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Piecewise Linear Income Tax Reforms

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Abstract

This paper addresses questions of the following nature: under what conditions does a welfare-improving reform of the existing piecewise linear income tax schedule necessitate a change in a particular agent type's marginal tax rate? Our analysis therefore addresses the sorts of questions typically debated by policy-makers, using a model of income taxation that resembles income tax systems used in practice. A locally optimal tax schedule is a special case of our tax reform analysis—the current piecewise linear income tax schedule is locally optimal if there does not exist an equilibrium-preserving and welfare-improving reform. We show that local optima involve progressive taxation, in that marginal tax rates are increasing in income. An extension of the model to include linear commodity taxation is also considered. In this case, local optima comprise positive commodity taxation and progressive income taxation.

Keywords: tax reform; piecewise linear income taxation.

JEL Classifications: H21, H24.

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1 Introduction

The aim of the optimal taxation literature is to determine the features of an optimal tax system. However, there are some long-standing criticisms of this approach to normative tax theory. In particular, the optimal tax approach implicitly assumes that the government is free to change all taxes, and that it is willing and able to implement the possibly large changes in taxes required to reach an optimum.[1] The characteristics of the current tax system are irrelevant and are ignored under the optimal tax approach. In practice, however, the government must take the existing tax system as its starting point, and actual changes in taxes tend to be "... slow and piecemeal" (Feldstein, 1976). Such observations motivate the tax reform approach, pioneered by Guesnerie (1977). Tax reform analysis takes the existing tax system as given, and then examines the conditions under which there exist small (modelled as differential) changes in taxes that are feasible (equilibrium-preserving) and desirable (welfare-improving).[2] The tax reform approach to policy-making therefore comes much closer than the optimal tax approach in capturing the actual behaviour of governments and the constraints that they face.

If one finds the preceding arguments reasonable, the question arises as to why the optimal tax approach continues to dominate the public finance literature, while tax reform papers are few and far between. At first thought, one may think that the tax reform approach is in some sense redundant—once the characteristics of the optimal tax system have been determined, the government should simply change taxes toward their optimal levels. However, it has been known for some time that changes "... in the right direction, but stop short of attaining the full optimum, can actually reduce welfare" (Dixit, 1975). Indeed, Guesnerie's (1977) temporary inefficiency result shows that an equilibrium-preserving and Pareto-improving policy reform may require a move from a production efficient allocation to a production inefficient allocation, even though production efficiency is desirable at an optimum (Diamond and Mirrlees, 1971). In our

[1] For example, two standard results in the optimal tax literature are that capital should not be taxed and that the highest-skilled workers should face a zero marginal tax rate on their labour income. These recommendations stand in stark contrast to the features of real-world tax systems, and implementing them would involve a major shock to the economy.

[2] For an excellent textbook treatment of the tax reform approach, see chapter 6 in Myles (1995).

opinion, the reason that the tax reform approach remains relatively neglected is because it is generally difficult to obtain clear-cut results. For example, the main result of Guesnerie (1977, Proposition 4) on the existence of equilibrium-preserving and Pareto-improving policy reforms is very technical, relating the position of a vector representing the equilibrium conditions to a cone representing Pareto improvements.[3] Diewert (1978) and Weymark (1979) use different mathematical techniques to Guesnerie,[4] but their results also tend to be quite technical. For the most part, the papers by Guesnerie, Diewert, and Weymark can be interpreted as providing empirically-testable formulae for the existence or otherwise of feasible and desirable tax reforms, rather than providing simple and clear results.[5] Other tax reform analyses, such as those by Hatta (1977), Konishi (1995), Brett (1998), Murty and Russell (2005), Krause (2007), and Duclos, et al. (2008), also tend to yield technical results that do not have a simple and intuitive economic interpretation.[6]

The main aim of this paper is to undertake a tax reform analysis which leads to clear-cut results. We examine income tax reforms using a model of income taxation that resembles the nonlinear income tax systems used in practice, i.e., piecewise linear income taxation, rather than the workhorse Mirrlees (1971) model of nonlinear income taxation.[7] In our model there are three types of agents (low-skill, middle-skill, and high-skill), and we assume that the utility function is quasi-linear in consumption. We think the assumption that there are only three types of agents is not too restrictive, since real-world income tax systems tend to be designed broadly around how low-income, middle-

[3]See also chapter 3 in Guesnerie (1995).

[4]In particular, they use Motzkin's Theorem of the Alternative to analyse tax reforms, as we do in this paper.

[5]An alternative way of analysing tax reform issues is to rely on numerical simulations. See, e.g., Creedy and Herault (2012) for a recent and interesting examination of welfare-improving income tax reforms using numerical methods.

[6]Tax reform techniques have also been used to revisit specific issues in optimal taxation, and in this case some clear conclusions can be reached. For example, Blackorby and Brett (2000) use tax reform techniques to examine the Diamond-Mirrlees production efficiency theorem. Fleurbaey (2006) takes a tax reform approach to examine the desirability of consumption taxation versus income taxation, while Krause (2009) undertakes a tax reform analysis of the Laffer argument.

[7]It should be noted that most of the tax reform literature examines linear commodity taxation rather than nonlinear income taxation, although Konishi (1995) is an exception. He examines a model with linear commodity taxation and nonlinear (Mirrlees-style) income taxation.

income, and high-income individuals should be taxed. The assumption that preferences are quasi-linear is perhaps more troubling, but quasi-linearity is a common assumption and seems necessary to obtain detailed and clear results. On the methodological side, we analyse tax reforms of a specific nature. That is, we examine the conditions under which a feasible welfare-improving tax reform requires a change in a particular agent type's marginal tax rate. While this approach is less general than that typically taken in the tax reform literature, it does have a real-world counterpart. For example, in recent years in the U.K. there has been much discussion over whether the top marginal income tax rate should be reduced.[8] In our model, this corresponds to asking under what conditions does an equilibrium-preserving and welfare-improving tax reform necessitate a reduction in the marginal tax rate faced by high-skill individuals. The existing piecewise linear income tax schedule can be considered locally optimal if there does not exist an equilibrium-preserving and welfare-improving tax reform. We show that such local optima involve progressive taxation, in the sense that marginal tax rates are increasing in income. When the model is extended to include linear commodity taxation, we show that local optima involve using positive commodity taxation alongside progressive income taxation. Therefore, we conclude that the main features of most real-world tax systems are broadly consistent with the characteristics for being locally optimal.

The remainder of the paper is organised as follows. Section 2 describes the basic model we use, and defines what we mean by equilibrium-preserving and welfare-improving tax reforms. Section 3 examines marginal income tax rate reforms, while Sections 4 and 5 consider an extension of our basic model to include linear commodity taxation alongside piecewise linear income taxation. Section 6 uses numerical methods to examine the comparative statics of a locally optimal tax system. Section 7 concludes, while proofs and some other mathematical details are contained in an appendix.

[8]Such discussion resulted in U.K. Chancellor George Osborne announcing on 21 March 2012 that the top marginal income tax rate would be reduced from 50% to 45% in April 2013.

2 The Model

There are three types of individual, and individuals are distinguished by their skill levels in employment or, equivalently, their wage rates. Type i's wage is denoted by w_i, where $w_3 > w_2 > w_1$ so that type 3 individuals are high-skill, type 2 individuals are middle-skill, and type 1 individuals are low-skill. We make the standard assumption that the economy's technology is linear, which implies that wages are fixed. Individuals have the same preferences, which are representable by the quasi-linear utility function:

$$x_i - \frac{l_i^{1+\gamma}}{1+\gamma} \tag{2.1}$$

where x_i is type i's consumption, l_i is type i's labour supply, and $\gamma > 0$ is a preference parameter. Let $y_i = w_i l_i$ denote type i's pre-tax labour income.

Figure 1 illustrates a piecewise linear income tax schedule (shown in bold lines), with the indifference curve of each type of individual being tangent to the tax schedule at the appropriate point for that type. The tax schedule is drawn under the assumption that the income tax system is "convex" or "progressive", i.e., marginal tax rates are increasing in income for the three types. Suppose the situation in Figure 1 represents the status quo situation. One can think of each type as having acted *as if* maximising utility subject to their personal a_i-a_i budget line, *and* where the parameters of their maximisation problem led them to a point on their budget line corresponding to the tax schedule. The same thinking would apply if the income tax system was "non-convex" or "regressive", as shown in Figure 2, which illustrates a tax schedule with decreasing marginal tax rates. Accordingly, given the status quo piecewise linear income tax schedule, with each type having chosen the appropriate point on this schedule, one can think of each type as having chosen x_i and l_i to maximise (2.1) subject to their budget constraint:

$$x_i \leq a_i + (1 - t_i) w_i l_i \tag{2.2}$$

under the implicit assumption that the parameters of the problem led them to a point on their budget line corresponding to the tax schedule. Note that a_i can be interpreted as type i's non-labour income, and t_i is the marginal tax rate faced by type i individuals. We assume that $t_i \in (0, 1)$ for all i under the current tax schedule. This implies that all types are taxed (not subsidised), and we rule out tax rates greater than or equal to 100% because individuals will not work for no income.

