Discussion Papers in Economics

No. 16/01

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Mauro Bambi, Alain Venditti

Department of Economics and Related Studies
University of York
Heslington
York, YO10 5DD

Time-varying Consumption Tax, Productive Government Spending, and Aggregate Instability*

Mauro BAMBI†

University of York, Department of Economics and Related Studies, York, UK

and

Alain VENDITTI‡

Aix-Marseille University (Aix-Marseille School of Economics)-CNRS-EHESS & EDHEC Business School, France.

First version: *July 2014*; Revised: *February 2016*

Abstract

In this paper we study the dynamics of an economy with productive government spending under the assumption that the government balances its budget by levying endogenous non-linear consumption taxes. For standard specification of the utility function and production function, we prove that under counter-cyclical consumption taxes, while there exists a unique balanced growth path, sunspot equilibria based on self-fulfilling expectations occur through a form of global indeterminacy.

JEL Classification C62, E32, H20, O41

Keywords Endogenous growth, time-varying consumption tax, global indeterminacy, self-fulfilling expectations, sunspot equilibria.

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*This work has been carried out thanks to the support of the A*MIDEX project (ANR-11-IDEX-0001-02) funded by the "Investissements d'Avenir" French Government program, managed by the French National Research Agency (ANR). We thank Yves Balasko, Jess Benhabib, Jean-Pierre Drugeon, Fausto Gozzi, Jean-Michel Grandmont, Herakles Polemarchakis, Neil Rankin, Xavier Raurich, Thomas Seegmuller, Mich Tvede, Yiannis Vailakis and Bertrand Wigniolle for useful comments and suggestions. We are also grateful to the seminar participants at the Newcastle-York Workshop on Economic Theory and at the internal seminar at the University of York. This paper benefited also from a presentation at the International Conference "Financial and Real Interdependencies: volatility, inequalities and economic policies", Católica Lisbon School of Business & Economics, Lisbon, May 28-30, 2015.

†E-mail: mauro.bambi@york.ac.uk

‡E-mail: alain.venditti@univ-amu.fr

1 Introduction

Since the paper of Schmitt-Grohé and Uribe [22], it is a well established fact that balanced budget rules may lead to belief-driven aggregate instability and endogenous sunspot fluctuations. However, depending on the fiscal policy, aggregate instability occurs under different types of preferences. While it requires a large enough income effect when labor income taxes are considered (see Abad *et al.* [1]),[1] low enough income effects are necessary under consumption taxes (see Nourry *et al.* [20]). Such a conclusion has strong policy implications as for a given specification of preferences, one type of fiscal policy must be preferred to the other if the government is willing to avoid endogenous fluctuations. For instance, under a standard additively-separable utility function, Giannitsarou [12] suggests that consumption taxes must be favoured with respect to income or capital taxes as they reduce the possible occurrence of aggregate instability.

These results can be criticized in two dimensions. First, as clearly mentioned by Schmitt-Grohé and Uribe [22], they are partially based on the assumption that tax rates are not predetermined,[2] while taxes are in practice typically set in advance.[3] Second, they are established within stationary models without long-run growth. The aim of this paper is to revisit the issue of aggregate instability coming from balanced budget rules focusing on consumption taxes compatible with endogenous growth. In practice, this requirement implies that the tax rate depends on detrended consumption to have a constant tax on a balanced growth path. As a consequence, we consider a time-varying consumption tax which is a predetermined variable and we are thus able to solve the two main weaknesses of the standard literature.

We consider a standard neoclassical growth model augmented with a government that provides a constant stream of expenditures financed through consumption taxes and a balanced budget rule. Endogenous growth is obtained from assuming a Barro-type [3] production function in which government spending acts as an external productive input. In order to have a constant tax on a balanced growth path, the

[1] Actually, local indeterminacy requires that consumption and labor are Edgeworth substitutes or weak Edgeworth complements (see also Linnemann [17]). These properties are associated to a Jaimovich-Rebelo [15] utility function characterized by a large enough income effect.

[2] The initial value of the tax rate is indeed a function of a forward variable (*i.e.*, consumption or labor).

[3] It is however claimed in Schmitt-Grohé and Uribe [22] that their main conclusions are robust to the consideration of a discrete-time reformulation of their model with tax rates set $k \geq 1$ periods in advance to that in each period $t \geq 0$, the tax rates for periods $t, \cdots, t+k-1$ are pre-determined.

tax rate needs to depend on de-trended consumption and thus becomes a state variable with a given initial condition. Finally, we consider a representative household characterized by a CRRA utility function and inelastic labor. Such a formulation is known to rule out the existence of endogenous fluctuations in exogenous growth models (see Giannitsarou [12]).

We first prove that there exists a unique Balanced Growth Path (BGP) along which the common growth rate of consumption, capital, GDP and government spending is constant. The particularity of such a BGP is that the equilibrium tax rate is just equal to its initial value. A consequence of this property is that, as in the Barro [3] model, there is no transitional dynamics with respect to this unique unstable BGP and, therefore, there exists a unique initial choice of consumption such that the economy evolves along its BGP. This conclusion is thus similar to the one reached by Giannitsarou [12]: there is a priori no room for endogenous fluctuations.

However, we can prove that the BGP is not the unique long run solution of our model. Indeed, if the tax rule is counter-cyclical with respect to consumption, for any arbitrary initial value of the tax rate, close enough to its initial condition, there exists a corresponding value for the tax rate, consumption, capital and the constant growth rate that can be an asymptotic equilibrium of our economy, namely an Asymptotic Balanced Growth Path (ABGP). An ABGP is not itself an equilibrium as it does not respect the initial conditions. However we prove that some transitional dynamics exist with a unique equilibrium path converging toward this ABGP. Moreover, we show that there exist a continuum of such ABGP and of equilibria each of them converging over time to a different ABGP.[4]

The existence of an equilibrium path converging to an ABGP is associated to the existence of consumers' beliefs that are different from those associated to the BGP. Indeed, they may believe that the consumption tax profile will not remain constant but rather change over time and eventually converge to a positive value different from the initial condition. A specific form of global indeterminacy emerges since from a given initial tax rate, the representative agent can choose an initial consumption to be immediately on the unique BGP or alternatively an initial consumption consistent with any other equilibrium converging to an ABGP. Again different choices reveal different consumers' beliefs of the long run outcome of the economy.

Because this specific form of global indeterminacy is fundamentally related to expectations, one may wonder about the possible existence of sunspot equilibria and

[4]On the other hand, we can prove that if the consumption tax is procyclical then the BGP is the unique equilibrium path.

endogenous fluctuations based on self-fulfilling beliefs. To this purpose, we adapt existing results (e.g. Shigoka [23], Benhabib *et al.* [7], Cazzavillan [9]) and we show that sunspot equilibria can be obtained by randomizing over the deterministic equilibria converging to the ABGPs. From an analytical viewpoint, we assume that the sunspot variable is a continuous time homogenous Markov chain and we use the generator of the chain as proposed by Grimmett and Stirzaker [13] to prove the existence of sunspot equilibria. We then conclude that in an endogenous growth framework, contrary to the conclusions of Giannitsarou [12], endogenous sunspot fluctuations may arise under a balanced budget rule and consumption taxes although there exists a unique underlying BGP equilibrium. It is also worth noting that contrary to Drugeon and Wigniolle [10] or Nishimura and Shigoka [19], our methodology allows to prove the existence of sunspots in a non-stationary economic environment while the steady state (BGP) is unstable.

Our results can be compared to some recent conclusions provided by Angeletos and La'O [2] and Benhabib *et al.* [5] within infinite horizon models with sentiments. They show that endogenous fluctuations, based on a certain type of extrinsic shocks called "sentiments", can be accommodated in unique-equilibrium, rational-expectations, macroeconomic models like those in the RBC/DSGE paradigm provided there is some mechanism that prevents the agents from having identical equilibrium expectations. Of course, our framework is still based on the existence of externalities as we need to generate a form of Ak technology to get endogenous growth. But, contrary to the standard literature which is based on the existence of local indeterminacy (see Benhabib and Farmer [4]), we find sunspot fluctuations while there exists a unique deterministic BGP without transitional dynamics. The existence of the continuum of ABGPs and of equilibrium paths converging to these, is also fundamentally based on the expectations of agents. From this point of view, our conclusions are also related to Farmer [11] where expectations-driven fluctuations are generated from the existence of a continuum of equilibrium unemployment rates in a dynamic general equilibrium model with search.

The rest of the paper is organized as follows. Section 2 presents the model and defines the intertemporal equilibrium. Section 3 proves the existence and uniqueness of a BGP. In Section 4 its stability is investigated and it is also shown that depending on agents' expectations, there may exist a continuum of other equilibria that converge toward some ABGPs. Based on this conclusion, we prove in Section 5 that sunspot equilibria and endogenous fluctuations based on self-fulfilling beliefs occur. Section 6 contains a conclusion and all the proofs are provided in a final Appendix.

2 Model Setup

In this section the endogenous growth model originally developed by Barro [3] is modified by assuming that the government levies a time-varying consumption tax to finance its spending. In this economy the government spending is productive since it is a public good provided by the government to the firms which use it as an essential input of production. For this reason, our paper is different from Giannitsarou [12] and Nourry *et al.* [20] where the government spending is just a pure waste of resources. As in Barro [3], productive government spending is the source of endogenous growth in our model.

2.1 Firms

A representative firm produces the final good y using a Cobb-Douglas technology with constant returns at the private level but which is also affected by a public good externality, $y = Ak^{\alpha}(L\mathcal{G})^{1-\alpha}$, where $\alpha \in (0,1)$ is the share of capital income in GDP, $\mathcal{G}$ is the per capita quantity of government purchases of goods and services and A is the constant TFP. We assume that population is normalized to one, $L = 1$, so that we get a standard Barro-type [3] formulation such that $y = Ak^{\alpha}\mathcal{G}^{1-\alpha}$. Profit maximization then respectively gives the rental rate of capital and the wage rate:

$$r = A\alpha\left(\tfrac{k}{\mathcal{G}}\right)^{\alpha-1}, \qquad w = A(1-\alpha)k\left(\tfrac{k}{\mathcal{G}}\right)^{\alpha-1} \tag{1}$$

2.2 Households

We consider a representative household endowed with a fixed amount of labor and an initial stock of private physical capital which depreciates at rate $\delta > 0$. His instantaneous utility function is consistent with endogenous growth and given by

$$u(c) = \tfrac{c^{1-\sigma}}{1-\sigma} \tag{2}$$

with $\sigma > 0$ the inverse of the elasticity of intertemporal substitution in consumption.

The representative household derives income from wage and capital. Denoting $\tau > 0$ the tax rate on consumption, his budget constraint is given by:

$$(1+\tau)c + \dot{k} = rk + w - \delta k \tag{3}$$

with r and w as given by (1).

