}\right)$. Thus, $\bar{m}_{1}$ is free at the last step of the procedure, and he must have proposed to woman $\bar{w}_{1}$ in the procedure. Since $\bar{m}_{1}$ is strongly acceptable to $\bar{w}_{1}$, the man $\bar{m}_{2}=\mu^{0}\left(\bar{w}_{1}\right)$ is also free at some step. Thus, $\bar{m}_{2}$ must have proposed to woman $\bar{w}_{2}$ because $\bar{w}_{2}$ is strongly acceptable to him and $\bar{w}_{2} \succ_{\bar{m}_{2}} \mu\left(\bar{m}_{2}\right)$. Iteratively, we can show that in this MP-DA procedure man $\bar{m}_{K}$ must have proposed to woman $\bar{w}_{K}$. Note that $\bar{w}_{K}$ is free under $\mu$. The woman $\bar{w}_{K}$ must be either a single woman or a married woman who has never received her husband's proposal in the procedure. Furthermore, note that $\bar{m}_{K}$ is strongly acceptable to $\bar{w}_{K}$ and $\bar{m}_{K} \succ_{\bar{w}_{K}} \mu\left(\bar{w}_{K}\right)$. Therefore, $\bar{w}_{K}$ should not have rejected the proposal from $\bar{m}_{K}$. This contradiction shows that matching $\mu$ is not blocked by any chain. As a result, the outcome $\mu$ is a weakly stable matching.

Proof of Proposition 2: Implementing the MP-TTC procedure and the WP-TTC procedure, check whether there exist any isolated single persons, isolated couples, MP-TTC cycles, WP-TTC cycles, or MP-TTC chains. Suppose that neither an isolated person, nor an isolated couple, nor an MP-TTC cycle, nor a WP-TTC cycle, nor an MP-TTC chain is found. We will prove there must exist at least one WP-TTC chain.

We first claim that in this case there must exist some free woman. Suppose not. Then, there is no single woman, and for each man $m$ it holds $\mu^{0}(m) \succeq_{m} m$. Thus in the MP-TTC procedure no married man has ever proposed to his wife, or else there must exist an isolated couple. Hence, at the end of the MP-TTC procedure, every married man $m$ keeps a mate $w$ such that $w \succ_{m} \mu^{0}(m) \succeq_{m} m$. In addition, every single man $m$ keeps a mate $w$ such that $w \succ_{m} m$, or else he is an isolated single man. Thus, there always exists an MP-TTC cycle in the outcome of the MP-TTC procedure, ${ }^{13}$ yielding a contradiction.

Choose any free woman $\bar{w}_{1}$. We next claim that, at the end of the WP-TTC procedure, the woman $\bar{w}_{1}$ always keeps a mate $\bar{m}_{1}$ such that $\bar{m}_{1} \succ_{\bar{w}_{1}} \mu^{0}\left(\bar{w}_{1}\right) \succeq_{\bar{w}_{1}} \bar{w}_{1}$. If $\bar{w}_{1}$ is a single

[^6]woman, then at the end of the WP-TTC procedure, $\bar{w}_{1}$ keeps a mate $\bar{m}_{1}$ because she is not an isolated single woman. Otherwise $\bar{w}_{1}$ is a married woman, her husband $\mu^{0}\left(\bar{w}_{1}\right)$ is not free, and so $\mu^{0}\left(\bar{w}_{1}\right) \succ_{\bar{w}_{1}} \bar{w}_{1}$. Note that $\bar{w}_{1}$ has never proposed to his husband $\mu^{0}\left(\bar{w}_{1}\right)$ in the WP-TTC procedure, or else $\left\{\bar{w}_{1}, \mu^{0}\left(\bar{w}_{1}\right)\right\}$ is an isolated couple. Thus, at the end of the WP-TTC procedure, $\bar{w}_{1}$ keeps a mate $\bar{m}_{1}$ such that $\bar{m}_{1} \succ_{\bar{w}_{1}} \mu^{0}\left(\bar{w}_{1}\right) \succeq_{\bar{w}_{1}} \bar{w}_{1}$.