Our approach has the advantage of being able to capture the essential features of piecewise linear income taxation while remaining simple, but we are not claiming it captures every feature. For example, most real-world tax systems exempt income from taxation up to a certain amount, and the higher marginal tax rates are applied to income earned above the corresponding tax-bracket thresholds. Nevertheless, our model of piecewise linear income taxation is suitable for our purposes, and it is a better approximation of the income tax systems used in practice than purely linear or purely nonlinear (Mirrleesian) models.

Social welfare is assumed to be measurable by the weighted utilitarian social welfare function:

$$\sum_{i=1}^{3} \pi_i n_i V_i(a_i, t_i, w_i, \gamma) \tag{2.3}$$

where $\pi_i > 0$ is type i's welfare weight, $n_i > 0$ is the number of type i individuals, and $V_i(\cdot)$ is type i's indirect utility function coming from programme (2.1) $-$ (2.2). We assume that the social welfare function is redistributive in the sense that the welfare weights are declining in skills, i.e., $\pi_1 > \pi_2 > \pi_3$.[9]

An equilibrium of our model corresponds to satisfaction of the government's budget constraint:

$$\sum_{i=1}^{3} n_i t_i w_i l_i(t_i, w_i, \gamma) - G \geq 0 \tag{2.4}$$

where $l_i(\cdot)$ is type i's labour supply function coming from programme (2.1) $-$ (2.2),[10] and $G > 0$ is the government's exogenously determined spending requirement. For analytical purposes, we assume that the status quo equilibrium is "tight", i.e., equation (2.4) holds with equality. This assumption allows us to differentiate equation (2.4).

[9] It is not necessary for our results that the welfare weights sum to one, although it would be natural to assume that they do. In our numerical simulations (Section 6), we choose weights that sum to one.

[10] The assumption that the utility function is quasi-linear in consumption ensures that each type's labour supply function is independent of a_i.

2.1 Tax Reforms

We define a *tax reform* as the vector $\mathrm{d}R := \langle \mathrm{d}t_1, \mathrm{d}t_2, \mathrm{d}t_3 \rangle$, which can be interpreted as the government implementing a small (modelled as differential) change in the piecewise linear income tax schedule. Starting in an initial tight equilibrium, a tax reform is said to be *equilibrium-preserving* if and only if:

$$\nabla Z \mathrm{d}R \geq 0 \tag{2.5}$$

where ∇Z is the gradient of equation (2.4) and is defined as:

$$\nabla Z := \langle n_1 w_1 \left(l_1 + t_1 \frac{\partial l_1}{\partial t_1} \right), n_2 w_2 \left(l_2 + t_2 \frac{\partial l_2}{\partial t_2} \right), n_3 w_3 \left(l_3 + t_3 \frac{\partial l_3}{\partial t_3} \right) \rangle \tag{2.6}$$

where all derivatives are evaluated in the status quo equilibrium. An equilibrium-preserving tax reform is a tax reform that moves the economy to a neighbouring equilibrium.

A tax reform is said to be *welfare-improving* if and only if:

$$\nabla W \mathrm{d}R > 0 \tag{2.7}$$

where:

$$\nabla W := \langle \pi_1 n_1 \frac{\partial V_1}{\partial t_1}, \pi_2 n_2 \frac{\partial V_2}{\partial t_2}, \pi_3 n_3 \frac{\partial V_3}{\partial t_3} \rangle \tag{2.8}$$

is the gradient of the weighted utilitarian social welfare function. A welfare-improving tax reform is a tax reform that increases social welfare, i.e., increases the value of the weighted utilitarian social welfare function.

3 Local Optima and Welfare-Improving Reforms

We first analyse the situation in which the current piecewise linear income tax schedule is locally optimal. Starting in an initial tight equilibrium, if there does not exist a tax reform $\mathrm{d}R$ such that:

$$\nabla Z \mathrm{d}R \geq 0 \tag{3.1}$$

$$\nabla W \mathrm{d}R > 0 \tag{3.2}$$

then there does not exist an equilibrium-preserving and welfare-improving tax reform, and the current tax system is locally optimal. By Motzkin's Theorem of the Alternative,[11] if there does not exist a tax reform $\mathrm{d}R$ that satisfies equations (3.1) and (3.2), there exist real numbers $\theta \geq 0$ and $\alpha > 0$ such that:

$$\theta \nabla Z + \alpha \nabla W = 0^{(3)} \tag{3.3}$$

The system of equations (3.3) characterises what the piecewise linear income tax schedule "looks like" when it is locally optimal. Let $\overline{z}$ denote the level of variable z when the tax schedule is locally optimal, and let T_i denote total income tax payments by type i individuals. Using (3.3) we obtain the following proposition (all proofs are provided in the appendix):

Proposition 1 *If the status quo piecewise linear income tax schedule is locally optimal, we have (i) no Laffer effect, i.e., $\partial \overline{T}_i / \partial \bar{t}_i > 0$ for all i, and (ii) progressive income taxation, i.e., $\bar{t}_1 < \bar{t}_2 < \bar{t}_3$.*

Part (i) of Proposition 1 is unsurprising, in that a locally optimal piecewise linear income tax schedule cannot be subject to the Laffer effect. Otherwise, the government could simultaneously reduce some type's marginal tax rate (making them better-off) and raise extra tax revenue. Part (ii) of Proposition 1 is interesting in light of the related literature which examines the characteristics of *globally* optimal piecewise linear income taxation.[12] For example, Sheshinski (1989) concludes that the globally optimal piecewise linear income tax schedule is always progressive, while Slemrod, et al. (1994) reach the opposite conclusion (regressive). More recently, Apps, et al. (2014) show that progressive or regressive piecewise linear income taxation may be globally optimal, but

[11] A statement of Motzkin's Theorem is provided in the appendix.

[12] Deriving the characteristics of a globally optimal piecewise linear income tax system is a challenging problem, because in principle both the number of tax brackets and the bracket thresholds must be chosen (although much of the literature has restricted attention to the two-bracket case). By contrast, the local approach is much simpler. We take the existing tax schedule as given, and non-existence of an equilibrium-preserving and welfare-improving tax reform implies that the current tax schedule is locally optimal.

they conclude that under reasonable parameter values it is progressive. In our model, a locally optimal piecewise linear income tax schedule is always progressive. However, it is straightforward to show that under strict utilitarianism ($\pi_1 = \pi_2 = \pi_3$) it would be linear ($\bar{t}_1 = \bar{t}_2 = \bar{t}_3$), and if the social welfare function was weighted towards the higher skilled ($\pi_1 < \pi_2 < \pi_3$) it would be regressive ($\bar{t}_1 > \bar{t}_2 > \bar{t}_3$).

We now consider the case in which the current piecewise linear income tax schedule is not locally optimal, and therefore there do exist equilibrium-preserving and welfare-improving tax reforms. To this end, note that $\mathrm{d}t_i \gtreqless 0$ if and only if $\nabla T_i \mathrm{d}R \gtreqless 0$, where $\nabla T_1 := \langle 1, 0, 0 \rangle$, $\nabla T_2 := \langle 0, 1, 0 \rangle$, and $\nabla T_3 := \langle 0, 0, 1 \rangle$.

Our tax reform methodology is based on the following reasoning. Starting in an initial tight equilibrium, if there does not exist a tax reform $\mathrm{d}R$ such that:

$$\nabla Z \mathrm{d}R \geq 0 \tag{3.4}$$

$$\nabla W \mathrm{d}R > 0 \tag{3.5}$$

$$\nabla T_i \mathrm{d}R \geq 0 \tag{3.6}$$

then there are two key possibilities: (i) There does not exist a tax reform that satisfies equations (3.4) and (3.5). In this case, there do not exist any equilibrium-preserving and welfare-improving tax reforms, so the status quo tax schedule is already locally optimal (see above) and equation (3.6) is redundant. (ii) There do exist tax reforms that satisfy (3.4) and (3.5), but all such reforms violate (3.6). In this case, the status quo tax schedule is suboptimal, and any move towards optimality requires a decrease in the marginal tax rate faced by type i individuals (i.e., a violation of equation (3.6)). As we are now interested in examining moves from a suboptimal towards an optimal tax schedule, we focus on this second possibility.

By Motzkin's Theorem, if there does not exist a tax reform $\mathrm{d}R$ that satisfies equations (3.4) – (3.6), there exist real numbers $\theta \geq 0$, $\alpha > 0$, and $\beta \geq 0$ such that:

$$\theta \nabla Z + \alpha \nabla W + \beta \nabla T_i = 0^{(3)} \tag{3.7}$$

The system of equations (3.7) characterises what the initial suboptimal tax schedule "looks like" when all equilibrium-preserving and welfare-improving tax reforms require a decrease in the marginal tax rate faced by type i individuals. For convenience, we restrict attention to those suboptimal tax schedules that are not subject to a Laffer effect. In other words, even though we are now considering suboptimal tax schedules, it is assumed that the tax schedule is not so far from optimality as to be subject to the Laffer effect. Using (3.7) we obtain the following result:

Proposition 2a *Consider an initial tight equilibrium of our model in which the piece-wise linear income tax schedule is suboptimal. If all equilibrium-preserving and welfare-improving tax reforms require a decrease in type i's marginal tax rate, then (i)* $t_i > \bar{t}_i$, $t_j < \bar{t}_j$ *and* $t_k < \bar{t}_k$, *and (ii)* $(1-t_j/(\gamma-\gamma t_j))/(1-t_k/(\gamma-\gamma t_k)) = (1-\bar{t}_j/(\gamma-\gamma\bar{t}_j))/(1-\bar{t}_k/(\gamma-\gamma\bar{t}_k))$.