The representative household then solves the following problem taking as given the prices r and w, and the time-varying paths of $\mathcal{G}$ and τ:

$$\begin{aligned}
&\max \int_0^\infty \frac{c^{1-\sigma}}{1-\sigma} e^{-\rho t} dt \\
&s.t.\ \ \dot{k} = rk + w - \delta k - (1+\tau)c \\
&\qquad k \geq 0,\ \ c \geq 0 \\
&\qquad k(0) = k_0 > 0\ \ given
\end{aligned}$$

where the set of admissible parameters is so defined

$$\mathbf{\Theta} \equiv \{(\alpha, \rho, \delta, \sigma, A):\ \ \alpha \in (0,1),\ \ \rho > 0,\ \ \delta > 0,\ \ \sigma > 0\ and\ A > 0\}.$$

Also, capital and consumption are continuous and differential functions in their domain $(0, \infty)$. The current value Hamiltonian associated to this problem is

$$\mathcal{H} = \frac{c^{1-\sigma}}{1-\sigma} + \lambda[rk + w - \delta k - (1+\tau)c]$$

where λ is the co-state variable. Considering (1), the first order conditions with respect to the control, c, and the state, k, write respectively

$$c^{-\sigma} = \lambda(1+\tau) \quad \Rightarrow \tag{4}$$

$$-\frac{\dot{\lambda}}{\lambda} = \alpha A \left(\frac{k}{\mathcal{G}}\right)^{\alpha-1} - \delta - \rho \tag{5}$$

Differentiation of equation (4) gives

$$\frac{\dot{c}}{c} = -\frac{1}{\sigma}\left[\frac{\dot{\lambda}}{\lambda} + \frac{\dot{\tau}}{1+\tau}\right] \tag{6}$$

Let us then substitute equation (5) into (6). It follows that, given an initial capital stock k_0, the tax and government spending path $(\tau(t), \mathcal{G}(t))_{t\geq 0}$, the representative household maximizes his/her utility by choosing any path $(c(t), k(t))_{t\geq 0}$ which solves the system of ODEs

$$\dot{k} = Ak^\alpha \mathcal{G}^{1-\alpha} - \delta k - (1+\tau)c \tag{7}$$

$$\frac{\dot{c}}{c} = \frac{1}{\sigma}\left[\alpha A \left(\frac{k}{\mathcal{G}}\right)^{\alpha-1} - \delta - \rho - \frac{\dot{\tau}}{1+\tau}\right] \tag{8}$$

respects the positivity constraints $k \geq 0,\ c \geq 0$, and the transversality condition

$$\lim_{t\to+\infty} \frac{k}{c^\sigma(1+\tau)} e^{-\rho t} = 0 \tag{9}$$

2.3 Government

The government balances its budget in every period:

$$\mathcal{G} = \tau(\tilde{c})c \tag{10}$$

where $\tilde{c} \equiv ce^{-\gamma t}$ indicates de-trended consumption with γ the (endogenous) asymptotic and constant growth rate of the economy.[5] The government spending $\mathcal{G}$ as well as the fiscal instrument τ are time-varying and endogenously determined as in Nourry *et al.* [20] among others.[6] In particular, Nourry *et al.* [20] study the case $\mathcal{G}(c) = \tau(c)c$ while we need that the tax rate depends on de-trended consumption to have a constant tax on a balanced growth path. From now on we will also assume the following:

Assumption 1. *The elasticity of the tax rate with respect to de-trended consumption is constant and given by*

$$\phi \equiv \frac{d\tau}{d\tilde{c}} \frac{\tilde{c}}{\tau} \tag{11}$$

It is worth noting that such a restriction is common in the literature. In their seminal contribution, Schmitt-Grohé and Uribe [22] consider a tax on labor income with constant government spending such that $\tau(wl) = \mathcal{G}/wl$ which has a constant elasticity with respect to its tax base equal to -1. The same property is assumed by Giannitsarou [12] with a consumption tax satisfying $\tau(c) = \mathcal{G}/c$. In Nourry *et al.* [20] however, the government spending is assumed to vary with consumption and the elasticity of the tax rate $\tau(c) = \mathcal{G}(c)/c$ is equal to $\eta - 1$ with η the constant elasticity of government spending with respect to consumption. Our formulation is very similar to this one with the exception that here we postulate a specific form of the tax function and government spending adjusts accordingly, while in Nourry *et al.* [20] the government spending rule is postulated and the tax rate adjusts accordingly.

As a consequence of Assumption 1 we have indeed that

$$\dot{\tau} \equiv \frac{d\tau}{dt} = \frac{d\tau}{d\tilde{c}}\dot{\tilde{c}} = \frac{d\tau}{d\tilde{c}}\tilde{c}\left(\frac{\dot{c}}{c} - \gamma\right) \tag{12}$$

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[5] More precisely, γ is the (constant) growth rate if the economy is on a BGP or is the asymptotic (constant) growth rate if the economy is not on a BGP but converges over time to an asymptotic BGP (see Definition 3). In fact, at this stage of the analysis, we cannot exclude *a priori* the existence of a subset of initial conditions such that the economy is not on a BGP at $t = 0$ but rather converges to it over time as it happens, for example, in an endogenous growth model with a Jones and Manuelli [16] production function.

[6] For a discussion on government spending to be an endogenous or exogenous variable, the interested reader may look at Blanchard and Fisher's textbook [8], page 591.

and therefore

$$\frac{\dot{\tau}}{\tau} \equiv \phi \left(\frac{\dot{c}}{c} - \gamma \right) \tag{13}$$

Integrating (13) leads to

$$\tau(t) \equiv B \left(c(t) e^{-\gamma t} \right)^{\phi} \tag{14}$$

with B a generic (and not exogenously given) constant. Therefore, the last expression (14) is rather a menu of fiscal policies. To select just one of them (and avoiding in this way to introduce a trivial form of indeterminacy in the model) we assume that $\tau(0) = \tau_0 > 0$ is exogenously given. By doing so, we may find the value of B and observe that the last expression, and therefore the fiscal rule (13), is equivalent to

$$\tau(t) \equiv \tau_0 \left(\frac{\tilde{c}(t)}{c_0} \right)^{\phi} = \tau_0 \left(\frac{c(t)}{c_0 e^{\gamma t}} \right)^{\phi} \tag{15}$$

Clearly the tax rate, τ, is a predetermined variable. This assumption is consistent with the fact that tax rates are typically set in advance (see for example Schmitt-Grohe and Uribe [22] - page 993) and it seems even more compelling in our model where the tax base is not predetermined since it depends on consumption. It is also worth to underline that identities (13) and (15) are a direct consequence of Assumption 1. Note that this formulation is consistent and indeed includes, under the restriction $\phi = 0$, the case of a constant and exogenously given tax rate $\tau = \tau_0$ briefly mentioned by Barro [3]. Moreover, the fiscal rule (13) is pro(counter)-cyclical if $\phi > 0$ ($\phi < 0$) since it increases (decreases) when consumption grows faster (slower) than γ.[7]

Before concluding this section, we notice that one could be tempted to assume an exogenous target value for γ. This would lead to two undesirable consequences: first, the model becomes an exogenous growth model since it emerges immediately from the fiscal rule that the growth rate of consumption (and therefore of capital) will not be anymore determined, as usual in an endogenous growth model, by a combination of parameters but rather by the target value itself otherwise the tax rate will be growing in the long run.[8] Secondly it can be easily proved that an equilibrium path will exist only for a zero-measure set of parameters.

[7] The definition of pro(counter)-cyclical is based on a comparison of the growth rates. This is consistent with the real business cycle literature where an economy is said to be in recession if it grows more slowly than at its trend.

[8] To see this point even more explicitly, observe that the fiscal rule could be rewritten as $\tau(t) = \tau_0 \left(c(t)/z(t) \right)^{\phi}$ with $z(t) = c_0 e^{\bar{\gamma} t}$. Then all the aggregate variables will grow at the rate of the variable $z(t)$ whose growth rate $\bar{\gamma}$ has been given exogenously.

2.4 Intertemporal equilibrium

Given an initial condition of capital $k_0 > 0$ and of the consumption tax $\tau_0 > 0$, an intertemporal equilibrium is any path $(c(t), k(t), \tau(t), \mathcal{G}(t))_{t\geq 0}$ which satisfies the system of equations (7), (8), (10) and (13), respects the inequality constraints $k \geq 0$, $c \geq 0$, and the transversality condition (9).

Therefore we may define the control-like variable $x \equiv \frac{c}{k}$ and observe that the intertemporal equilibrium can be derived studying the following system of nonlinear ODEs in the variables (x, τ):

$$\begin{aligned} \frac{\dot{x}}{x} &= \frac{[(1+\tau)(1-\sigma)-\phi\tau]\left[\alpha A(x\tau)^{1-\alpha}-\delta-\rho-\sigma\gamma\right]}{\sigma(1+\tau)+\phi\tau} \\ &+ \gamma(1-\sigma)+(1+\tau)x-(1-\alpha)A(x\tau)^{1-\alpha}-\rho \qquad (16) \\ \frac{\dot{\tau}}{\tau} &= \frac{\phi(1+\tau)}{\sigma(1+\tau)+\phi\tau}\left[\alpha A(x\tau)^{1-\alpha}-\delta-\rho-\sigma\gamma\right] \qquad (17) \end{aligned}$$

The interested reader may find in Appendix A the detailed procedure to obtain this system starting from equations (7), (8), (10) and (13). It is also worth noting that x_0 is not predetermined since it depends on c_0 while τ_0 is exogenously given and therefore predetermined.

3 Balanced growth paths

A balanced growth path (BGP) is an intertemporal equilibrium where consumption, and capital are purely exponential functions of time t, namely:

$$k(t) = k_0 e^{\gamma t} \qquad \textit{and} \qquad c(t) = c_0 e^{\gamma t} \qquad \forall t \geq 0. \qquad (18)$$

From equation (13) it follows immediately that along a BGP the consumption tax is constant and equal to

$$\tau(t) = \hat{\tau} = \tau_0 \qquad \forall t \geq 0$$

with the hat symbol indicating, from now on, the value of a variable on a BGP. Along the BGP, the tax rate is therefore constant and equal to its initial value. Also, government spending will be purely exponential with a growth rate equal to γ consistently with the balanced-budget rule (10). Therefore, equations (16) and (17) rewrite

$$\begin{aligned} 0 &= (\alpha-\sigma)A(x\hat{\tau})^{1-\alpha}+\sigma(1+\hat{\tau})x-\rho-\delta(1-\sigma) \equiv g(x) \qquad (19) \\ \gamma &= \frac{1}{\sigma}\left[\alpha A(x\hat{\tau})^{1-\alpha}-\delta-\rho\right]. \qquad (20) \end{aligned}$$

Studying the zeros of the first equation is the necessary step to prove existence and uniqueness of a balanced growth path. Moreover, along a BGP, the transversality condition (9) becomes

$$\lim_{t \to +\infty} \frac{k_0}{c_0^\sigma (1+\tau_0)} e^{-[\rho - \gamma(1-\sigma)]t} = 0 \tag{21}$$

It follows that condition (21) holds if and only if $\rho - \gamma(1-\sigma) > 0$. Therefore, any value of γ solution of equation (19) needs to satisfy $\gamma \geq 0$ when $\sigma \geq 1$ and $\gamma \in [0, \rho/(1-\sigma))$ when $\sigma < 1$.

Proposition 1 (Existence and Uniqueness of a BGP). *Given any initial condition of capital $k_0 > 0$ and the tax rate $\tau_0 > 0$, there exist $\underline{A} > 0$, $\underline{\tau} > 0$ and $\bar{\tau}(\sigma) \in (0, +\infty]$ with $\bar{\tau}(\sigma) > \underline{\tau}$ such that when $A > \underline{A}$ and one of the following conditions holds:*

i) $\sigma \geq 1$ and $\tau_0 > \underline{\tau}$,

ii) $\sigma \in (0,1)$ and $\tau_0 \in (\underline{\tau}, \bar{\tau}(\sigma))$,

there is a unique balanced growth path where the ratio of consumption over capital is constant and equal to $\hat{x}$ – the unique positive root of equation (19) – and the growth rate of the economy is

$$\hat{\gamma} = \alpha A (\hat{x}\hat{\tau})^{1-\alpha} - \delta - \rho > 0, \tag{22}$$

with $\hat{\tau} = \tau_0$.

Proof. See Appendix A.2. ■

Discussion of these conditions is in order. The requirements of a level of technology greater than $\underline{A}$ and of a tax rate larger than $\underline{\tau}$ guarantee positive economic growth. The first one is indeed a condition similar to the one in the AK model while the second one allows to provide a large enough government spending to sustain growth through the technology $y = Ak^\alpha \mathcal{G}^{1-\alpha}$. Also the condition of a tax rate lower than $\bar{\tau}(\sigma)$ when $\sigma < 1$ guarantees that the transversality condition is respected and therefore the utility is bounded. Therefore, Proposition 1 shows that if $A > \underline{A}$ and one of the conditions $i) - ii)$ holds, a unique BGP exists. Of course, uniqueness depends on the existence of a unique value $\hat{x}$ which implies a unique specification of initial consumption for any exogenously given initial condition of the capital stock and consumption tax. For any given k_0 and τ_0, we have indeed $c_0 = k_0\hat{x}$ and $\tau(\tilde{c}) = \tau_0$ so that the stationary value of de-trended consumption $\tilde{c}$ corresponding to the BGP is derived from (18) and such that $\tilde{c} = c_0 = k_0\hat{x}$.