If $\bar{m}_{1}$ is a free man, then we obtain a WP-TTC chain $\left(\bar{w}_{1}, \bar{m}_{1}\right)$. Suppose $\bar{m}_{1}$ is not a free man and let $\bar{w}_{2}=\mu^{0}\left(\bar{m}_{1}\right)$ denote his wife. Then, $\bar{m}_{1}=\mu^{0}\left(\bar{w}_{2}\right) \succ_{\bar{w}_{2}} \bar{w}_{2}$, and similarly, at the end of WPTTC procedure the woman $\bar{w}_{2}$ also keeps a mate $\bar{m}_{2}$ such that $\bar{m}_{2} \succ_{\bar{w}_{2}} \bar{m}_{1} \succeq_{\bar{w}_{2}} \bar{w}_{2}$. Since there is no WP-TTC cycle, it satisfies that $\bar{m}_{2} \notin\left\{\mu^{0}\left(\bar{w}_{1}\right), \mu^{0}\left(\bar{w}_{2}\right)\right\}$. Iteratively, with the assumption that there is no WP-TTC cycle, we can obtain a WP-TTC chain ( $\bar{w}_{1}, \bar{m}_{1}, \bar{w}_{2}, \bar{m}_{2}, \cdots, \bar{w}_{K}, \bar{m}_{K}$ ) for some $K \geq 1$.

Proof of Theorem 1: Let $\mathcal{M}=\left(A, \mu^{0}, \succ\right)$ stand for an arbitrary general marriage model. For any permissible coalition $S \subseteq A$, we can define a sub-model $\mathcal{M}(S)=\left(S, \mu_{S}^{0}, \succ^{S}\right)$ on $S$ as follows: (i) $\mu_{S}^{0}(x)=\mu^{0}(x)$ for all $x \in S$; and (ii) for any people $x, y, z \in S, y \succ_{x}^{S} z$ if and only if $y \succ_{x} z$. In addition, for convenience, for any chain or cycle of distinct people $\tau=\left(x_{1}, x_{2}, \cdots, x_{K}\right)$, we will use $A(\tau)=\left\{x_{1}, x_{2}, \cdots, x_{K}\right\}$ to denote the set of all people in $\tau$.

We will prove the theorem by induction. It is easy to check that the strict core of every marriage model with one or two people is non-empty. Assume now that in every marriage model with no more than $t$ people, the strict core is non-empty. We will show that the strict core is also non-empty in every marriage model with $t+1$ people. Let $\mathcal{M}=\left(A, \mu^{0}, \succ\right)$ be the general marriage model with $t+1$ people, i.e., $|A|=t+1$. In order to show that the strict core is not empty in $\mathcal{M}$, by Proposition 2 we need to consider the following four cases:

Case (1): There is at least one isolated single person. Choose any such an isolated single person $x^{*}$. Set $A^{\prime}=A \backslash\left\{x^{*}\right\}$. Then, $A^{\prime}$ is a permissible coalition, and the sub-model $\mathcal{M}\left(A^{\prime}\right)$ is well defined. Since $\left|A^{\prime}\right|=|A|-1=t$, the strict core of $\mathcal{M}\left(A^{\prime}\right)$ is not empty. Choose an arbitrary matching $\rho$ from the strict core of $\mathcal{M}\left(A^{\prime}\right)$. Then, we will show that the proper matching $\mu$ defined by

$$
\mu(x)= \begin{cases}\rho(x) & \text { if } x \in A^{\prime} \\ x^{*} & \text { if } x=x^{*}\end{cases}
$$

is in the strict core of $\mathcal{M}$. Assume by way of contradiction that $\mu$ is improved upon by a permissible coalition $S$ with a matching $\tau$ on $S$. Then, $x^{*} \in S$, or else $\rho$ itself is also improved upon by $S$ with $\tau$ in $\mathcal{M}\left(A^{\prime}\right)$, and $\tau\left(x^{*}\right)=x^{*}$ because $x^{*}$ is an isolated single person. Thus, $\rho$ is
also improved upon by $S \backslash\left\{x^{*}\right\}$ with the same matching $\tau$ confined on $S \backslash\left\{x^{*}\right\}$. This contradicts the assumption that $\rho$ is in the strict core of $\mathcal{M}\left(A^{\prime}\right)$.