By reversing the inequality in equation (3.6), we obtain:

Proposition 2b *Consider an initial tight equilibrium of our model in which the piece-wise linear income tax schedule is suboptimal. If all equilibrium-preserving and welfare-improving tax reforms require an increase in type i's marginal tax rate, then (i)* $t_i < \bar{t}_i$, $t_j > \bar{t}_j$ *and* $t_k > \bar{t}_k$, *and (ii)* $(1-t_j/(\gamma-\gamma t_j))/(1-t_k/(\gamma-\gamma t_k)) = (1-\bar{t}_j/(\gamma-\gamma\bar{t}_j))/(1-\bar{t}_k/(\gamma-\gamma\bar{t}_k))$.

Part (i) of Proposition 2a indicates that if all equilibrium-preserving and welfare-improving tax reforms require a decrease in type i's marginal tax rate, then type i's marginal tax rate must currently be higher than its locally optimal level. While this is intuitive and immediately implies that *total* tax payments by the other two types (types j and k) must be less than locally optimal, part (i) indicates that the marginal tax rates of *both* other types must be less than locally optimal. This is related to part (ii) of Proposition 2a, which shows that the relationship between the marginal tax rates of types j and k in the suboptimal tax schedule must be the same as when the tax schedule is locally optimal. The intuition is that if the relationship was not the same as in a local optimum, then an equilibrium-preserving and welfare-improving tax reform could be implemented by moving their marginal tax rates towards their locally optimal ratios. That is, an equilibrium-preserving and welfare-improving reform could be implemented without decreasing type i's marginal tax rate. Therefore, if all equilibrium-preserving and welfare-improving tax reforms require a decrease in type i's marginal tax rate, the relationship between types j and k's marginal tax rates must be the same as that in a local optimum.

Proposition 2b covers the case when all equilibrium-preserving and welfare-improving tax reforms require an increase in type i's marginal tax rate. As intuition would suggest, this case is simply the mirror image of that for a required decrease in type i's marginal tax rate.

4 Extension: Incorporating a Linear Commodity Tax

As most real-world tax systems include linear commodity taxation alongside piecewise linear income taxation, in this section we incorporate commodity taxation into our model. To this end, we introduce two consumption goods, z and x, where z will be treated as the numeraire good (price normalised to one) and x will be the taxed good. The consumer price of x is equal to $(1+\tau)p$, where p is the (fixed) producer price and τ is the commodity tax. We assume that $\tau > -1$ to eliminate the possibility of a non-positive consumer price.

As above, one can think of the status quo equilibrium as the outcome of each type of individual having chosen z_i, x_i, and l_i to maximise their utility function:

$$z_i + \frac{x_i^{1-\sigma}}{1-\sigma} - \frac{l_i^{1+\gamma}}{1+\gamma} \tag{4.1}$$

subject to their budget constraint:

$$z_i + (1+\tau)px_i \leq a_i + (1-t_i)w_i l_i \tag{4.2}$$

where $\sigma > 0$ is a preference parameter. When $\sigma = 1$, the utility function becomes logarithmic in consumption of x_i.

Social welfare is again assumed to be measurable by a weighted utilitarian social welfare function:

$$\sum_{i=1}^{3} \pi_i n_i V_i(a_i, t_i, w_i, \gamma, \tau, p, \sigma) \tag{4.3}$$

where $V_i(\cdot)$ is now type i's indirect utility function coming from programme (4.1) – (4.2).

The government's budget constraint now includes receipts from commodity taxation:

$$\sum_{i=1}^{3} n_i \tau p x_i(\tau, p, \sigma) + \sum_{i=1}^{3} n_i t_i w_i l_i(t_i, w_i, \gamma) - G \geq 0 \tag{4.4}$$

where $x_i(\cdot)$ and $l_i(\cdot)$ are, respectively, type i's commodity demand and labour supply functions coming from programme (4.1) – (4.2).

4.1 Tax Reforms

With the introduction of commodity taxation, we now define a tax reform as $\mathrm{d}R := \langle \mathrm{d}\tau, \mathrm{d}t_1, \mathrm{d}t_2, \mathrm{d}t_3 \rangle$, which represents small changes in the commodity tax and the piecewise linear income tax schedule. Starting in an initial tight equilibrium, a tax reform is equilibrium-preserving if and only if:

$$\nabla Z \mathrm{d}R \geq 0 \tag{4.5}$$

where ∇Z is now the gradient of equation (4.4):

$$\nabla Z := \langle \sum_{i=1}^{3} n_i p \left(x_i + \tau \frac{\partial x_i}{\partial \tau} \right), n_1 w_1 \left(l_1 + t_1 \frac{\partial l_1}{\partial t_1} \right), n_2 w_2 \left(l_2 + t_2 \frac{\partial l_2}{\partial t_2} \right), n_3 w_3 \left(l_3 + t_3 \frac{\partial l_3}{\partial t_3} \right) \rangle \tag{4.6}$$

where all derivatives are evaluated in the status quo equilibrium.

A tax reform is welfare-improving if and only if:

$$\nabla W \mathrm{d}R > 0 \tag{4.7}$$

where:

$$\nabla W := \langle \sum_{i=1}^{3} \pi_i n_i \frac{\partial V_i}{\partial \tau}, \pi_1 n_1 \frac{\partial V_1}{\partial t_1}, \pi_2 n_2 \frac{\partial V_2}{\partial t_2}, \pi_3 n_3 \frac{\partial V_3}{\partial t_3} \rangle \tag{4.8}$$

is the gradient of the weighted utilitarian social welfare function.

5 Local Optima and Desirable Tax Reforms

We again begin by characterising the tax system at a local optimum. Starting in an initial tight equilibrium, if there does not exist a tax reform $\mathrm{d}R$ such that:

$$\nabla Z \mathrm{d}R \geq 0 \tag{5.1}$$

$$\nabla W \mathrm{d}R > 0 \tag{5.2}$$

then by Motzkin's Theorem there exist real numbers $\theta \geq 0$ and $\alpha > 0$ such that:

$$\theta \nabla Z + \alpha \nabla W = 0^{(4)} \tag{5.3}$$

The system of equations (5.3) characterises what the tax system—comprising linear commodity taxation and piecewise linear income taxation—looks like when it is locally optimal. Let T_c denote total commodity tax payments. Using (5.3) we obtain:

Proposition 3 *If the status quo commodity tax and piecewise linear income tax schedule is locally optimal, we have (i) no Laffer effect, i.e., $\partial\overline{T}_c/\partial\overline{\tau} > 0$ and $\partial\overline{T}_i/\partial\bar{t}_i > 0$ for all i, (ii) progressive income taxation, i.e., $\bar{t}_1 < \bar{t}_2 < \bar{t}_3$, and (iii) positive commodity taxation, i.e., $\overline{\tau} > 0$.*

Part (i) of Proposition 3 is analogous to part (i) of Proposition 1, in that a tax system that is locally optimal cannot be subject to a Laffer effect, now with respect to both the commodity tax and the marginal tax rates. Part (ii) of Proposition 3 is identical to part (ii) of Proposition 1, in that a locally optimal income tax schedule is progressive. Part (iii) of Proposition 3 shows that it is locally optimal to use commodity taxation alongside piecewise linear income taxation. This stands in contrast to the well-known Atkinson and Stiglitz (1976) result that commodity taxation is redundant alongside optimal nonlinear (Mirrlees-style) income taxation, provided the utility function of all individuals is the same and separable in labour. Our quasi-linear utility function is the same for all individuals and is separable in labour, but it is locally optimal to employ positive commodity taxation. It is also worth noting that this result is independent of how the social welfare weights are distributed. The intuition is simple; the government needs to raise a certain amount of revenue, from taxing labour and/or consumption. By taxing both labour and consumption it is broadening the tax base, which is less distortionary than taxing labour or consumption alone.

We now examine reforms of a non-optimal commodity tax and piecewise linear income tax schedule. With the introduction of commodity taxation into the model, $\mathrm{d}t_i \gtreqless 0$ if and only if $\nabla T_i \mathrm{d}R \gtreqless 0$, where $\nabla T_1 := \langle 0, 1, 0, 0\rangle$, $\nabla T_2 := \langle 0, 0, 1, 0\rangle$, and $\nabla T_3 := \langle 0, 0, 0, 1\rangle$. Starting in an initial tight equilibrium, if there does not exist a tax reform $\mathrm{d}R$ such that:

$$\nabla Z \mathrm{d}R \geq 0 \tag{5.4}$$

$$\nabla W \mathrm{d}R > 0 \tag{5.5}$$

$$\nabla T_i \mathrm{d}R \geq 0 \tag{5.6}$$

then all reforms that satisfy (5.4) and (5.5) must violate (5.6), i.e., require a decrease in type i's marginal tax rate.