Clearly the value of the positive real zero $\hat{x}$ and of $\hat{\gamma}$ depend on the exogenously given parameters but also on the initial condition of the consumption tax τ_0.[9] For this reason we may explicitly write $\hat{x}$ and $\hat{\gamma}$ as continuous and differentiable functions of these values, i.e. $\hat{x} = \hat{x}(\alpha, \tau_0, \rho, \delta, \sigma)$ and $\hat{\gamma}(\alpha, \tau_0, \rho, \delta, \sigma)$. Given a generic capital stock k_0, the balanced growth path is

$$\hat{k} = k_0 e^{\hat{\gamma}(\alpha,\tau_0,\rho,\delta,\sigma)t} \qquad \textit{and} \qquad \hat{c} = \hat{x}(\alpha, \tau_0, \rho, \delta, \sigma) k_0 e^{\hat{\gamma}(\alpha,\tau_0,\rho,\delta,\sigma)t}$$

where the growth rate is positive if and only if the conditions of Proposition 1 hold. For example , it is easy to check numerically that given $\hat{\tau} = \tau_0 = 0.2$, the following parameter's values, $\alpha = 1/3$, $\rho = 0.01$, $\delta = 0.025$ and $A = 2.2$, imply $\underline{\tau} = 0.15$ and $(\hat{x}, \hat{\gamma}) = (0.123, 0.0135)$ when $\sigma = 2$, $(\hat{x}, \hat{\gamma}) = (0.096, 0.0175)$ when $\sigma = 1$, and $(\hat{x}, \hat{\gamma}) = (0.088, 0.018)$ with $\bar{\tau}(\sigma) = 0.279$ when $\sigma = 0.8$.

It is also worth noting that at the BGP the consumption tax is constant and therefore not distorting (i.e. lump-sum). Therefore the BGP analysis just done is identical to an economy where the consumption tax is set to be constant over time.

We conclude this section with some comparative statics results that provide sufficient conditions for the growth rate $\hat{\gamma}$ and welfare to be increasing functions of the tax rate $\tau_0 = \hat{\tau}$. Indeed, we can easily compute welfare along the BGP characterized by the stationary values of the growth rate $\hat{\gamma}$ and the ratio of consumption over capital $\hat{x}$, namely

$$W(\hat{\gamma}, \hat{x}) = \frac{(\hat{x}k_0)^{1-\sigma}}{(1-\sigma)[\rho - \hat{\gamma}(1-\sigma)]} \tag{23}$$

We then get the following result:

Corollary 1. *Let the conditions of Proposition 1 hold. There exist $A_{min} > \underline{A}$ and $\underline{\sigma} > 0$ such that when $\tau_0 < (1-\alpha)/\alpha$, $\sigma > \underline{\sigma}$ and $A > A_{min}$, then*

$$\frac{d\hat{x}}{d\tau_0} > 0, \quad \frac{d\hat{\gamma}}{d\tau_0} > 0 \quad \textit{and} \quad \frac{dW(\hat{\gamma}, \hat{x})}{d\tau_0} > 0$$

Proof. See Appendix A.3. ■

Obviously, as shown by expression (23), along the BGP welfare is an increasing function of both the growth rate $\hat{\gamma}$ and the ratio of consumption over capital $\hat{x}$. Because of the public good externality in the production function, a large growth factor allows to generate an increasing amount of public good which improves the

[9] The fact that the taxation enters in the equation of the consumption-capital ratio and therefore affects also the growth rate is not surprising and consistent with previous contributions (e.g. Barro [3]).

aggregate production level and thus consumption. Corollary 1 then provides conditions for a positive impact of the tax rate τ_0 on the growth rate, consumption over capital and welfare. In particular, such a conclusion requires a low enough elasticity of intertemporal substitution in consumption $1/\sigma$ which prevents a too large consumption smoothing over time in order to ensure a larger consumption in the long run, i.e. along the BGP

4 Transitional dynamics

4.1 Local determinacy of the steady state $(\hat{x}, \hat{\tau})$

In this section we start by investigating the local stability properties of the steady state $(\hat{x}, \hat{\tau})$ (with $\hat{\tau} = \tau_0$) which characterizes the unique BGP of our economy. Let us recall that in the formulation considered by Barro [3] where the tax rate τ is constant, there is no transitional dynamics and the economy directly jumps on the BGP from the initial date $t = 0$. In our framework we get similar conclusions:

Proposition 2. *Consider the steady state $(\hat{x}, \hat{\tau})$ (with $\hat{\tau} = \tau_0$) which characterizes the unique BGP of our economy. For any given initial conditions (k_0, τ_0), there is no transitional dynamics, i.e. there exists a unique $c_0 = k_0\hat{x}$ such that the economy directly jumps on the BGP from the initial date $t = 0$.*

Proof. See Appendix A.4. ■

As shown in the proof of Proposition 2, the steady state $(\hat{x}, \hat{\tau})$ is locally saddle-path stable if and only if $\phi \in (-\sigma(1+\hat{\tau})/\hat{\tau}, 0)$ and locally unstable otherwise.[10] However, in both cases, we find the same conclusion as in Barro [3]: there is no transitional dynamics with respect to the BGP as any initial choice of $c(0)$ different from $c_0 = k_0\hat{x}$ leads to trajectories diverging from $(\hat{\tau}, \hat{x})$. This is not really surprising since the tax rate on the BGP is exactly $\hat{\tau} = \tau_0$ which is the initial condition of the state variable of our problem.

To make this argument more explicit, consider Figure 1 which illustrates Proposition 2 and shows the phase diagrams when the parameters are set as in the previous section. The initial conditions are $k_0 = 1$ and $\tau_0 = 0.2$, $\sigma = 1$ and ϕ is equal to 0.5

[10]Note that the change in stability at $\phi = -\sigma(1+\hat{\tau})/\hat{\tau}$ occurs through a discontinuity in a similar way as in the model of Benhabib and Farmer [4] (see for example figure 2, page 34) since one of the eigenvalues of the Jacobian matrix changes its sign from $+\infty$ to $-\infty$.

(left diagram) and -0.01 (right diagram). According to the directions of the arrows it is clear that in both phase diagrams, any choice of $x_0 \neq \hat{x}_0$ along the vertical line $\hat{\tau} = \tau_0$ leads to paths which cannot converge to $(\hat{x}, \hat{\tau})$. In this case we have local determinacy of the steady state $(\hat{x}, \hat{\tau})$ since given any k_0 and τ_0 satisfying the conditions of Proposition 1, there exists a unique choice of $x_0 = \hat{x}_0$ and therefore of $c_0 = k_0\hat{x}$ which pins down an equilibrium path corresponding to the BGP described in the previous section.

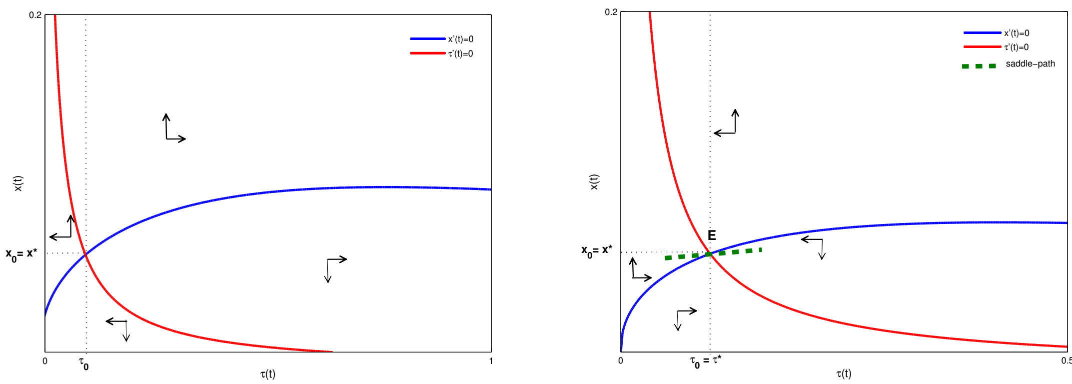


Figure 1: Phase Diagrams when $\phi = 0.5$ (left) and $\phi = -0.01$ (right) and $\gamma = \hat{\gamma}^0$

4.2 Existence of other equilibria

As we have shown in the previous subsection, the unique steady state $(\hat{x}, \hat{\tau})$ may be a saddle-point or totally unstable depending on whether $\phi \in (-\sigma(1+\hat{\tau})/\hat{\tau}, 0)$ or not. While these two possible configurations do not alter the fact that when $\tau_0 = \hat{\tau}$, the economy directly jumps on the BGP from the initial date $t = 0$, we can prove that contrary to Barro [3], some particular transitional dynamics may occur in our model. Indeed, the BGP as defined by $(\hat{x}, \hat{\tau})$ and $\hat{\gamma}$ is not the unique possible equilibrium of our economy. In this section, depending on the value of ϕ, we look for the existence of equilibrium paths $(x_t, \tau_t)_{t\geq 0}$ which may eventually converge to an Asymptotic BGP, denoted from now on ABGP, defined as follows:

Definition 1 (ABGP). *An ABGP is any path $(x(t), \tau(t))_{t\geq 0} = (x^*, \tau^*)$ such that:*

a) τ^* is a positive arbitrary constant sufficiently close to (but different from) τ_0;

b) (x^*, τ^*) is a steady state of (16)-(17) with $x^* > 0$ and $\gamma^* > 0$ solution of

$$0 = (\alpha - 1)A(x\tau^*)^{1-\alpha} + (1+\tau^*)x - \rho \qquad (24)$$

$$\gamma = \alpha A(x\tau^*)^{1-\alpha} - \delta - \rho. \qquad (25)$$

c) (x^*, τ^*) satisfies the transversality condition.

Crucially an ABGP is not an equilibrium since it does not satisfy the initial condition $\tau(0) = \tau_0$. An ABGP in terms of the original variables is a path

$$k^* = k_0 e^{\gamma^*(\alpha,\tau^*,\rho,\delta,\sigma)t} \qquad \text{and} \qquad c^* = x^*(\alpha,\tau^*,\rho,\delta,\sigma) k_0 e^{\gamma^*(\alpha,\tau^*,\rho,\delta,\sigma)t} \tag{26}$$

which is defined as a steady state (x^*, τ^*) of the system (16)-(17) but is *not an equilibrium* because τ^* is generically different from the exogenously given initial condition of the consumption tax, τ_0. If such an ABGP exists, the asymptotic value (x^*, τ^*) as well as the asymptotic growth rate of the economy γ^* will not be pinned down by τ_0 through equations (19) and (20) as before, but rather from the asymptotic value of the consumption tax τ^* and then by equations

$$0 = (\alpha - 1)A(x\tau^*)^{1-\alpha} + (1+\tau^*)x - \rho \tag{27}$$

$$\gamma = \alpha A(x\tau^*)^{1-\alpha} - \delta - \rho. \tag{28}$$

The existence of an equilibrium path converging to an ABGP is associated to the existence of consumers' beliefs that are different from those associated to the BGP. Indeed, they may believe that the consumption tax profile will not remain constant but rather change over time and eventually converge to a positive value $\tau^* \neq \tau_0$. Therefore, the consumption over capital ratio and the growth rate will converge to x^* and γ^* respectively. Based on that we will prove in the next Proposition that under some conditions on ϕ the consumers may indeed decide a consumption path which makes this belief self-fulfilling.

Building on Propositions 1 and 2 we can prove the following result:

Proposition 3. *Given any initial condition $k_0 > 0$ and $\tau_0 > 0$, consider $\underline{\tau}$ and $\bar{\tau}(\sigma)$ as defined by Proposition 1. There exist $\underline{A} > 0$ such that when $A > \underline{A}$, there is a unique equilibrium path $(x_t, \tau_t)_{t \geq 0}$ converging over time to the ABGP (x^*, τ^*) if and only if $\phi \in (-\sigma(1+\tau^*)/\tau^*, 0)$ and one of the following conditions holds:*

i) $\sigma \geq 1$ and $\tau^* > \underline{\tau}$,

ii) $\sigma \in (0,1)$ and $\tau^* \in (\underline{\tau}, \bar{\tau}(\sigma))$.

Proof. See Appendix A.5. ■

Discussion of these conditions is again in order. First of all, if the consumers believe that the asymptotic tax rate will be τ^* then a unique ABGP exists if one of the conditions i)-ii) holds. As already discussed previously, the requirement of a level of technology greater than $\underline{A}$ is a standard condition for AK models and a tax rate larger than $\underline{\tau}$ provides a large enough government spending to sustain growth through the technology $y = Ak^\alpha \mathcal{G}^{1-\alpha}$. Also the condition $\tau < \bar{\tau}(\sigma)$ when $\sigma < 1$ guarantees that the transversality condition is respected and thus the utility is bounded.