Case (2): There are some isolated couples. Choose an arbitrary isolated couple $\left\{m^{*}, w^{*}\right\}$. Set $A^{\prime}=A \backslash\left\{m^{*}, w^{*}\right\}$. Then, $A^{\prime}$ is a permissible coalition, and the sub-model $\mathcal{M}\left(A^{\prime}\right)$ is well defined. Since $\left|A^{\prime}\right|=|A|-2=t-1$, the strict core of $\mathcal{M}\left(A^{\prime}\right)$ is not empty. Choose any matching $\rho$ from the strict core of the sub-market $\mathcal{M}\left(A^{\prime}\right)$. Then, similarly we can show that the proper matching $\mu$ defined by

$$
\mu(x)= \begin{cases}\rho(x) & \text { if } x \in A^{\prime} \\ \mu^{0}(x) & \text { if } x \in\left\{m^{*}, w^{*}\right\}\end{cases}
$$

is in the strict core of $\mathcal{M}$.
Case (3): There are some MP-TTC cycles or WP-TTC cycles. Without loss of generality, assume there are some MP-TTC cycles. Choose an arbitrary MP-TTC cycle $\tau=$ $\left(\bar{m}_{1}, \bar{w}_{1}, \bar{m}_{2}, \bar{w}_{2}, \cdots, \bar{m}_{K}, \bar{w}_{K}\right)(K \geq 2)$. By the definition of MP-TTC cycle, $A(\tau)$ is a permissible coalition. Set $A^{\prime}=A \backslash A(\tau)$. Note that $A^{\prime}$ can be empty. The case of $A^{\prime}=\emptyset$ is trivial. Actually, in this case the matching defined by MP-TTC cycle $\tau$ is in the strict core matching of $\mathcal{M}$. We will focus on $A^{\prime} \neq \emptyset$ which is a permissible coalition. The sub-model $\mathcal{M}\left(A^{\prime}\right)$ is well defined. Since $\left|A^{\prime}\right|<|A|=t+1$, the strict core of $\mathcal{M}\left(A^{\prime}\right)$ is not empty. Choose any matching $\rho$ from the strict core of $\mathcal{M}\left(A^{\prime}\right)$, and define a proper matching $\mu$ as follows:

$$
\mu(x)= \begin{cases}\rho(x) & \text { if } x \in A^{\prime}, \\ \bar{w}_{k} & \text { if } x=\bar{m}_{k}, k=1,2, \cdots, K \\ \bar{m}_{k} & \text { if } x=\bar{w}_{k}, k=1,2, \cdots, K\end{cases}
$$

We claim that this matching $\mu$ is a strict core matching of $\mathcal{M}$. By Lemma 2 , to prove $\mu$ is a strict core matching of $\mathcal{M}$, it is sufficient to show that $\mu$ is neither improved upon by a cycle nor by a chain.

First, assume to the contrary that $\mu$ is improved upon by a cycle $\nu=\left(x_{1}, y_{1}, x_{2}, y_{2}, \cdots, x_{K^{\prime}}, y_{K^{\prime}}\right)$. Then, $A(\tau) \cap A(\nu) \neq \emptyset$, or else cycle $\nu$ also improves upon the strict core matching $\rho$ in the submodel $\mathcal{M}\left(A^{\prime}\right)$. From the definition of cycle, we see that each $z \in A(\tau) \cap A(\nu)$ is a married person and his or her mate $\mu^{0}(z) \in A(\tau) \cap A(\nu)$. Thus, without loss of generality, we may assume that $\bar{w}_{1}=y_{1}$ and $\bar{m}_{2}=x_{2}$. Since $\nu$ improves upon $\mu$, we see that (i) $x_{2}$ is acceptable to $y_{2}$ because $x_{2} \succeq_{y_{2}} \mu\left(y_{2}\right) \succeq_{y_{2}} \mu^{0}\left(y_{2}\right)$, and (ii) $y_{2} \succeq_{x_{2}} \mu\left(x_{2}\right)=\mu\left(\bar{m}_{2}\right)=\bar{w}_{2}$. On the other hand, it follows from the definition of MP-TTC cycle that $\bar{w}_{2}$ is $\bar{m}_{2}$ 's favorite amongst those of his strongly acceptable women who accept him. This implies that $\bar{w}_{2}=y_{2}$, and hence $\bar{m}_{3}=\mu^{0}\left(\bar{w}_{2}\right)=\mu^{0}\left(y_{2}\right)=x_{3}$. In such a way, we can iteratively show that $\bar{w}_{k}=y_{k}$ and $\bar{m}_{k+1}=x_{k+1}$, for all $k=2, \cdots, K$,
where $K+1$ denotes 1 . Thus, these two cycles $\tau$ and $\nu$ must be the same. Therefore, we have $A(\nu)=A(\tau)$ and $\nu(x) \equiv \mu(x)$ for all $x \in A(\nu)$. This implies that matching $\mu$ is not improved upon by cycle $\tau$, leading to a contradiction.