By Motzkin's Theorem, if there does not exist a tax reform $\mathrm{d}R$ that satisfies equations (5.4) – (5.6), there exist real numbers $\theta \geq 0$, $\alpha > 0$, and $\beta \geq 0$ such that:

$$\theta \nabla Z + \alpha \nabla W + \beta \nabla T_i = 0^{(4)} \tag{5.7}$$

It then follows from the system of equations (5.7) that:

Proposition 4a *Consider an initial tight equilibrium of our extended model in which the tax system is suboptimal. If all equilibrium-preserving and welfare-improving tax reforms require a decrease in type i's marginal tax rate, then (i)* $t_i > \bar{t}_i$, $t_j < \bar{t}_j$, $t_k < \bar{t}_k$ *and* $\tau < \bar{\tau}$, *(ii)* $(1 - t_j/(\gamma - \gamma t_j))/(1 - t_k/(\gamma - \gamma t_k)) = (1 - \bar{t}_j/(\gamma - \gamma \bar{t}_j))/(1 - \bar{t}_k/(\gamma - \gamma \bar{t}_k))$, *and (iii)* $n(1 - \tau/(\sigma - \sigma\tau))/(1 - t_h/(\gamma - \gamma t_h)) = n(1 - \bar{\tau}/(\sigma - \sigma\bar{\tau}))/(1 - \bar{t}_h/(\gamma - \gamma \bar{t}_h))$ *for all* $h \neq i$, *where* $n = n_1 + n_2 + n_3$.

By reversing the inequality in (5.6), we obtain:

Proposition 4b *Consider an initial tight equilibrium of our extended model in which the tax system is suboptimal. If all equilibrium-preserving and welfare-improving tax reforms require an increase in type i's marginal tax rate, then (i)* $t_i < \bar{t}_i$, $t_j > \bar{t}_j$, $t_k > \bar{t}_k$ *and*

$\tau > \overline{\tau}$, *(ii)* $(1 - t_j/(\gamma - \gamma t_j))/(1 - t_k/(\gamma - \gamma t_k)) = (1 - \overline{t}_j/(\gamma - \gamma\overline{t}_j))/(1 - \overline{t}_k/(\gamma - \gamma\overline{t}_k))$, *and* *(iii)* $n(1 - \tau/(\sigma - \sigma\tau))/(1 - t_h/(\gamma - \gamma t_h)) = n(1 - \overline{\tau}/(\sigma - \sigma\overline{\tau}))/(1 - \overline{t}_h/(\gamma - \gamma\overline{t}_h))$ *for all* $h \neq i$, *where* $n = n_1 + n_2 + n_3$.

The intuition underlying Propositions 4a and 4b is similar to that behind Propositions 2a and 2b. If all equilibrium-preserving and welfare-improving reforms require a reduction in type i's marginal tax rate (Proposition 4a), then it is currently higher than its locally optimal level. Accordingly, total tax collections from the other sources—commodity taxation and income taxation of types j and k—must be less than locally optimal. Indeed, income tax payments by types j and k, as well as commodity tax payments, must each be less than locally optimal. This is because the relationships between type j's marginal tax rate, type k's marginal tax rate, and the commodity tax rate must be the same as at a local optimum (parts (ii) and (iii) of Proposition 4a). Otherwise, an equilibrium-preserving and welfare-improving reform could be implemented by moving these taxes toward their optimal ratios, i.e., a reduction in type i's marginal tax rate would not be necessary.

Lastly, we examine the case when all equilibrium-preserving and welfare-improving tax reforms require a decrease in the commodity tax rate. Note that $\mathrm{d}\tau \gtreqless 0$ if and only if $\nabla C \mathrm{d}R \gtreqless 0$, where $\nabla C := \langle 1, 0, 0, 0 \rangle$. Starting in an initial tight equilibrium, if there does not exist a tax reform $\mathrm{d}R$ such that:

$$\nabla Z \mathrm{d}R \geq 0 \tag{5.8}$$

$$\nabla W \mathrm{d}R > 0 \tag{5.9}$$

$$\nabla C \mathrm{d}R \geq 0 \tag{5.10}$$

then all reforms that satisfy (5.8) and (5.9) must violate (5.10), i.e., require a decrease in the commodity tax rate.

By Motzkin's Theorem, if there does not exist a tax reform $\mathrm{d}R$ that satisfies equations

(5.8) − (5.10), there exist real numbers $\theta \geq 0$, $\alpha > 0$, and $\beta \geq 0$ such that:

$$\theta \nabla Z + \alpha \nabla W + \beta \nabla C = 0^{(4)} \tag{5.11}$$

Using (5.11) we obtain the following result:

Proposition 5a *Consider an initial tight equilibrium of our extended model in which the tax system is suboptimal. If all equilibrium-preserving and welfare-improving tax reforms require a decrease in the commodity tax rate, then (i)* $\tau > \overline{\tau}$ *and* $t_i < \overline{t}_i$ *for all i, and (ii)* $(1 - t_i/(\gamma - \gamma t_i))/(1 - t_j/(\gamma - \gamma t_j)) = (1 - \overline{t}_i/(\gamma - \gamma \overline{t}_i))/(1 - \overline{t}_j/(\gamma - \gamma \overline{t}_j))$ *for all i, j.*

By reversing the inequality in (5.10), we obtain:

Proposition 5b *Consider an initial tight equilibrium of our extended model in which the tax system is suboptimal. If all equilibrium-preserving and welfare-improving tax reforms require an increase in the commodity tax rate, then (i)* $\tau < \overline{\tau}$ *and* $t_i > \overline{t}_i$ *for all i, and (ii)* $(1 - t_i/(\gamma - \gamma t_i))/(1 - t_j/(\gamma - \gamma t_j)) = (1 - \overline{t}_i/(\gamma - \gamma \overline{t}_i))/(1 - \overline{t}_j/(\gamma - \gamma \overline{t}_j))$ *for all i, j.*

The intuition for Propositions 5a and 5b is similar to that for earlier propositions. If all equilibrium-preserving and welfare-improving reforms require a decrease in the commodity tax rate (Proposition 5a), then the commodity tax rate must be higher than locally optimal, the marginal tax rates of all types must be less than locally optimal, and the relationships between the marginal tax rates in the current suboptimal tax system must be the same as at a local optimum.

6 Comparative Statics

In this section, we use numerical simulations to explore the comparative statics of a locally optimal tax system, for the general case of linear commodity taxation and piecewise linear income taxation. To this end, we first calibrate the model using empirically plausible parameter values. These baseline parameter values are presented in Table 1.

The OECD (2012) reports that on average across OECD countries, 26% of all adults have below upper-secondary education, 44% have upper-secondary education, and the remaining 30% have tertiary education. We therefore normalise the size of the population to unity, and assume that 26% of individuals are low-skill, 44% are middle-skill, and 30% are high-skill; i.e., we set $n_1 = 0.26$, $n_2 = 0.44$, and $n_3 = 0.30$. In the utility function (4.1), σ can be interpreted as the individuals' coefficient of relative risk aversion. Chetty (2006) concludes that a reasonable estimate of the coefficient of relative risk aversion is one. We therefore set $\sigma = 1$, so that the utility function becomes logarithmic in x_i.[13] Likewise, $1/\gamma$ can be interpreted as the individuals' labour supply elasticity. We set $\gamma = 2$ as this implies a labour supply elasticity of 0.5, which is consistent with empirical estimates (see, e.g., Chetty et al. (2011)). Goldin and Katz (2007) estimate that the college wage premium is approximately 60%, and that the high-school wage premium is approximately 30%. We therefore normalise the low-skill type's wage to unity ($w_1 = 1$), and set the middle-skill type's wage at $w_2 = 1.30$ and the high-skill type's wage at $w_3 = 2.08$. We choose the social welfare weights to be modestly declining in skills ($\pi_1 = 0.36$, $\pi_2 = 0.34$, and $\pi_3 = 0.30$), so that the social welfare function is approximately utilitarian. Finally, we set $G = 0.60$, because this implies that government spending as a share of national income (G/Y where $Y = \sum n_i y_i$) is approximately 40%, which is consistent with empirical evidence. Given these baseline parameter values, a locally optimal tax system has a commodity tax rate of $\bar{\tau} = 0.21$, and marginal income tax rates of $\bar{t}_1 = 0.18$, $\bar{t}_2 = 0.24$, and $\bar{t}_3 = 0.34$.

Figures $3 - 5$ show how the locally optimal tax rates change in response to changes in key model parameters. These are obtained by varying the parameter in question around its baseline value, while holding all other parameters at their baseline levels. Figure 3 shows how the tax rates change in response to changes in the welfare weights. As intuition would suggest, t_i is decreasing in π_i, while the marginal tax rates of the other two types are increasing to offset the loss of tax revenues collected from type i individuals. Interestingly, the response of τ to changes in π_i is non-monotonic. This is because there are two opposing forces at work. On the one hand, an increase in τ raises tax revenues, which allows a further reduction in t_i. But on the other hand, an increase in τ hurts all individuals, including type i individuals. Figure 4 shows the responses to

[13] Setting $\sigma = 1$ also has the convenient property that it means we do not have to specify a value for the producer price (p) of the consumption good, because p drops out of the calculations when $\sigma = 1$.

changes in the population parameters. All tax rates are declining in each population parameter, because the larger population means the government can tax each individual less and still meet its revenue requirement. Likewise, all tax rates are declining in each type's wage rate (Figure 5), because an increase in a wage rate implies, *ceteris paribus*, an increase in economy-wide income. Accordingly, the government can reduce tax rates and still meet its revenue requirement.