Furthermore, given (k_0, τ_0) there exists a unique equilibrium path converging to the ABGP (x^*, τ^*) if and only if $\phi \in (-\sigma(1+\tau^*)/\tau^*, 0)$. To provide an intuition for this result let us assume for simplicity that $\sigma = 1$. Considering the expression of the tax rate as given by (15), let us denote $g(c) \equiv (1+\tau(\tilde{c}))c$ with $\tilde{c} = ce^{-\gamma t}$. It follows that the elasticity of $g(c)$ is given by

$$\varepsilon_{gc} \equiv \tfrac{g'(c)c}{g(c)} = 1 + \tfrac{\phi\tau}{1+\tau}$$

Since $\phi > -(1+\tau^*)/\tau^*$, we get $\varepsilon_{gc} > 0$. Consider then the system of ODEs (7)-(8) with $\sigma = 1$ which can be written as

$$\dot{k} = Ak^\alpha \mathcal{G}^{1-\alpha} - \delta k - g(c) \quad (29)$$

$$\frac{\dot{c}}{c} = r - \delta - \rho - \frac{\dot{\tau}}{1+\tau} \quad (30)$$

If households expect that in the future the consumption tax rate will be above average, then they expect to consume less in the future and thus, considering that $\varepsilon_{gc} > 0$, we derive from (29) that $g(c)$ is decreasing and thus investment is increasing. This implies that the rental rate of capital r is decreasing and thus through equation (30) that consumption is also decreasing. We conclude from the balanced budget rule that the tax rate is decreasing and that the initial expectation is self-fulfilling. Of course this mechanism requires a low enough elasticity of intertemporal substitution in consumption, i.e. a large enough value of σ, to avoid intertemporal consumption's compensations associated to the initial expected decrease of c.

Figure 2 shows the phase diagrams which implicitly account for the consumers' beliefs of a steady state where the same parameters' values as in Figure 1 except that here $(x^*, \tau^*) = (0.123, 0.25)$ and therefore a growth rate $\gamma = 0.037$.

Observe that both the locus $\dot{x} = 0$ and $\dot{\tau} = 0$ are shifted with respect to the previous case to reflect the different beliefs.[11] According to the directions of the arrows it is clear that in the case of a pro-cyclical consumption tax (i.e. $\phi = 0.5$), there does not exist an equilibrium path which makes this belief self-fulfilling. On the other hand, in the case of a counter-cyclical consumption tax (i.e. $\phi = -0.01$) an equilibrium path converging to the steady state may exist as shown by the golden path in the Figure. In this case the consumers' belief is indeed self-fulfilling.

[11] This is indeed obvious from equations (16) and (17) since the growth rate, γ, enters explicitly in both of them.

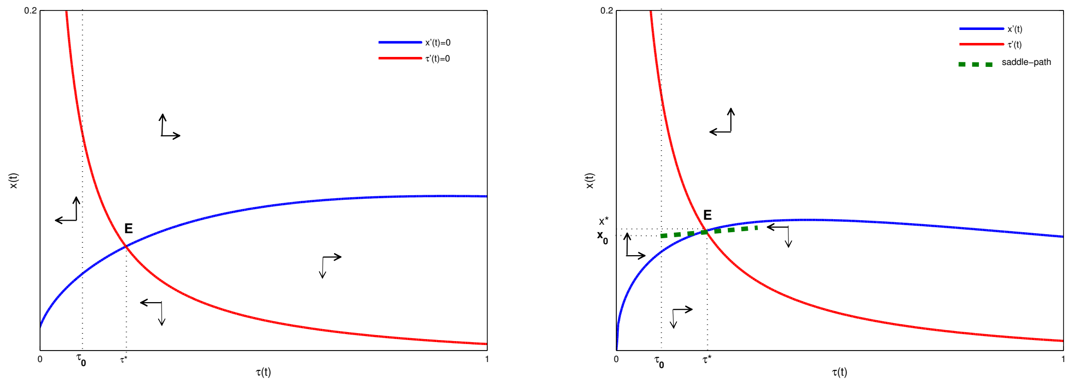


Figure 2: Phase Diagrams when $\phi = 0.5$ (left) and $\phi = -0.01$ (right) and $\gamma = \hat{\gamma}$[1]

4.3 Overall dynamics

Proposition 3 actually proves that there exists a continuum of equilibria each of them converging to a different ABGP. In fact, any value of τ^* in a neighborhood of the given initial value τ_0 can be a self-fulfilling belief for the consumers if the conditions of the Proposition are met. Of course this implies a form of global indeterminacy since from a given τ_0, one can select either the unique BGP by jumping on it from the initial date or select any other equilibrium converging to an ABGP. Again different choices reveal different consumers' beliefs of the long run outcome of the economy.

Combining the results found in section 3 and subsections 4.1 and 4.2 allows to state the following Theorem which fully characterizes the dynamics of the economy.

Theorem 1. *Given the initial conditions k_0 and τ_0, let $\tau_{inf} = \tau_0 - \epsilon > 0$ and $\tau_{sup} = \tau_0 + \varepsilon$ with $\epsilon, \varepsilon > 0$ small enough. Consider $\underline{\tau}$ and $\bar{\tau}(\sigma)$ as defined by Proposition 1. There exists $\underline{A} > 0$ such that if $A > \underline{A}$, $\phi \in \left(-\frac{\sigma(1+\tau_{sup})}{\tau_{sup}}, 0\right)$ and one of the following conditions holds:*

i) $\sigma \geq 1$ and $\tau_{inf} > \underline{\tau}$,

ii) $\sigma \in (0,1)$, $\tau_{inf} > \underline{\tau}$ and $\tau_{sup} < \bar{\tau}(\sigma)$,

then there is a continuum of equilibrium paths, indexed by the letter j, departing from (τ_0, x_0^j), each of them converging to a different ABGP (τ^{*j}, x^{*j}) with $\tau^{*j} \in (\tau_{inf}, \tau_{sup})$, i.e. the dynamics of the economy is globally, but not locally, indeterminate. On the other hand, the dynamics of the economy is globally and locally determinate if $\phi \in (0, +\infty)$.

Proof. See Appendix A.6. ∎

To fully understand the dynamic behavior of the economy we can write explicitly the solution of the linearized system:

$$\tilde{\tau} = b_1 v_{11} e^{\lambda_1 t} + b_2 v_{21} e^{\lambda_2 t} \tag{31}$$

$$\tilde{x} = b_1 v_{12} e^{\lambda_1 t} + b_2 v_{22} e^{\lambda_2 t} \tag{32}$$

where $\mathbf{v_i} \equiv (v_{i1}, v_{i2})^T$ is the eigenvector associated to the eigenvalue λ_i, with $i = 1, 2$ while b_i are arbitrary constants. If $\phi \in (-\sigma(1+\tau^*)/\tau^*, 0)$ and assuming without loss of generality that $\lambda_2 > 0$, the saddle-path solution can be easily found imposing $b_2 = 0$. Combining (31) and (32) and imposing $b_2 = 0$ we get that

$$\tilde{\tau}_0 = \frac{v_{11}}{v_{12}} \tilde{x}_0 \tag{33}$$

with $\tilde{\tau}_0 = \tau_0 - \tau^*$ and $\tilde{x}_0 = x_0 - x^*$. Therefore, given any initial condition k_0 and τ_0 we have the following solution of x converging over time to (x^*, τ^*):

$$x = x^* + \frac{v_{12}}{v_{11}} \tilde{\tau}_0 e^{\lambda_1 t} \tag{34}$$

Of course as $t \to \infty$ we have that $\tilde{\tau} \to \tau^*$, meaning that c converges to the corresponding ABGP since $x \to x^*$ also converges to the corresponding ABGP. Observe also that the initial level of consumption for this equilibrium path can be obtained from (34) evaluated at $t = 0$, taking into account (26), and it is equal to

$$c_0 = c^* + \frac{v_{12}}{v_{11}} (\tau_0 - \tau^*) k_0 \tag{35}$$

Remark 1. *Note that we have a constraint on the initial choice of* c_0 *(and therefore on* x_0*) because initial consumption cannot be higher than the initial wealth,* $c_0 \leq y_0 - \delta k_0 - \tau_0 c_0$ *which at the equilibrium implies that*

$$x_0 \leq A \frac{A\tau_0^{\frac{1-\alpha}{\alpha}}}{(1+\tau_0)^{\frac{1}{\alpha}}} - \delta$$

Figure 3 shows the presence of global indeterminacy in the phase diagram $(x(t), \tau(t))$.

The initial tax rate is assumed to be equal to $\tau_0 = 0.2$, and the parameters are chosen as in the balanced growth path section with ϕ set to -0.5. Different initial choices of c_0 pins down different equilibrium paths of the tax rate and of the consumption-capital ratio, each of them converging to a different steady state, characterized by a different growth rate of consumption and capital. Similarly Figure 4 illustrates the emergence of global indeterminacy in the spaces $(t, x(t))$ and $(t, \tau(t))$.

On the other hand, local and global determinacy arise when the government spending uses pro-cyclical consumption tax. In fact, in this case we have two strictly

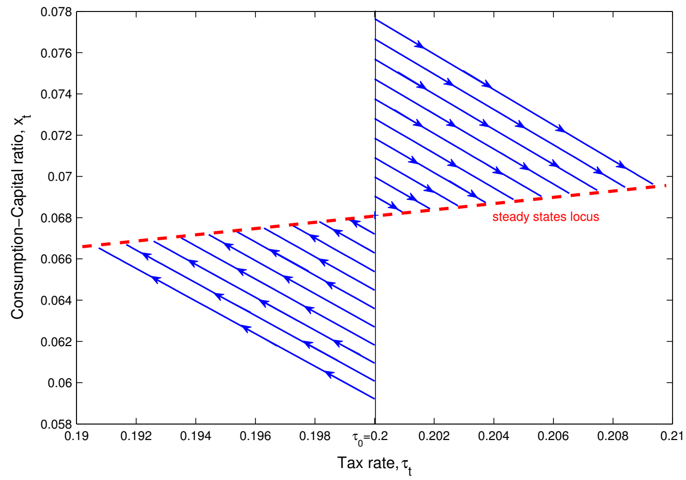
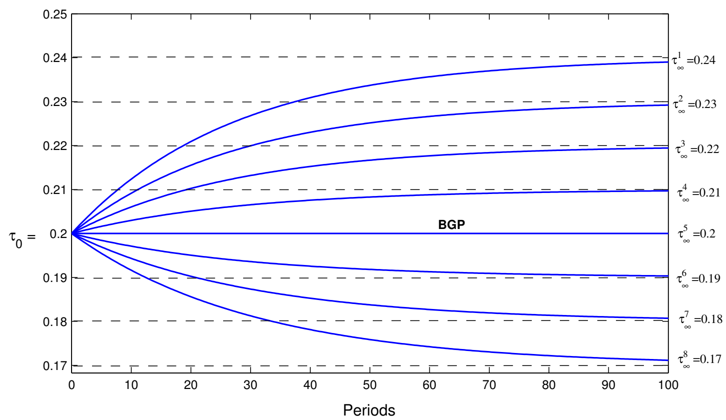


Figure 3: Converging equilibria for different initial values of c_0.



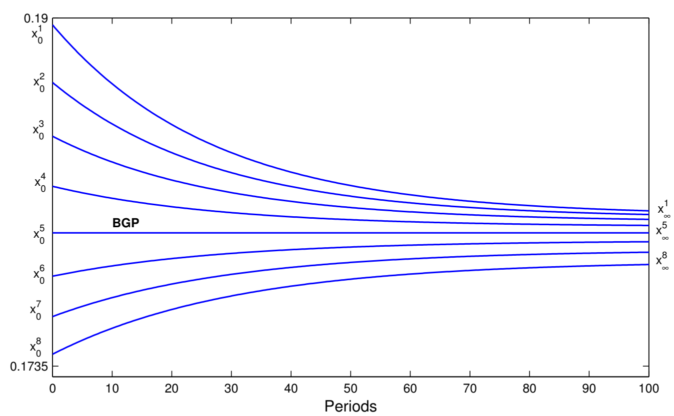


Figure 4: Dynamics of the Tax Rate and of the Consumption-capital Ratio for Fiscal Policy (13)

positive eigenvalues and therefore two explosive paths to be ruled out by setting $b_1 = b_2 = 0$. In this case the economy has no transitional dynamics and the only solution is the balanced growth path solution described in the previous section. It is also worth noting that the dynamics in the case with pro-cyclical time-varying consumption-tax, i.e. $\phi \in (0, +\infty)$, is equivalent to the dynamics of an economy with constant consumption tax.