Next, assume to the contrary that $\mu$ is improved upon by a chain $\nu=\left(x_{1}, x_{2}, \cdots, x_{K^{\prime}}\right)$. Then, $A(\tau) \cap A(\nu) \neq \emptyset$, or else chain $\nu$ also improves upon the strict core matching $\rho$ in the sub-model $\mathcal{M}\left(A^{\prime}\right)$. Since $A(\nu)$ is a permissible coalition, each $x \in A(\tau) \cap A(\nu)$ is also a married person and his or her mate $\mu^{0}(x) \in A(\tau) \cap A(\nu)$. Recall that the sequence ( $x_{K^{\prime}}, x_{K^{\prime}-1}, \cdots, x_{1}$ ) improves upon $\mu$. Then, without loss of generality, we may assume that $\bar{w}_{1}=x_{\bar{k}}$ and $\bar{m}_{2}=x_{\bar{k}+1}$ for some $\bar{k}<K^{\prime}$. Thus, by the definition of improvement chain and MP-TTC cycle, as in the previous case we can prove that $\bar{w}_{2}=x_{\bar{k}+2}$ and $\bar{m}_{3}=x_{\bar{k}+3}$. Iteratively, we can further show that the chain $\nu$ must be the cycle $\tau$, leading to a contradiction.

Case (4): There are some MP-TTC chains or WP-TTC chains. Implement the MP-DA procedure, by Proposition 3, we obtain a weakly stable matching $\tau \neq \mu^{0}$. Set $A^{\prime}=\{x \in$ $\left.A \mid \tau(x)=\mu^{0}(x)\right\}$. Then, $A^{\prime} \subset A\left(A^{\prime} \neq A\right),\left|A^{\prime}\right|<|A|=t+1$. If $A^{\prime}=\emptyset$, then every person is free under $\tau$. And hence $\tau$ is stable and in the core in the sense of Gale and Shapley. Thus, the proper matching $\tau$ is not improved upon by any coalition, and so is in the strict core of $\mathcal{M}$. In the following, we will assume $A^{\prime} \neq \emptyset$. Thus, $A^{\prime}$ is a permissible coalition, the sub-model $\mathcal{M}\left(A^{\prime}\right)$ is well defined, and its the strict core of $\mathcal{M}\left(A^{\prime}\right)$ is not empty. Choose any matching $\rho$ from the strict core of $\mathcal{M}\left(A^{\prime}\right)$, and define a proper matching $\mu$ by

$$
\mu(x)= \begin{cases}\rho(x) & \text { if } x \in A^{\prime} \\ \tau(x) & \text { if } x=A \backslash A^{\prime}\end{cases}
$$

We will prove by way of contradiction that the matching $\mu$ is a strict core matching of $\mathcal{M}$. Suppose that $\mu$ is not in the strict core of $\mathcal{M}$. Then, by Lemma $1, \mu$ must be improved upon by a cycle or a chain.