7 Closing Remarks

We have suggested that the tax reform approach to analyse policy-making comes much closer than the optimal tax approach in capturing the actual behaviour of governments and the constraints that they face. We have also suggested that the relative scarcity of the tax reform literature may follow from the fact that tax reform analyses tend to produce results that are quite technical in nature and difficult to interpret. In this paper, we have analysed tax reforms using a model and methodology that lead to a relatively clear description of locally optimal tax schedules, and of when specific tax reforms are required to improve suboptimal tax schedules. Moreover, the types of tax reform questions we have addressed correspond quite closely to those actually faced by policy-makers, which typically revolve around whether a specific piecemeal reform—such as reducing the top marginal tax rate—should be implemented. The price paid for the clarity achieved in this paper is that we have used a simple model, and we have assumed that preferences are quasi-linear. That said, our model of linear commodity taxation and piecewise linear income taxation resembles the tax systems used in practice, and the assumption that preferences are quasi-linear is not uncommon. Nevertheless, in future research it may be worth exploring the extent to which the model can be generalised, while still allowing a tax reform analysis that leads to clear conclusions.

8 Appendix

A.1 Motzkin's Theorem of the Alternative

Let A, C, and D be $c_1 \times m$, $c_2 \times m$, and $c_3 \times m$ matrices, respectively, where A is non-vacuous (not all zeros). Then *either*:

$$Az \gg 0^{(c_1)} \qquad Cz \geq 0^{(c_2)} \qquad Dz = 0^{(c_3)}$$

has a solution $z \in \mathbb{R}^m$, *or*:

$$b_1 A + b_2 C + b_3 D = 0^{(m)}$$

has a solution $b_1 > 0^{(c_1)}$, $b_2 \geq 0^{(c_2)}$, and b_3 sign unrestricted, *but never both.* A proof of Motzkin's Theorem can be found in Mangasarian (1969).

A.2 Proof of Proposition 1

If there exist real numbers $\theta \geq 0$ and $\alpha > 0$ such that system (3.3) is satisfied, then there must also exist real numbers under the same sign restrictions that satisfy (3.3), but with $\alpha = 1$. Thus, without loss of generality, we set $\alpha = 1$. Expanding (3.3) now yields:

$$\bar{\theta} n_1 w_1 \left[\bar{l}_1 + \bar{t}_1 \frac{\partial \bar{l}_1}{\partial t_1} \right] + \pi_1 n_1 \frac{\partial \overline{V}_1}{\partial t_1} = 0 \tag{A.1}$$

$$\bar{\theta} n_2 w_2 \left[\bar{l}_2 + \bar{t}_2 \frac{\partial \bar{l}_2}{\partial t_2} \right] + \pi_2 n_2 \frac{\partial \overline{V}_2}{\partial t_2} = 0 \tag{A.2}$$

$$\bar{\theta} n_3 w_3 \left[\bar{l}_3 + \bar{t}_3 \frac{\partial \bar{l}_3}{\partial t_3} \right] + \pi_3 n_3 \frac{\partial \overline{V}_3}{\partial t_3} = 0 \tag{A.3}$$

As the second terms in (A.1), (A.2), and (A.3) are negative, the first terms must be positive. This implies that $\bar{\theta} > 0$ and $\bar{l}_i + \bar{t}_i \frac{\partial \bar{l}_i}{\partial t_i} > 0$. Therefore, because $T_i = n_i t_i w_i l_i(\cdot)$ and $\partial T_i / \partial t_i = n_i w_i \left(l_i + t_i \frac{\partial l_i}{\partial t_i} \right)$, a locally optimal income tax schedule cannot be subject to the Laffer effect.

The Lagrangian corresponding to programme (2.1) – (2.2) is:

$$\mathcal{L} = x_i - \frac{l_i^{1+\gamma}}{1+\gamma} + \lambda \left[a_i + (1 - t_i) w_i l_i - x_i \right] \tag{A.4}$$

where $\lambda > 0$ is the Lagrange multiplier. The relevant first-order conditions are:

$$1 - \lambda = 0 \tag{A.5}$$

$$-l_i^\gamma + \lambda(1 - t_i)w_i = 0 \tag{A.6}$$

Note also that by the Envelope Theorem:

$$\frac{\partial V_i}{\partial t_i} = \frac{\partial \mathcal{L}}{\partial t_i} = -\lambda w_i l_i = -w_i l_i \tag{A.7}$$

Using (A.5), (A.6), and (A.7), equations (A.1) – (A.3) can be simplified to:

$$\bar{\theta}\left[1 - \frac{\bar{t}_1}{\gamma(1-\bar{t}_1)}\right] - \pi_1 = 0 \tag{A.8}$$

$$\bar{\theta}\left[1 - \frac{\bar{t}_2}{\gamma(1-\bar{t}_2)}\right] - \pi_2 = 0 \tag{A.9}$$

$$\bar{\theta}\left[1 - \frac{\bar{t}_3}{\gamma(1-\bar{t}_3)}\right] - \pi_3 = 0 \tag{A.10}$$

Dividing (A.9) by (A.8) yields:

$$\frac{1 - \frac{\bar{t}_2}{\gamma(1-\bar{t}_2)}}{1 - \frac{\bar{t}_1}{\gamma(1-\bar{t}_1)}} = \frac{\pi_2}{\pi_1} \in (0, 1) \tag{A.11}$$

which establishes that $\bar{t}_2 > \bar{t}_1$. Analogous manipulations of (A.9) and (A.10) establish that $\bar{t}_3 > \bar{t}_2$. ■

A.3 Proof of Proposition 2a

We prove Proposition 2a for the case when all equilibrium-preserving and welfare-improving tax reforms require a decrease in the high-skill type's marginal tax rate, as the proofs for the other two types are analogous. If there exist real numbers $\theta \geq 0$, $\alpha > 0$, and $\beta \geq 0$ such that system (3.7) is satisfied, then there must also exist real numbers under the same sign restrictions that satisfy (3.7), but with $\alpha = 1$. Thus, without loss of generality, we set $\alpha = 1$. Also, if $\beta = 0$ the status quo tax schedule is already locally optimal. Therefore, we consider the case in which $\beta > 0$. Expanding

(3.7) now yields:

$$\theta n_1 w_1 \left[l_1 + t_1 \frac{\partial l_1}{\partial t_1} \right] + \pi_1 n_1 \frac{\partial V_1}{\partial t_1} = 0 \tag{A.12}$$

$$\theta n_2 w_2 \left[l_2 + t_2 \frac{\partial l_2}{\partial t_2} \right] + \pi_2 n_2 \frac{\partial V_2}{\partial t_2} = 0 \tag{A.13}$$

$$\theta n_3 w_3 \left[l_3 + t_3 \frac{\partial l_3}{\partial t_3} \right] + \pi_3 n_3 \frac{\partial V_3}{\partial t_3} + \beta = 0 \tag{A.14}$$

Using (A.5), (A.6), and (A.7), equation (A.14) can be simplified to:

$$\theta \left[1 - \frac{t_3}{\gamma(1-t_3)} \right] - \pi_3 + \frac{\beta}{n_3 w_3 l_3} = 0 \tag{A.15}$$

By applying the Implicit Function Theorem to (A.15), we obtain:

$$\frac{\partial \theta}{\partial \beta} = \frac{\frac{-1}{n_3 w_3 l_3}}{1 - \frac{t_3}{\gamma(1-t_3)}} < 0 \tag{A.16}$$

Equation (A.16) implies that $\theta < \overline{\theta}$, i.e., the value of θ under existing suboptimal taxation is lower than that when the tax schedule is locally optimal. It now follows from (A.1) and (A.12), after algebraic simplification, that:

$$\frac{1 - \frac{t_1}{\gamma(1-t_1)}}{1 - \frac{\overline{t}_1}{\gamma(1-\overline{t}_1)}} = \frac{\overline{\theta}}{\theta} > 1 \tag{A.17}$$

which implies that $t_1 < \overline{t}_1$. Analogous manipulations of (A.2) and (A.13) yield $t_2 < \overline{t}_2$, and $t_1 < \overline{t}_1$ and $t_2 < \overline{t}_2$ together imply that $t_3 > \overline{t}_3$. Finally, (A.12) and (A.13) yield:

$$\frac{1 - \frac{t_2}{\gamma(1-t_2)}}{1 - \frac{t_1}{\gamma(1-t_1)}} = \frac{\pi_2}{\pi_1} \tag{A.18}$$

which is the same ratio as when the tax schedule is locally optimal (see (A.11)). ■

A.4 Proof of Proposition 2b

The proof of Proposition 2b is the mirror image of that for Proposition 2a, and is therefore omitted. ■