Finally global indeterminacy arises when the government uses counter-cyclical consumption tax without having a predetermined target growth rate. Under these circumstances, the long run growth rate as well as the consumption over capital ratio cannot be univocally determined within the model. Therefore an economy characterized by these features and an initial value of the tax rate $\tau(0)$ can remain on a balanced growth path but can also follows alternative paths towards different ABGPs each of them characterized by a different (asymptotic) growth rate.

Remark 2. *One could think that our main results strongly rely on the consideration of a consumption tax. This is actually not the case. If we assume that government expenditures are financed through a tax on income, i.e.* $\mathcal{G} = \tau(\tilde{y})y$, *with* $\tilde{y} = ye^{\gamma t}$, *and that the elasticity of the tax rate with respect to detrended output is constant and equal to* ϕ, *then we find similar results. Indeed, the model is basically the same except that the capital accumulation equation becomes now*

$$\dot{k} = (1-\tau)Ak^{\alpha}\mathcal{G}^{1-\alpha} - \delta k - c$$

Moreover, solving $\mathcal{G} = \tau y = \tau A k^{\alpha}\mathcal{G}^{1-\alpha}$ *with respect to* $\mathcal{G}$ *gives* $\mathcal{G} = (\tau A)^{1/\alpha}k$. *From the corresponding first order conditions, straightforward computations then lead to the following dynamical system*

$$\frac{\dot{x}}{x} = \frac{1}{\sigma}\left[(1-\tau)A(\tau A)^{\frac{1-\alpha}{\alpha}}(\alpha-\sigma) - \delta(1-\sigma) - \rho + \sigma x\right] \tag{36}$$

$$\frac{\dot{\tau}}{\tau} = \frac{\phi\alpha}{\alpha - \phi(1-\alpha)}\left[(1-\tau)A(\tau A)^{\frac{1-\alpha}{\alpha}} - \delta - \gamma - x\right] \tag{37}$$

with $x \equiv \frac{c}{k}$. *Along a BGP as defined by (18), we find again that* τ *is constant and equal to its initial value* τ_0. *It follows that there exists a unique steady-state of the dynamical system characterized by stationary values of* x *and* γ, *i.e.* $\hat{x}$ *and* $\hat{\gamma}$, *that depend on* τ_0. *But there also exist a continuum of ABGP characterized by an asymptotic value of* τ, *namely* $\tau^* \neq \tau_0$ *with* τ^* *sufficiently close to* τ_0, *and of equilibrium path each of them converging to a different ABGP provided the saddle-point property holds.*

5 Aggregate instability

Suppose that the households choose an initial value of consumption such that they are at $t = 0$ in (τ_0, x_0^j). From Theorem 1, we know that if some conditions on parameters are respected then the deterministic dynamical system (16)-(17) has a unique solution around the steady state (τ^{*j}, x^{*j}) converging to it over time. Let us call this path $(x, \tau) = \boldsymbol{\phi}_j(t) \equiv (\phi_{j,x}(t), \phi_{j,\tau}(t))$ as shown on the following Figure 5:

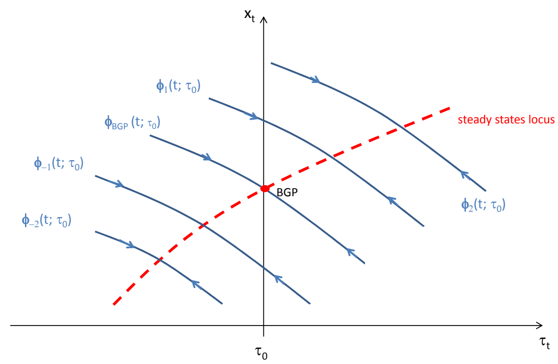


Figure 5: Examples of Saddle-Paths

As observed before, this is indeed the unique equilibrium consistent with an asymptotic growth rate γ^j. Clearly aggregate instability cannot emerge unless we introduce changes in the households beliefs. In the following we do that by introducing extrinsic uncertainty and we prove that sunspot equilibria, and therefore aggregate instability, may emerge in our framework.

5.1 An illustrative example

Before studying the existence of sunspot equilibria in our continuous time environment, it is useful to provide an example in discrete time assuming a period length equal to a positive integer, h.[12] Due to the presence of extrinsic uncertainty, the discrete time counterpart of the dynamical system (36)-(37) takes the form:

$$\begin{pmatrix} \mathbb{E}_t(\Delta x_{t+h}) \\ \Delta \tau_{t+h} \end{pmatrix} = F(x_t, \tau_t)h \qquad \textit{with } (x_0, \tau_0) \textit{ given}$$

where $\Delta\chi_{t+h} \equiv \chi_{t+h} - \chi_t$ with $\chi = x, \tau$, $\mathbb{E}_t$ is a conditional expectation operator, and $F(x_t, \tau_t)$ is the left hand side of the continuous time dynamical system, after (36) has been multiplied by x and (37) by τ. Since there is no intrinsic uncertaity

[12] In the following we assume that the dynamics is indeed invariant to the choice of time.

in our model, then such system can be rewritten as

$$\begin{pmatrix} \Delta x_{t+h} \\ \Delta \tau_{t+h} \end{pmatrix} = F(x_t, \tau_t)h + s \begin{pmatrix} \Delta \varepsilon_{t+h} \\ 0 \end{pmatrix} \qquad \textit{with } (x_0, \tau_0) \textit{ given} \tag{38}$$

where $\mathbb{E}_t(\Delta\varepsilon_{t+h}) = 0$ with ε_t the sunspot variable (see Shigoka [23] or Benhabib and Wen [6], among others).

Suppose now that the sunspot variable, ε_t takes the values $(0, z_1, z_2)$ at dates $(0, t_1+h, t_2+h)$ respectively.[13] Therefore, the dynamics of the system in the interval of time $t \in [0, t_1]$ will be described by the initial value problem (from now on IVP)

$$\begin{pmatrix} \Delta x_{t+h} \\ \Delta \tau_{t+h} \end{pmatrix} = F(x_t, \tau_t)h \qquad \textit{with } (x_0, \tau_0) \textit{ given}$$

which we know from Theorem 1 to have a unique solution in a neighbourhood of the steady state. In fact, such theorem tells us that *given an initial condition* the economy is on its BGP or there is a unique equilibrium path converging to an ABGP (see Figure 5). Let us indicate this equilibrium path with

$$\{x_t, \tau_t\}_{t=0}^{t_1} = \{\phi_{1x}(t), \phi_{1\tau}(t)\}_{t=0}^{t_1}.$$

Between date t_1 and $t_1 + h$, the second sunspot arrives and, therefore, we have that

$$\begin{pmatrix} \Delta x_{t_1+h} \\ \Delta \tau_{t_1+h} \end{pmatrix} = F(x_{t_1}, \tau_{t_1})h + s \begin{pmatrix} z_1 - 0 \\ 0 \end{pmatrix}$$

which clearly implies that

$$\begin{pmatrix} x_{t_1+h} \\ \tau_{t_1+h} \end{pmatrix} = \begin{pmatrix} \phi_{1x}(t_1) \\ \phi_{1\tau}(t_1) \end{pmatrix} + F(\phi_{1x}(t_1), \phi_{1\tau}(t_1))h + s \begin{pmatrix} z_1 \\ 0 \end{pmatrix} \tag{39}$$

where the first two terms on the RHS are obtained considering the equilibrium path found previously. Based on this observation, it follows that the dynamics of the system in the interval of time $t \in [t_1 + h, t_2]$ will be given by the IVP

$$\begin{pmatrix} \Delta x_{t+h} \\ \Delta \tau_{t+h} \end{pmatrix} = F(x_t, \tau_t)h \qquad \textit{with } (x_{t_1+h}, \tau_{t_1+h}) \textit{ given by } (39). \tag{40}$$

[13] Two remarks are in order. First, a sunspot is, as usual, an unanticipated random shock from the household's perspective. Second, the values taken by the sunspot variable as well as the time of the arrival of a sunspot, are exogenously given in the example. This will be relaxed in the next section.

Again from Theorem 1, we know that there exists a unique solution of this IVP in a neighborhood of the steady state. In fact, we proved that the dynamical system has a continuum of solutions, each of them associated with a different inital condition and, crucially, converging to a different ABGP with a different asymptotic growth rate. The presence of the sunspot variable has just modified the deterministic framework by allowing a "jump" at date $t_1 + h$ of size sz_1 in the no-predetermined variable, as it emerges from (39),[14] while the dynamics of the economy is still described by $F(.)$ since the uncertainty is extrinsic and does not affect the fundamentals. The equilibrium path in the interval $t \in [t_1 + h, t_2]$ will be the solution of (40):

$$\{x_t, \tau_t\}_{t=t_1+h}^{t_2} = \{\phi_{2x}(t), \phi_{2\tau}(t)\}_{t=t_1+h}^{t_2}. \tag{41}$$

Of course, the realizations of the sunspot variable and of the parameter, s, have to be choosen, as usual, sufficiently small to guarantee that the resulting equilibrium respect all the inequalities constraints.

Similarly the equilibrium path in the interval $t \in [t_2 + h, \infty]$, will be derived as the solution of the same dynamical system $F(x_t, \tau_t)h$, where this time the given initial condition, $(x_{t_2+h}, \tau_{t_2+h})$, was obtained as it follows

$$\begin{pmatrix} x_{t_2+h} \\ \tau_{t_2+h} \end{pmatrix} = \begin{pmatrix} \phi_{2x}(t_2) \\ \phi_{2\tau}(t_2) \end{pmatrix} + F(\phi_{2x}(t_2), \phi_{2\tau}(t_2))h + s \begin{pmatrix} z_2 - z_1 \\ 0 \end{pmatrix} \tag{42}$$

Again, the presence of the sunspot variable and specifically of the change in its realization at date $t_2 + h$ implies a "jump" of size $s(z_2 - z_1)$ in the no-predetermined variable but no change in the fundamentals and hence no change in $F(.)$. The equilibrium path in the interval $t \in [t_2 + h, \infty]$ will be:

$$\{x_t, \tau_t\}_{t=t_2+h}^{\infty} = \{\phi_{3x}(t), \phi_{3\tau}(t)\}_{t=t_2+h}^{\infty}. \tag{43}$$

Under the conditions in Theorem 1, this equilibrium path will converge over time to an ABGP with an asymptotic growth rate γ_3. An example of the resulting path is shown in Figure 6. From this example, two considerations: first, a sunspot equilibrium will be indeed a randomization over the deterministic equilibrium paths found in the previous sections; secondly, the continuous time case can be naturally derived by considering the limit $h \to 0$.

In the next section, we describe more formally the existence of sunspot equilibria by explicitly describing the stochastic process governing the sunspot variable.

[14] In continuous time, the jump of the no-predetermined (control) variable at time t_1 can be easily derived from equation (39) considering the limit $h \to 0^+$.

5.2 Sunspot Equilibria

We begin our analysis on the existence of sunspot equilibria by describing more formally the sunspot variables. Let us consider the probability space $(\Omega, B_\Omega, \mathbb{P})$ where Ω is the sample space, B_Ω is a σ-field associated to Ω, and $\mathbb{P}$ is a probability measure. Assume also that the state space is a countable subset of $\mathbb{R}$:

$$Z \equiv \{z_1, ..., z_{\bar{\imath}}, ..., z_N\} \subset \mathbb{R}$$

with $-\underline{\varepsilon} \leq z_1 < ... < z_{\bar{\imath}} = 0 < ... < z_N \leq \bar{\varepsilon}$. Then each random variable ε_t is a function from $\Omega \to Z$ which we assume to be B_Ω-measurable.[15]

Moreover we assume that the family of random variables $\{\varepsilon_t\}_{t\geq 0}$ is a continuous-time homogeneous Markov chain with $p_{ij}(t-s)$ indicating the transition probability to move from state i at time s to state j at time t with $s \leq t$ (see Appendix A.7 for more details) while the initial probability distribution of ε_0 is denoted by $\pi = (\pi_1, ..., \pi_N)$ with $\pi_j = \mathbb{P}(\varepsilon_0 = z_j)$.