Case (4-1): $\mu$ is assumed to be improved upon by a cycle $\nu=\left(x_{1}, y_{1}, x_{2}, y_{2}, \cdots, x_{K}, y_{K}\right)$. In this case we will derive a contradiction that the weakly stable matching $\tau$ is blocked by a chain contained by cycle $\nu$. We first have that: (i) there is some $\bar{k}=1, \cdots, K$ such that $y_{\bar{k}} \succ_{x_{\bar{k}}} \mu\left(x_{\bar{k}}\right)$ and $x_{\bar{k}} \succ_{y_{\bar{k}}} \mu\left(y_{\bar{k}}\right)$, because every person has a strict preference relation; (ii) $A(\nu) \cap A^{\prime} \neq \emptyset$ is a permissible coalition, or else $x_{\bar{k}}$ and $y_{\bar{k}}$ are both free under $\tau$ and $\tau$ is blocked by the pair (chain) $\left(x_{\bar{k}}, y_{\bar{k}}\right)$; (iii) $A(\nu) \backslash A^{\prime} \neq \emptyset$ is a permissible coalition, or else cycle $\nu$ improves upon the strict core matching $\rho$ in the sub-model $\mathcal{M}\left(A^{\prime}\right)$. The case (4-1) is illustrated in Figure 3.

Since $\nu$ is a cycle, we can without loss of generality assume that $x_{1} \notin A^{\prime}$ and $y_{K}=\mu^{0}\left(x_{1}\right) \notin A^{\prime}$.

$$
x_{1}=x_{i^{\prime}}=x_{i}
$$

Finally we consider the case in which $x_{i}=x_{1}$ or $x_{j}=x_{K}$ is a single person. Recall that $\tau$ is a weakly stable matching and $\nu$ improves upon $\mu$. It is clear that $x_{i+1} \succ_{x_{i}} \tau\left(x_{i}\right)=x_{i}$ if $x_{i}=x_{1} \in A^{\prime}$. If $x_{i}=x_{1} \notin A^{\prime}$, then $x_{i+1}=x_{2} \in A^{\prime}$ and $x_{i+1} \neq \tau\left(x_{i}\right)$. And so $x_{i+1} \succ_{x_{i}} \tau\left(x_{i}\right) \succ_{x_{i}} x_{i}$. Similarly, we can show $x_{j-1} \succ_{x_{j}} \tau\left(x_{j}\right)$ and $x_{j-1} \succ_{x_{j}} x_{j}$ when $x_{j}=x_{K}$. Similarly we can further prove that $\bar{\nu}$ is also a blocking chain of the weakly stable matching $\tau$, yielding a contradiction.

By now we have proved that the strict core of the marriage model $\mathcal{M}$ is not empty.

## REFERENCES

Abdulkadiroğlu, Atila, Parag A. Pathak, and Alvin E. Roth. 2005. "The New York city high school match." American Economic Review, 95: 364-367.

Abdulkadiroğlu, Atila, and Tayfun Sönmez. 1999. "House allocation with existing tenants." Journal of Economic Theory, 88: 233-260.

Abdulkadiroğlu, Atila, and Tayfun Sönmez. 2003. "School choice: A mechanism design approach." American Economic Review, 93: 729-747.

Ausubel, LaWrence. 2004: "An efficient ascending-bid auction for multiple objects." American Economic Review, 94: 1452-1475.

Ausubel, Lawrence. 2006. "An efficient dynamic auction for heterogeneous commodities." American Economic Review, 96: 602-629.

Becker, Gary S. 1973."A theory of marriage: Part I." Journal of Political Economy, 81: 813-846.

Becker, Gary S. 1974. "A theory of marriage: Part II." Journal of Political Economy, 82: 11-26.

Becker, Gary S. 1981. A Treatise on the Family, Harvard University Press, Cambridge, Massachusetts.

Chen, Yan, and Onur Kesten. 2016. "Chinese college admission and school choice reforms: A theoretical analysis." forthcoming in Journal of Political Economy. Chung, Kim-Sau. 2000. "On the existence of stable roommate matchings." Games and Economic Behavior, 33: 206-230.

Crawford, Vincent P., and Elsie Marie Knoer. 1981. "Job matching with heterogeneous firms and workers." Econometrica, 49: 437-450.

Debreu, Gerard, and Herbert Scarf. 1963. "A limit theorem on the core of an economy." International Economic Review, 4: 235-246.