A.5 Proof of Proposition 3

If there exist real numbers $\theta \geq 0$ and $\alpha > 0$ such that system (5.3) is satisfied, then there must also exist real numbers under the same sign restrictions that satisfy (5.3), but with $\alpha = 1$. Thus, without loss of generality, we set $\alpha = 1$. Expanding (5.3) now yields:

$$\overline{\theta}\sum_{i=1}^{3} n_i p \left[\overline{x}_i + \overline{\tau}\frac{\partial \overline{x}_i}{\partial \tau}\right] + \sum_{i=1}^{3} \pi_i n_i \frac{\partial \overline{V}_i}{\partial \tau} = 0 \tag{A.19}$$

$$\overline{\theta} n_1 w_1 \left[\overline{l}_1 + \overline{t}_1 \frac{\partial \overline{l}_1}{\partial t_1}\right] + \pi_1 n_1 \frac{\partial \overline{V}_1}{\partial t_1} = 0 \tag{A.20}$$

$$\overline{\theta} n_2 w_2 \left[\overline{l}_2 + \overline{t}_2 \frac{\partial \overline{l}_2}{\partial t_2}\right] + \pi_2 n_2 \frac{\partial \overline{V}_2}{\partial t_2} = 0 \tag{A.21}$$

$$\overline{\theta} n_3 w_3 \left[\overline{l}_3 + \overline{t}_3 \frac{\partial \overline{l}_3}{\partial t_3}\right] + \pi_3 n_3 \frac{\partial \overline{V}_3}{\partial t_3} = 0 \tag{A.22}$$

Notice that (A.20) – (A.22) are the same as (A.1) – (A.3). Therefore, we again have $\overline{t}_1 < \overline{t}_2 < \overline{t}_3$ and no Laffer effect *vis-a-vis* marginal tax rates. Regarding commodity taxation, the second term in (A.19) is negative, implying that the first term is positive. Therefore, since $T_c = \sum n_i \tau p x_i(\cdot)$ and $\partial T_c/\partial\tau = \sum n_i p \left(x_i + \tau \frac{\partial x_i}{\partial \tau}\right)$, there cannot be a commodity tax Laffer effect at a local optimum.

The Lagrangian corresponding to programme (4.1) – (4.2) is:

$$\mathcal{L} = z_i + \frac{x_i^{1-\sigma}}{1-\sigma} - \frac{l_i^{1+\gamma}}{1+\gamma} + \lambda\left[a_i + (1-t_i)w_i l_i - z_i - (1+\tau)p x_i\right] \tag{A.23}$$

where $\lambda > 0$ is the Lagrange multiplier. The relevant first-order conditions are:

$$1 - \lambda = 0 \tag{A.24}$$

$$x_i^{-\sigma} - \lambda(1+\tau)p = 0 \tag{A.25}$$

$$-l_i^{\gamma} + \lambda(1-t_i)w_i = 0 \tag{A.26}$$

Note also that by the Envelope Theorem:

$$\frac{\partial V_i}{\partial \tau} = \frac{\partial \mathcal{L}}{\partial \tau} = -\lambda p x_i = -p x_i \quad \text{and} \quad \frac{\partial V_i}{\partial t_i} = \frac{\partial \mathcal{L}}{\partial t_i} = -\lambda w_i l_i = -w_i l_i \tag{A.27}$$

Equation (A.19) can now be simplified to:

$$\overline{\theta}(n_1 + n_2 + n_3)\left[1 - \frac{\overline{\tau}}{\sigma(1+\overline{\tau})}\right] - (\pi_1 n_1 + \pi_2 n_2 + \pi_3 n_3) = 0 \tag{A.28}$$

By adding (A.20) – (A.22) after undertaking appropriate algebraic simplifications, we obtain:

$$\overline{\theta}\sum_{i=1}^{3} n_i \left[1 - \frac{\overline{t}_i}{\gamma(1-\overline{t}_i)}\right] - (\pi_1 n_1 + \pi_2 n_2 + \pi_3 n_3) = 0 \tag{A.29}$$

Equations (A.28) and (A.29) imply that:

$$\frac{\overline{\tau}}{(1+\overline{\tau})} = \sum_{i=1}^{3} \frac{n_i \sigma \overline{t}_i}{(n_1 + n_2 + n_3)\gamma(1-\overline{t}_i)} \tag{A.30}$$

which establishes that $\overline{\tau} > 0$. ■

A.6 Proof of Proposition 4a

We prove Proposition 4a for the case when all equilibrium-preserving and welfare-improving tax reforms require a decrease in the high-skill type's marginal tax rate, as the proofs for the other two types are analogous. If there exist real numbers $\theta \geq 0$, $\alpha > 0$, and $\beta \geq 0$ such that system (5.7) is satisfied, then there must also exist real numbers under the same sign restrictions that satisfy (5.7), but with $\alpha = 1$. Thus, without loss of generality, we set $\alpha = 1$. Also, if $\beta = 0$ the status quo tax system is already locally optimal. Therefore, we consider the case in which $\beta > 0$. Expanding (5.7) now yields:

$$\theta\sum_{i=1}^{3} n_i p\left[x_i + \tau\frac{\partial x_i}{\partial \tau}\right] + \sum_{i=1}^{3}\pi_i n_i \frac{\partial V_i}{\partial \tau} = 0 \tag{A.31}$$

$$\theta n_1 w_1\left[l_1 + t_1\frac{\partial l_1}{\partial t_1}\right] + \pi_1 n_1\frac{\partial V_1}{\partial t_1} = 0 \tag{A.32}$$

$$\theta n_2 w_2\left[l_2 + t_2\frac{\partial l_2}{\partial t_2}\right] + \pi_2 n_2\frac{\partial V_2}{\partial t_2} = 0 \tag{A.33}$$

$$\theta n_3 w_3\left[l_3 + t_3\frac{\partial l_3}{\partial t_3}\right] + \pi_3 n_3\frac{\partial V_3}{\partial t_3} + \beta = 0 \tag{A.34}$$

As in the proof of Proposition 2a, application of the Implicit Function Theorem to equation (A.34) yields $\partial\theta/\partial\beta < 0$, implying that $\theta < \overline{\theta}$ and therefore $t_1 < \bar{t}_1$ and $t_2 < \bar{t}_2$. Using (A.24), (A.25), and (A.27), equation (A.31) can be simplified to:

$$\theta(n_1 + n_2 + n_3)\left[1 - \frac{\tau}{\sigma(1+\tau)}\right] - (\pi_1 n_1 + \pi_2 n_2 + \pi_3 n_3) = 0 \tag{A.35}$$

It then follows from (A.28) and (A.35) that $\tau < \overline{\tau}$, and $t_1 < \bar{t}_1$, $t_2 < \bar{t}_2$, and $\tau < \overline{\tau}$ together imply that $t_3 > \bar{t}_3$.

As in the proof of Proposition 2a, equations (A.32) and (A.33) imply that:

$$\frac{1 - \frac{t_2}{\gamma(1-t_2)}}{1 - \frac{t_1}{\gamma(1-t_1)}} = \frac{\pi_2}{\pi_1} \tag{A.36}$$

which is the same ratio as when the tax system is locally optimal. Likewise, by combining (A.32) with (A.35), and (A.33) with (A.35), we obtain:

$$\frac{\sum n_i \left[1 - \frac{\tau}{\sigma(1+\tau)}\right]}{1 - \frac{t_1}{\gamma(1-t_1)}} = \frac{\sum \pi_i n_i}{\pi_1} \quad \text{and} \quad \frac{\sum n_i \left[1 - \frac{\tau}{\sigma(1+\tau)}\right]}{1 - \frac{t_2}{\gamma(1-t_2)}} = \frac{\sum \pi_i n_i}{\pi_2} \tag{A.37}$$

which are the same relationships that prevail when the tax system is locally optimal. ■

A.7 Proof of Proposition 4b

The proof of Proposition 4b is the mirror image of that for Proposition 4a, and is therefore omitted. ■

A.8 Proof of Proposition 5a

If there exist real numbers $\theta \geq 0$, $\alpha > 0$, and $\beta \geq 0$ such that system (5.11) is satisfied, then there must also exist real numbers under the same sign restrictions that satisfy (5.11), but with $\alpha = 1$. Thus, without loss of generality, we set $\alpha = 1$. Also, if $\beta = 0$ the status quo tax system is already locally optimal. Therefore, we consider the case in which $\beta > 0$. Expanding (5.11) now yields:

$$\theta \sum_{i=1}^{3} n_i p\left[x_i + \tau\frac{\partial x_i}{\partial \tau}\right] + \sum_{i=1}^{3} \pi_i n_i \frac{\partial V_i}{\partial \tau} + \beta = 0 \tag{A.38}$$

$$\theta n_1 w_1 \left[l_1 + t_1 \frac{\partial l_1}{\partial t_1}\right] + \pi_1 n_1 \frac{\partial V_1}{\partial t_1} = 0 \tag{A.39}$$

$$\theta n_2 w_2 \left[l_2 + t_2 \frac{\partial l_2}{\partial t_2} \right] + \pi_2 n_2 \frac{\partial V_2}{\partial t_2} = 0 \tag{A.40}$$

$$\theta n_3 w_3 \left[l_3 + t_3 \frac{\partial l_3}{\partial t_3} \right] + \pi_3 n_3 \frac{\partial V_3}{\partial t_3} = 0 \tag{A.41}$$

Using (A.24), (A.25), and (A.27), equation (A.38) can be manipulated to yield:

$$\theta(n_1 + n_2 + n_3) \left[1 - \frac{\tau}{\sigma(1+\tau)} \right] - (\pi_1 n_1 + \pi_2 n_2 + \pi_3 n_3) + \frac{\beta}{p x_i} = 0 \tag{A.42}$$

By applying the Implicit Function Theorem to (A.42), we obtain:

$$\frac{\partial \theta}{\partial \beta} = \frac{\frac{-1}{p x_i}}{(n_1 + n_2 + n_3) \left[1 - \frac{\tau}{\sigma(1+\tau)} \right]} < 0 \tag{A.43}$$

Equation (A.43) implies that $\theta < \overline{\theta}$. It then follows from (A.20) and (A.39), after undertaking some algebraic simplifications, that $t_1 < \overline{t}_1$. Likewise, from (A.21) and (A.40) we obtain $t_2 < \overline{t}_2$, and from (A.22) and (A.41) we obtain $t_3 < \overline{t}_3$. These inequalities on marginal tax rates together imply that $\tau > \overline{\tau}$.