Differently from the discrete-time case, the evolution of a continuous-time Markov chain cannot be described by the initial distribution π and the $n-step$ transition probability matrix, $\mathbf{P}^n$, since there is no implicit unit length of time. However, it is possible to define a matrix $\mathbf{G}$ (generator of the chain) which takes over the role of $\mathbf{P}$. This procedure can be found in Grimmett and Stirzaker [13] among others and, as far as we know, our paper represents the first economic application of this procedure.

Let $\mathbf{P}_t$ be the $N \times N$ matrix with entries $p_{ij}(t)$. The family $\{\mathbf{P}_t\}_{t\geq 0}$ is the *transition stochastic semigroup* of the Markov chain (see Appendix A.7) and the evolution of $\{\varepsilon_t\}_{t\geq 0}$ depends on $\{\mathbf{P}_t\}_{t\geq 0}$ and the initial distribution π of ε_0. Let us also assume from now on that the transition stochastic semigroup $\{\mathbf{P}_t\}_{t\geq 0}$ is standard, i.e. $\lim_{t\to 0} \mathbf{P}_t - \mathbf{I} = \mathbf{0}$ or

$$\lim_{t\to 0} p_{ii}(t) = 1 \qquad and \qquad \lim_{t\to 0} p_{ij}(t) = 0 \; for \; i \neq j$$

Under these assumptions on the semigroup the following result can be proved:

Proposition 4. *Consider the interval $(t, t+h)$ with h small. Then*

$$\lim_{h\to 0} \frac{1}{h}(\mathbf{P}_h - \mathbf{I}) = \mathbf{G}$$

i.e. there exists constants $\{g_{ij}\}$ such that

$$p_{ii}(h) \simeq 1 + g_{ii}h \qquad and \qquad p_{ij}(h) \simeq g_{ij}h \;\; if \; i \neq j \tag{44}$$

[15] The function ε_t is measurable if $\{\omega \in \Omega : \varepsilon_t(\omega) \leq z\} \in B_\Omega$ for each $z \in Z$.

with $g_{ii} \leq 0$ and $g_{ij} > 0$ for $i \neq j$. The matrix $\mathbf{G} = (g_{ij})$ is called the generator of the Markov chain $\{\varepsilon_t\}_{t \geq 0}$.

Proof. See Grimmett and Stirzaker [13], Chapter VI, page 256-258. ■

Therefore, the continuous-time Markov chain $\{\varepsilon_t\}_{t \geq 0}$ has a generator $\mathbf{G}$ which can be used together with the initial probability distribution π to describe the evolution of the chain. For this purpose, the following definition will turn out to be useful:

Definition 2. *Let $\varepsilon_s = z_i$, we define the "holding time" as*

$$\mathcal{T}_i \equiv \inf\{t \geq 0 : \varepsilon_{s+t} \neq z_i\}$$

Therefore the "holding time" is a random variable describing the further time until the Markov chain changes its state. The following Proposition is crucial to understand the evolution of the chain from a generic initial state $\varepsilon_s = z_i$.

Proposition 5. *Under the assumptions on the Markov chain introduced so far, the following results hold:*

1) *The random variable $\mathcal{T}_i$ is exponentially distributed with parameter g_{ii}. Therefore,*
$$p_{ii}(t) = \mathbb{P}(\varepsilon_{s+t} = z_i | \varepsilon_s = z_i) = e^{g_{ii}t}.$$

2) *If there is a jumps, the probability that the Markov chain jumps from z_i to $z_j \neq z_i$ is $-\frac{g_{ij}}{g_{ii}}$.*

Proof. See Grimmett and Stirzaker [13], Chapter VI, page 259-260. ■

Through the last Proposition we can fully describe the evolution of the Markov chain and therefore we have all the ingredients to build sunspot equilibria. Before doing that we define a sunspot equilibrium as it follows:

Definition 3 (Sunspot Equilibrium). *A sunspot equilibrium is a stochastic process $\{(\tau_t, x_t, \varepsilon_t)\}_{t \geq 0}$ such that $\{(\tau_t, x_t)\}_{t \geq 0}$ solves the continuous-time counterpart of the stochastic system (38), respect the inequality constraints $\tau_t, x_t > 0$ and the transversality condition.*

As shown in our example, sunspot equilibria can be built up randomizing over the deterministic equilibria.

Theorem 2 (Existence of Sunspot Equilibria). *Assume that all the conditions for indeterminacy in Theorem 1 hold as well as all the assumptions on* $\{\varepsilon_t\}_{t\geq 0}$ *introduced so far. Then sunspot equilibria exist.*

Proof. See Appendix A.8. ■

An example of a sunspot equilibrium is drawn in Figure 6.

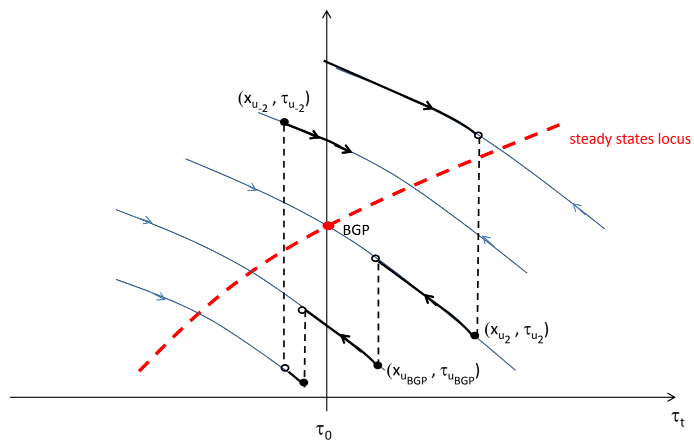


Figure 6: Example of a Sunspot Equilibrium

Moreover, the following additional remarks are also useful.

Remark 3. *If all the states are assumed to be absorbing* ($g_{ii} = 0$*, for all* i*) then the exogenous uncertainty plays a role only at* $t = 0$ *where the economy starting at* (τ_0, x_0) *jumps on the deterministic equilibrium path* $(x, \tau) = \boldsymbol{\phi}_j(t)$ *with probability* π_j *and remains there forever.*

Remark 4 (Stationary Sunspot Equilibrium). *If the transition probabilities in the stochastic semigroup are specified such that there exists a vector* $\boldsymbol{\mu}$ *with* $\boldsymbol{\mu} = \boldsymbol{\mu}\mathbf{P}_t$ *for all* $t \geq 0$ *then* $\boldsymbol{\mu}$ *is a stationary distribution of the Markov chain* $\{\varepsilon_t\}_{t\geq 0}$ *and the sunspot equilibrium is stationary.*

Theorem 1 shows how to build sunspot equilibria starting from our deterministic model characterized by a unique deterministic and locally determinate BGP and a contiuum of other equilibria each of them converging over time to a different ABGP.

Contrary to the standard literature where sunspot equilibria are based on the existence of local indeterminacy (see Benhabib and Farmer [4]), we find sunspot fluctuations while there exists a unique unstable BGP as well as a continuum of other equilibria converging to the ABGPs. From this point of view, our conclusions share some similarities with Farmer [11] where expectations-driven fluctuations, in an economy with a continuum of steady states, are generated from the existence of a continuum of equilibrium unemployment rates.

6 Conclusion

We have considered a Barro-type [3] endogenous growth model in which a government provides as an external productive input a constant stream of expenditures financed through consumption taxes and a balanced budget rule. In order to have a constant tax on a balanced growth path, the tax rate needs to depend on de-trended consumption and thus becomes a state variable with a given initial condition. We also consider a representative household characterized by a CRRA utility function and inelastic labor. Such a formulation is known to rule out the existence of endogenous fluctuations in a standard stationary framework (see Giannitsarou [12]).

We have proved that there exists a unique Balanced Growth Path (BGP) along which the common growth rate of consumption, capital, GDP and government spending is constant. Moreover, as in the Barro [3] model, there is no transitional dynamics with respect to this unique BGP. However, we have shown that the BGP is not the unique long run solution of our model. Indeed, if the tax rule is countercyclical with respect to consumption, for any arbitrary initial value of the tax rate, close enough to its initial condition, there exists a corresponding value for the tax rate, consumption, capital and the constant growth rate that can be an asymptotic equilibrium of our economy, namely an Asymptotic Balanced Growth Path (ABGP). An ABGP is not itself an equilibrium as it does not respect the initial conditions. However, some transitional dynamics exist with a unique equilibrium path converging toward this ABGP, and we prove that there exist a continuum of such ABGP and of equilibria each of them converging over time to a different ABGP.

The existence of an equilibrium path converging to an ABGP is associated to the existence of consumers' beliefs that are different from those associated to the BGP. Indeed, they may believe that the consumption tax profile will not remain constant but rather change over time and eventually converge to a positive value different from the initial condition. Based on this property, we prove the existence sunspot equilibria and thus that endogenous sunspot fluctuations may arise under a balanced budget rule and consumption taxes although there exists a unique underlying BGP equilibrium.

A Appendix

A.1 Derivation of equations (16) and (17)

Dividing equation (7) by k and using the balanced-budget rule we may rewrite the system of equations (7), (8) as :

$$\begin{aligned} \frac{\dot{k}}{k} &= \left(\frac{k}{c\tau}\right)^{\alpha-1} - \delta - (1+\tau)\frac{c}{k} && (45) \\ \frac{\dot{c}}{c} &= \frac{1}{\sigma}\left[\alpha\left(\frac{k}{c\tau}\right)^{\alpha-1} - \delta - \rho - \frac{\dot{\tau}}{1+\tau}\right] && (46) \end{aligned}$$

Substituting equation (46) into the fiscal policy rule (13) and solving for $\dot{\tau}/\tau$, we derive

$$\frac{\dot{\tau}}{\tau} = \frac{\phi(1+\tau)}{\sigma(1+\tau)+\phi\tau}\left[\alpha A\left(\frac{k}{\tau c}\right)^{\alpha-1} - \delta - \rho - \sigma\gamma\right] \tag{47}$$

Now let us define the control-like variable $x \equiv \frac{c}{k}$ which implies that $\frac{\dot{x}}{x} = \frac{\dot{c}}{c} - \frac{\dot{k}}{k}$. Subtracting equation (45) from equation (46) and using the definition of the new variable x together with equation (47) we find

$$\begin{aligned} \frac{\dot{x}}{x} &= \frac{(1+\tau)(1-\sigma)-\phi\tau}{\sigma(1+\tau)+\phi\tau}\left[\alpha A\left(\tau x\right)^{1-\alpha} - \delta - \rho - \sigma\gamma\right] \\ &+ \gamma(1-\sigma) + (1+\tau)x - (1-\alpha)A(\tau x)^{1-\alpha} - \rho && (48) \\ \frac{\dot{\tau}}{\tau} &= \frac{\phi(1+\tau)}{\sigma(1+\tau)+\phi\tau}\left[\alpha A\left(\tau x\right)^{1-\alpha} - \delta - \rho - \sigma\gamma\right] && (49) \end{aligned}$$

■

A.2 Proof of Proposition 1

The proof is articulated in four steps.