Demange, Gabrielle, David Gale, and Marilda Sotomayor. 1986. "Multi-item auctions." Journal of Political Economy, 94: 863-872.

Dubins, L. E., and D.A. Freedman. 1981. "Machiavelli and the Gale-Shapley alGorithm." The American Mathematical Monthly, 88: 485-494.

Edgeworth, Francis. Y. 1881. Mathematical Psychics, Kegan Paul Publishers, London. Gale, David, and Lloyd S. Shapley. 1962. "College admissions and the stability of marriage." American Mathematical Monthly, 69: 9-15.

Gillies, D. B. 1959. "Solutions to general non-zero-sum games," in A. W. Tucker and R. D. Luce (Eds.) Contributions to the Theory of Games IV, Princeton University Press, 47-85.

Gul, Faruk, and Ennio Stacchetti. 2000. "The English auction with differentiated commodities." Journal of Economic Theory, 92: 66-95.

Hatfield, John William, and Paul R. Milgrom. 2005. "Matching with contracts." American Economic Review, 95: 913-35.

Hitsch, G̈unter, Ali, Horta̧su, and Dan, Arirly. 2010. "Matching and sorting in online dating." American Economic Review, 100: 130-163.

Ju, Yuan, and Zaifu Yang. 2016 "The English housing market mechanism," unpubLISHED MANUSCRIPT.

Kamada, Yuichiro, and Fuhito Kojima. 2015. "Efficient matching under distributional Constraints: theory and applications." American Economic Review, 105: 67-99. Kelso, Alexander S.Jr., and Vincent P. Crawford. 1982. "Job matching, coalition formation, and gross substitutes." Econometrica, 50: 1483-1504.

Kojima, Fuhito. 2011. "Robust stability in matching markets." Theoretical Economics, 6: 257-267.

Kojima, Fuhito, and Mihai Manea. 2010. "Axioms for deferred acceptance." Econometrica, 78: 633-653.

Koopmans, Tjalling C., and Martin Beckmann. 1957. "Assignment problems and the location of economic activities." Econometrica 25: 53-76.

Ma, Jinpeng. 1994. "Strategy-proofness and the strict core in a market with indivisibilities." International Journal of Game Theory, 23: 75-83.

Milgrom, Paul. 2000. "Putting auction theory to work: the simultaneous ascending auction." Journal of Political Economy, 108: 245-272.

Ostrovsky, Michael. 2008. "Stability in supply Chain network." American Economic Review, 98: 897-923.

Pathak, Parag, and Tayfun Sönmez. 2008. "Leveling the playing field: Sincere and sophisticated players in the Boston mechanism." American Economic Review, 98: 1636-52.

Pathak, Parag, and Tayfun Sönmez. 2013. "School admissions reform in Chicago and England: Comparing mechanisms by their vulnerability to manipulation." American Economic Review, 103: 80-106.

Perry, Motty, and Philip Reny. 2005. "An efficient multi-unit ascending auction." Review of Economic Studies, 72: 567-592.

Predtetchinski, Arkadi, and P. Jean-Jacques Herings. 2004. "A necessary and SUFFICIENT CONDITION FOR THE NON-EMPTINESS OF THE CORE OF A NON-TRANSFERABLE utility game." Journal of Economic Theory, 116: 84-92.

Quinzii, Martine. 1984. "Core and competitive equilibria with indivisibilities. International Journal of Game Theory, 13: 41-60.

Roth, Alvin E. 1984. "The evolution of the labor market for medical interns and Residents: A case study in game theory." Journal of Political Economy, 92: 991-1016. Roth, Alvin E., Tayfun Sönmez, and Utku Ünver. 2004. "Kidney exchange." Quarterly Journal of Economics, 119: 457-488.

Roth, Alvin E., and Marilda A. O. Sotomayor. 1990. Two-Sided Matching, Cambridge University Press, New York.

Scarf, Herbert. 1967. "The core of an $n$-person game." Econometrica, 35: 50-69.
Shapley, Lloyd S. 1967. "On balanced sets and cores." Naval Research Logistics Quarterly, 14: 453-460.