Finally, equations (A.39), (A.40), and (A.41) can be manipulated to yield:

$$\frac{1 - \frac{t_2}{\gamma(1-t_2)}}{1 - \frac{t_1}{\gamma(1-t_1)}} = \frac{\pi_2}{\pi_1}, \quad \frac{1 - \frac{t_3}{\gamma(1-t_3)}}{1 - \frac{t_1}{\gamma(1-t_1)}} = \frac{\pi_3}{\pi_1} \quad \text{and} \quad \frac{1 - \frac{t_3}{\gamma(1-t_3)}}{1 - \frac{t_2}{\gamma(1-t_2)}} = \frac{\pi_3}{\pi_2} \tag{A.44}$$

which are the same relationships that hold when the tax system is locally optimal. ■

A.9 Proof of Proposition 5b

The proof of Proposition 5b is the mirror image of that for Proposition 5a, and is therefore omitted. ■

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FIGURE 1

Convex (progressive) piecewise linear income taxation

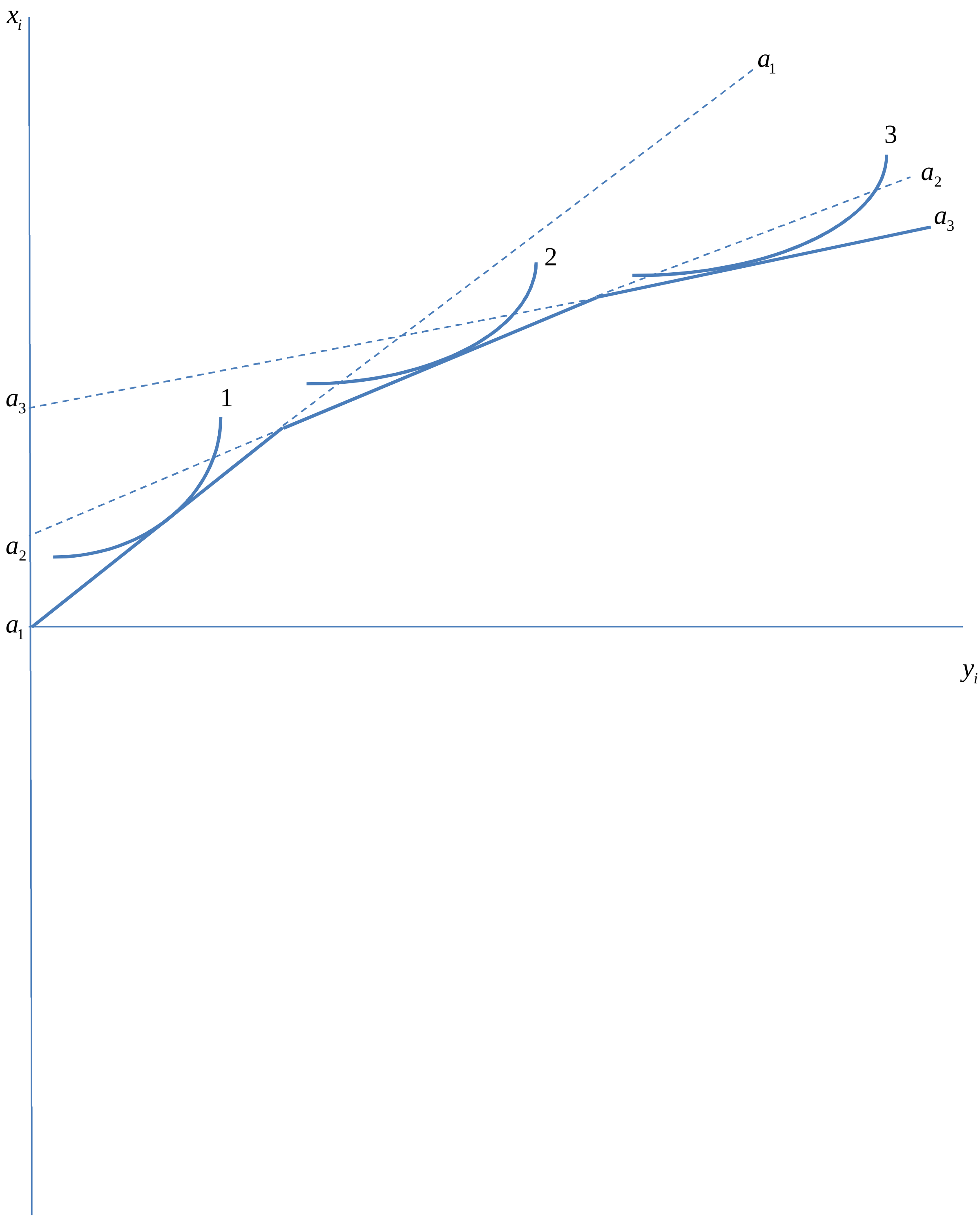

FIGURE 2

Non-convex (regressive) piecewise linear income taxation

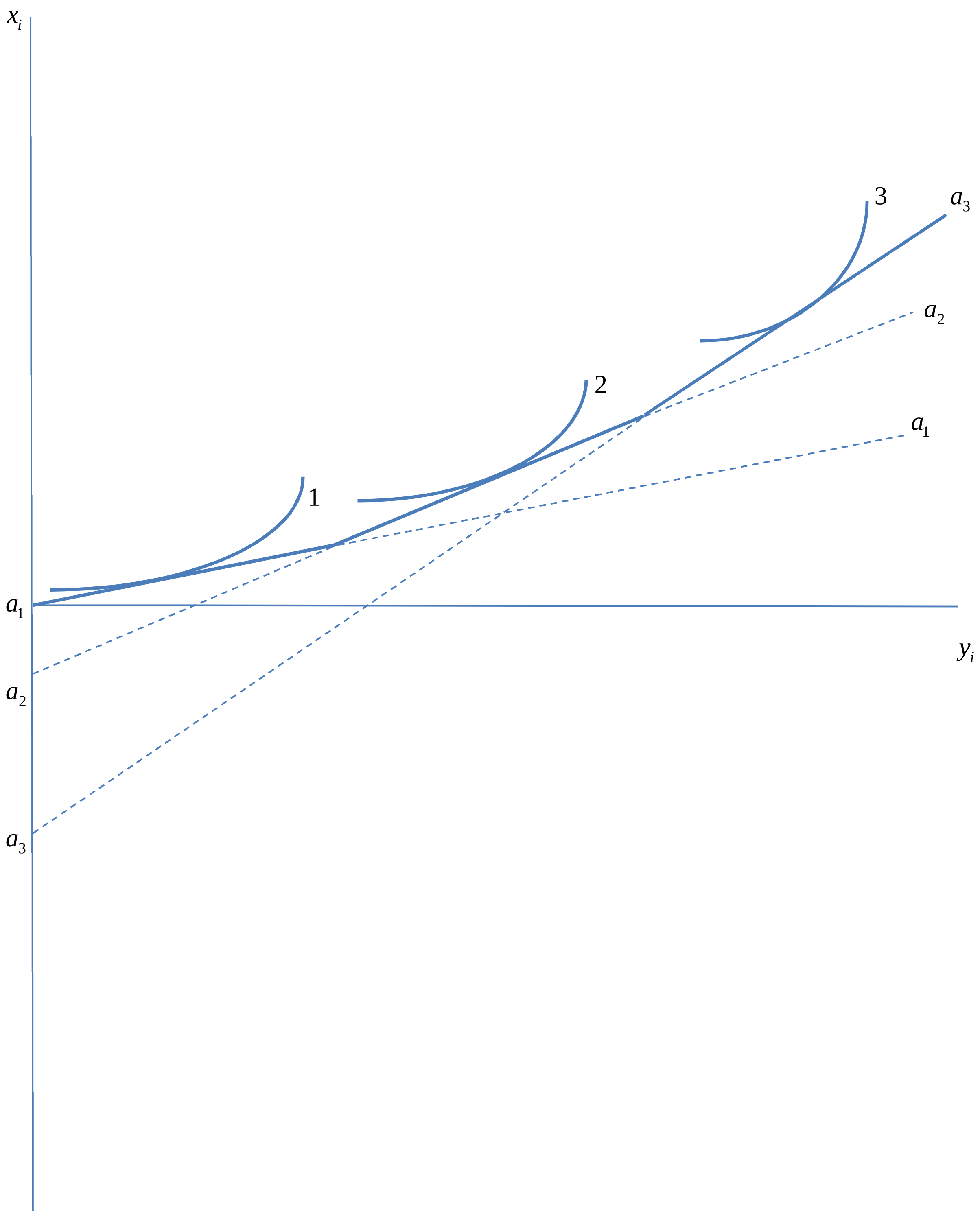

TABLE 1

Baseline Parameter Values and Locally Optimal Tax Rates

σ	1.00	π_1	0.36	n_1	0.26	w_1	1.00
γ	2.00	π_2	0.34	n_2	0.44	w_2	1.30
G	0.60	π_3	0.30	n_3	0.30	w_3	2.08
$\bar{\tau}$	0.21	$\bar{t}_1$	0.18	$\bar{t}_2$	0.24	$\bar{t}_3$	0.34
G/Y	0.39						