The first step of the proof consists in showing that there exists a positive solution of the equation $g(x) = 0$. Note first that that $g(0) = -\rho - \delta(1-\sigma) < 0$ if and only if $\sigma < \sigma_0 \equiv (\delta+\rho)/\delta$, $\lim_{x\to+\infty} g(x) = +\infty$, and $g'(x) = (\alpha-\sigma)(1-\alpha)A\hat{\tau}^{1-\alpha}x^{-\alpha} + \sigma(1+\hat{\tau})$. If $\alpha \geq \sigma$, $g'(x) > 0$ for any x and the uniqueness of the solution is ensured. On the contrary, if $\sigma > \alpha$ we get $g'(x) = 0$ if and only if

$$x = x_{min} = \left(\frac{(1-\alpha)(\sigma-\alpha)A\hat{\tau}^{1-\alpha}}{\sigma(1+\hat{\tau})}\right)^{\frac{1}{\alpha}} > 0$$

Since $g(x)$ is a continuous function, we conclude that $g'(x) < 0$ when $x \in (0, x_{min})$ and $g'(x) > 0$ when $x > x_{min}$. Moreover, we get

$$g(x_{min}) = -\sigma \left[\frac{\alpha(1+\hat{\tau})}{1-\alpha} \left[A\hat{\tau}^{1-\alpha}(1-\alpha)\frac{1-\frac{1}{\sigma}}{1+\hat{\tau}} \right]^{\frac{1}{\alpha}} + \delta\left(\frac{1}{\sigma}-1\right) + \frac{\rho}{\sigma} \right]$$

with $\partial g(x_{min})/\partial\sigma < 0$, $g(x_{min})|_{\sigma=1} < 0$ and

$$\lim_{\sigma\to+\infty} \left[\frac{\alpha(1+\hat{\tau})}{1-\alpha} \left[A\hat{\tau}^{1-\alpha}(1-\alpha)\frac{1-\frac{1}{\sigma}}{1+\hat{\tau}} \right]^{\frac{1}{\alpha}} + \delta\left(\frac{1}{\sigma}-1\right) + \frac{\rho}{\sigma} \right] = \alpha A^{\frac{1}{\alpha}} \left(\frac{\hat{\tau}(1-\alpha)}{1+\hat{\tau}}\right)^{\frac{1-\alpha}{\alpha}} - \delta \tag{50}$$

It follows that when $A > A_1$ with

$$A_1 \equiv \left(\frac{\delta}{\alpha}\right)^{\alpha} \left(\frac{1+\hat{\tau}}{\hat{\tau}(1-\alpha)}\right)^{1-\alpha},$$

the expression (50) is positive and $\lim_{\sigma\to+\infty} g(x_{min}) = -\infty$ so that $g(x_{min}) < 0$ for any $\sigma > \alpha$. Therefore, from all these results we conclude the following:

- if $\sigma \in (\alpha, \sigma_0)$ then $g(0) < 0$ and there also exists a unique $\hat{x}$ solution of $g(x) = 0$;

- if $\sigma > \sigma_0$, then $g(0) > 0$, $g(x_{min}) < 0$ and there exists two solutions of $g(x) = 0$, namely $\tilde{x}$ and $\hat{x}$ with $\tilde{x} < \hat{x}$.

The second step of the proof is to verify that the steady state value of x, in particular in the case of multiplicity, leads to a constant growth rate γ which is positive and satisfies the transversality condition (21). We need to check that $\gamma > 0$ when $\sigma \leq 1$ and $\gamma \in (0, \rho/(1-\sigma))$ when $\sigma < 1$. Since

$$\gamma = \frac{1}{\sigma}\left[\alpha A(x\hat{\tau})^{1-\alpha} - \delta - \rho\right]$$

the inequality $\gamma > 0$ is equivalent to

$$x > \frac{1}{\hat{\tau}}\left(\frac{\delta+\rho}{\alpha A}\right)^{\frac{1}{1-\alpha}} \equiv \underline{x}$$

A sufficient condition for the existence of a steady state value for x, $\hat{x}$, such that $\gamma(\hat{x}) > 0$ is $\hat{x} > \underline{x}$. Since $\hat{\tau} = \tau_0$, this inequality is obtained if $g(\underline{x}) < 0$, i.e.

$$\left(\frac{\delta+\rho}{\alpha A}\right)^{\frac{1}{1-\alpha}} < \tau_0 \left[\frac{\delta(1-\alpha)+\rho}{\alpha} - \left(\frac{\delta+\rho}{\alpha A}\right)^{\frac{1}{1-\alpha}}\right] \tag{51}$$

It follows that when $A > A_2$ with

$$A_2 \equiv \left(\frac{\alpha}{\delta(1-\alpha)+\rho}\right)^{1-\alpha} \frac{\delta+\rho}{\alpha}$$

the right-hand-side of (51) is positive. Then, in this case, $g(\underline{x}) < 0$ if and only if

$$\tau_0 > \frac{\left(\frac{\delta+\rho}{\alpha A}\right)^{\frac{1}{1-\alpha}}}{\frac{\delta(1-\alpha)+\rho}{\alpha} - \left(\frac{\delta+\rho}{\alpha A}\right)^{\frac{1}{1-\alpha}}} \equiv \underline{\tau} \tag{52}$$

It is worth noting that when $g(\underline{x}) < 0$, uniqueness of $\hat{x}$ is also ensured. Indeed, in the case where $\sigma > \sigma_0$, we have shown previously that a second solution $\tilde{x}$ of $g(x) = 0$ occurs with $\tilde{x} < \hat{x}$. It is obvious to derive that if $g(\underline{x}) < 0$ then $\underline{x} > \tilde{x}$ and $\tilde{x}$ is characterized by a negative growth rate.

Let us consider finally the restriction to satisfy the transversality condition when $\sigma < 1$, namely

$$\gamma = \frac{1}{\sigma}\left[\alpha A(x\hat{\tau})^{1-\alpha} - \delta - \rho\right] < \rho/(1-\sigma)$$

This inequality is equivalent to

$$x < \tfrac{1}{\hat{\tau}}\left(\tfrac{\delta(1-\sigma)+\rho}{\alpha A(1-\sigma)}\right)^{\frac{1}{1-\alpha}} \equiv \bar{x}$$

We then need to check that $\hat{x} < \bar{x}$. This inequality is obtained if $g(\bar{x}) > 0$, i.e.

$$1 > \tau_0\left[(1-\alpha)A^{\frac{1}{1-\alpha}}\left(\tfrac{\delta(1-\sigma)+\rho}{\alpha(1-\sigma)}\right)^{\frac{-\alpha}{1-\alpha}} - 1\right] \tag{53}$$

The right-hand-side of this inequality is decreasing with respect to σ and negative when $\sigma = 1$. Moreover we get

$$\lim_{\sigma\to 0}\left[(1-\alpha)A^{\frac{1}{1-\alpha}}\left(\tfrac{\alpha}{\delta+\rho}\right)^{\frac{1}{1-\alpha}} - 1\right]$$

which can be positive or negative. When this expression is negative, then (53) holds for any τ_0. When this expression is positive, there exists $\sigma_1 \in (0,1)$ such that if $\sigma > \sigma_1$, (53) again holds for any τ_0. On the contrary, when $\sigma \in (0, \sigma_1)$, (53) holds if $\tau_0 < \bar{\tau}(\sigma)$ with

$$\bar{\tau}(\sigma) \equiv \frac{1}{(1-\alpha)A^{\frac{1}{1-\alpha}}\left(\frac{\delta(1-\sigma)+\rho}{\alpha(1-\sigma)}\right)^{\frac{-\alpha}{1-\alpha}} - 1}$$

To simplify the formulation, we have then proved that when $\sigma < 1$, there exists $\bar{\tau}(\sigma) \in (0, +\infty]$ such that $\hat{x} < \bar{x}$ and the corresponding growth rate $\hat{\gamma}$ satisfies the transversality condition if and only if $\tau_0 < \bar{\tau}(\sigma)$. But to complete the proof, we need to show in this case that $\underline{\tau} < \bar{\tau}(\sigma)$. We know that $\bar{\tau}(\sigma)$ is an increasing function of σ over $(0, \sigma_1)$ and that $\bar{\tau}(\sigma_1) = +\infty$. Moreover straightforward computations show that $\underline{\tau} < \bar{\tau}(0)$. Therefore, $\underline{\tau} < \bar{\tau}(\sigma)$ for any $\sigma \in (0,1)$. The conclusions of the Proposition follow denoting $\underline{A} = \max\{A_1, A_2\}$. ■

A.3 Proof of Corollary 1

Under the conditions of Proposition 1, consider $\hat{x} = \hat{x}(\alpha, \tau_0, \rho, \delta, \sigma)$ the solution of equation (19) and recall that $\hat{\tau} = \tau_0$. Let us also denote equation (19) as follows

$$h(x, \tau_0) \equiv (\alpha - \sigma)A(x\tau_0)^{1-\alpha} + \sigma(1+\tau_0)x - \rho - \delta(1-\sigma) = 0 \tag{54}$$

We first get

$$\frac{\partial h}{\partial x} = (1-\alpha)A\frac{(x\tau_0)^{1-\alpha}}{x}(\alpha-\sigma)+\sigma(1+\tau_0)$$

From equation (16) evaluated along the steady state $(\hat{x},\hat{\gamma})$ we derive

$$(1+\tau_0)\hat{x} = (1-\alpha)A(\hat{x}\tau_0)^{1-\alpha}+\rho-\hat{\gamma}(1-\sigma) = \tfrac{(1-\alpha)A(\delta+\rho+\sigma\hat{\gamma})}{\alpha}+\rho-\hat{\gamma}(1-\sigma) \tag{55}$$

and thus

$$\left.\frac{\partial h}{\partial x}\right|_{x=\hat{x}} = \frac{(1-\alpha)A(\delta+\rho+\sigma\hat{\gamma})+\alpha\sigma(1+\tau_0)[\rho-\hat{\gamma}(1-\sigma)]}{\alpha\hat{x}} > 0$$

as the transversality condition (21) holds.

Second we also compute from (54)

$$\frac{\partial h}{\partial \tau_0} = (1-\alpha)A\frac{(x\tau_0)^{1-\alpha}}{\hat{\tau}}(\alpha-\sigma)+\sigma x$$

Using again (54) evaluated along the steady state $(\hat{x},\hat{\gamma})$ we get

$$(\alpha-\sigma)A(x\tau_0)^{1-\alpha} = \rho+\delta(1-\sigma)-\sigma(1+\tau_0)\hat{x}$$

and thus

$$\left.\frac{\partial h}{\partial \tau_0}\right|_{x=\hat{x}} = \frac{(1-\alpha)(\delta+\rho)-\sigma\{\hat{x}[(1-\alpha)-\tau_0\alpha]+\delta(1-\alpha)\}}{\tau_0} \tag{56}$$

If $\tau_0 < (1-\alpha)/\alpha$ and $\sigma > \underline{\sigma}$ with

$$\underline{\sigma} \equiv \tfrac{(1-\alpha)(\delta+\rho)}{\hat{x}[(1-\alpha)-\tau_0\alpha]+\delta(1-\alpha)}$$

then the expression (56) is negative. To be consistent with Proposition 1, we need now to check that $\underline{\tau} < (1-\alpha)/\alpha$. Using (52), we conclude that this inequality holds if and only if $A > A_{min}$ with

$$A_{min} \equiv \left(\tfrac{\alpha}{(1-\alpha)[\delta(1-\alpha)+\rho]}\right)^{1-\alpha}\tfrac{\delta+\rho}{\alpha} > \underline{A}$$

Finally, under all these conditions, we conclude from the implicit function theorem that

$$\frac{d\hat{x}}{d\tau_0} = -\frac{\partial h/\partial\tau_0|_{x=\hat{x}}}{\partial h/\partial x|_{x=\hat{x}}} > 0$$

The result on the growth rate can be found immediately by differentiating equation (22) with respect to $\hat{\tau}$:

$$\frac{d\hat{\gamma}}{d\tau_0} = \alpha(1-\alpha)(\hat{x}\tau_0)^{-\alpha}\left(\left.\frac{d\hat{x}}{d\tau_0}\right|_{x=\hat{x}}+\hat{x}\right)$$

Let us finally consider the expression of welfare along the BGP as given by (23). We derive

$$\frac{dW(\hat{\gamma},\hat{x})}{d\tau_0} = (\hat{x}k_0)^{1-\sigma}\left[\frac{d\hat{x}}{d\tau_0}\frac{\rho-\hat{\gamma}(1-\sigma)}{\hat{x}}+\frac{d\hat{\gamma}}{d\tau_0}\right] > 0$$

and the result follows. ■

A.4 Proof of Proposition 2

Let us consider the system (16)-(17)

$$\begin{aligned} \dot{x} &= \left\{ \frac{[(1+\tau)(1-\sigma)-\phi\tau]\,[\alpha A(x\tau)^{1-\alpha}-\delta-\rho-\sigma\gamma]}{\sigma(1+\tau)+\phi\tau} \right. \\ &+ \left. \gamma(1-\sigma)+(1+\tau)x-(1-\alpha)A(x\tau)^{1-\alpha}-\rho \right\} x \equiv \varphi(\tau,x) \qquad (57) \\ \dot{\tau} &= \frac{\phi\tau(1+\tau)}{\sigma(1+\tau)+\phi\tau}\left[\alpha A(x\tau)^{1-\alpha}-\delta-\rho-\sigma\gamma\right] \equiv \phi(\tau,x) \qquad (58) \end{aligned}$$

Linearizing this system around the steady state $(\hat{x},\hat{\tau})$ gives

$$\begin{pmatrix} \dot{x} \\ \dot{\tau} \end{pmatrix} = \begin{pmatrix} \frac{[(1+\hat{\tau})(\alpha-\sigma)-\phi\hat{\tau}](1-\alpha)(\delta+\rho+\sigma\hat{\gamma})}{\alpha[\sigma(1+\hat{\tau})+\phi\hat{\tau}]}+(1+\hat{\tau})\hat{x} & \frac{[(1+\hat{\tau})(\alpha-\sigma)-\phi\hat{\tau}](1-\alpha)(\delta+\rho+\sigma\hat{\gamma})}{\alpha[\sigma(1+\hat{\tau})+\phi\hat{\tau}]\hat{\tau}}\hat{x}+\hat{x}^2 \\ \frac{\phi\hat{\tau}(1+\hat{\tau})(1-\alpha)(\delta+\rho+\sigma\hat{\gamma})}{[\sigma(1+\hat{\tau})+\phi\hat{\tau}]\hat{x}} & \frac{\phi(1+\hat{\tau})(1-\alpha)(\delta+\rho+\sigma\hat{\gamma})}{\sigma(1+\hat{\tau})+\phi\hat{\tau}} \end{pmatrix} \begin{pmatrix} \tilde{x} \\ \tilde{\tau} \end{pmatrix}$$

where $\tilde{\tau} \equiv \tau - \hat{\tau}$ and $\tilde{x} \equiv x - \hat{x}$.