Shapley, Lloyd S. 1973. "On balanced games without side payments." In T.C. Hu and S.M. Robinson (Eds.), Mathematical programming, Academic Press, New York, 261-290.

Shapley, Lloyd S, and Herbert Scarf. 1974. "On cores and indivisibilities." Journal of Mathematical Economics, 1: 23-37.

Shapley, Lloyd S, and Martin Shubik. 1972. "The assignment game I: The core."

International Journal of Game Theory, 1: 111-130.
Sönmez, Tayfun. 2013. "Bidding for army career specialties: Improving the ROTC branching mechanism," Journal of Political Economy, 121: 186-219.

Sönmez, Tayfun, and Tobiaz Switzer. 2013. "Matching with (branch-of-choice) contracts at the United States Military Academy." Econometrica, 81: 451-488.

Sun, Ning, and Zaifu Yang. 2009. "A double-track adjustment process for discrete markets with substitutes and complements." Econometrica, 77: 933-952.

Sun, Ning, and Zaifu Yang. 2014. "An efficient and incentive compatible dynamic auction for multiple complements." Journal of Political Economy, 122: 422-466.

Sun, Ning, and Zaifu Yang. 2016. "On the existence of stable social economic networks," in preparation.

Voena, Alessandra. 2015. "Yours, mine, and ours: does divorce laws affect the intertemporal behavior of married couples?" American Economic Review, 105: 22952332.


[^0]:    ${ }^{1}$ We wish to thank participants at several workshops and seminars for their feedback.
    ${ }^{2}$ N. Sun, School of Economics, Shanghai University of Finance and Economics, Shanghai 200433, China; nsun@mail.shufe.edu.cn.
    ${ }^{3}$ Z. Yang, Department of Economics and Related Studies, University of York, York YO10 5DD, UK; zaifu.yang@york.ac.uk.

[^1]:    ${ }^{4}$ Legislation concerning divorce varies from one country to another and in some countries the rules may change from male to female. For instance, divorce in the United States is a matter of each state and can therefore vary from one state to another.
    ${ }^{5}$ While it is true that unilateral divorce has becoming easier than it used to be, spouses in particular women and their children have also generally gotten better protection by divorce laws than in the past and in some sense unilateral divorce has become more costly.

[^2]:    ${ }^{6}$ Subsequent papers on auction and matching design include Demange et al. (1986), Gul and Staachetti (2000), Milgrom (2000), Ausubel (2004, 2006), Hatfield and Milgrom (2005), Perry and Reny (2005), Ostrovsky (2008), Sun and Yang (2009, 2014), and Kamada and Kojima (2015) amongst others.
    ${ }^{7}$ See Ju and Yang (2016) for a related study on the English housing market mechanism.
    ${ }^{8}$ See also Abdulkadiroğlu, Pathak and Roth (2005), Pathak and Sönmez (2008, 2013), Sönmez (2013), Sönmez and Switzer (2013), and Chen and Kesten (2016).
    ${ }^{9}$ Sun and Yang (2016) generalize Ostrovsky's vertical chain model to allow a variety of structures including cycles.

[^3]:    ${ }^{10}$ The general case of allowing indifference in preferences will be discussed in Section 4.

[^4]:    ${ }^{11}$ We say a man $m$ is free at a step, if he is a free man or his wife $\mu^{0}(m)$ has being engaged with some other man at the beginning of the current step.

[^5]:    ${ }^{12}$ The set of all remaining people is a permissible coalition.

[^6]:    ${ }^{13}$ Consider a directed bipartite graph $G$ on $A$ defined by: there is an arrow from a man $m \in A$ to a woman $w \in A$ if and only if $m$ is kept by $w$ at the end of the MP-TTC procedure, there is an arrow from a woman $w \in A$ to a man $m \in A$ if and only if $\mu^{0}(w)=m$. Since there is neither isolated man nor isolated married couple, in graph $G$ there is one and only one arrow from a man to some woman, and there is one arrow from a married woman to her husband. Therefore, there must at least exist a directed cycle in $G$. It can be checked further that such a directed cycle is an MP-TTC cycle.