FIGURE 3

Comparative statics with respect to the welfare weights

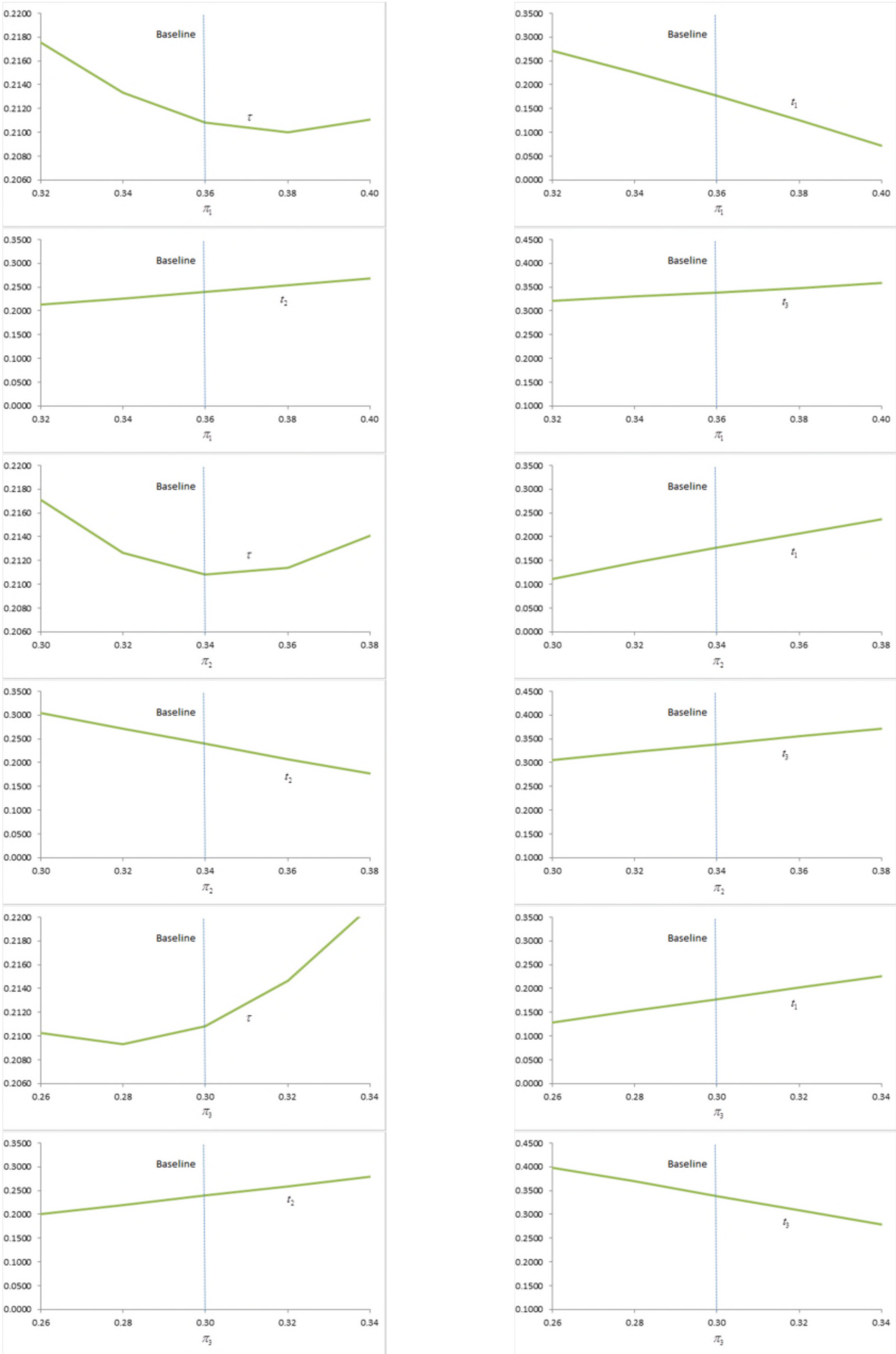

FIGURE 4

Comparative statics with respect to population size

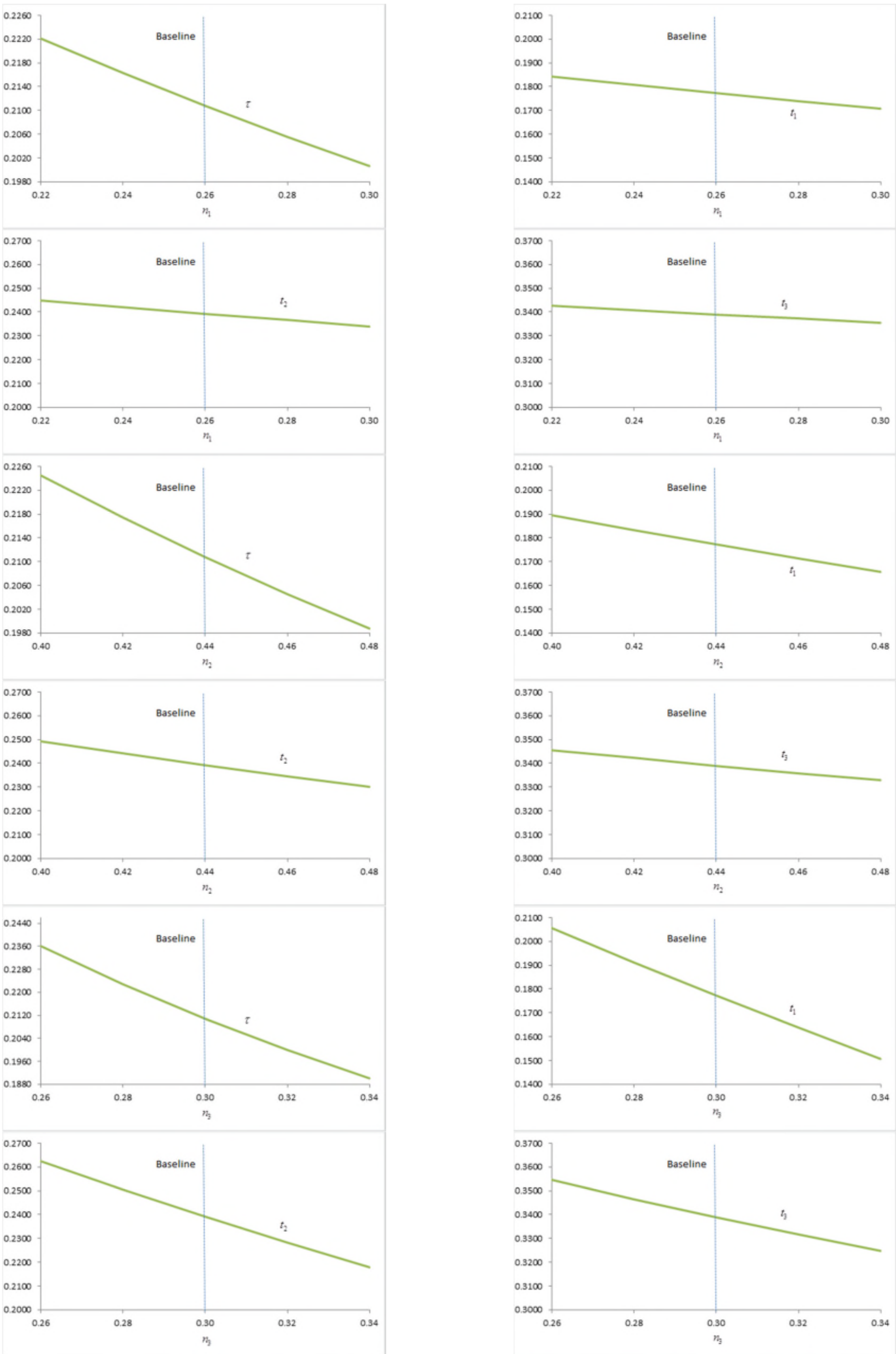

FIGURE 5

Comparative statics with respect to wage rates