The proof consists in studying the determinant and trace of the Jacobian matrix, $\mathbf{J}$, which is the coefficient's matrix of the linearized system. After some tedious but straightforward computations, the trace and determinant of the Jacobian can be found to be equal to

$$\begin{aligned} \det(\mathbf{J}) &= \frac{\phi(1+\hat{\tau})(1-\alpha)(\delta+\rho+\sigma\hat{\gamma})}{[\sigma(1+\hat{\tau})+\phi\hat{\tau}]}\hat{x} \\ \mathrm{tr}(\mathbf{J}) &= \frac{\phi(1+\hat{\tau})(1-\alpha)(\delta+\rho+\sigma\hat{\gamma})}{\sigma(1+\hat{\tau})+\phi\hat{\tau}}+\frac{[(1+\hat{\tau})(\alpha-\sigma)-\phi\hat{\tau}](1-\alpha)(\delta+\rho+\sigma\hat{\gamma})}{\alpha[\sigma(1+\hat{\tau})+\phi\hat{\tau}]}+(1+\hat{\tau})\hat{x} \end{aligned}$$

It follows immediately that

$$\det(\mathbf{J})<0 \quad \Leftrightarrow \quad \frac{\phi}{\sigma(1+\hat{\tau})+\phi\hat{\tau}}<0 \quad \Leftrightarrow \quad \phi \in \left(-\frac{\sigma(1+\hat{\tau})}{\hat{\tau}},0\right).$$

In this case the steady state $(\hat{x},\hat{\tau})$ is a saddle-point. If on the contrary, $\phi \in \left(-\infty,-\frac{\sigma(1+\hat{\tau})}{\hat{\tau}}\right) \cup (0,\infty)$ we need to study the sign of the trace of the Jacobian. Assume first that $\phi < -\sigma(1+\hat{\tau})/\hat{\tau}$. We then get

$$(1+\hat{\tau})(\alpha-\sigma)-\phi\hat{\tau} > \alpha(1+\hat{\tau})$$

which implies that $\mathrm{tr}(\mathbf{J})>0$ and the steady state $(\hat{x},\hat{\tau})$ is totally unstable. Assume finally that $\phi>0$. From equations (57)-(58) evaluated at the steady state $(\hat{x},\hat{\tau})$ we get

$$(1+\tau)\hat{x} = (1-\alpha)A(\hat{x}\hat{\tau})^{1-\alpha}+\rho-\gamma(1-\sigma) = \frac{(1-\alpha)(\delta+\rho+\sigma\hat{\gamma})}{\alpha}+\rho-\gamma(1-\sigma)$$

From this expression we derive that

$$\frac{[(1+\hat{\tau})(\alpha-\sigma)-\phi\hat{\tau}](1-\alpha)(\delta+\rho+\sigma\hat{\gamma})}{\alpha[\sigma(1+\hat{\tau})+\phi\hat{\tau}]}+(1+\hat{\tau})\hat{x}=\frac{(1+\hat{\tau})(1-\alpha)(\delta+\rho+\sigma\hat{\gamma})+[\rho-\hat{\gamma}(1-\sigma)][\sigma(1+\hat{\tau})+\phi\hat{\tau}]}{\sigma(1+\hat{\tau})+\phi\hat{\tau}}>0$$

since the transversality condition implies $\rho - \hat{\gamma}(1-\sigma) > 0$. It follows again that $\mathrm{tr}(\mathbf{J}) > 0$ and the steady state $(\hat{x}, \hat{\tau})$ is totally unstable.

We have then proved that for any value of σ, the steady state $(\hat{x}, \hat{\tau})$ is either a saddle-point or totally unstable. Since the stationary value of the tax rate $\hat{\tau}$ is given by its initial value τ_0, in both of these configurations, the only initial value of $x(t)$ compatible with the transversality condition is $x(0) = \hat{x}$ and the economy immediately jumps on the BGP from the initial date $t = 0$. ■

A.5 Proof of Proposition 3

Given a τ^* the proof and existence and uniqueness of an ABGP is basically the same as the proof of Proposition 1 once we have substituted τ^* to $\hat{\tau}$. In particular the value of A_1 is now substituted by

$$A_1^* \equiv \left(\frac{\delta}{\alpha}\right)^{\alpha}\left(\frac{1+\tau^*}{\tau^*(1-\alpha)}\right)^{1-\alpha} \tag{59}$$

The critical values $\underline{\tau}$ and $\bar{\tau}(\sigma)$ have the same expressions as in the proof of Proposition 1 while $\underline{A} = \max\{A_1^*, A_2\}$.

Concerning the asymptotic stability of (x^*, τ^*), the computations given in the proof of Proposition 2 applies so that (x^*, τ^*) is saddle-path stable if and only if $\phi \in (-\sigma(1+\tau^*)/\tau^*, 0)$. In this case, for a given τ_0 close enough to τ^*, there exists a unique value of $x(0)$ such that the equilibrium path $(x(t), \tau(t))$ converges towards (x^*, τ^*). Note that if on the contrary, $\phi \in (-\infty, -\sigma(1+\tau^*)/\tau^*) \cup (0, +\infty)$, then (x^*, τ^*) is totally unstable and, therefore, an equilibrium $(x_t, \tau_t)_{t\geq 0}$ converging to the ABGP does not exist. Indeed, in this case, as $\tau(0) = \tau_0 \neq \tau^*$, the only equilibrium path is to jump on the unique BGP as given by $(\hat{x}, \hat{\tau})$. ■

A.6 Proof of Theorem 1

The result follows from Propositions 1, 2 and 3. Note that the value of A_1 or A_1^* is now substituted by

$$A_1^{max} \equiv \min_{\tau\in(\tau_{inf},\tau_{sup})}\left(\frac{\delta}{\alpha}\right)^{\alpha}\left(\frac{1+\tau}{\tau(1-\alpha)}\right)^{1-\alpha} = \left(\frac{\delta}{\alpha}\right)^{\alpha}\left(\frac{1+\tau_{inf}}{\tau_{inf}(1-\alpha)}\right)^{1-\alpha} \tag{60}$$

The critical values $\underline{\tau}$ and $\bar{\tau}(\sigma)$ have the same expressions as in the proof of Proposition 1 while $\underline{A} = \max\{A_1^{max}, A_2\}$. Moreover, the condition on ϕ becomes $\phi \in (-\underline{\phi}, 0)$

with

$$\underline{\phi} = \min_{\tau \in (\tau_{inf}, \tau_{sup})} \frac{\sigma(1+\tau)}{\tau} = \frac{\sigma(1+\tau_{sup})}{\tau_{sup}}$$

Assuming also $\tau_{inf} > \underline{\tau}$ and $\tau_{sup} < \bar{\tau}(\sigma)$, we get that for any $\tau^{*j} \in (\tau_{inf}, \tau_{sup})$, the conditions for the existence of a solution of system (27)-(28) as given in Propositions 1 and 3, and the restriction on ϕ as given in Proposition 2 hold.

When $\phi \in \left(-\frac{\sigma(1+\tau_{sup})}{\tau_{sup}}, 0\right)$, local determinacy means that for any given τ_0 there is a unique equilibrium path either jumping on the unique BGP $(\hat{x}, \hat{\tau})$ or converging toward some ABGP (τ^{*j}, x^{*j}) with $\tau^{*j} \in (\tau_{inf}, \tau_{sup})$. Global indeterminacy means that while the initial tax rate τ_0 is given, the economy may converge to different asymptotic equilibria depending on the beliefs of the agents. Also we have not only multiple equilibria but a continuum of them. In fact, under the condition on ϕ we have that variations of τ^* make x, $\det(\mathbf{J})$, $\mathrm{tr}(\mathbf{J})$ change continuously. The last two changes imply a continuous changes of the eigenvalues, λ_1, λ_2, and of the associated eigenvectors. Therefore, the solution of x will also change continuously.

Finally, when $\phi \in (0, +\infty)$, the unique equilibrium path consists in jumping on the unique BGP from the initial date. There is thus local and global determinacy. ■

A.7 Further content on Section 5

Definition 4 (Markov property). *A family of random variables $\{\varepsilon_t\}_{t \geq 0}$ satisfies the Markov property if*

$$\mathbb{P}(\varepsilon_{t_n} = z_j \mid \varepsilon_{t_1} = z_1, ..., \varepsilon_{t_{n-1}} = z_{n-1}) = \mathbb{P}(\varepsilon_{t_n} = z_j \mid \varepsilon_{t_{n-1}} = z_{j_{n-1}})$$

for all $z_1, ..., z_{n-1} \in Z$ and any sequence $t_1 < t_2 < ... < t_n$ of times.

Definition 5 (Homogeneity). *A Markov chain is homogenous if given the transition probability*

$$p_{ij}(s,t) \equiv \mathbb{P}(\varepsilon_t = z_j | \varepsilon_s = z_i) \qquad for \quad s \leq t$$

we have that

$$p_{ij}(s,t) = p_{ij}(0, t-s) \qquad \forall i, j, s, t,$$

and we write $p_{ij}(t-s)$ for $p_{ij}(s,t)$.

Definition 6 (Stochastic semigroup). *A transition semigroup $\{\mathbf{P}_t\}_{t \geq 0}$ is stochastic if it satisfies the following:*

i) $\mathbf{P}_0 = \mathbf{I}$*;*

ii) $\mathbf{P}_t$ *is stochastic (i.e.* $p_{ij}(t) \geq 0$ *and* $\sum_{j=1}^{N} p_{ij}(t) = 1$*)*

iii) $\mathbf{P}_{s+t} = \mathbf{P}_s \mathbf{P}_t$ *if* $s, t \geq 0$

A.8 Proof of Theorem 2

Taking into account the example in section 5.1 and assuming that all the conditions in Theorem 1 are respected, a sunspot equilibrium starting at $t = 0$ in state j with probability π_j can be summed up as follows:

$$(x_t, \tau_t) = \boldsymbol{\phi}_j(t), \qquad for\ t \in [0, t_j) \tag{61}$$

$$(x_{t_j}, \tau_{t_j}) = (\phi_{i,x}(t_j), \phi_{j,\tau}(t_j)), \qquad with\ probability\ -\frac{g_{ji}}{g_{jj}} \tag{62}$$

$$(x_t, \tau_t) = \boldsymbol{\phi}_i(t), \qquad for\ t \in [t_j, t_i) \tag{63}$$

with t_m a value taken by the random variable $\mathcal{T}_m \sim e^{g_{mm}t}$ with $m = i, j$ for any $i, j = 1, ..., N$ and $i \neq j$. The jump in the control-like variables at date t_i have size $s(z_i - z_j)$, and the resulting path respects the inequalities constraints as long as $\underline{\varepsilon}, \bar{\varepsilon}$ and s are sufficiently small. An example of a sunspot equilibrium is drawn in Figure 6. ■

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